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Revision 1

Power Spectral Density Analysis

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Abstract

This Technical Report develops the generalized techniques for determining the equation describing the power spectral density function (G^2/cps versus frequency, etc.) and the equation for determining the root mean square of a power spectral density function. Examples of both types of equations are included in the Appendix.

Power Spectral Density Analysis

I. Introduction

Environmental test specifications require an understanding of the theory and the functional (testing) techniques of power spectral density (PSD) analysis. These specifications will define a PSD function over some given frequency band. The ordinate of this function will be some quantity which is proportional to power, such as V^2/cps or G^2/cps (where V = the voltage and G = the ratio of the test acceleration to the acceleration of gravity).

In order to define accurately the actual tests from the test specifications and to evaluate the test results, a general procedure will be given that covers all possible present and future test specifications.

II. Theory

Figure 1 represents a generalized PSD function $Y = Y$ (frequency), where Y is a quantity that is proportional to power. In this particular analysis, the PSD function will have the form Y (decibels) versus log (frequency). (Note that no particular scale is shown, since this figure represents only the most generalized case.) In terms of V or G, the expressions for Y are:

$$Y = 10 \log_{10} \left[\frac{(G^2/\text{cps})}{(G_0^2/\text{cps})} \right] \tag{1a}$$

or

$$Y = 10 \log_{10} \left[\frac{(V^2/\text{cps})}{(V_0^2/\text{cps})} \right] \tag{1b}$$

Note that (G_0^2/cps) and (V_0^2/cps) are the reference levels for their corresponding PSD function. The decibel scale is based on this reference level; therefore, the reference must always be given for any absolute level measurements or calculations, as seen in Eqs. (1c) and (1d).

A graph of Y versus f is therefore made on log-log paper; the following derivations are based on a graph consisting of straight line segments, as plotted on log-log paper (Fig. 2). The derivations will be done in terms of $Y = \log y$, and the derivation will be the same for a PSD of any power-like quantity.

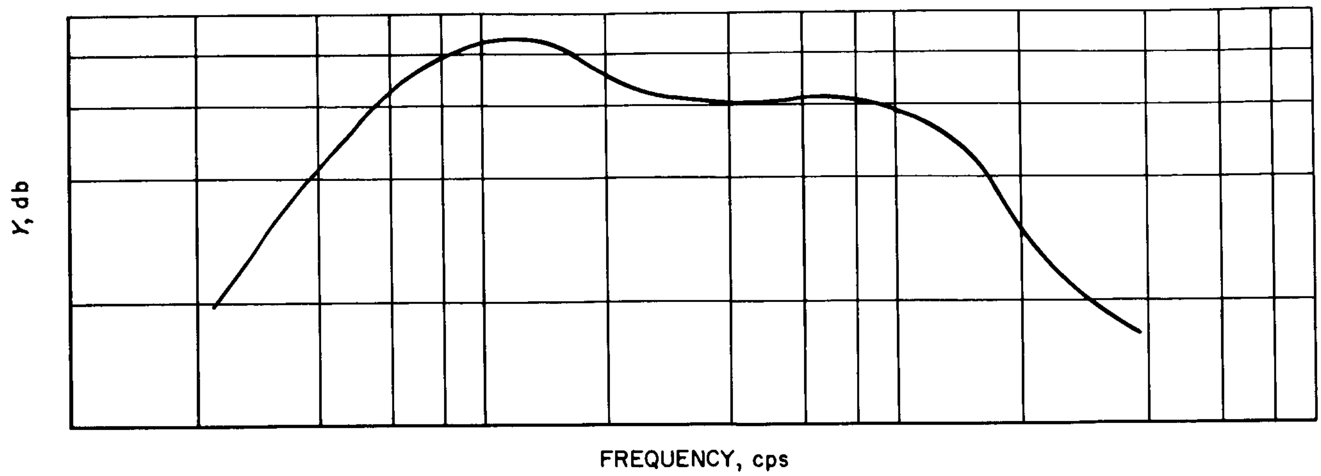

Fig. 1. Generalized PSD function

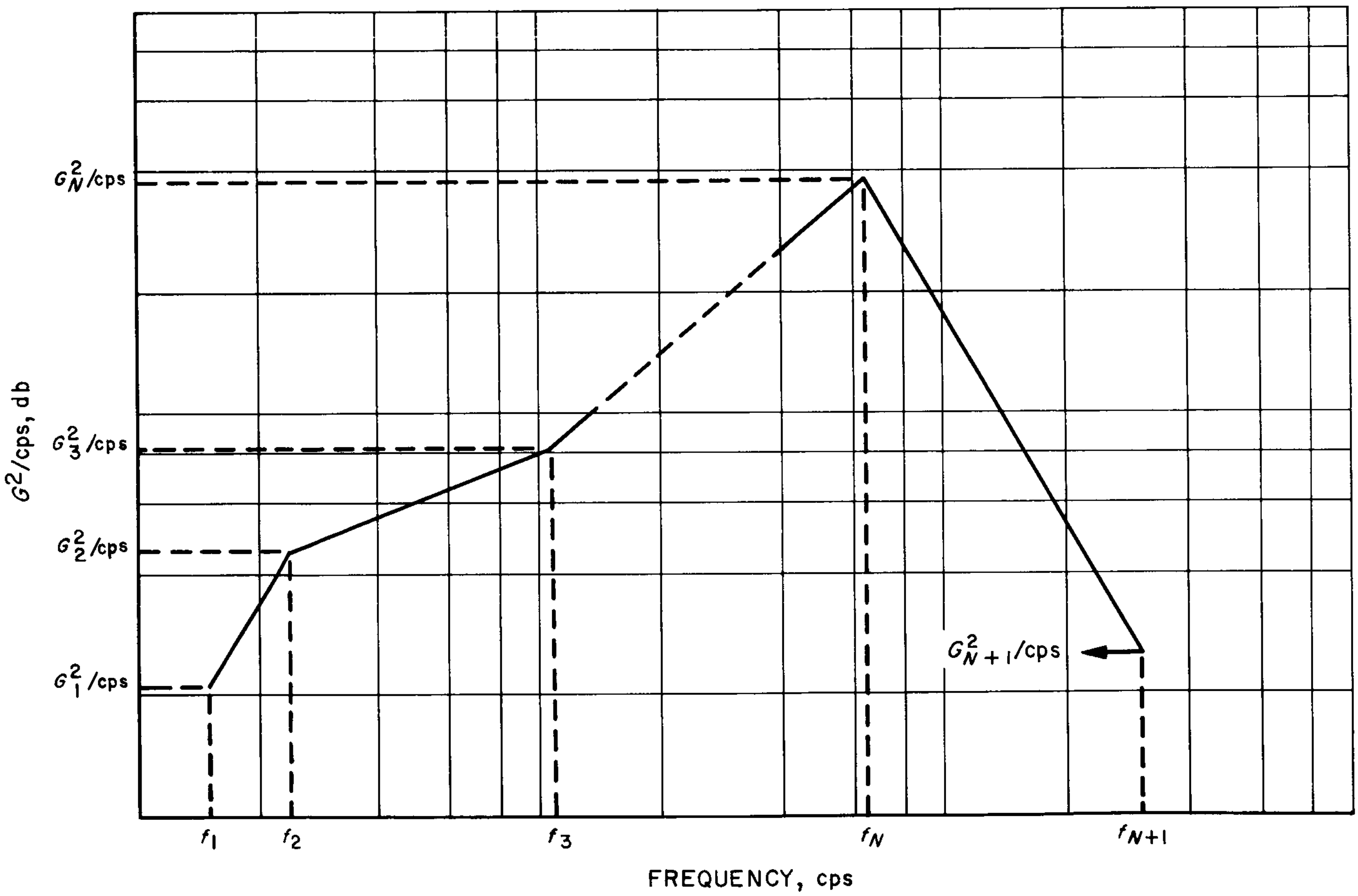

Fig. 2. PSD function with constant decibel-per-octave slopes

It should be noted that the value of $G^2/\text{cps} \equiv G_i^2/\text{cps}$, corresponding to a particular decibel reading Y_i, would be given by [starting with Eq. (1a)]

$$Y_i\,(\text{db}) = 10 \log \left[\frac{(G_i^2/\text{cps})}{(G_0^2/\text{cps})}\right] \quad (1c)$$

Therefore

$$G_i^2/\text{cps} = (G_0^2/\text{cps})\, 10^{(0.10)Y_i(\text{db})} \quad (1d)$$

and

$$V_i^2/\text{cps} = (V_0^2/\text{cps})\, 10^{(0.10)Y_i(\text{db})} \quad (1e)$$

The equation of the "line segment" (between Frequencies 1 and 2) of Fig. 2 is given by

$$Y - Y_1 = M\,(X - X_1) \quad (2a)$$

where

$$M = \left(\frac{Y_2 - Y_1}{X_2 - X_1}\right) \quad (2b)$$

Since Y is plotted in decibels, then the value of Y which corresponds directly to a power quantity is given by[1]

$$Y = \log_{10}(y) \qquad \text{and} \qquad X = \log_{10} f \quad (3)$$

where

$$y = G^2/\text{cps} \qquad \text{or} \qquad V^2/\text{cps, etc.} \quad (4)$$

Therefore

$$\log y - \log y_1 = \left(\frac{\log y_2 - \log y_1}{\log f_2 - \log f_1}\right)(\log f - \log f_1) \quad (5)$$

or

$$\log\left(\frac{y}{y_1}\right) = \left[\frac{\log(y_2/y_1)}{\log(f_2/f_1)}\right] \log\left(\frac{f}{f_1}\right) \quad (6)$$

Therefore

$$y = y_1\,(f/f_1)^M = (y_1 f_1^{-M})\, f^M \quad (7a)$$

where

$$M = \frac{\log(y_2/y_1)}{\log(f_2/f_1)} \quad (7b)$$

[1]For the remainder of this report, the following notation will be used: $\log_{10} N = \log N$.

Equations (7a) and (7b) represent the general expressions for describing the PSD function.

For the special case where the slope (M) is given in terms of A in decibels/octave (where A may be positive, negative, or zero)

$$10 \log\left(\frac{y_2}{y_1}\right) = A_1, \quad \text{db/octave} \quad (8)$$

We obtain the following expressions for Eqs. (7a) and (7b). For the octave condition given in Eq. (8)

$$f_2 = 2 f_1 \quad (9)$$

Therefore

$$M = \frac{(0.10)\,A_1}{\log(2)} = (0.3322)\,A_1 \quad (10)$$

Thus, Eq. (7a) becomes, in terms of the lower limits

$$y = (y_1 f_1^{-0.3322A_1})\, f^{0.3322A_1} \quad (11)$$

The relationship between y_1 and y_2 is given by

$$y_2 = y_1 \left(\frac{f_2}{f_1}\right)^{0.3322A_1} \quad (12)$$

Therefore, Eq. (11) may be given, in terms of the upper limits, as

$$y = (y_2 f_2^{-0.3322A_1})\, f^{0.3322A_1} \quad (13)$$

Equation (11) represents any of the line segments of Fig. 2 with the following values for y_1, f_1, and A_1:

f_1 = lowest frequency over which the particular line segment is defined (14a)

y_1 = value of the ordinate ("power-like" quantity) which corresponds to f_1 (14b)

A_1 = value of the constant decibel-per-octave slope of the particular line segment (Fig. 2) (14c)

Any of the line segments of Fig. 2 may also be represented with the following values for y_2, f_2, and A_1 in Eq. (13):

f_2 = highest frequency over which the particular line segment is defined (15a)

y_2 = value of the ordinate ("power-like" quantity) which corresponds to f_2 (15b)

A_1 = same value as in Eq. (14c) (15c)

For the special case where $y = G^2/\text{cps}$, we obtain the following expressions:

$$G^2/\text{cps} = [(G_1^2/\text{cps})\, f_1^{-0.3322A}]\, f^{0.3322A} \quad (16a)$$

$$G^2/\text{cps} = [(G_2^2/\text{cps})\, f_2^{-0.3322A}]\, f^{0.3322A} \quad (16b)$$

with

$$(G_2^2/\text{cps}) = (G_1^2/\text{cps})\left(\frac{f_2}{f_1}\right)^{0.3322A} \quad (17)$$

Equations (16a), (16b), and (17) summarize the equations describing the PSD functions. (See the Appendix for special cases of A.) Table 1 gives values of 0.3322A versus A for common PSD slopes.

The following derivation describes the method of determining the RMS of a PSD graph.

The generalized form for the (RMS) of a value of $G = H(f)$ is given by (Fig. 2).

$$G\,(RMS) = \left\{\int_{f_1}^{f_{N+1}} [G^2(f)/\text{cps}]\, df\right\}^{1/2} \quad (18)$$

with an equivalent form for V_{RMS} etc. For simplification, Eqs. (16a) and (16b) will be rewritten:

$$G^2/\text{cps} = [(G_1^2/\text{cps})\, f_1^{-A_1/3.01}]\, f^{A_1/3.01} \quad (19)$$

in terms of the lower limit values and

$$G^2/\text{cps} = [(G_2^2/\text{cps})\, f_2^{-A_1/3.01}]\, f^{A_1/3.01} \quad (20)$$

Table 1. Values of 0.3322A versus A for common PSD decibel/octave slopes

A, db/octave	0.3322A
0	0
±3	±0.9966
±6	±1.9932
±9	±2.9898
±12	±3.9864
±15	±4.9830
±18	±5.9796
±21	±6.9762
±24	±7.9728
±48	±15.9496

in terms of the upper limit values. Therefore, Eq. (18) becomes

$$G\,(RMS) = \left(\sum_{N=1}^{P} B_N\right)^{1/2} \quad (21)$$

where P = the number of line segments of the PSD curve, and

$$B_N = \int_{f_N}^{f_{N+1}} [(G_N^2/\text{cps})\, f_N^{-A_N/3.01}]\, f^{A_N/3.01}\, df \quad (22a)$$

Therefore

$$B_N = \frac{3.01}{A_N + 3.01}\Big)\,(G_N^2/\text{cps}) \times (f_N^{-A_N/3.01})\,[f_{N+1}^{(A_N+3.01)/3.01} - f_N^{(A_N+3.01)/3.01}] \quad (22b)$$

which is in the terms of the lower limit values, or

$$B_N = \left(\frac{3.01}{A_N + 3.01}\right)(G_{N+1}^2/\text{cps}) \times (f_{N+1}^{-A_N/3.01})\,[f_{N+1}^{(A_N+3.01)/3.01} - f_N^{(A_N+3.01)/3.01}] \quad (22c)$$

which is in terms of the upper limit values. Note that Eqs. (22b) and (22c) hold for all values of A except $A = -3.01$. The equation for this case follows. Starting with Eq. (22a) with $A = -3.01$:

$$B_N = \int_{f_N}^{f_{N+1}} [(G_N^2/\text{cps})\, f_N]\, f^{-1}\, df$$

$$= [(G_N^2/\text{cps})\, f_N] \int_{f_N}^{f_{N+1}} \left(\frac{df}{f}\right) \qquad (23)$$

Therefore

$$B_N = [(G_N^2/\text{cps})\, f_N]\, \log_\varepsilon \left(\frac{f_{N+1}}{f_N}\right)$$

$$= [(G_N^2/\text{cps})\, f_N]\; 2.30 \log_{10} \left(\frac{f_{N+1}}{f_N}\right) \qquad (24)$$

which is in terms of the lower limit values, or

$$B_N = [(G_{N+1}^2/\text{cps})\, f_{N+1}]\, \log_\varepsilon \left(\frac{f_{N+1}}{f_N}\right)$$

$$= [(G_{N+1}^2/\text{cps})\, f_{N+1}]\; 2.30 \log_{10} \left(\frac{f_{N+1}}{f_N}\right) \qquad (25)$$

which is in terms of the upper limit values.

III. Summary of Equations

This section of this Technical Report consists of a summary of the equations needed to:

(1) Describe the PSD function (given in terms of G^2/cps as a representative "power-like" quantity).

(a) The general equation for a straight-line-segmented PSD curve on log-log graph paper:

$$G^2/\text{cps} = [(G_1^2/\text{cps})\, f_1^{-M}]\, f^M \qquad (1)$$

in terms of lower limit values, or

$$G^2/\text{cps} = [(G_2^2/\text{cps})\, f_2^{-M}]\, f^M \qquad (2)$$

in terms of upper limit values, where

$$M = \log \left[\frac{(G_2^2/\text{cps})}{(G_1^2/\text{cps})}\right] \qquad (3)$$

and

$$G_2^2/\text{cps} = (G_1^2/\text{cps})\, (f_2/f_1)^M \qquad (4)$$

is the relationship between G_1^2/cps and G_2^2/cps for a given line segment.

(b) The special case of the slope of a straight-line-segmented PSD curve expressed in A_1, db/octave, on log-log graph paper:

$$G_i^2/\text{cps} = (G_0^2/\text{cps})\, 10^{0.10 Y_i\,(\text{db})} \qquad (1)$$

where G_0^2/cps is the decibel reference level.

$$M = 0.3322A \qquad (2)$$

Therefore

$$G^2/\text{cps} = [(G_1^2/\text{cps})\, f_1^{-0.3322A}]\, f^{0.3322A} \qquad (3)$$

in terms of lower limit values, or

$$G^2/\text{cps} = [(G_2^2/\text{cps})\, f_2^{-0.3322A}]\, f^{0.3322A} \qquad (4)$$

in terms of upper limit values.

$$G_2^2/\text{cps} = (G_1^2/\text{cps})\, (f_2/f_1)^{0.3322A} \qquad (5)$$

(2) Determine the RMS of a PSD function.

(a) General equation for RMS:

$$G_{RMS} = \left\{\int_{f_1}^{f_{N+1}} [G^2(f)/\text{cps}]\, df\right\}^{1/2} \qquad (6)$$

(b) The special case of the slope of a straight-line-segmented PSD curve expressed in A, db/octave, on log-log graph paper:

$$G_{RMS} = \left(\sum_{N=1}^{P} B_N\right)^{1/2} \qquad (7)$$

where P = the number of sections (line segments); therefore

$$B_N = \left(\frac{3.01}{A_N + 3.01}\right)(G_N^2/\text{cps})$$

$$\times \left(f_N^{-A_N/3.01}\right)\left[f_{N+1}^{(A_N+3.01)/3.01} - f_N^{(A_N+3.01)/3.01}\right] \qquad (8)$$

in terms of lower limit values, or

$$B_N = \left(\frac{3.01}{A_N + 3.01}\right)(G_{N+1}^2/\text{cps})$$

$$\times \left(f_{N+1}^{-A_{N-1}/3.01}\right)\left[f_{N+1}^{(A_N+3.01)/3.01} - f_N^{(A_N+3.01)/3.01}\right] \qquad (9)$$

in terms of upper limit values. The special case of the slope = −3.01:

$$B_N = [G_N^2/\text{cps})\, f_N] \log_\varepsilon \left(\frac{f_{N+1}}{f_N}\right)$$

$$= [(G_N^2/\text{cps})\, f_N]\; 2.30 \log_{10} \left(\frac{f_{N+1}}{f_N}\right) \qquad (10)$$

in terms of lower limit values, or:

$$B_N = [(G_{N+1}^2/\text{cps})\, f_{N+1}] \log_\varepsilon \left(\frac{f_{N+1}}{f_N}\right)$$

$$= [(G_{N+1}^2/\text{cps})\, f_{N+1}]\; 2.30 \log_{10} \left(\frac{f_{N+1}}{f_N}\right) \qquad (11)$$

in terms of upper limit values.

Appendix
Special Case Values

I. Special Case Values for Fig. 1

$$G_0^2/\text{cps} = 1.00 \qquad (1)$$

$$f_1 = 50 \text{ cps} \qquad (2)$$

$$f_2 = 100 \text{ cps} \qquad (3)$$

$$f_3 = 1000 \text{ cps} \qquad (4)$$

$$f_4 = 2000 \text{ cps} \qquad (5)$$

$$A_1 \text{ (db/octave)} = +3 \qquad (6)$$

$$A_2 \text{ (db/octave)} = 0 \qquad (7)$$

$$A_3 \text{ (db/octave)} = -12 \qquad (8)$$

The examples of equations on p. 8 will be given with reference to Fig. A-1.

Table A-1. Summary for special case of Fig. 1

Curve data	Area under curve sections			G_{RMS}
	B_1: f_1 to f_2	B_2: f_2 to f_3	B_3: f_3 to f_4	
G_0^2/cps = 1.00 f_1 = 50 cps f_2 = 100 cps f_3 = 1000 cps f_4 = 2000 cps A_1 = + 3 db/octave A_2 = 0 db/octave A_3 = − 12 db/octave	18.80	450.90	146.53	24.8

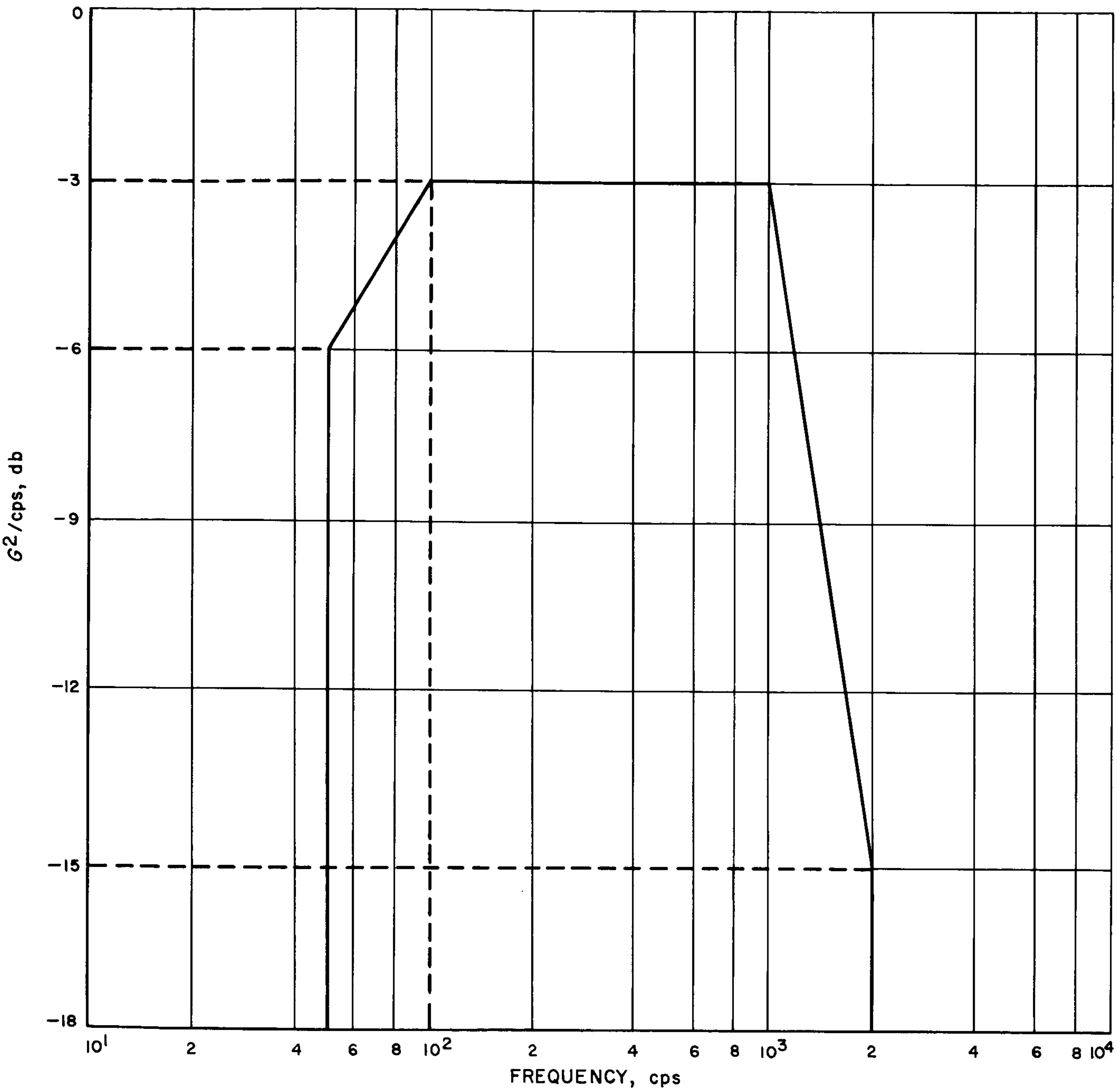

Fig. A-1. PSD function for example solved in Appendix

A. Examples of Equations Describing the PSD Function for Special Case of Straight-Line-Segmented PSD of Fig. A-1

1. *Line segment between f_1 and f_2 ($A_1 = +3$ db/octave).*

$$G_i^2/\text{cps} = (G_0^2/\text{cps})\, 10^{0.10 Y_i (\text{db})} = (1.0)\, 10^{+(0.10)(-6)} = 0.251 \quad (1a)$$

where $Y_i = -6$ db.

$$\underline{\underline{G_i^2/\text{cps} \equiv G_1^2/\text{cps} = 0.251}} \quad (1b)$$

$$\begin{aligned} G^2/\text{cps} &= [(G_1^2/\text{cps})\, f_1^{-0.3322A}\, f^{0.3322A} \\ &= (0.251)\,(50)^{(-0.3322)(3)}\, f^{(0.3322)3} \\ &= (5.10)\, 10^{-3}\, f^{0.9966} \end{aligned} \quad (2a)$$

Therefore

$$\underline{\underline{G^2/\text{cps} = (5.10)\, 10^{-3}\, f^{0.9966}}} \quad (2b)$$

the equation of the PSD curve between f_1 and f_2.

2. *Line segment between f_2 and f_3 ($A_2 = 0$).*

$$G_i^2/\text{cps} \equiv G_2^2/\text{cps} = (1.0)\, 10^{+(0.10)(-3)} = 0.501 \quad (3)$$

where $Y_i = -3$ db. Therefore

$$\underline{\underline{G_2^2/\text{cps} = 0.501}} \quad (4a)$$

$$G^2/\text{cps} = [(G_2^2/\text{cps})\, f_2^{-(0.3322)(0)}]\, f^{(0.3322)(0)} = G_2^2/\text{cps} \quad (4b)$$

Therefore

$$\underline{\underline{G^2/\text{cps} = 0.501 \text{ (constant)}}} \quad (4c)$$

the equation of the PSD curve between f_2 and f_3.

3. *Line segment between f_3 and f_4 ($A_3 = -12$ db/octave).*

$$G_i^2/\text{cps} \equiv G_3^2/\text{cps} = (1.0)\, 10^{(0.10)(-3)} = 0.501 \quad (5)$$

$$\begin{aligned} G^2/\text{cps} &= [(G_3^2/\text{cps})\, f_3^{(-0.3322)(-12)}]\, f^{(0.3322)(-12)} \\ &= [(4.561)\, 10^{11}]\, f^{-3.9864} \end{aligned} \quad (6a)$$

Therefore

$$G^2/\text{cps} = [(4.561)\, 10^{11}]\, f^{-3.9864} \quad (6b)$$

the equation of the curve between f_3 and f_4, etc.

B. Example of Determining the (RMS) for Special Case of Straight-Line-Segmented PSD of Fig. A-1

1. *Values of B_N ($N = 1,2,3$), where*

$$B_N = \left(\frac{3.01}{A_N + 3.01}\right)(G_N^2/\text{cps}) \times (f_N^{-A_N/3.01})\, [f_{N+1}^{(A_N+3.01)/3.01} - f_N^{(A_N+3.01)/3.01}] \quad (7)$$

Here the "lower limits" equation is being used.

($N = 1$):

$$\begin{aligned} B_1 &= \left(\frac{3.01}{A_1 + 3.01}\right)(G_1^2/\text{cps}) \times (f_1^{-A_1/3.01})\, [f_2^{(A_1+3.01)/3.01} - f_1^{(A_1+3.01)/3.01}] \\ &= \left(\frac{3.01}{3.00 + 3.01}\right)(0.251)\,(50^{-3/3.01}) \times [100^{(3.00+3.01)/3.01} - 50^{(3.00+3.01)/3.01}] \end{aligned} \quad (8a)$$

Therefore

$$B_1 = 18.80 \quad (8b)$$

($N = 2$):

$$\begin{aligned} B_2 &= \left(\frac{3.01}{A_2 + 3.01}\right) G_2^2/\text{cps}\, (f_2^{-A_2/3.01}) \times [f_3^{(A_2+3.01)/3.01} - f_2^{(A_2+3.01)/3.01}] \\ &= \left(\frac{3.01}{0 + 3.01}\right)(0.501)\,(f_2^{0/3.01}) \times [f_3^{(0+3.01)/3.01} - f_2^{(0+3.01)/3.01}] \end{aligned} \quad (9a)$$

Therefore

$$B_2 = 450.90 \quad (9b)$$

• $(N = 3)$:

$$B_3 = \left(\frac{3.01}{A_3 + 3.01}\right) (G_3^2/\text{cps}) (f_3^{-A_3/3.01})$$
$$\times [f_4^{(A_3+3.01)/3.01} - f_3^{(A_3+3.01)/3.01}]$$
$$= \left(\frac{3.01}{-12.00 + 3.01}\right) (0.501) [1000^{-(-12)/3.01}]$$
$$\times [2000^{(-12.00+3.01)/3.01} - 1000^{(-12.00+3.01)/3.01}] \quad (10a)$$

Therefore

$$B_3 = 146.53 \quad (10b)$$

2. *Value of G_{RMS}, where*

$$G_{RMS} = \left(\sum_{i=1}^{N} B_N\right)^{1/2} = (B_1 + B_2 + B_3)^{1/2} \quad (11)$$

$$G_{RMS} = [(18.80) + (450.90) + (146.53)]^{1/2}$$
$$= (616.23)^{1/2} \quad (12a)$$

Therefore

$$G_{RMS} = 24.8 \quad (12b)$$