Additive Property of Modal Density for a Composite Structure

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Summary

The additive property of modal density for composite structures is discussed. It is demonstrated analytically that the property is valid for a particular system. The composite structure analyzed consists of two beams joined at right angles to form an L-shaped frame. The receptance approach is used to determine the frequency equation and its asymptotic representation is presented. Graphical presentation of the cumulative number of modes for the composite and its constituents give rise to a verification of the additive property of modal density for a composite structure.

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Introduction

The problem of finding the number of natural frequencies of an elastic structure that are contained in a given frequency interval occurs in the analysis of structural vibration under the action of a random load with broad spectral content [1, 2]. Knowledge in this problem area has been contributed by several investigators [3, 4, 5].

The studies that have been reported on the determination of the density of the distribution of natural frequencies (or what is also termed the modal density) have been concerned with classic structural shapes such as beams, plates, and cylinders. However, these shapes rarely occur in a real application in engineering as separate elements. Thus in applying, for example, the statistical energy analysis to a complex system, the modal density of composite structures must be considered.

A composite structure is composed of a number of substructures or components each of which may be identified (ideally) with some classic shape. Assuming the modal density of each substructure is known, it is postulated that the modal density of the composite structure is equal to the sum of the modal densities of its components [2]. If the jth component of the composite exhibits N_j modes within the frequency interval $\Delta\omega$, then its modal density at the center of the band $\Delta\omega$ is defined as

$$n_j(\omega) = \frac{N_j}{\Delta\omega} \tag{1}$$

Thus the modal density of the composite at ω would be given by

$$n(\omega) = \frac{1}{\Delta\omega} \sum_{j=1}^{m} N_j \qquad (2)$$

where the summation extends over the total number of elements, m, that give rise to the composite structure.

The additive property of modal densities expressed by equation (2) has not been proven analytically in a general form nor has it been demonstrated analytically for particular composite structures [2]. Several experiments, however, have indicated general agreement with the postulate [6]. The physical reasoning reported in [2] serves as a basis for acceptance of the concept.

If a lumped mass system with k degrees of freedom is joined to another lumped mass system that has λ degrees of freedom, then the resulting lumped mass system will in general possess $m = k + \lambda$ degrees of freedom. Since a resonant frequency exists for each degree of freedom, the combined system will exhibit m resonant modes. The frequencies at which resonant modes occur for the combined system will be different from those of the subsystems. In a given frequency interval the combined system will contain modes equal in number to the sum of the modes in that same frequency interval for the separate lumped mass systems - according to the additive postulate.

A consideration of two simplified subsystems further illustrates the point. If the two single mass systems shown in Figure 1 are connected, a two degree of freedom system is obtained, Figure 2. The resonant frequencies of the subsystems are ω_1 and ω_2, and for the combined system the natural frequencies can be expressed as

$$\omega^2 = \frac{1}{2}\left\{\left[\omega_1^2 + (1+\alpha)\,\omega_2^2\right] \pm \sqrt{\left[\omega_1^2 + (1+\alpha)\,\omega_2^2\right]^2 - (4\omega_1^2\,\omega_2^2)}\right\} \quad (3)$$

where α is the mass ratio m_2/m_1. If $\alpha = 1$ and $\omega_1 = \omega_2$, equation (3) gives the following for the resonant frequencies of the system of Figure 2.

$$\omega_{s1} = 0.667\omega_1 \quad , \quad \omega_{s2} = 1.65\omega_1 \quad (4)$$

It is noted from (4) that one of the natural frequencies for the combined system is above that for either of subsystems while the other is below. Thus combining the systems causes an upward and downward shift for the resonant frequencies as compared to the values for the individual systems. If this observation is extrapolated to systems with many degrees of freedom, it would be expected that an equal number of modes would be shifted into and out of a given frequency interval upon combining two systems so that on the average the total number of modes in this interval would remain unchanged.

While a general proof of the additive property of modal densities is not given here, it is demonstrated below that it holds true for a particular composite structure.

The Composite Structure

In order to investigate the postulate (3) for a particular system, the modal density of the composite as well as its subsystems must be amenable to analytical treatment. Thus the resonant modes must be determined for the composite and its constituents and this requires consideration of a relatively simple system. One system that can be treated is illustrated in Figure 3. This structure can be imagined to be constructed by bending a single beam at a right angle to form the two lengths ℓ_1 and ℓ_2 and cantilevered at one end. Alternatively, two beams of lengths ℓ_1 and ℓ_2 may be welded at right angles to form the composite structure. The subsystems may then be supposed to be two beams as illustrated in Figure 4.

In Reference [7], the frequency equation for the composite structure of Figure 3 is obtained through a consideration of the two beams shown in Figure 4 with the two coupling coordinates q_1, q_2. The boundary conditions for the two subsystems are fixed-free and pinned-free. These details are not important insofar as the modal densities of the individual beams are concerned since the boundary conditions only affect the modal spacing for the first few modes. The modeling of the composite structure of Figure 3 as illustrated in Figure 4 is convenient in determining the frequency equation through the receptance procedure and gives very close agreement between theoretical and experimental results as observed by the writer and also reported in [7] for the low modes.

If the composite structure, Figure 3, is excited vertically in the plane of the paper at the anchor point, then longitudinal vibrational modes can occur in the vertical member. However, since the purpose of this exercise is to compare spacing of the bending modes for the substructures with those for the composite, the vertical member is considered to be a rigid body insofar as longitudinal motion is concerned [7]. Thus in subsequent analysis, only bending modes in the plane of the paper will be examined for the composite and for the two substructures as well.

Frequency Equations and Modal Spacing

In order to determine the modal densities of the structures under consideration, the modal spacing must be found as a function of frequency. This can be accomplished by solving the frequency equations to ascertain the total number of modes that can occur up to some frequency Ω and allowing Ω to vary from zero to an arbitrary upper limit. When this information is presented in graphical form, an examination of the slopes of the resulting curves will yield the modal densities.

Without loss in generality of results, a simplification in the analysis can be effected by assuming that $\ell_1 = \ell_2 = \ell$ and that both members are of the same material. With these simplifying conditions, the frequency equation for the composite structure, A, becomes

$$F(\lambda\ell) = \left[\frac{\cos\lambda\ell - \sin\lambda\ell\ \operatorname{Cosh}\lambda\ell}{\cos\lambda\ell\ \operatorname{Cosh}\lambda\ell + 1} + \frac{1}{\lambda\ell}\right] \times$$

$$\left[\frac{\cos\lambda\ell\ \operatorname{Sinh}\lambda\ell + \sin\lambda\ell\ \operatorname{Cosh}\lambda\ell}{\cos\lambda\ell\ \operatorname{Cosh}\lambda\ell + 1} + \frac{\cos\lambda\ell\ \operatorname{Cosh}\lambda\ell + 1}{\cos\lambda\ell\ \operatorname{Sinh}\lambda\ell - \sin\lambda\ell\ \operatorname{Cosh}\lambda\ell}\right]$$

$$+ \left[\frac{\sin\lambda\ell\ \operatorname{Sinh}\lambda\ell}{\cos\lambda\ell\ \operatorname{Cosh}\lambda\ell + 1}\right] = 0 \qquad (5)$$

The resonant frequencies are then determined according to the relation

$$\omega_j = \frac{(\lambda_j\ell)^2}{\ell^2}\, c_\ell K \qquad (6)$$

where K is the radius of gyration and C_{ℓ} is the longitudinal wave velocity. The frequency equations for the substructures are

$$\text{(free-pinned)} \qquad \cos \lambda\ell \ \operatorname{Sinh} \lambda\ell - \sin \lambda\ell \ \operatorname{Cosh} \lambda\ell = 0 \qquad (7)$$

$$\text{(clamped-free)} \qquad \cos \lambda\ell \ \operatorname{Cosh} \lambda\ell + 1 = 0 \qquad (8)$$

Equation (6) may be used again to compute the natural frequencies corresponding to the eigenvalues as reckoned from equations (7) and (8).

An examination of equation (5) shows that the values of $(\lambda\ell)$ which satisfy either equation (7) or (8) correspond to the occurrence of an infinite value in the frequency equation for the composite. This condition can pose a slight inconvenience if equation (5) is simply programmed to determine $F(\lambda\ell) = 0$, according to an observation in a sign change followed by a finer iterative process. If this procedure is followed it is found that sign changes occur under three conditions: (1) for the true eigenvalues, $(\lambda_j\ell)$, for the composite, (2) for the eigenvalues of the free-pinned beam, and (3) for the eigenvalues of the clamped-free beam. Since the values for the second and third cases can be computed accurately, these are discounted and the remaining $(\lambda_j\ell)$ that cause a sign change are the real roots to equation (5). This scheme is satisfactory up through about the first ten modes for the composite and the results are listed in Table 1. Beyond the tenth mode, the eigenvalues for the composite are close to those values which gives rise to an infinite value for $F(\lambda\ell)$ and the iterative process becomes inefficient.

The eigenvalues that correspond to the higher modes for the composite can be determined by rearranging equation (5) and using the approximations that

$$\text{Tanh } \lambda\ell \approx 1 \quad , \quad \text{Sech } \lambda\ell \approx 0 \tag{9}$$

With these observations implemented, equation (5) can be written as

$$2\,(\lambda\ell) = \frac{\tan^2(\lambda\ell) - 2}{1 - \tan(\lambda\ell)} \tag{10}$$

Equation (10) is essentially an asymptotic version of equation (5) and it is shown graphically in Figure 5. In Figure 5, the intersection points of the equations $G_1\,(\lambda\ell) = 2\,(\lambda\ell)$ and $G_2\,(\lambda\ell)$ defined by (11) represent the solutions to equation (10).

$$G_2\,(\lambda\ell) = \frac{\tan^2(\lambda\ell) - 2}{1 - \tan\lambda\ell} \tag{11}$$

With the information gained from equation (10) and Figure 5, the listing of eigenvalues in Table 1 can be completed. As $\lambda\ell$ increases, the regularity in modal spacing becomes apparent.

Cumulative Number of Modes and Modal Density

The cumulative number of modes that exist up to an arbitrary value for $(\lambda_j\ell)$ can be ascertained for the composite and the substructures from Table 1. This gives rise to the mode tabulation given in Table 2.

The information in Table 2 has been illustrated graphically in Figure 6. In Figure 6 the cumulative number of modes for the composite and the substructures have been plotted against the eigenvalues ($\lambda\ell$). If the additive postulate holds, then it must be true that

$$N_A\,(\lambda\ell) = N_B\,(\lambda\ell) + N_C\,(\lambda\ell) \tag{12}$$

An examination of Figure 6 shows that equation (12) is satisfied so that the additive property of modal densities for the particular composite under consideration is verified. From Figure 6, it is found that the relation between N_A and ($\lambda\ell$) is given by

$$N_A\,(\lambda\ell) = \frac{16}{25}\,(\lambda\ell) \tag{13}$$

Introducing the quantities $\bar{N} = N_A\sqrt{K/\ell}$ and $\nu = \omega\ell/C_\ell$, equation (13) becomes

$$\bar{N}\,(\nu) = \frac{16}{25}\,\nu^{1/2} \tag{14}$$

Thus the modal density for the composite is given by

$$\bar{n}\,(\nu) = \frac{d\bar{N}(\nu)}{d\nu} = \frac{8}{25}\,\nu^{-1/2} \tag{15}$$

The modal density for a beam, irrespective of the boundary conditions, can be written as

$$n\,(\nu) = \frac{1}{2\pi}\,\nu^{-1/2} \tag{16}$$

Since the composite is constructed of two identical beams, the sum of the modal densities of the substructures is

$$n_B(\nu) + n_C(\nu) = \frac{1}{\pi}\nu^{-1/2} \tag{17}$$

Observation shows that the coefficient $(1/\pi)$ in (17) compares favorably with the coefficient (8/25) in equation (15). Equations (14) and (15) are shown graphically in Figures 7 and 8, respectively.

Conclusion

The additive property of modal densities for composite structures has been discussed. It is verified analytically that this property is true for a particular structure composed of two identical beams welded at right angles.

References

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Table 1. Tabulation of eigenvalues for the composite structure and the substructures

Free-pinned beam	Clamped-free beam	Composite frame
3.9266	1.8751	1.0823
7.0686	4.6941	1.7864
10.2102	7.8548	3.9691
13.3518	10.9955	4.8049
16.4934	14.1372	7.0984
19.6363	17.2800	7.9126
22.7791	20.4229	10.2318
25.9219	23.5657	11.0386
29.0648	26.7086	13.3688
32.2076	29.8514	14.1706
35.3505	32.9943	16.4997
38.4933	36.1372	17.2854
41.6362	39.2800	19.6425
44.7790	42.4228	20.4282
47.9219	45.5657	22.7853
51.0647	48.7085	23.5710
54.2076	51.8514	25.9281
57.3504	54.9942	26.7138
60.4933	58.1371	29.0709
63.6361	61.2799	29.8566
66.7790	64.4228	32.2137
69.9218	67.5656	32.9994

Table 2. Tabulation of the number of modes for the composite and substructures

Eigenvalue λ	Number of modes free-pinned beam N_C	Number of modes clamped-free beam N_B	Number of modes composite N_A
11	3	4	7
20	6	7	13
30	9	10	20
40	12	13	26
50	15	16	32
60	18	19	39
70	22	22	45
80	25	25	51
90	28	29	58
100	31	32	64
125	39	40	80
150	47	48	96
175	55	56	112

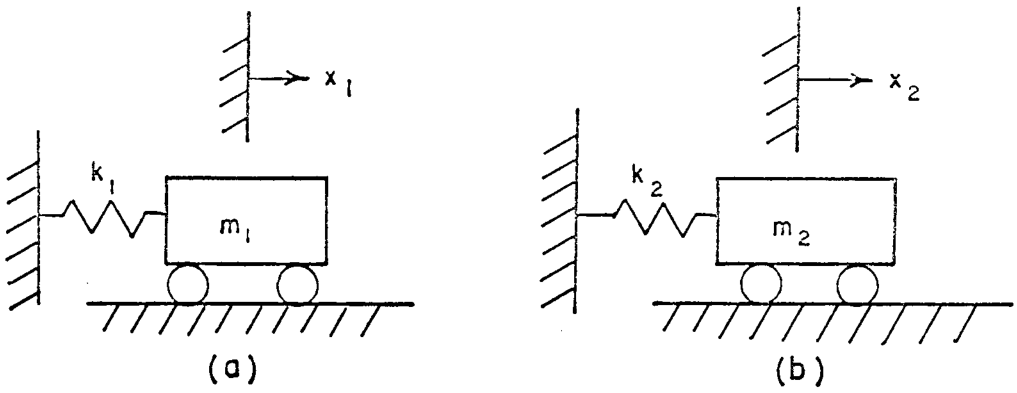

FIGURE 1. LUMPED MASS SYSTEM

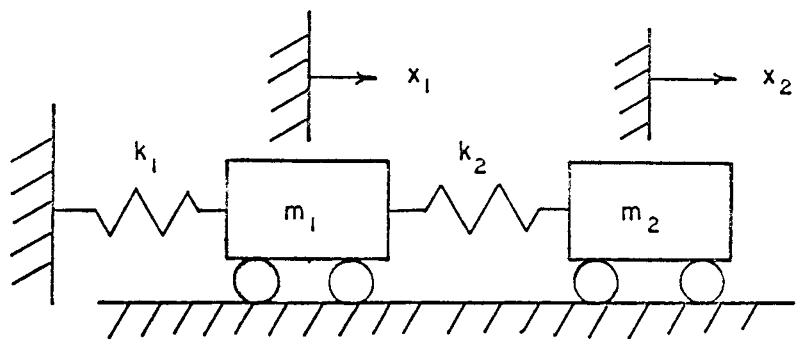

FIGURE 2. TWO DEGREE OF FREEDOM LUMPED MASS SYSTEM

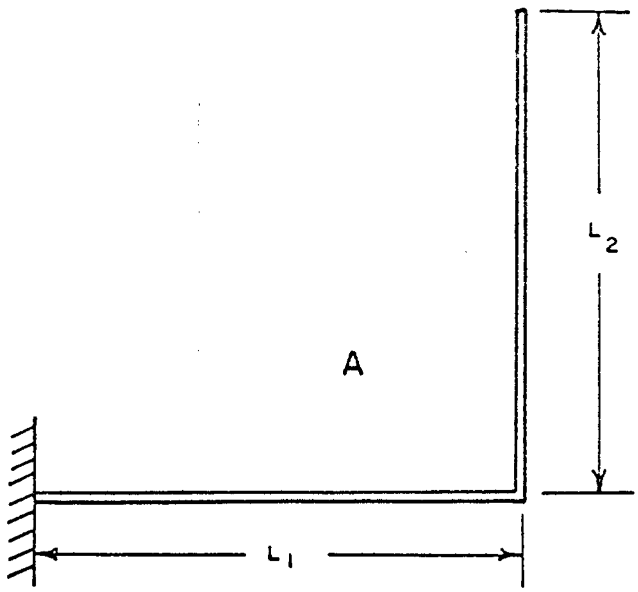

FIGURE 3. COMPOSITE STRUCTURE

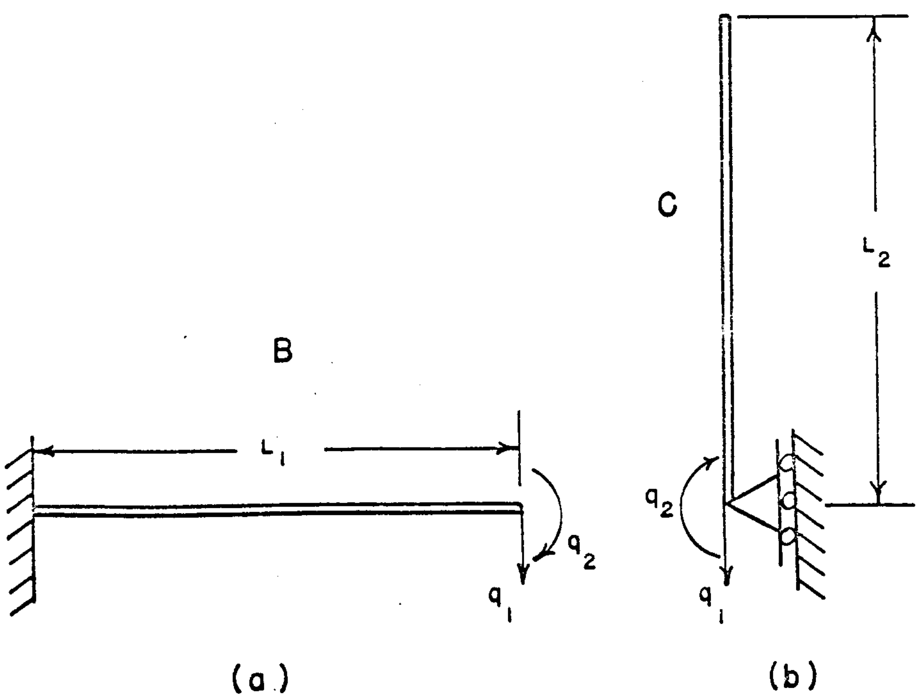

FIGURE 4. TWO SUBSTRUCTURES

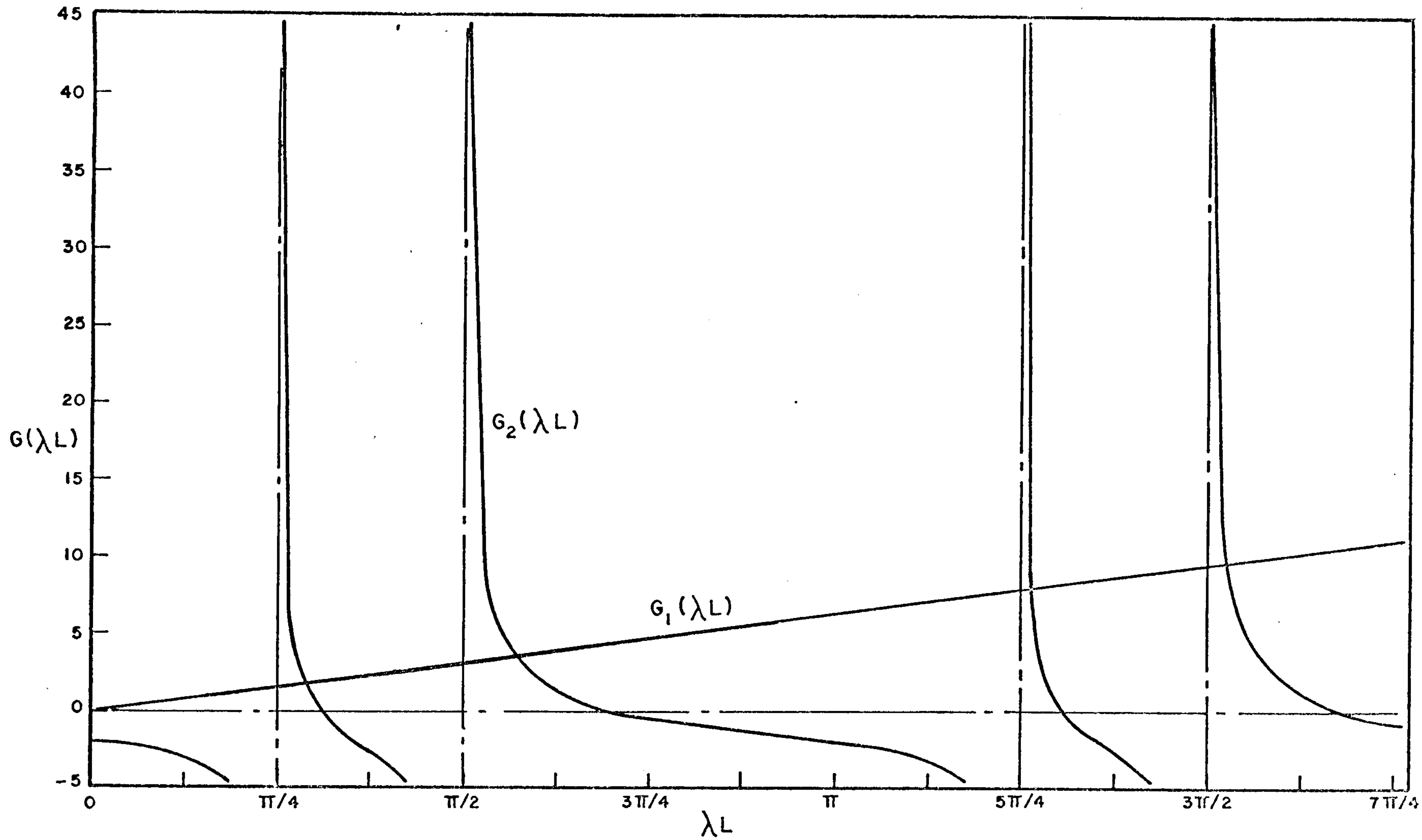

FIGURE 5. APPROXIMATE SOLUTION TO FREQUENCY EQUATION

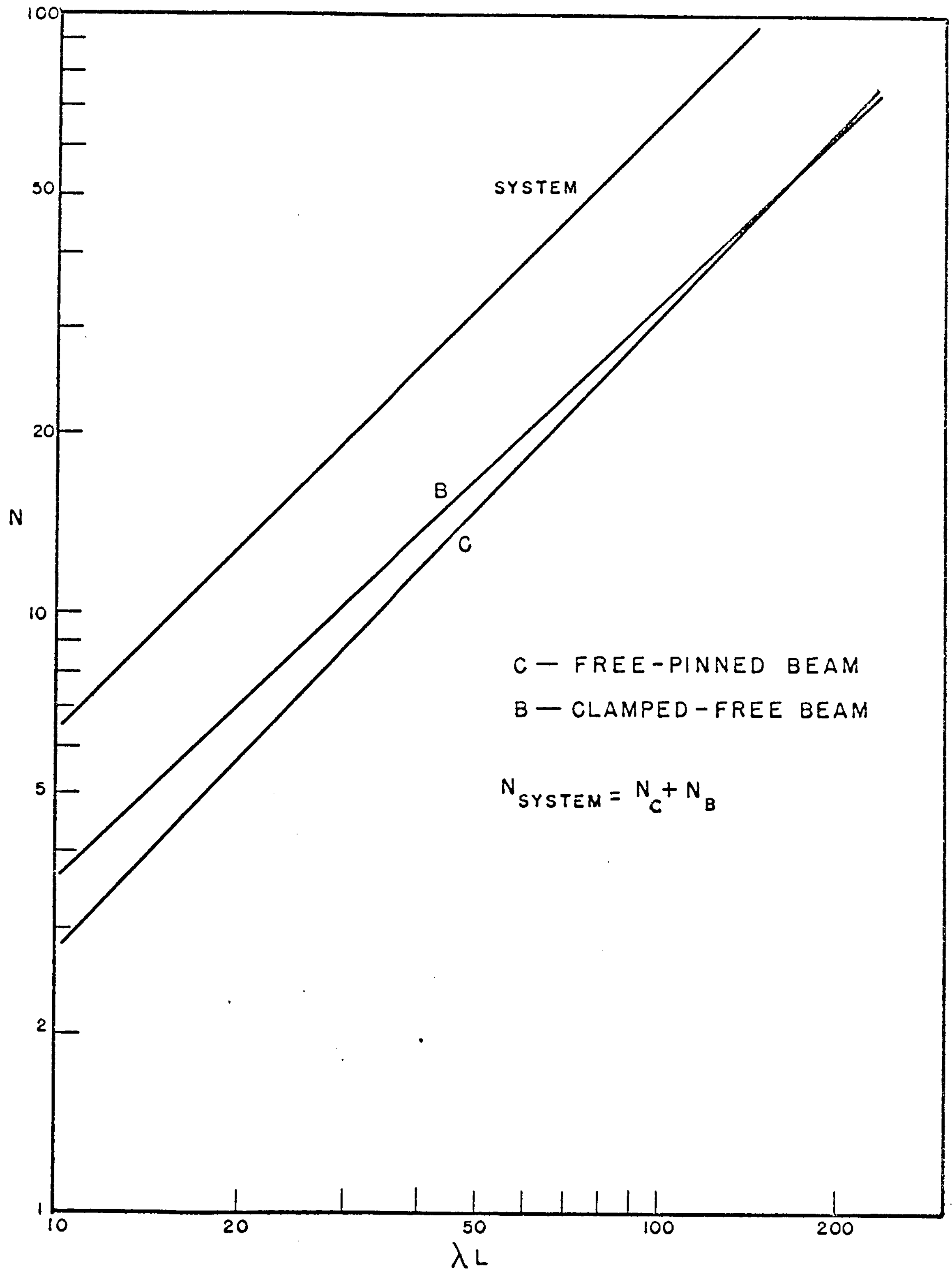

FIGURE 6. CUMULATIVE NUMBER OF MODES

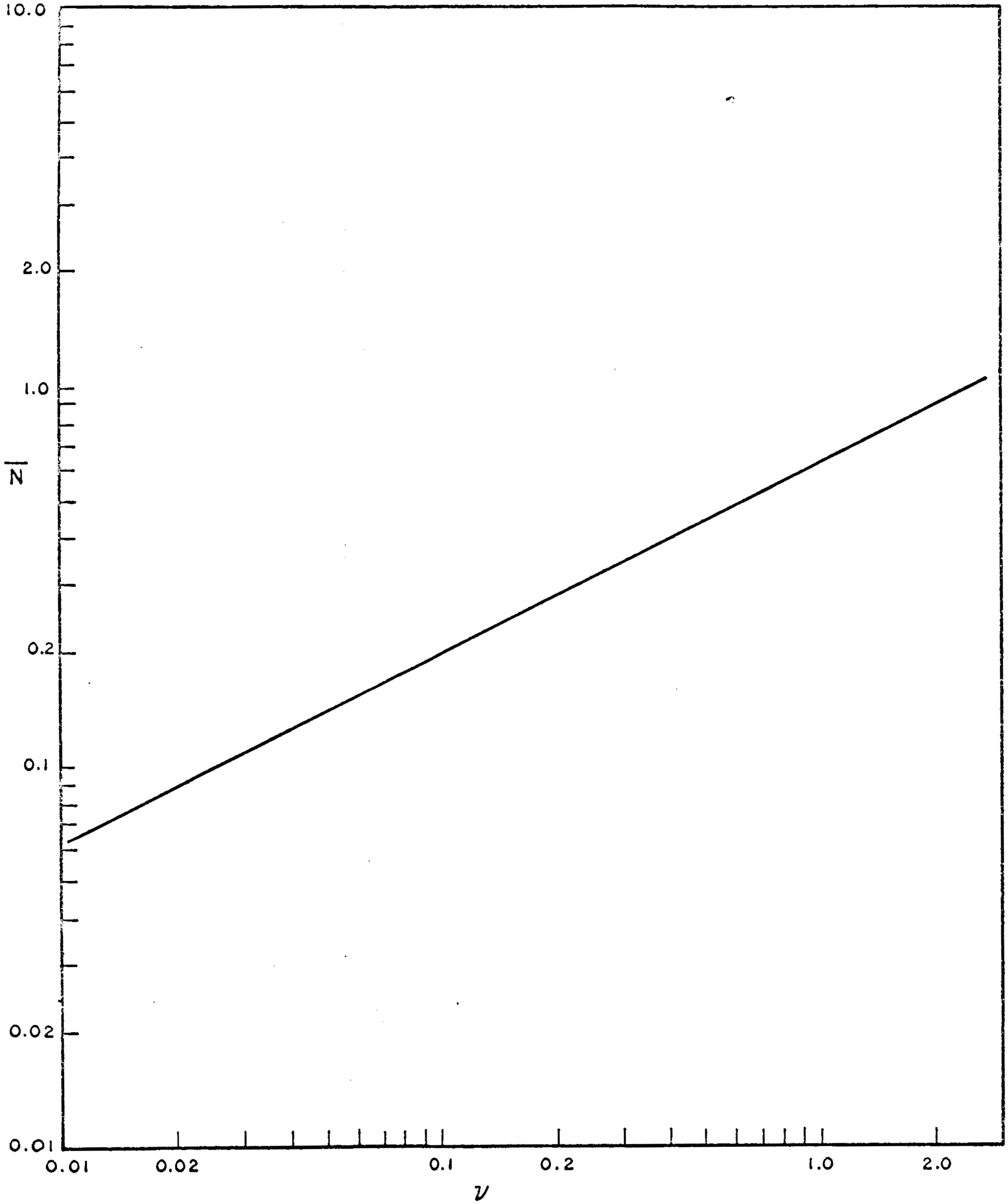

FIGURE 7. NUMBER OF MODES VERSUS FREQUENCY

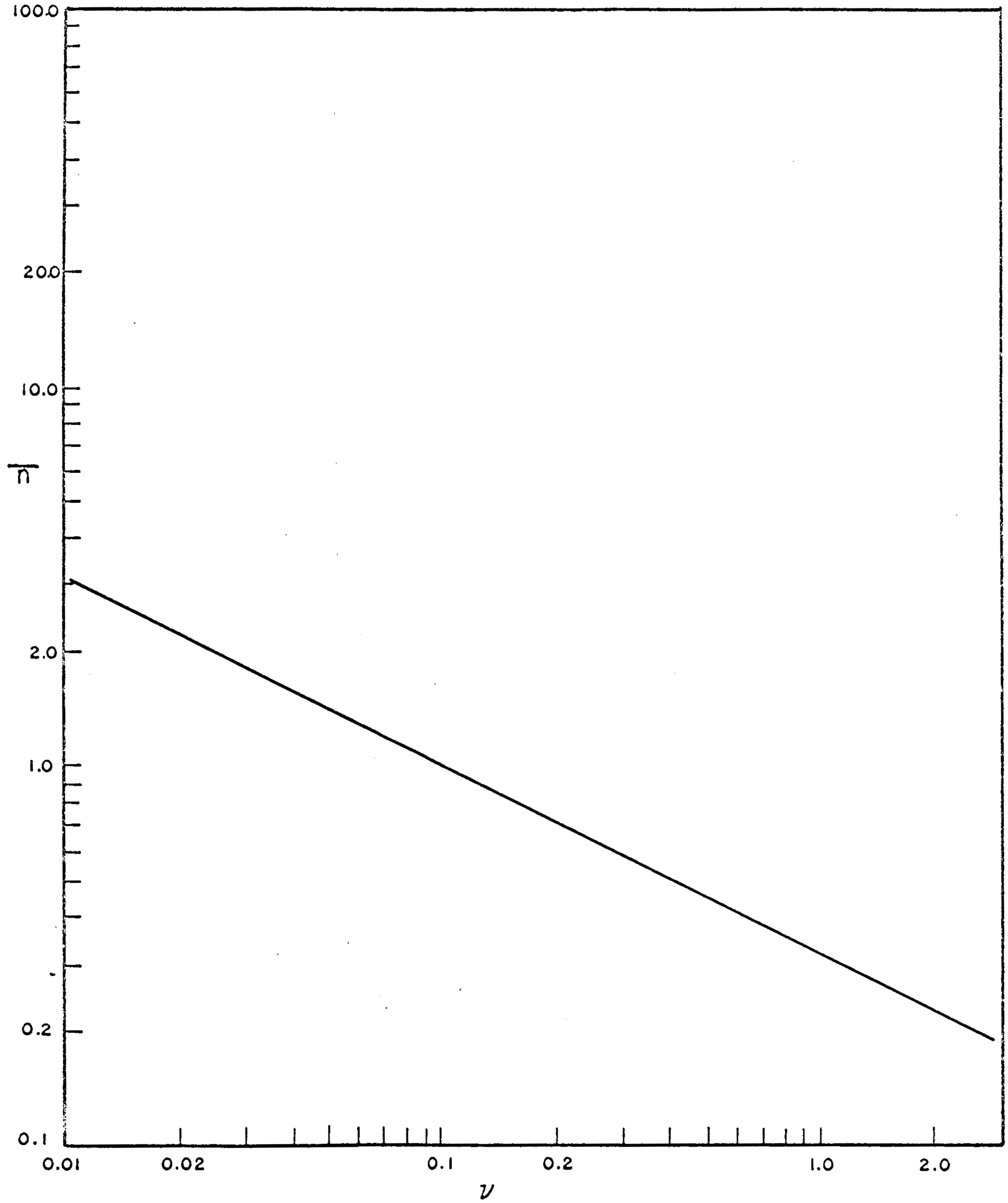

FIGURE 8. MODAL DENSITY VERSUS FREQUENCY