General Disclaimer

One or more of the Following Statements may affect this Document

- This document has been reproduced from the best copy furnished by the organizational source. It is being released in the interest of making available as much information as possible.
- This document may contain data, which exceeds the sheet parameters. It was furnished in this condition by the organizational source and is the best copy available.
- This document may contain tone-on-tone or color graphs, charts and/or pictures, which have been reproduced in black and white.
- This document is paginated as submitted by the original source.
- Portions of this document are not fully legible due to the historical nature of some of the material. However, it is the best reproduction available from the original submission.

OPERATIONS RESEARCH GROUP

Columbia University

School of Engineering and Applied Science

NEW YORK, N. Y. 10027

Operations Research Group
Columbia University
New York, New York

OPTIMAL CONTINUOUS REVIEW POLICIES FOR TWO PRODUCT INVENTORY SYSTEMS WITH JOINT SETUP COSTS

by

Edward Ignall

Technical Report Number 35
December 15 1966

This research was supported by the
Army, Navy, Air Force and N.A.S.A.
under a contract administered by
the Office of Naval Research

Contract Number Nonr 266(55)
Task Number NR 042-099

OPTIMAL CONTINUOUS REVIEW POLICIES FOR TWO PRODUCT INVENTORY SYSTEMS WITH JOINT SETUP COSTS

Edward Ignall
Columbia University*

Introduction

In this paper we are concerned with finding optimal stationary ordering policies for a two product inventory system in which the inventory position is under continuous review. The times at which demands for the products, A and B, occur are assumed to be determined by two independent Poisson processes with intensities λ_A and λ_B respectively. At the time of each demand, immediate delivery of one unit of one of the products is requested.[1] We assume that a holding cost of h_A per unit time is charged for each unit of product A in inventory, with a similar

---

[1] That is, the time between successive demands for product A is an exponential random variable with mean $1/\lambda_A$, and similarly for B.

* This research was supported in part by the Office of Naval Research under contract Nonr 266(55). Reproduction in whole or in part is permitted for any purpose of the United States Government.

rate h_B for product B inventory. We assume that when an order is placed, a setup cost is incurred, and that this cost is K if only one product is ordered and mK ($1 \leq m \leq 2$) if both products are ordered. To simplify the problem and highlight the influence of the possible saving in setup cost on the form of the optimal policy, we assume that delivery of the order is instantaneous. In this spirit we also assume that runout is not permitted, so that an order must be placed whenever the inventory of either product drops to zero. Any policy that minimizes long run average costs will be called optimal. With this criterion, since all demands are met, it is clear that the unit purchase price is irrelevant for the ordering decision.

Our interest in this problem stems from the "random joint order policy" proposed by Balintfy [1] and from a two product model treated by Silver [4]. For a 2 product inventory system, a random joint policy requires specification of three points for each product: a reorder point, an "order-to" point, and a "can-order" point which lies between the other two. Whenever the inventory of any product drops to its reorder point, all products with inventory less than or equal to their can-order points are ordered, and the inventory of each product that is ordered is brought up to its order-to point.[2]

---

[2] Balintfy proposed this policy for an n product system. The name random joint was chosen because the number of products requested on a single order is not fixed but is a random variable. The reorder point and order to point are the s and S of periodic review.

While the addition of can-order points makes this type of policy more complicated than policies which treat each product independently, a random joint policy is relatively easy to implement, and it does attempt to take advantage of the setup cost saving that results from joint ordering. It was our hope that random joint policies would in fact be optimal for many classes of problems (non-linear costs, delivery lags, etc.). Unfortunately, and this is our main result, they are not always optimal even for our simple two product problem.

Silver [4] analyzed a two product system for two types of policies when the products were identical: $\lambda_A = \lambda_B$, $h_A = h_B$. The first policy was treating the products independently and corresponds to setting can-order points equal to reorder points. The second was always ordering to a particular two dimensional point (when an order was placed) and corresponds to setting the can-order points equal to the order-to points. Restricting himself to these types of policies, Silver found the region in the $\lambda K/h$ vs. $(m-1)\lambda K/h$ plane where each policy (a policy being a specification of a type and either an order quantity or an order-to point) was optimal. Our starting point was to allow other types of policies, including random joint ones, and attempt to divide the plane in the same way. Our results are pictured in Figures 1, 2, and 3. After interpreting them, we will describe how they were obtained.

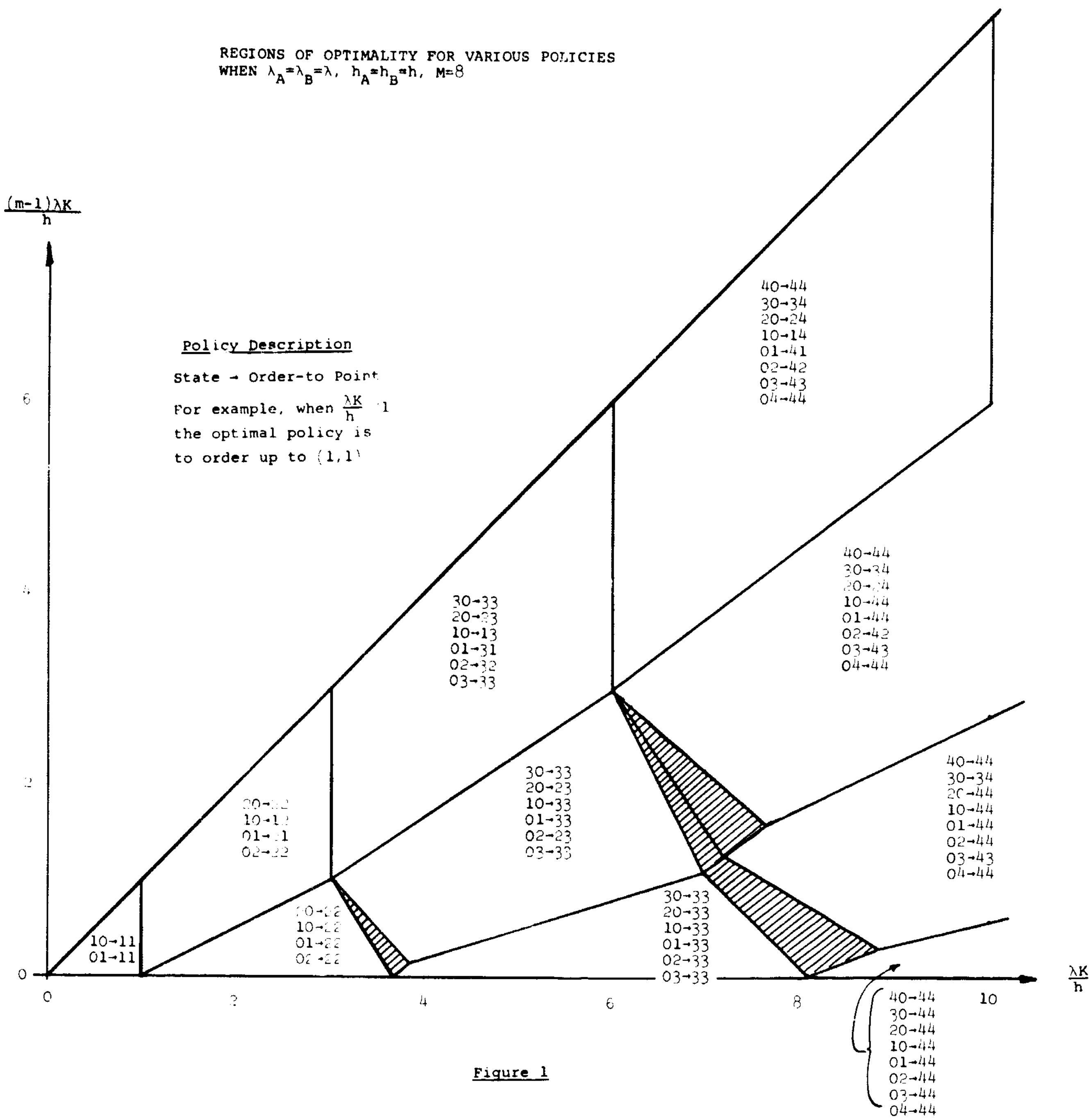
REGIONS OF OPTIMALITY FOR VARIOUS POLICIES
WHEN $\lambda_A = \lambda_B = \lambda$, $h_A = h_B = h$, M=8
$\frac{(m-1)\lambda K}{h}$
Policy Description
State → Order-to Point
For example, when $\frac{\lambda K}{h}$ 1
the optimal policy is
to order up to (1,1)
40→44
30→34
20→24
10→14
01→41
02→42
03→43
04→44
30→33
10→13
01→31
02→32
03→33
30→33
20→23
10→33
01→33
02→23
03→33
10→11
01→11
30→33
20→33
10→33
01→33
02→33
03→33
40→44
30→34
10→44
01→44
02→44
03→43
04→44
40→44
30→44
20→44
10→44
01→44
02→44
03→44
04→44
$\frac{\lambda K}{h}$
0
2
4
6
8
10

Figure 1

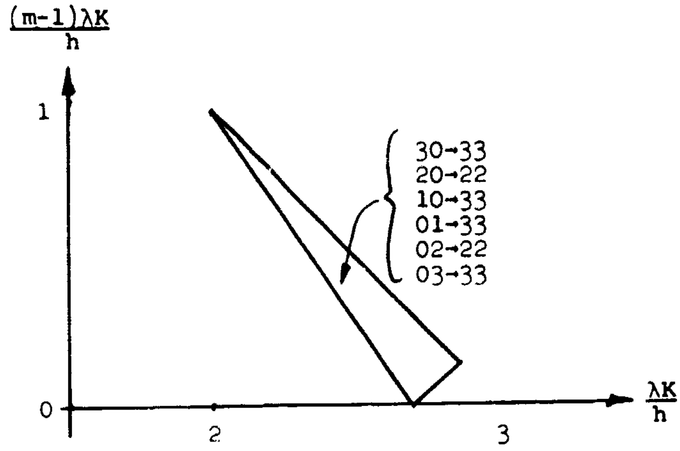

Figure 2

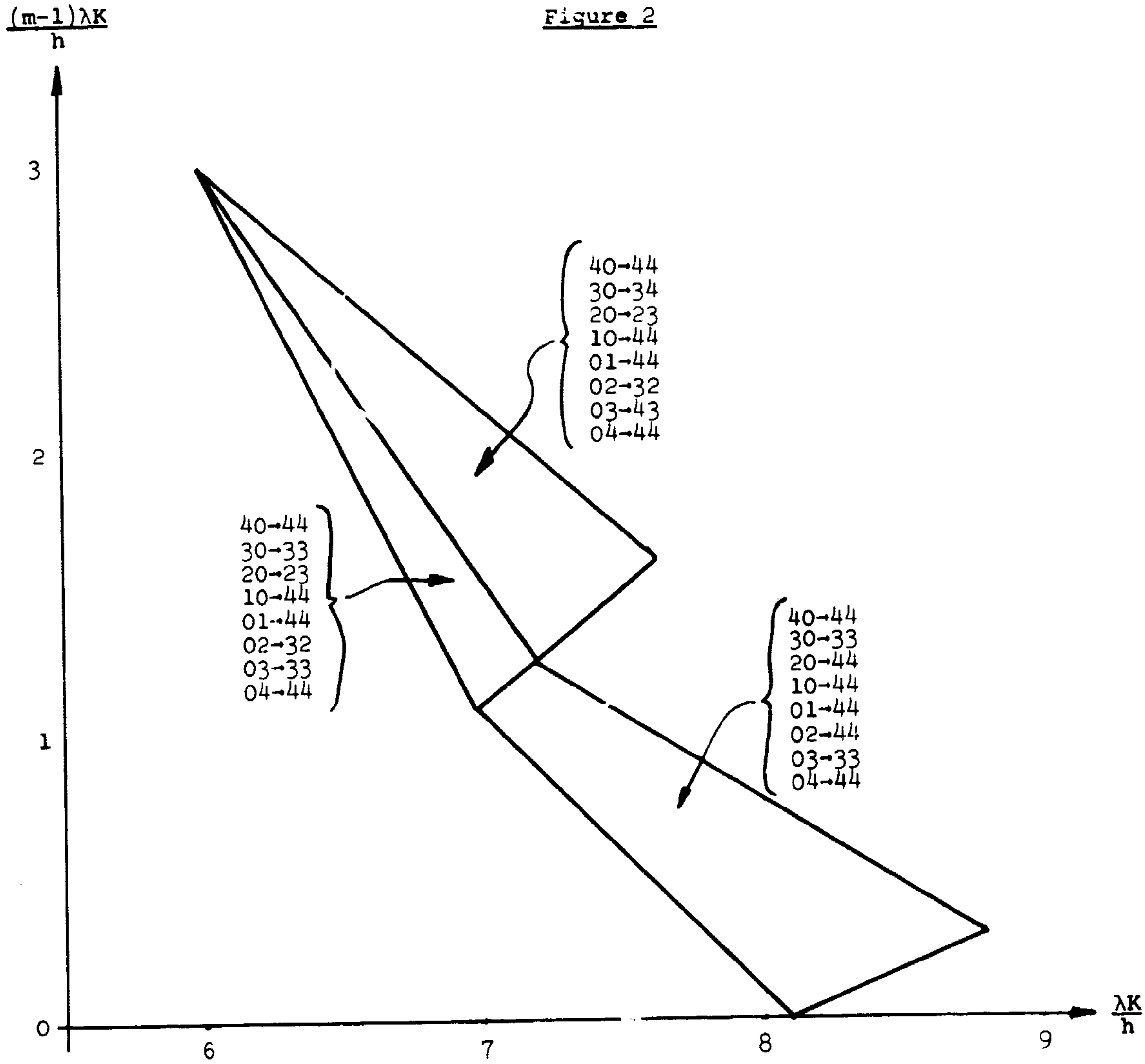

Figure 3

Results for a Symmetric Case

The results pictured in Figure 1 are for $\lambda_A = \lambda_B = \lambda$ and $h_A = h_B = h$. (For computational purposes, a maximum inventory, M, equal to 8 has been chosen; see the next section.) The regions of the $\lambda K/h$ vs. $(m-1)\lambda K/h$ plane where non-random joint policies are optimal are shaded in that figure. The non-random joint policies are given in Figures 2 and 3. These policies would not be as easy to implement as random joint policies: For example, in the policy pictured in Figure 2, the quantity ordered of the product which triggers the order depends on the inventory of the other product.

In the non-shaded regions, the results are consistent with our intuition. For example, for $\frac{\lambda K}{h} = 9$, as $\frac{(m-1)\lambda K}{h}$ increases from 0, the can-order point increases from the reorder point to the order-to point.

Obtaining Optimal Policies

In general, an ordering policy is a prescription for when and how much to order, given the history of the system up to the current time. The existence of positive holding costs and the no runout assumption imply that, under an optimal policy, an order will be placed when and only when the inventory of one of the products drops to zero. We will further restrict ourselves to policies that are stationary: that is, where the amount ordered depends only on the current inventory position and not on calendar time or previous inventory positions or previous orders.

Consider any stationary policy where orders are placed whenever the inventory of some product drops to zero. Then, under our assumption of two independent Poisson demand processes, the times at which orders are placed and the state (inventory position for each product) at each such time form a Markov Renewal Process. That is, given that an order has been placed from state (i_1, i_2) at time t_0, the probability that the next order will be placed from state (j_1, j_2) before time t_0+t depends not on the state prior to t_0, but only on (i_1, i_2), (j_1, j_2), and t. (Note that we are considering only inventory positions from which orders are placed as states: Other inventory positions, including order to points, are not considered as states. This reduction is computationally important.) It is clear that this conditional probability will be independent of the condition of the system prior to t_0 if, for the product whose inventory was not zero at t_0, the distribution of the time from t_0 to its next demand is independent of the time of its last demand prior to t_0. But this is true because the demand process in question is Poisson.[3]

Therefore, for any set of parameter values λ_A, λ_B, h_A, h_B, m, K an optimal stationary policy can be found by Markov Renewal Programming (Jewell [3], DeCani [2]). Assume that there is some maximum allowable total inventory, M. Then there are $2 \times (M-1)$ states that need be of interest, namely, (0, M-1), (0, M-2), . . . , (0, 1), (1, 0), . . . , (M-1, 0). The two dimensional vectors I and J will denote states and the two dimensional vector Z will denote an order-to point. From a state J for which $j_2 = 0$, it is possible to order only to those points Z which satisfy $z_1 \geq j_1$, $z_2 \geq 1$, $z_1 + z_2 \leq M$. A similar restriction applies when $j_1 = 0$.

Using DeCani's notation [2], define

p_{IJ}^Z = the probability that the next order will be from state J, given we now order from state I to point Z.

e_I^Z = the expected time until the next order is placed, given we now order from state I to point Z.

---

3 It will also be true if there is a single demand process with interdemand times being arbitrary independent identically distributed random variables, if, at each demand event, the probability that the demand is for A is $\lambda_A/(\lambda_A+\lambda_B)$, independent of the previous demand history. However, the appropriateness of this kind of assumption is open to question.

q_I^Z = the expected reward from now until the next order is placed, given we now order from state I to point Z.

For our problem, p_{IJ}^Z and e_I^Z are independent of I, while q_I^Z is not (since setup cost depends on whether or not both products are ordered in getting to Z from I).

To calculate p_J^Z when $j_2 = 0$, observe that an order is placed from $J = (j_1, 0)$ if and only if inventory first drops to $(j_1, 1)$ and the next demand is for product B. Therefore,

$$p_J^Z = \binom{z_1 + z_2 - j_1 - 1}{z_2 - 1} \quad \lambda_A^{z_1 - j_1} \; \lambda_B^{z_2 - 1} \qquad \lambda_B \quad .$$

Similarly, when $j_1 = 0$,

$$p_J^Z = \binom{z_1 + z_2 - j_2 - 1}{z_1 - 1} \; \lambda_A^{z_1} \, \lambda_B^{z_2 - j_2} \qquad .$$

Consider the following fact, which we wish to take advantage of. The two independent Poisson demand processes that have been assumed are indistinguishable from a single Poisson process with intensity $\lambda_A + \lambda_B$ for which there is probability $\lambda_A/(\lambda_A+\lambda_B)$ that the demand is for A at each event. Consequently, the expected time for n demands is $n/(\lambda_A+\lambda_B)$, independent of the identity of the products demanded. Since there are $z_1+z_2-j_1-j_2$ demands in going from Z to J,

$$e^Z = \sum_J p_J^Z (z_1 + z_2 - j_1 - j_2)/(\lambda_A + \lambda_B) \quad .$$

Let H_J^Z be defined as the expected inventory cost incurred in going from Z to J. Then, using indistinguishability,

$$H_J^Z = \frac{1}{2}\left(h_A(z_1+j_1) + h_B(z_2+1)\right)\left(z_1+z_2-j_1\right)/\left(\lambda_A+\lambda_B\right) \qquad \text{if } j_2=0$$

$$\frac{1}{2}\left(h_A(z_1+1) + h_B(z_2+j_2)\right)\left(z_1+z_2-j_2\right)/\left(\lambda_A+\lambda_B\right) \qquad \text{if } j_1=0 \quad .$$

Define $\delta(I,Z)$ to be 1 if $z_1 > i_1$ and $z_2 > i_2$ and 0 if not. Then

$$q_I^Z = -\left(K + \delta(I,Z)\cdot (m-1)K + \sum_J p_J^Z H_J^Z\right) \quad .$$

If policy C specifies that an order is placed to Z from I, then we define $e_I^C \equiv e_I^Z$, $q_I^C = q_I^Z$, and $p_{IJ}^C = p_{IJ}^Z$. For policy C, the long run average cost is given by $-g_C$ where g_C and $\{u_I^C\}$ solve

$$(1) \qquad g_C e_I^C + u_I^C = q_I^C + \sum_J p_{IJ}^C u_J^C \qquad \text{for all i} \quad .$$

For C to be optimal, it must be true for every state (including those that are transient under C) that the "test quantity" for the point specified by C is no smaller than the test quantity for any other point (DeCani [2]). In other words, it is required that, for all I and Z,

$$(2) \qquad \frac{q_I^C + \sum p_{IJ}^C u_J^C - u_I^C}{e_I^C} \geq \frac{q_I^Z + \sum p_{IJ}^Z u_J^C - u_I^C}{e_I^Z}$$

If time units are defined so that $\lambda_A = 1$ and then cost units are defined so that $h_A = 1$, and λ_B and h_B are fixed, we note that e_I^C and p_{IJ}^C do not depend on K and m and that q_I^C is linear in K and in (m-1)K. Therefore (1) implies that g_C and the $\{u_I^C\}$ are linear in K and (m-1)K, so that for policy C, the line determined by making (2) an equality divides the K, (m-1)K plane into two regions: In one the inequality is true, in the other it is not. Consequently the region in this plane where C is optimal is the intersection of the half spaces found by making (2) an equality for alll I and Z. The results reported earlier were obtained by specifying several policies and, for each one, examining all the test quantities to find the region where it was optimal.

Because of the linearity, for a given policy, finding g and the $\{u_I\}$ for three different values of (K, (m-1)K) enables computation of the slope and intercept of each of the lines which make (2) an equality. Our computer program did this, eliminated some redundant lines, and printed the slopes and intercepts of the remaining lines. This took about three seconds per policy for the $\lambda_A = \lambda_B$, $h_A = h_B$, M = 8 case on an IBM 7094.

References

1. Joseph L. Balintfy, "On a Basic Class of Multi-Item Inventory Problems," Management Science, Vol. 10, No. 2, January 1964, pp. 287-97.

2. John S. DeCani, "A Dynamic Programming Algorithm for Embedded Markov Chains when the Planning Horizon Is at Infinity," Management Science, Vol. 10, No. 4, July 1964, pp. 716-33.

3. William S. Jewell, "Markov Renewal Programming, I and II," Operations Research, Vol. 11, No. 6, November 1963, pp. 938-71.

4. Edward A. Silver, "Some Characteristics of a Special Joint-Order Inventory Model," Operations Research, Vol. 13, No. 2, March 1965, pp. 319-22.

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY *(Corporate author)* Columbia University	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

OPTIMAL CONTINUOUS REVIEW POLICIES FOR TWO PRODUCT INVENTORY SYSTEMS WITH JOINT SETUP COSTS

4. DESCRIPTIVE NOTES *(Type of report and inclusive dates)*

Technical Report

5. AUTHOR(S) *(Last name, first name, initial)*

IGNALL, Edward

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 15, 1966	13	4

8a. CONTRACT OR GRANT NO. Nonr 266(55) b. PROJECT NO. c. Task No. d. NR 042-099	9a. ORIGINATOR'S REPORT NUMBER(S) Operations Research Group, Columbia University, School of Engineering and Applied Science 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Logistics and Mathematical Statistics Branch, Office of Naval Research, Washington, D.C. 20360

13. ABSTRACT

Minimum average cost ordering policies for continuously reviewed two product inventory systems with joint setup costs are sought. Disappointly, the optimal policy, even in a simple symmetric case, is not always simple: For some values of cost and demand parameters, a policy that would be difficult to implement is optimal. Markov Renewal Programming is used to find the region in parameter space where a given policy is optimal.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Markovian decision processes						
Inventory theory						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ______________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ______________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ______________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS), (S), (C), or (U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.