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"Steepest Descent and the Merging of Several Waves"

by

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ABSTRACT

The question of how to obtain an analytic description of the merging of several waves is studied. It is assumed that the wave structure is given as a complex integral (say a Fourier integral) and an asymptotic approximation is desired. Motivated by the work of Ursell we study the asymptotic properties of integrals of the form $\int \exp ip(t)\,dt$ where $\rho(t)$ is a polynomial in t. Explicit asymptotic formulas are obtained for the two types of polynomials $p(t) = zt + t^{n+1}/n+1$ and $p(t) = a_1 t + a_2 t^2 + \ldots + a_{n-1} t^{n-1} + t^{n+1}/(n+1)$.

Section 0 - Previous Work

In studying the problem of wave propagation in various media, the solution is often written as an inverse Fourier transform. If the problem is somewhat complex the transform must often be evaluated by the methods of steepest descent or stationary phase. In this paper we wish to investigate a problem which can arise in the use of these methods, namely the case where the integrand has several saddle points at approximately the same point. The situation we have in mind is one where several waves are interacting to give some form of cancellation.

A case of two waves interacting, of the kind we have in mind, has been thoroughly studied by Ursell (1964). The particular case he considers is the Kelvin ship wave problem, where in an arc of angle 39 1/2 degrees astern of a ship there are two waves. At an angle of 19 3/4 degrees either side of the stern line the two waves merge in a strange fashion, and outside this angle there are no waves. Ursell, following previous work by Chester, Friedman and Ursell (1957), showed that this phenomenon can be adequately described by the use of a modification of the method of steepest descent, which is now briefly described.

In the methods of steepest descent or stationary phase one is interested in evaluating integrals of the form

$$f(\lambda) = \int_c g(z) \exp\left\{\lambda\, f(z)\right\} dz \qquad (1)$$

where the path c is sufficiently long (compared to something) to be considered infinite, and λ is large. The procedure is to look for points z_o in the complex z plane such that $f'(z_o) = 0$. The path c is then deformed to pass through the points z_o (if this can be done) so that the major contribution to the integral is given by the values of the integrand near z_o.

In this way the exponential is given essentially by

$$\exp\left\{\lambda f(z)\right\} \approx \exp \lambda \left\{ f(z_o) + \frac{f''(z_o)}{2}(z-z_o)^2 \right\} . \tag{2}$$

In the ship wave problem there are two such points, whose location is dependent on the angle φ astern of the ship. (Thus $f = f(z;\varphi)$) . However, as φ approaches the critical angle φ_o of 19 3/4, the two saddle points z_o approach each other and the second derivative in (2) vanishes. Thus at the critical angle φ_o we have

$$\exp\left\{\lambda f(z;\varphi_o)\right\} \approx \exp \lambda \left\{ f(z_o;\varphi_o) + \frac{f'''(z_o;\varphi_o)}{3!}(z-z_o)^3 \right\} \tag{3}$$

Thus the exponent is no longer a quadratic of the form $a + bx^2$ but a cubic of the form $a + cx^3$.

The question answered by Chester, Friedman and Ursell, and in more detail by Ursell was that of how to obtain one representation of the exponential which reduces to the two separated waves of the form (2) for $\varphi \sim 0^o$ and which reduces to the single cubic (3) as $\varphi \to \varphi_o$. Their solution was that $f(z;\varphi)$ could be adequately represented in the entire region by

$$f(z;\lambda) \approx a + bx + x^3 \tag{4}$$

This is perhaps a bit surprising because the essential nature of both (2) and (3) is that the first order term is missing. They show that it is not only adequate but extremely convenient to use the form (4).

In this paper we consider what is the logical extension of their ideas to the case of several saddle points merging at some point.

Section 1 - Generalized Airy Functions

The apparent extension of the idea of Chester, Friedman and Ursell would be to consider approximating (1) by an integral of the form

$$I_n(z) = \int_c \exp\left\{zt - t^{n+1}/(n+1)\right\}dt \tag{5}$$

where c is some contour in the complex plane for which the integral converges for all z. We investigate the functions defined by (1) which are generalizations of the Airy functions.

First we have

Lemma 1 Let c be any path such that (5) converges absolutely. Then (1) satisfies the equation

$$y^{(n)} - zy = 0 \tag{6}$$

Proof Under the hypothesis differentiation under the integral sign is valid. Then if $y = I_n$

$$\begin{aligned} y^{(n)} - zy &= \int_c (t^n - z) \exp\left\{zt - t^{n+1}/(n+1)\right\}dt \\ &= -\exp\left\{tz - t^{n+1}/(n+1)\right\}\Big|_a^b \\ &= 0, \end{aligned}$$

where a and b are the end points of the path c, since the integrand must vanish at the end points. Thus the lemma is proved.

Let α_{ni} be the angle associated with the i^{th} member (counting counterclockwise from the positive real axis) of the (n+1) roots, $\sqrt[n+1]{1}$. Let c_i be a path from infinity to infinity starting on a ray with angle $\alpha_{n,i+1}$ and ending on a ray with angle α_{ni}. Then these (n+1) paths define (n+1) functions $I_{n,i}$.

Lemma 2 These (n+1) functions are linearly dependent.

Proof The sum $I_{n,o} + I_{n,1} + \,..I_{n,n+1}$ form an integral around a closed contour. The integrand is everywhere analytic. Thus by Cauchy's theorem the sum is zero.

Thus the functions $I_{n,i}$ form at most n linearly independent solutions of (6). To prove that they form exactly n solutions we need the following:

Lemma 3 All solutions of (6) are entire functions of z.

Proof We need only note that (6) has no singular points except $z = \infty$.

We now proceed to show that the first n of these solutions have, on some ray, distinct different asymptotic behaviors. Our contention will then be proved. We consider

$$I_{n,k} \equiv \int_{b_k}^{a_k} \exp\left\{ zt - t^{n+1}/(n+1) \right\} dt \qquad (7)$$

where $a_k = \infty\, e^{2\pi i k/(n+1)}$ and $b_k = a_{k+1}$, and the path is shown in figure 1. The integrand has saddle points at

$$\frac{d}{dt}\,(t - t^{n+1}/z(n+1)) = 1 - t^n/z = 0,$$

or

$$t_k = \sqrt[n]{z}\ . \qquad (8)$$

We must distinguish $I_{n,o}$ and $I_{n,n+1}$ from the others. In these special two cases the saddle point (8) is on the boundary of the region defined by the rays at the end points. In the remaining cases the saddle point (8) is essentially on the path defining the function (for real positive z). The path of steepest descent is given by

$$\mathrm{Im}\,(zt - t^{n+1}/(n+1) = \mathrm{Im}\, z^{n+1/n}(\frac{n}{n+1}) \ . \qquad (9)$$

For z real and positive this reduces to

$$\mathrm{Im}\ (t - t^{n+1}/(n+1)\ z) = \frac{n}{n+1}\ \mathrm{Im}\ \ z^{1/n}. \qquad (9a)$$

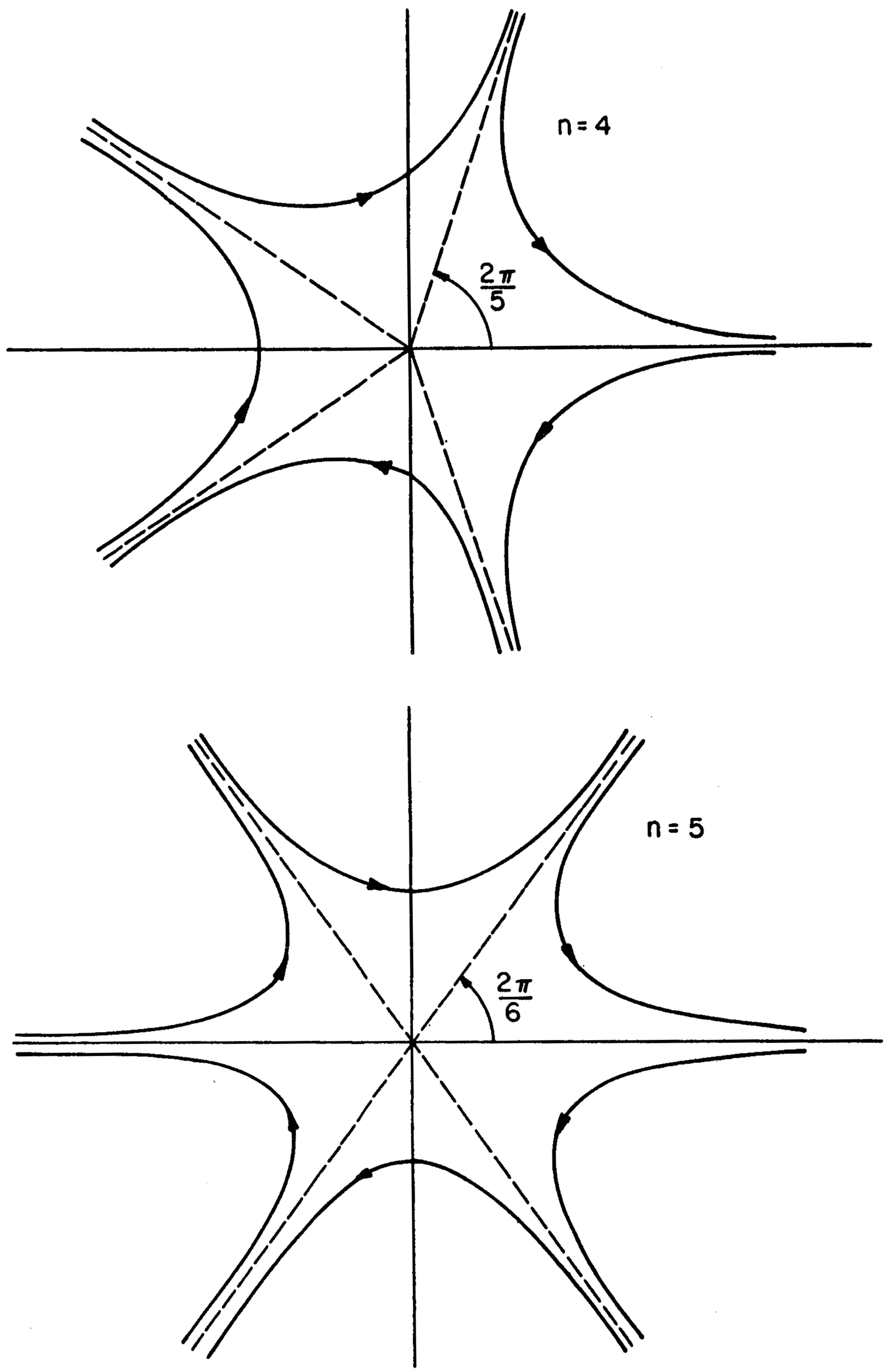


FIGURE 1. CURVES DEFINING SOLUTIONS TO EQUATION FOR REAL, POSITIVE Z.

The actual calculation of the path defined by (9a) is not necessary. One can show trivially that the paths must necessarily be near to the rays α_k, by writing $t = re^{i\varphi}$ and observing in (10) that as $r \to \infty$, $\sin (n+1)\varphi$ must $\to 0$, for fixed z.

$$r \left[\sin \varphi - \frac{r^n}{z(n+1)} \sin (n+1)\varphi \right] = \frac{n}{n+1} \operatorname{Im} z^{1/n}. \tag{10}$$

In the vicinity of the saddle point we must inspect the second derivative of the exponential,

$$h'' = -nt^{n-1}/z \Big|_{t=\sqrt[n]{z}} = -n/\sqrt[n]{z} \tag{11}$$

Then the direction of the steepest descent path through the saddle point t_k is given by the requirement that the angle φ_k be such that the exponent

$$F = -z\, 1/2 \frac{n}{\sqrt[n]{z}} (t-t_k)^2 = -\frac{z}{2} \frac{n}{\sqrt[n]{z}} (\rho e^{i\varphi_k})^2$$

be real and negative. Thus if

$$\sqrt[n]{z} = r^{1/n} e^{i\Theta_k} ,$$

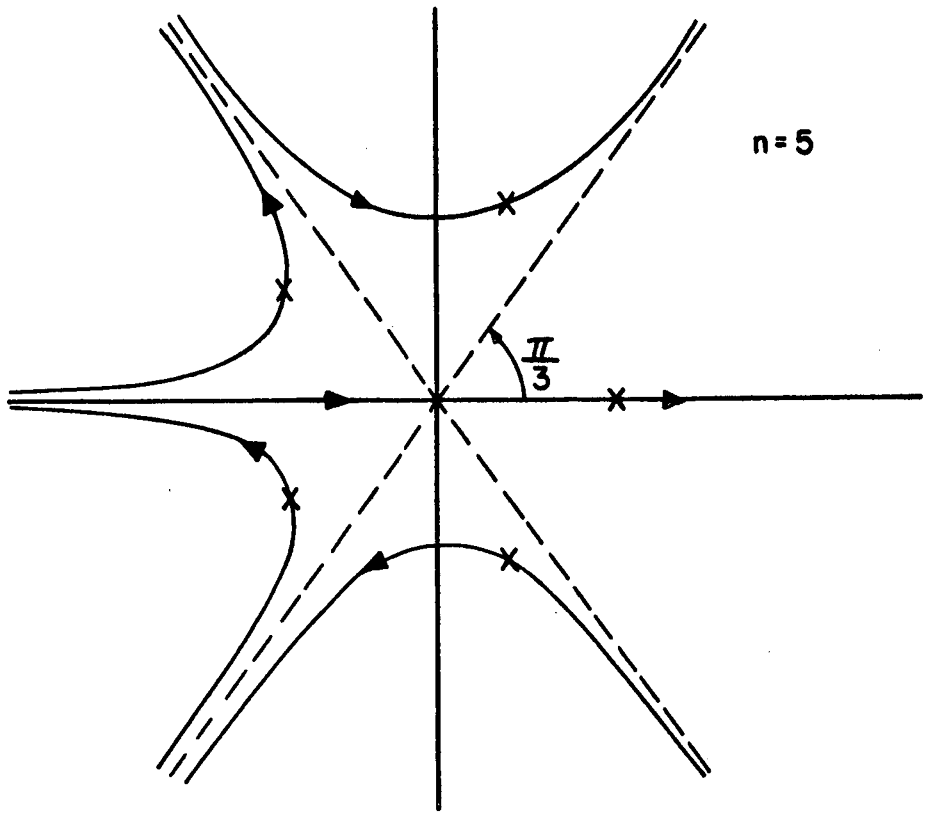
then

$$\varphi_k = \Theta_k / 2 \tag{12}$$

From figure 2 it is clear that, for $k \neq 0$, $(n+1)$, the direction given by (12) is essentially that of the original path near the saddle point. Thus the original path of integration c_i, for $i \neq 0$, $(n+1)$, can be deformed into the path of steepest descent. Thus we obtain the result, for real positive z,

$$I_{n,k} \sim \exp \left\{ z^{(n+1)/n} \left(\frac{n}{n+1} \right) + 1/2\, i\, \Theta_k \right\} \sqrt{\frac{2\pi}{n\, |z|^{(n-1)/n}}} , \tag{13}$$

where the k^{th} root of z (counterclockwise) is taken for $I_{n,k}$. For k = 0



Saddle points: $t_o = \sqrt[n]{z}$

Angle of steepest descent = $1/2 \arg(t_o)$

FIG. 2 SADDLE POINTS AND STEEPEST DESCENT CURVES FOR n=5 or (n+1) the saddle point is on the real axis, and from (12) the path of steepest descent goes through this point along the real axis, and in fact, is the entire real axis. (If n is even, this path goes through the saddle point on the negative real axis and is a path of steepest ascent for that point. Otherwise it is a path of pure descent.) Thus the path defining the integral $I_{n,o}$ or $I_{n,n+1}$ can not be deformed into a path of steepest descent with negligible contribution from the deformation. However, the path of definition can be deformed into a path through the saddle point at θ_1 or θ_{n-1} respectively plus a path through the saddle point at θ_o on the real axis. The path through the saddle point at θ_1 or θ_{n-1} is a path of stationary phase. We shall follow this path to its intersection with the real axis, then turn and proceed on the path of steepest descent. That this can be done is obvious from the following. From (12) it follows that the contribution of the saddle point at θ_1 or θ_{n-1} is smaller in amplitude than that of the saddle point on the real axis. A path of stationary phase is one of constant amplitude, so that somewhere the path of steepest descent reaches the same amplitude as the pathes of stationary phase through the points at θ_1 and θ_{n-1}. The path of stationary phase form an angle of 45^o with the path of steepest descent through the saddle point, and thus there is a path of stationary phase through the points at θ_1 and θ_{n-1} which proceeds to the real axis. Thus the integrals $I_{n,o}$ and $I_{n,n+1}$ have two contributions. We chose to take as a fundamental function a linear combination of these two which eliminates some of the second contribution. Let

$$f_{n,n}(z) = I_{n,o} - I_{n,n+1} \qquad (14)$$

then

$$f_{n,n}(z) \sim - \sqrt{\frac{2\pi}{n\,|z|^{(n-1)/n}}}\ \exp\left\{ z^{(n+1)/n} \left(\frac{n}{n+1}\right) + \frac{i}{2}\,\theta_1 \right\}$$

$$+ \sqrt{\frac{2\pi}{n\,|z|^{(n-1)/n}}}\ \exp\left\{z^{(n+1)/n}\left(\frac{n}{n+1}\right) + \frac{i}{2}\ \theta_o\right\}$$
$$+ \sqrt{\frac{2\pi}{n\,|z|^{(n-1)/n}}}\ \exp\left\{z^{(n+1)/n}\left(\frac{n}{n+1}\right) + \frac{i}{2}\ \theta_{n+1}\right\} \qquad (15)$$
$$+ \sqrt{\frac{2\pi}{n\,|z|^{(n-1)/n}}}\ \exp\left\{z^{(n+1)/n}\left(\frac{n}{n+1}\right) + \frac{i}{2}\ \theta_n\right\}$$

The middle two terms add, and, for real positive z, the first and last terms have the same real part, and opposite signs on their imaginary parts. Now if we define

$$f_{n,k} = I_{n,k} \qquad 1 \leq k \leq n-1 \qquad (16)$$

we have

Theorem 1 The n functions $f_{n,k}$ given by (14) and (16) form a linearly independent set of solutions to (6), and thus all solutions.

Proof The fact that they form no more than n follows from Lemma 2. The fact that they form exactly n independent ones follows from the asymptotic expansions (13) and (15). The fact that they form all solutions follows from the fact that (6) is linear.

Our results may be put on a form more useable for analytic continuation if we agree to the following: any root of z will be the principal root, and θ is the corresponding angle. Then we may write

$$f_{n,k}(z) \sim \sqrt{\frac{2\pi}{n\,z^{(n-1)/n}}}\ e^{\pi ik/n} \exp\left\{\frac{n}{n+1}\ z^{(n+1)/n}\ e^{2\pi ik/n}\right\} \qquad (14a)$$
$$1 \leq k \leq n-1$$

$$f_{n,n}(z) \sim \sqrt{\frac{2\pi}{nz^{(n-1)/n}}}\ \Big[\, 2\exp\left\{\frac{n}{n+1}\ z^{(n+1)/n}\right\}$$
$$+\ e^{-\pi i/n} \exp\left\{\frac{n}{n+1}\ z^{(n+1)/n}\ e^{-2\pi i/n}\right\} \qquad (16a)$$
$$-\ e^{\pi i/n} \exp\left\{\frac{n}{n+1}\ z^{(n+1)/n}\ e^{2\pi i/n}\right\}\Big]$$

Having shown that these integrals represent all solutions of (6) we may use the results of Turritin (1950) or Heading (1957) which establishes Stokes' multipliers for the connection of the different asymptotic solutions in various regions of the complex plane to various series solutions about the origin. The computations involved are rather tedeous, and from the point of view of applications, not very informative. The integral representations give much simpler and more useful forms for the answer.

Of particular interest for many applications is the behavior of a given solution for negative z, when its behavior for positive z is known. This information is needed for boundary value problems. As we shall see, one must distinguish between n even or odd. A change of variable $z \to -z$ rotates the saddle points counterclockwise by π/n. For most of the functions $f_{n,k}$ the new position of the saddle points are such that the calculations proceed exactly as before. The exceptions are those functions whose defining curves go through, or are tangent to, the real axis.

The path of steepest descent, from (12), goes through the saddle point at an angle given

$$\varphi_k = (\theta_k + \pi/n - \pi)/2, \tag{17}$$

where θ_k is the angle of the saddle point for z real and positive. It is clear, see figure 3, that the direction of the path of steepest descent is essentially that of the curve defining the function, except for certain cases listed below.

The exceptional cases are, for n even, $f_{n,n/2}$ and $f_{n,o}$, and for n odd, $f_{n,(n-1)/2}$, $f_{n,(n+1)/2}$ and $f_{n,o}$. For $f_{n,o}$ the defining curves

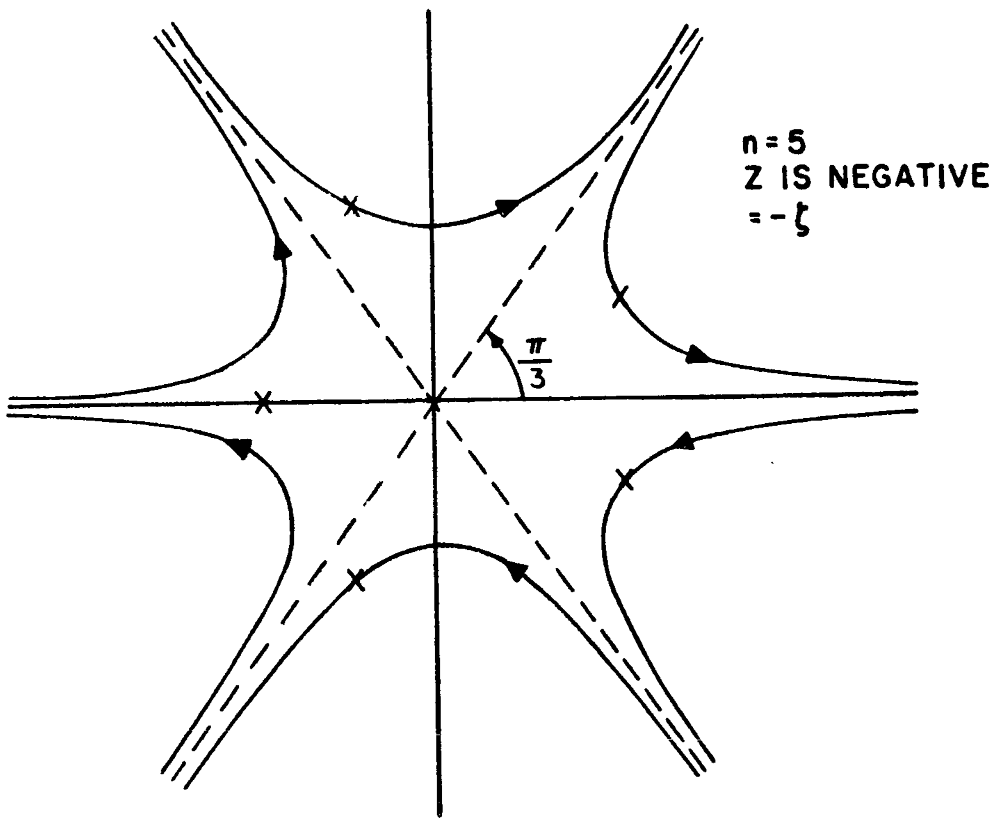


FIG. 3 LOCATION OF SADDLE POINTS AND DEFINING CURVES FOR FIG. 2 IF Z IS REAL AND NEGATIVE

now have only a single saddle point to determine their behavior, and for the other integrals there are two saddle points instead of one.

For both odd and even n to find $f_{n,o}$ we take the contribution from the two saddle points and obtain, for $z = re^{+\pi i} = -\xi$

$$f_{n,o}(-z) \sim -i\sqrt{\frac{2\pi}{n\xi^{(n-1)/n}}}\; e^{\pi i/n}\left\{\exp\left[\frac{-n}{n+1}\;\xi^{(n+1)/n}\,e^{\pi i/n} - \frac{\pi i}{2n}\right] + \exp\left[\frac{-n}{n+1}\;\xi^{(n+1)/n}\,e^{-\pi i/n} + \frac{\pi i}{2n}\right]\right\}. \tag{18}$$

For $f_{n,n/2}$ the principal domain of the defining curve contains no saddle points. The defining curve may be deformed into the curve of steepest descent only from the two saddle points of the adjoining region outward (to the left). However, the two saddle points can be joined by paths of stationary phase. Thus we obtain

$$f_{n,n/2} \sim \sqrt{\frac{2\pi}{n\,z^{(n-1)/n}}}\left\{e^{\pi i-\pi i/2n}\exp\left[\frac{n}{n+1}\;z^{(n+1)/n}\;e^{(n-2)\pi i/n}\right] + e^{\pi i/2n}\exp\left[\frac{-n}{n+1}\;z^{(n+1)/n}\right]\right\} \tag{19}$$

where again the prinicpal root is meant. If we write

$$z = e^{\pi i}\,\xi\,,$$

so that ξ is real and positive, then

$$f_{n,n/2}(-z) \sim -i\sqrt{\frac{2\pi e^{\pi i/n}}{n\,\xi^{(n-1)/n}}}\left\{\exp\left[\frac{n}{n+1}\;\xi^{(n+1)/n}\,e^{\pi i/n} + \frac{\pi i}{2n}\right] - \exp\left[\frac{n}{n+1}\,\xi^{(n+1)/n}\;e^{-\pi i/n} - \frac{\pi i}{2n}\right]\right\}$$

Thus

$$f_{n,n/2}(-z) \sim \sqrt{\frac{8\pi e^{\pi i/n}}{n\,\xi^{(n-1)/n}}}\;\exp(\alpha\,\xi^{(n+1)/n})\,\sin\left(\beta\,\xi^{(n+1)/n} + \frac{\pi}{2n}\right), \tag{19a}$$

where $\alpha = \frac{n}{n+1} \quad e^{\pi i/n}$ and $\beta = \frac{n}{n+1} \quad e^{\pi i/n}$. Since α is positive this solution is exponentially growing, and also oscillatory.

We may then conclude that

Theorem 3 The only functions of this class which change from purely exponential to purely oscillatory upon a change of z to minus z are the Airy functions (n = 2).

We are also able to study the small arguement properties of these functions very easily. It is well known that there exists a set of solutions

$$\begin{aligned} y_1 &= (1 + \alpha_1 x^{n+1} + \beta_1 x^{2n+1} + \dots) \\ y_2 &= x\,(1 + \alpha_2 x^{n+1} + \beta_2 x^{2n+1} + \dots) \\ y_n &= x^{n-1}\,(1 + \alpha_n x^{n+1} + \beta_n x^{2n+1} + \dots), \end{aligned} \tag{20}$$

which solutions may be found by the classical method of Frobenius. From lemma 3 these series have infinite radii of convergence. The question is how do these solutions connect with our solutions $f_{n,k}$. This problem has been answered by Turritin (1950). We show how to rederive his answers in a much easier fashion. In each integral we let

$$u = \frac{t^{n+1}}{n+1} \tag{21}$$

so that

$$\begin{aligned} I_{n,k} = {} & \frac{e^{2\pi i(k+1)/(n+1)}}{(n+1)^{n/n+1}} \int_{\infty}^{o} \exp z\,[(n+1)\,e^{2\pi i(k+1)}u]^{\frac{1}{n+1}} \,\frac{e^{-u}du}{u^{n/n+1}} \\ & + \frac{e^{2\pi ik/(n+1)}}{(n+1)^{n/n+1}} \int_{o}^{\infty} \exp z\,[(n+1)\,e^{2\pi ik}\,u]^{\frac{1}{n+1}} \,\frac{e^{-u}du}{u^{n/n+1}} \end{aligned} \tag{22}$$

We now expand the exponential in (22) in powers of z and integrate.

From (20) it is clear that only the first n terms must be obtained. The remaining terms may be obtained more easily by the method of Frobenius and a uniqueness arguement. Each of the integrals in (22) are Gamma functions and thus the coefficient of the n^{th} power of z in (22) is

$$z^m: \exp\left[2\pi i k \left(\frac{1+m}{1+n}\right)\right] \frac{\left\{1 - e^{2\pi i\left(\frac{1+m}{1+n}\right)}\right\}}{m!\ (n+1)^{(n-m)/(n+1)}} \Gamma\left(\frac{1+m}{1+n}\right) \tag{23}$$

Thus, for example,

$$I_{n,1} = e^{2\pi i d}\left\{1 - e^{2\pi i d}\right\} d^{-d}\,\Gamma(d) + z e^{4\pi i d}\left\{1 - e^{4\pi i d}\right\} d^{-2d}\,\Gamma(2d)$$
$$+ z^2 e^{6\pi i d}\left\{1 - e^{6\pi i d}\right\} d^{-3d}\,\Gamma(3d) + \ldots \tag{24}$$

where $d = \frac{1}{n+1}$. By comparison to (20) we see that

$$I_{n,1} = e^{2\pi i d}\left\{1 - e^{2\pi i d}\right\} d^{-d}\,\Gamma(d)\, y_1$$
$$+ e^{4\pi i d}\left\{1 - e^{2\pi i d}\right\} d^{-2d}\,\Gamma(2d)\, y_2 \tag{24a}$$
$$+ \ldots$$

Thus the Stokes' multipliers may be calculated out for each solution. The results agree with those of Turritin (1950) and Heading (1957).

For the purpose of studying the merging of several waves, theorem 3 shows that these solutions are inappropriate. Only one or at most two of the saddle points of (5) are on the imaginary axis, where they must be for steady waves. Thus another integral is appropriate.

Section 2 Merging Waves

It is fairly clear that the correct extension of the Airy function which represents several waves is

$$F(z) = \int_c \exp i \left\{ A + z\, t + a_2 t^2/2! + a_3 t^3/3! + \ldots + a_{n-2}\, t^{n-2}/(n-2)! + t^n/n! \right\} dt \qquad (1)$$

The function F(z) satisfies the differential equation

$$y^{(n-1)}(z) + a_{n-2}\, y^{(n-3)} + \ldots + a_3\, y'' + a_2\, y' + zy = 0 \qquad (2)$$

if c is any path whose end points eventually follow a radial path to infinity in a sector such that t^n has positive imaginary part. The real axis is the limit of such a path in the sense that the axis is taken as the limit as $\varepsilon \to O^+$ of a path from $r \exp i(\pi + (-1)^n \varepsilon)$, where n is the exponent in t^n, to $r \exp i\varepsilon$. Then the problem of representing several waves by (1) is reduced basically to the question of studying the real roots of the derivative of the polynomial in the exponent of the integrand of (1).

The use of (1) as the basic tool for an asymptotic expansion can be approached in several ways. One can try to obtain a uniformly valid approximation to a system of several waves simultaneously. This is basically the approach of Chester, Friedman and Ursell for the case of two waves. The idea proceeds basically as follows. Given that we have a system of several waves described by

$$W(zr) = \int_c G(x) \exp(i z f(x, r))\, dx \qquad (3)$$

where $\frac{\partial f}{\partial k}$ has several real zeros, f tends to ∞ as z tends to ∞ and c is some path deformable to the real axis, we ask how we can uniformly write W(z), asymptotically for large z, as a series of terms involving (1) and its derivatives, taking (1) as a known function. The answer obtained by Chester, Friedman and Ursell for two waves was that the desired representation could be obtained by chosing the coefficients in (1) so that the zeros of $\frac{\partial f}{\partial x}$ corresponded to the zeros of the derivative of the polynomial

$$a_o + a_1 t + t^3/3 \quad .$$

In this manner the implicit change of variables from x to t, as given by

$$z\, f(x, r) = a_o(r) + a_1 t + t^3/3 \quad , \tag{4}$$

is a one-to-one change of variables uniformly for all x and t, which they showed in detail.

It is fairly clear that for the case of n nearby saddle points the appropriate change of variables is given by

$$z\, f(x, r) = a_o(r) + a_1 t + \frac{a_2 t^2}{2} + \ldots + \frac{a_{n-1}}{(n-1)} t^{n-1} + \frac{t^{n+1}}{(n+1)} \tag{5}$$

where the change of variables would be uniformly one-to-one if the coefficients in (5) were chosen so that the zeros of $\frac{\partial f}{\partial x}$ would agree with the zeros of the polynomial

$$t^n + a_{n-1} t^{n-2} + \ldots + a_2 t + z = 0. \tag{6}$$

The proof that (4) is one-to-one and the computation of the coefficients is quite tedeous. The corresponding calculations for (5) would be even more difficult. In fact, for $n > 3$ the problem is unsolvable, in general, for no explicit formula exists for the zeros of an arbitrary quadratic polynomial in terms of the coefficients. However, there is no reason to believe that the transformation would not be one-to-one, despite the computational difficulties. For the case of three merging waves (n = 3) the known formulas for the roots of a cubic would enable one to determine the coefficients of (6) in terms of the zeros of $\frac{\partial f}{\partial x}$.

There is, however, an inherent disadvantage in using the approach of Chester, Friedman and Ursell from the standpoint of relating the expansions thus obtained to the properties of the original function f(x, r). First, for widely separated saddle points, the usual expression for W(r) is in terms of the sum of several integrals, whereas only one integral is obtained (to first order) in this uniform approach. Second, the usual steepest descent methods give the answers directly in terms of the second (or higher) derivatives of f at each saddle point. This uniform approach gives the answers directly in terms of the values of f only, at several different points. While it is possible to then derive expressions for the various derivatives, the calculations are tedeous.

For the purpose of studying the wave behavior as the waves merge it is more convenient to adopt a non-uniform approach. In the vicinity of n merging saddle points the ideas of the uniform approach show that the function f must be approximated by an $(n+1)^{th}$ order polynomial. The question is how to chose the coefficients. The uniform approach is to fit the slope of the polynomial to that of f at n points. We chose instead to match one point as well as possible, by matching the Taylor series expansion. We may thus study the behavior of that particular wave.

Let the saddle point be at the origin x = 0. Then we have

$$z f(x, r) = z\left(f(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots\right) \qquad (7)$$

We wish to approximate this by

$$g(t, r) = a_o + a_1 (t - t_o) + \ldots + \frac{a_{n-1}}{(n-1)} (t - t_o)^{n-1} + \frac{(t - t_o)^{n+1}}{n+1} . \qquad (5a)$$

Our approximation procedure is to close the variable t, the expansion point t_o and the coefficients a_n to match the first n terms of (7) exactly. Expanding (5a) and equating like powers, the coefficients of t^{n+1} and t^n give

$$t = \sqrt[n+1]{\frac{z f^{(n+1)}(0)}{n!}} \; x, \qquad (9)$$

$$t_o = \sqrt[n+1]{\frac{z}{n!}} \quad \frac{f^{(n)}(0)}{[f^{(n+1)}(0)]^{n/(n+1)}} . \qquad (10)$$

The expressions for the coefficients a_i are more complicated. We note that for a_o we

$$a_o = z f(o) + z_1 t_o - a_2 t^2_o + \ldots + \frac{(-1)^{n+1}}{n+1} t_o^{n+1} . \qquad (11)$$

Thus from (10) we see that if the n^{th} derivative vanishes, $t_o = o$ and so (11) reduces to the usual simple form $a_o = z f(o)$. It is clear that the coefficients a_n can be computed iteratively and thus any degree polynomial can be solved for. (This is in contrast to the uniform approach which requires that the roots of an n^{th} degree polynomial be found -- a procedure generally impossible for $n \geq 5$.) We exhibit these coefficients explicitly for n = 4.

$$a_2 = 3 \left(\frac{z}{3!}\right)^{1/2} \left\{ 2 f'' f^{(iv)} - (f''')^2 \right\} (f^{iv})^{-6/4}$$

$$a_1 = 2 \left(\frac{z}{3!}\right)^{3/4} \left\{ 3 f'' f^{(iv)} - (f''')^2 \right\} f''' (f^{iv})^{-9/4} \qquad (12)$$

$$a_o = z \left\{ f(o) + \frac{1}{8} \frac{(f''')^2}{(f^{(iv)})^3} [4 f'' f^{(iv)} - (f''')^2] \right\}$$

Thus the study of (3), to first order, reduces to the study of (for these waves)

$$h(z) = \int \exp i \left\{ a_o + a_1 t + a_2 t^2/2 + t^4/4 \right\} \frac{dx}{dt} \, dt \tag{13}$$

$$\approx \left(\frac{3!}{z f^{(iv)}} \right)^{1/4} \exp ia_o \int \exp i \left\{ a_1 t + \frac{a_2}{2} t^2 + t^4/4 \right\} dt \; .$$

A closed form evaluation of the integral on the right hand side of (13) is not known. However, from (12) we can see what are the various possible limits which will give expressions such that the integral can be evaluated conveniently.

First we note that if $f''(o)$ and $f'''(o)$ are both zero, that is the three waves have merged, then we have $a_1 = a_2 = 0$, $a_o = z\, f(o)$ and thus we have the expected answer,

$$h(z) \approx c \left(\frac{3!}{z f^{(iv)}(o)} \right)^{1/4} \exp i\, z\, f(o), \tag{14}$$

where c is the constant $(= \frac{1}{2} (\frac{4}{i})^{1/4} \Gamma(\frac{1}{4}))$ gotten by evaluating $\int \exp(\frac{it^4}{4})dt$, in terms of the gamma function.

For sufficiently small $f''(o)$ and $f'''(o)$ (measured against powers of z according to (12)) we may evaluate

$$F(z) = \int \exp i \left\{ a_1 t + \frac{a_2}{2} t^2 + t^4/4 \right\} dt \tag{15}$$

by

$$F(z) \approx \int \left\{ 1 + ia_1 t - \frac{a_1^2 t^2}{2} + \ldots + \frac{ia_2 t^2}{2} + \ldots \right\} \exp \frac{it^4}{4} \; dt \; .$$

Letting

$$i \frac{t^4}{4} = - u$$

then

$$F(z) \approx \frac{1}{2}\left(\frac{4}{i}\right)^{1/4}\left\{\Gamma\left(\frac{1}{4}\right) + \frac{(-4i)^{5/4}}{4}\, a_1\,\Gamma\left(\frac{1}{2}\right) + \frac{(-4i)^{3/2}}{8}\,(a_2 + i a_1{}^2)\,\Gamma\left(\frac{3}{4}\right) + \dots\right\}$$

We have then

<u>Theorem 4</u> For sufficiently small f'' and f''',

$$h(z) \approx \frac{1}{2}\left(\frac{4!}{izf^{iv}}\right)^{1/4} \exp(i\, a_o)\left\{\Gamma\left(\frac{1}{4}\right) - 8\left(\frac{\pm izf^{iv}}{4!}\right)^{3/4} \frac{[3f''f^{iv} - (f''')^2]}{(f^{iv})^3}\,\Gamma\left(\frac{1}{2}\right)f'''\right.$$

$$-6\left(\frac{+izf^{iv}}{4!}\right)^{1/2} [2f''f^{iv} - (f''')^2]\;\frac{\Gamma(3/4)}{(f^{iv})^2} \qquad (16)$$

$$\left. -32\left(\frac{+izf^{iv}}{4!}\right)^{3/2} [3f''f^{iv} - (f''')^2]^2\;\frac{(f''')^2\,\Gamma(3/4)}{(f^{iv})^6} + \dots\right\}.$$

For the case f''(o) = 0 then (16) reduces to

$$h(z) \approx \frac{1}{2}\left(\frac{4!}{izf^{iv}}\right)^{1/4}\left\{\exp i\, a_o\right\}\left\{\Gamma\left(\frac{1}{4}\right) + 6\left(\frac{i\,(z^{1/4} f''')^4}{4!\,(f^{iv})^3}\right)^{1/2}\Gamma\left(\frac{3}{4}\right)\right.$$

$$\left. + 8\left(\frac{i\,(z^{1/4} f''')^4}{4!\,(f^{iv})^3}\right)^{3/4}\Gamma(1/2) + \dots\right\} \qquad (17)$$

It is clear that the approach to the strictly merged case goes as a power of the ratio $\left(\frac{(z^{1/4} f''')^4}{(f^{iv})^3}\right)$. And from (12) the expression for a_o depends also upon this ratio. This is our measure of smallness referred to above. If f''' tends to zero fast enough so that this ratio remains less than 1 as z gets large then (17) is an adequate approximation.

On the other hand, consider the case where two saddle points approach the third symmetrically so that f'''(o) = 0. Then (16) reduces to

$$h(z) \approx \frac{1}{2}\left(\frac{4!}{izf^{iv}}\right)^{1/4} \exp izf(o)\left\{\Gamma\left(\frac{1}{4}\right) - 12\left(\frac{iz(f'')^2}{4!\;f^{iv}}\right)^{1/2}\Gamma\left(\frac{3}{4}\right) + \dots\right\} \qquad (18)$$

In contrast to the previous case where $f''' = o(z^{-1/4})$ for an adequate approximation, we need $f'' = o(z^{-1/2})$. Also, for real values of $f^{(n)}$, the initial correction to the strictly merged value was independent of the sign of f''' in the previous case. For this case the correction depends upon the sign of f''.

We now consider what happens if the saddle points are not too close. We may evaluate (13) itself by a simple saddle point method. Let

$$k(t) = a_1 t + a_2 t^2/2 + t^4/4 \tag{19}$$

Then the saddle points of k(t) are at

$$a_1 + a_2 t + t^3 = 0 \tag{20}$$

Using standard methods for solving a cubic we see that the nature of the roots depend upon the sign of the discriminant

$$D = \frac{a_1^2}{4} + \frac{a_2^3}{27}$$

$$= \left(\frac{z}{3!}\right)^{3/2} \left(f^{(iv)}\right)^{-5/2} \left\{ 8 f'' f^{(iv)} - 3 (f''')^2 \right\} (f'')^2 . \tag{21}$$

If the quantities in (21) are real, then it is well known (see any algebra text) that (20) has 3 real unequal roots if $D < 0$, three real roots -- at least two of which are equal -- if $D = 0$ and one real root (2 complex) if $D > 0$. A complex root gives an exponentially varying solution. The quantity (19) represents an expansion of f(t) about one of the three waves. Thus initially we must have $D < 0$. One of these three roots represents the point about which the expansion is made, and accurately represents the wave in question. As shown in the case of two waves in Comstock (1966), the other two roots represent the other two waves, much less accurately. At the merging we have f'' and f''' both vanishing and thus D vanishes. We may ask what happens after the merging. The answer depends upon whether the derivatives change sign or not. Curiously enough, a change in sign of (f''') alone does not, from (21), change the nature of the results. There are still 3 real roots. It is the sign of f'' and the relative magnitudes of f'' and f''' which determine whether "after" the merging there will be three waves or one. Since

$$k(t_o) = \frac{1}{4} t_o (3a_1 + a_2 t_o), \tag{22}$$

then as long as any of the saddle points are on the real axis, that axis is a path of stationary phase. If two of the saddle points are complex, then the major contribution to the integral is just from the one real root and there is just one wave. By tracing the roots through zero one can determine which of the waves survives. In this case

$$h(z) \approx \left(\frac{3!}{zf^{iv}}\right)^{1/4} \exp i\left(a_o + \frac{3}{4} a_1 t_o + \frac{1}{4} a_2 t_o^2\right) \sqrt{\frac{-2i\pi}{a_2 + 3t_o^2}} \tag{23}$$

where t_o is the real solution of (20). Since a_2 and t_o^2 both behave as $\sqrt{z}$ then (23) behaves as $(\sqrt{z})^{-1}$, as expected.

We have seen how, after the merging of the three waves, there may emerge one wave or three, depending upon the values of the derivatives of f near the merger point.

If D does not change sign at the merging point, then all three waves survive. However, there may be an interchange of energy. One can trace the three waves by following the roots through the zero. The necessary formulas to do so are given below in (24). One must remember that only one of the roots represents accurately the amplitude of the wave, since the expansion is about one of the waves.

Is it possible that no wave will emerge? For this to happen, all three roots of (20) must be complex. This can happen only if the coefficients of (20) are complex. Since the formula (23) is valid for any of the saddle points, we look at it. The combination of the coefficients a_i and the saddle point t_o in the exponent are such that the exponent is proportional to z. Thus while a change in sign of z makes a_i complex, it still does not make the exponent complex. Complex values of the derivatives of f are sufficient. It is unlikely, however, that this would happen. Thus one would not expect a complete cancellation of all of the waves.

To see in more detail the behavior for the waves, one needs the complete formulae for the saddle point. We give them here.

$$A = \sqrt[4]{z/3!}\,(f^{iv})^{-3/4}\sqrt{f'''\left\{3f''f^{iv} - (f''')^2\right\} + f''f^{iv}\sqrt{8f''f^{iv} - 3(f''')^2}}\,,$$

$$B = \sqrt[4]{z/3!}\,(f^{iv})^{-3/4}\sqrt{f'''\left\{3f''f^{iv} - (f''')^2\right\} - f''f^{iv}\sqrt{8f''f^{iv} - 3(f''')^2}}\,,$$

$$t_o = A + B,\ -\left(\frac{A+B}{2}\right) + \left(\frac{A-B}{2}\right) i\sqrt{3},\ -\left(\frac{A+B}{2}\right) - \left(\frac{A-B}{2}\right) i\sqrt{3} \quad . \tag{24}$$

These formulae are sufficiently complicated that an analysis for general functions f is not particularly valuable. We have discussed those aspects which are readily amenable to discussion, and have indicated how one might study the merging of several waves. We have the further details to a reader who has a specific case to investigate.

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