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# LIQUID SLOSHING IN ELASTIC CONTAINERS

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## ABSTRACT

The coupled oscillations of containers, partially filled with an inviscid and incompressible liquid, are investigated. The following container configurations are considered: (1) a long, rectangular tank with an elastic bottom, (2) a long, rectangular tank with elastic walls, and (3) a cylindrical tank with an elastic wall. Linear theories are used for the description of the motion of the fluid, and that of the container. Numerical values of the natural frequencies are presented for various liquid heights and tank configurations.

## SYMBOLS

| | |
|---|---|
| $D$ | Plate regidity |
| $E$ | Modulus of elasticity |
| $g$ | Gravitiational acceleration |
| $h$ | Plate thickness |
| $H$ | Liquid depth |
| $L$ | Tank width |
| $p$ | Pressure |
| $t$ | Time |
| $w$ | Transverse displacement of plate |
| $x,y,z$ | Cartesian coordinates |
| $\varphi$ | Velocity potential |
| $\omega$ | Natural circular frequency |
| $\nu$ | Poisson's ratio |
| $\rho_f$ | Density of liquid |
| $\rho$ | Density of container material |
| $\Omega_m$ | Natural frequencies for rigid bottom tank |

# INTRODUCTION

Numerous authors have used linear and non-linear theories in studies of liquid oscillations in rigid containers. However, work involving coupled oscillations of liquid and elastic containers, is quite limited. Miles [1] accounted for the flexibility of the tank by considering specific natural modes of the tank. Bleich [2] used approximate methods to include the elastic properties of the bottom of the tank and obtained a solution which is valid for large depths of liquid. Bhuta and Koval [3] accounted for the flexibility of the bottom by treating it as a membrane. Siekmann and Chang [4] calculated the natural frequencies of a liquid in a cylindrical tank with elastic bottom, by using a method similar to that of Bhuta and Koval. Huang [5] and [6] studied the longitudinal sloshing of a liquid in an elastic, hemispherical tank, with contradictory results. Coale and Nagano [7] presented an approximate solution for the axisymmetric modes of an elastic cylindrical-hemispherical tank, partially filled with liquid. Only membrane theory is used for the tank and some boundary conditions are not completely satisfied.

The present study is concerned with the coupled oscillation of a long, elastic, rectangular tank, and a cylindrical tank with elastic wall, both partially filled with liquid. The analyses of the containers are based on bending theory.

## BASIC EQUATIONS

The governing equation for the motion of an inviscid and incompressible liquid is

$$\nabla^2 \varphi = 0 \, , \tag{1}$$

where $\varphi$ is the velocity potential, related to the velocity of the particle by

$$\vec{v} = \overline{\nabla}\varphi \, . \tag{2}$$

At the free surface of the liquid the boundary condition is

$$\left[\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z}\right]_{z=H} = 0 \, , \tag{3}$$

where z is the vertical coordinate, as measured from the bottom of the tank, and where H is the height of the liquid. The pressure at any point in the liquid is given by

$$P = \rho_f \frac{\partial \varphi}{\partial t} + \rho_f g(H-Z) \tag{4}$$

where $\rho_f$ is the mass density of the liquid.

The equations of motion of the containers may be written in the following general form

$$[L_{\alpha\beta}]\{u_i\} + \rho h\{\ddot{u}_i\} = \{p_i\} \, , \tag{5}$$

where the $L_{\alpha\beta}$ are spatial differential operators, $u_i$ are the components of displacement for the container with respect to static equilibrium configuration, $\rho$ is the density of the container material, h is the thickness of the tank wall, and $p_i$ is the loading function. For the problem under study only normal loading component exist and is related to the velocity potential as follows:

$$p_i = \rho_f \left.\frac{\partial\varphi}{\partial t}\right|_{\text{at the interface of liquid and tank.}}$$

Equation (5) couples the liquid velocity potential $\varphi$ and the container displacements $u_i$.

As usual, harmonic motion is assumed for the system. The general solutions for $\varphi$ and $u_i$ are then obtained by solving the coupled differential equations (1) and (5) in conjunction with the condition, equation (3), at the free surface of the liquid. The requirement of compatible motion of liquid and container then results in the frequency equation. An iterative procedure is used to obtain the numerical values for the natural frequencies.

## LONG RECTANGULAR TANK WITH ELASTIC BOTTOM

In Cartesian coordinates the motion of an inviscid and incompressible liquid is governed by

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \ . \qquad (1)$$

The geometry of the system under study is shown in Figure 1. For a long, rectangular tank, the fluid particles are assumed to move in vertical planes. The motion of the liquid is therefore independent of y. The boundary conditions at the free surface of the liquid, and at the interfaces of the tank are

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=0} = 0 \ , \qquad (2)$$

$$\left.\frac{\partial \varphi}{\partial x}\right|_{x=L} = 0 \ , \qquad (3)$$

$$\left[\frac{\partial^2 \varphi}{\partial t^2} + g \frac{\partial \varphi}{\partial z}\right]_{z=H} = 0 \ , \qquad (4)$$

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = 0 \text{ (Rigid bottom)} \ , \qquad (5a)$$

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = \frac{\partial w}{\partial t} \text{ (Elastic bottom)} \ , \qquad (5b)$$

where w is the transverse displacement of the plate, g is the gravitational acceleration, and t is the time.

When harmonic motion is assumed for the system, the general solution of Equation (1), in conjunction with the boundary conditions (2), (3), and (4) is found to be

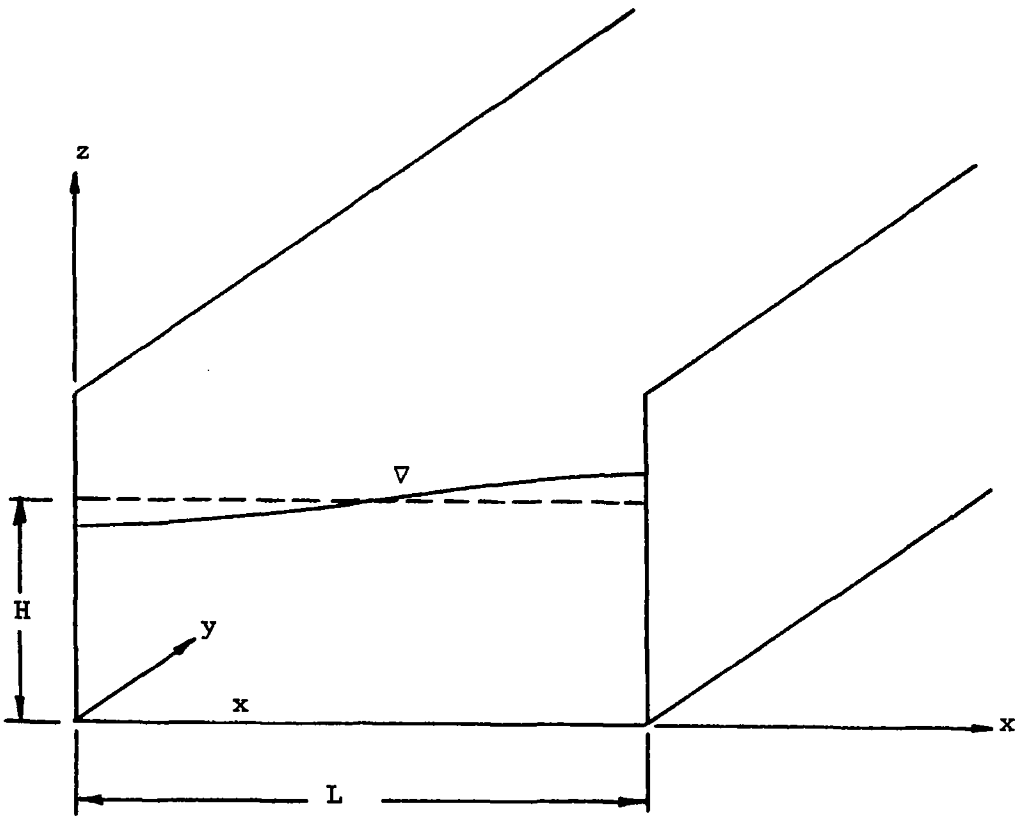


Figure 1. Long Rectangular Tank.

$$\varphi(x,z,t) = e^{i\omega t} \sum_{m=o}^{\infty} A_m \left[ \cosh \frac{m\pi z}{L} + \gamma_m \sinh \frac{m\pi z}{L} \right] \cosh \frac{m\pi x}{L} , \qquad (6)$$

where

$$\gamma_m = \frac{g\left(\frac{m\pi}{L}\right) \sin \frac{m\pi H}{L} - \omega^2 \cosh \frac{m\pi H}{L}}{\omega^2 \sinh \frac{m\pi H}{L} - g\left(\frac{m\pi}{L}\right) \cosh \frac{m\pi H}{L}} , \qquad (7)$$

and where $\omega$ is the natural circular frequency of the system.

For a tank with a rigid bottom the satisfaction of Equation (5a) results in the frequencies $\Omega_m$ of the liquid oscillation

$$\Omega_m{}^2 = g \left(\frac{m\pi}{L}\right) \tanh \frac{m\pi H}{L} . \qquad (8)$$

For a tank with an elastic bottom the elastokinetic behavior of the bottom is governed by the plate equation:

$$D\nabla^4 w + \rho h \frac{\partial^2 w}{\partial t^2} = p\Big|_{z=0} , \qquad (9)$$

where w is the transverse displacement with respect to static equilibrium position, D is the plate rigidity, $\rho$ is the plate density, h is the plate thickness, and the pressure p, on the plate is

$$p\Big|_{z=0} = \rho_f \frac{\partial\varphi}{\partial t}\Big|_{z=0} \qquad (10)$$

Again, for a long tank, Equation (10) is independent of y. Substitution of Equation (6) into Equation (9), in conjunction with Equation (10) yields:

$$D \frac{\partial^4 w}{\partial x^4} + \rho h \frac{\partial^2 w}{\partial t^2} = \rho_f i\omega e^{i\omega t} \sum_{m=0}^{\infty} A_m \cos \frac{m\pi x}{L} . \qquad (11)$$

The boundary conditions along the edges of the plate, or $x = 0$ and $x = L$, are:

a) Simply supported case

$$w = 0 , \tag{12}$$

$$\frac{\partial^2 w}{\partial x^2} = 0 . \tag{13}$$

b) Clamped case

$$w = 0 , \tag{14}$$

$$\frac{\partial w}{\partial x} = 0 . \tag{15}$$

The general solution of Equation (11) is assumed as

$$w(x,t) = W(x)e^{i\omega t} . \tag{16}$$

Substitution in Equation (11) results in the ordinary differential equation

$$D \frac{d^4 W}{dx^4} - \rho h \, \omega^2 W = \rho_f \, i\omega \sum_{m=0}^{\infty} A_m \cos \frac{m\pi x}{L} . \tag{17}$$

The general solution of Equation (17) is found to be

$$W(x) = B_1 \cosh \lambda x + B_2 \sinh \lambda x + B_3 \cos \lambda x + B_4 \sin \lambda x$$

$$+ i\omega \sum_{m=0}^{\infty} \alpha_m A_m \cos \frac{m\pi x}{L} , \tag{18}$$

where

$$\lambda^2 = \omega \sqrt{\frac{\rho h}{D}} , \qquad \alpha_m = \frac{1}{D} \frac{\rho_f}{\left(\frac{m\pi}{L}\right)^2 - \lambda^4} \tag{19}$$

The respective application of the boundary conditions (12), (13) or (15) and (16) then has the result that the arbitrary constants $B_1$, $B_2$, $B_3$ and $B_4$ may be expressed in terms of the $A_m$ as

$$B_j = i\omega \sum_{m=0}^{\infty} \beta_{jm}\, \alpha_m\, A_m \quad (j = 1,2,3, \text{ and } 4) \ , \tag{20}$$

in which the $\beta_{jm}$ have the forms:

a) Simply supported case

$$\beta_{1m} = -\frac{1}{2}\left[1 - \frac{1}{\lambda^2}\left(\frac{m\pi}{L}\right)^2\right] , \tag{21a}$$

$$\beta_{2m} = \frac{1}{2}\left[\coth \lambda L - \frac{\cos m\pi}{\sinh \lambda L}\right]\left[1 - \frac{1}{\lambda^2}\left(\frac{m\pi}{L}\right)^2\right] , \tag{21b}$$

$$\beta_{3m} = -\frac{1}{2}\left[1 + \frac{1}{\lambda^2}\left(\frac{m\pi}{L}\right)^2\right] , \tag{21c}$$

$$\beta_{4m} = \frac{1}{2}\left[\cot \lambda L - \frac{\cos m\pi}{\sin \lambda L}\right]\left[1 + \frac{1}{\lambda^2}\left(\frac{m\pi}{L}\right)^2\right] \tag{21d}$$

b) Clamped case

$$\beta_{jm} = B_{j1} + B_{j2} \cos m\pi \ , \tag{22}$$

where

$$B_{11} = \frac{1 - \cosh \lambda L \cos \lambda L - \sinh \lambda L \sin \lambda L}{2(\cosh \lambda L \cos \lambda L - 1)} , \tag{23a}$$

$$B_{12} = \frac{\cosh \lambda L - \cos \lambda L}{2(\cosh \lambda L \cos \lambda L - 1)} , \tag{23b}$$

$$B_{31} = \frac{1 - \cosh \lambda L \cos \lambda L + \sinh \lambda L \sin \lambda L}{2(\cosh \lambda L \cos \lambda L - 1)} , \tag{23c}$$

$$B_{32} = -B_{12} \,, \tag{23d}$$

$$B_{21} = -B_{41} = \frac{B_{31} \sin \lambda L - B_{11} \sinh \lambda L}{\cosh \lambda L - \cos \lambda L} \,, \tag{23e}$$

$$B_{22} = -B_{42} = \frac{B_{32} \sin \lambda L - B_{12} \sinh \lambda L}{\cosh \lambda L - \cos \lambda L} \,, \tag{23f}$$

The functions cosh $\lambda x$, sinh $\lambda x$, cos $\lambda x$ and sin $\lambda x$ can be expanded as Fourier cosine series in $\frac{m\pi x}{L}$. As a result, the general solution for the transverse deformation of the plate may be written as:

$$w(x,t) = i\omega e^{i\omega t}\left\{\sum_{m=o}^{\infty} \alpha_m \left[\frac{1}{2}\left(\beta_{1m}a_o + \beta_{2m}b_o + \beta_{3m}c_o + \beta_{4m}d_o\right) + \sum_{n=1}^{\infty}\left(\beta_{1m}a_n + \beta_{2m}b_n + \beta_{3m}c_n + \beta_{4m}d_n\right)\cos\frac{n\pi x}{L}\right] + \sum_{n=0}^{\infty} \alpha_n \cos\frac{n\pi x}{L}\right\}, \tag{24}$$

where

$$a_o = \frac{2}{\lambda L} \sinh \lambda L \,, \tag{25a}$$

$$a_n = \frac{2\lambda \sinh \lambda L \cos n\pi}{L\left[\lambda^2 + \left(\frac{n\pi}{L}\right)^2\right]} \,, \tag{25b}$$

$$b_o = \frac{2}{\lambda L} (\cosh \lambda L - 1) \,, \tag{25c}$$

$$b_n = \frac{2\lambda}{L\left[\lambda^2 + \left(\frac{n\pi}{L}\right)^2\right]} [\cosh \lambda L \cos n\pi - 1] \,, \tag{25d}$$

$$c_o = \frac{2}{\lambda L} \sin \lambda L \,, \tag{25e}$$

$$c_n = \frac{2\lambda \sin \lambda L \cos n\pi}{L\left[\lambda^2 - \left(\frac{n\pi}{L}\right)^2\right]} \,, \tag{25f}$$

$$d_o = \frac{2}{\lambda L}(1 - \cos\lambda L) \tag{25g}$$

$$d_n = \frac{2\lambda}{L\left[\lambda^2 - \left(\frac{n\pi}{L}\right)^2\right]}[1 - \cos\lambda L \cos n\pi] \tag{25h}$$

For a tank with an elastic bottom, the satisfaction of the boundary conditions (5b) in conjunction with Equations (6), (18) and (20) results in a doubly infinite system of simultaneous, homogeneous algebraic equations for the $A_m$, i.e.,

$$[a_{nm}]\{A_m\} = \{0\} , \tag{26}$$

where

$$a_{om} = -\frac{1}{2}\omega^2\alpha_m(\beta_{1m}a_o + \beta_{2m}b_o + \beta_{3m}c_o + \beta_{4m}d_o) + \frac{\rho_f}{\rho_h}\delta_{om} , \tag{27a}$$

$$a_{nm} = -\omega^2\alpha_m(\beta_{1m}a_n + \beta_{2m}b_n + \beta_{3m}c_n + \beta_{4m}d_n) - \left(\frac{\rho_f\frac{\omega^2}{D}}{\left(\frac{n\pi}{L}\right)^4 - \lambda^4} + \frac{n\pi}{L}\gamma_n\right)\delta_{nm} , \tag{27b}$$

$\delta_{nm}$ being the Kronecker delta. For a nontrivial solution of Equation (26) the determinant of the coefficient matrix must vanish. The frequency equation thence becomes

$$|a_{nm}| = 0 . \tag{28}$$

## Numerical Examples

The following data is used to calculate the frequencies for the liquid in a partially filled tank with elastic bottom:

$$\nu = 0.3;\ g = 32.2\ \text{ft/sec}^2;\ L = 2.0\ \text{ft}.$$

The bottom is considered to be clamped along its edges. The lowest circular frequencies in radians per sec for various liquid depths are calculated by taking twenty terms in the series solution. The numerical results are shown in Table 1 and also plotted in Figure 2.

TABLE 1

FUNDAMENTAL FREQUENCIES "$\omega$"

| H/L | Rigid Bottom | d = 77*<br>D = 49 in-lb | d = 110<br>D = 16.8 in-lb | d = 193<br>D = 3.13 in-lb |
|---|---|---|---|---|
| 1/8 | 4.3842 | 4.2779 | 4.1485 | 3.4800 |
| 1/4 | 5.7557 | 5.6970 | 5.5781 | 4.7859 |
| 3/8 | 6.4560 | 6.4275 | 6.3518 | 5.6484 |
| 1/2 | 6.8104 | 6.7898 | 6.7421 | 6.2109 |
| 5/8 | 6.9731 | 6.9629 | 6.9454 | 6.5859 |

* $d = \rho_f L/(\rho h)$ and D is the plate rigidity

These numerical values were obtained by using an iterative procedure on a Burrough 220 computer. For $H/L \geq 3/4$, the difference between the frequencies for the tank with elastic bottom and that with a rigid bottom was undetectable.

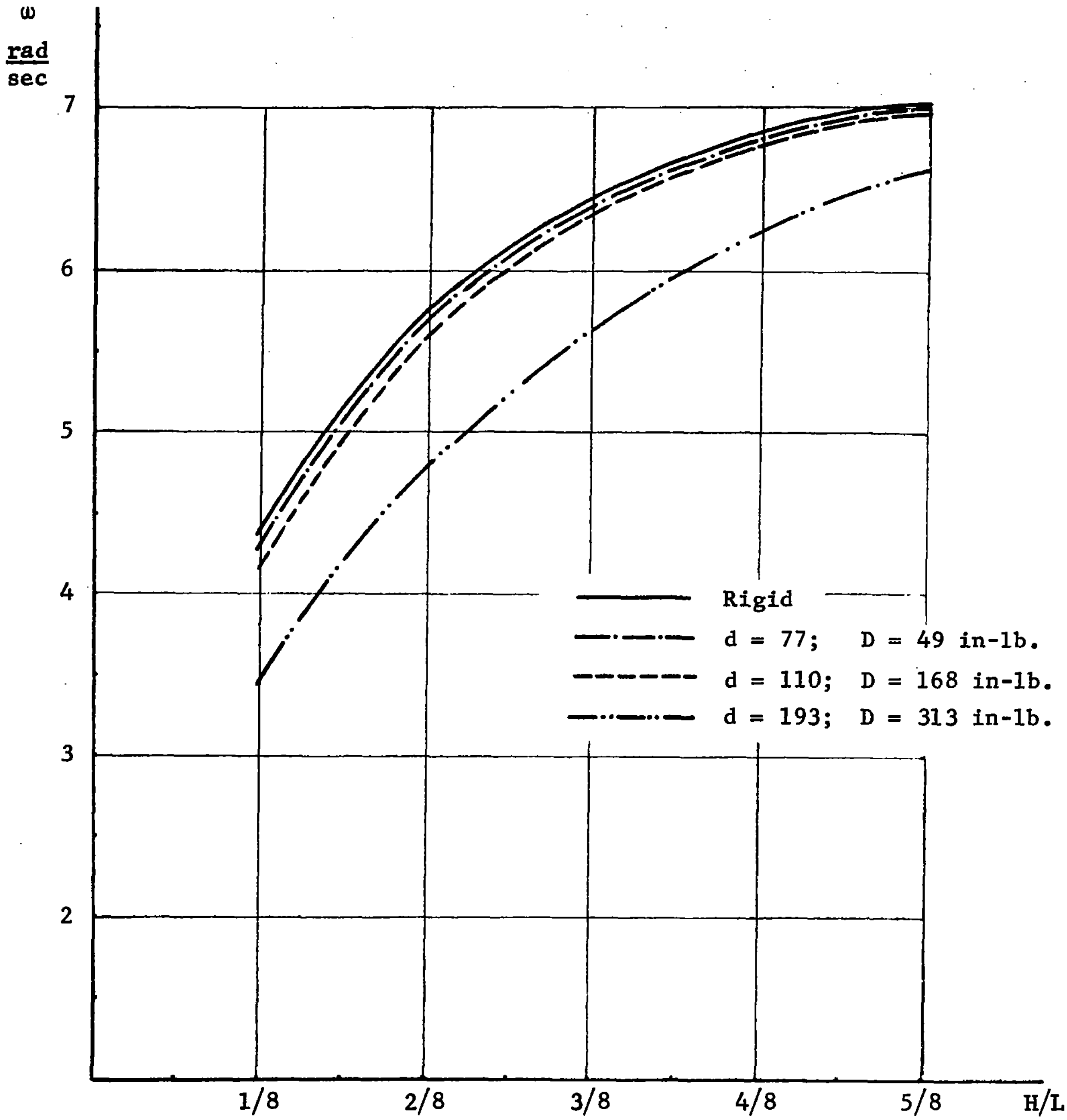


Figure 2. Fundamental Frequencies of Long Rectangular Tank with Elastic Bottom.

LONG, RECTANGULAR TANK WITH ELASTIC WALL

For an infinitely long tank the equation governing the motion of the liquid in the tank is

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \ . \tag{1}$$

The geometry of the system under consideration is shown in Figure (1). The boundary conditions associated with Equation (1) are

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=0} = 0 \ , \tag{2}$$

$$\left[\frac{\partial^2\varphi}{\partial t^2} + g\,\frac{\partial\varphi}{\partial z}\right]_{z=H} = 0 \ , \tag{3}$$

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=0} = \frac{\partial u}{\partial t} \ , \tag{4}$$

$$\left.\frac{\partial\varphi}{\partial x}\right|_{x=L} = \frac{\partial v}{\partial t} \ . \tag{5}$$

The general solution for the velocity potential is assumed to be

$$\varphi(x,z,t) = X(x)Z(z)e^{i\omega t} \ , \tag{6}$$

where $\omega$ represents the natural circular frequency of the system. The substitution of Equation (6) in Equation (1) and the satisfaction of the boundary conditions (2) and (3) have the result

$$\varphi(x,z,t) = e^{i\omega t} \sum_{n=1}^{\infty} (A_n \cosh \epsilon_n x + B_n \sinh \epsilon_n x)\cos \epsilon_n z \ , \tag{7}$$

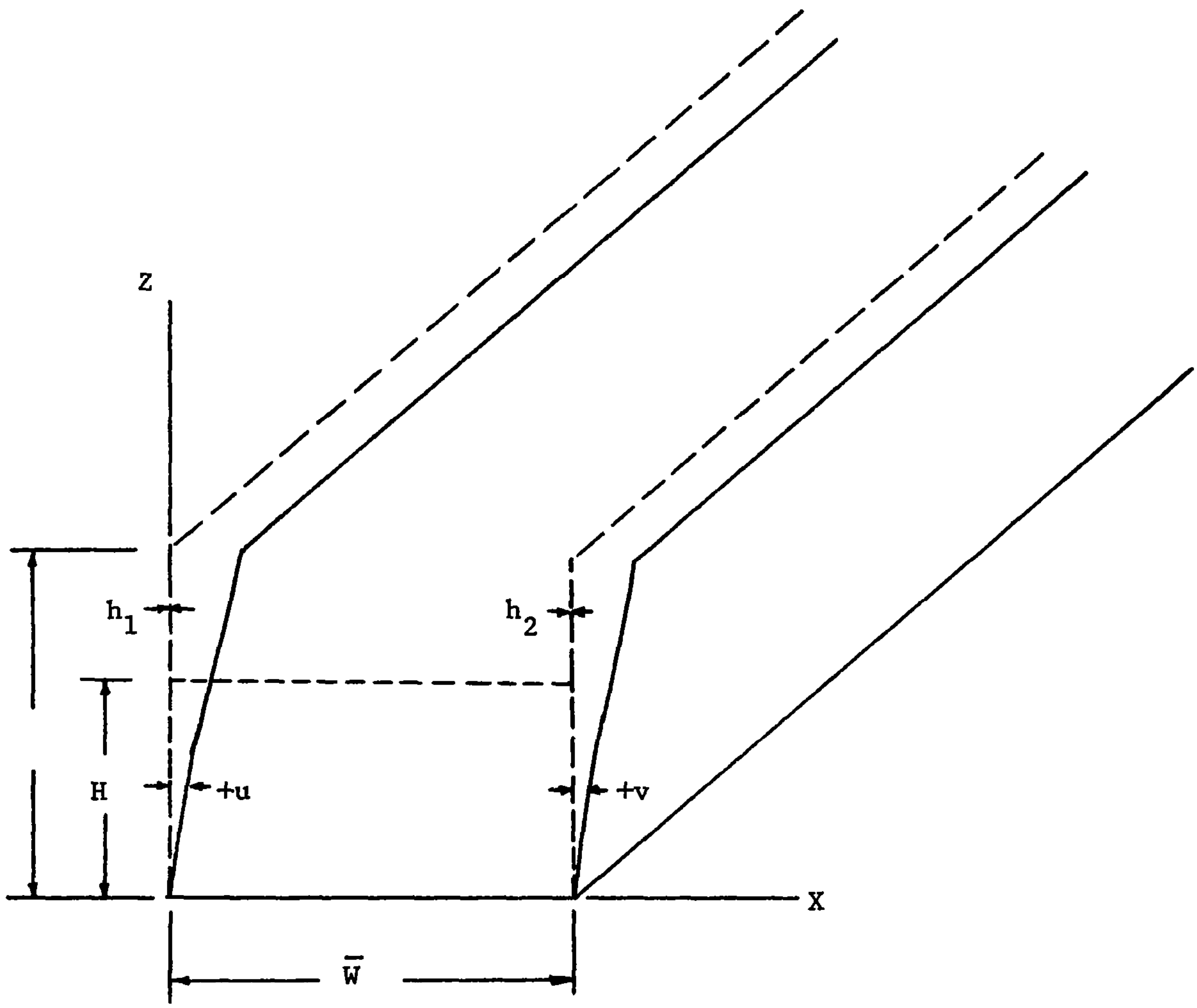


Figure (3)

where the $\epsilon_n$ are the roots of the equation

$$\epsilon H \tan \epsilon H = -\frac{\omega^2 H}{g} . \tag{8}$$

The equation of motion of the wall along x=0 is

$$\frac{\partial^4 \mu}{\partial z^4} + \frac{\rho_1 h_1}{D_1} \frac{\partial^2 \mu}{\partial t^2} = \frac{\rho_f}{D_1} \frac{\partial \varphi}{\partial t}\bigg|_{x=0} \tag{9}$$

The assumption of harmonic motion of the form

$$\mu(z,t) = U(z) e^{i\omega t} , \tag{10}$$

and substitution of this expression in Equations (1) and (9), has the result

$$\frac{d^4 U}{dz^4} - \lambda_1{}^4 U = \frac{\rho_f}{D_1} (i\omega) \sum_{n=1}^{\infty} A_n \cos \epsilon_n z . \tag{11}$$

The general solution of this equation is

$$U = B_1 \cosh\lambda_1 z + B_2 \sinh \lambda_1 z + B_3 \cos \lambda_1 z + B_4 \sin \lambda_1 z$$

$$+ \frac{\rho_f}{D_1} (i\omega) \sum_{n=1}^{\infty} \frac{A_n}{\epsilon_n{}^4 - \lambda_1{}^4} \cos \epsilon_n z , \tag{12}$$

which must satisfy the possible boundary conditions:

a) Simply supported

$$\begin{aligned} \mu(0) &= \mu(L) = 0 , \\ \mu''(0) &= \mu''(L) = 0 . \end{aligned} \tag{13}$$

b) Both edges fixed

$$\begin{aligned} \mu(0) &= \mu(L) = 0 , \\ \mu'(0) &= \mu'(L) = 0 . \end{aligned} \tag{14}$$

Both edges of the walls will be considered fixed here. The integration constants $B_j$ ($j = 1$ to 4) can be expressed in terms of $A_n$ by applying the boundary conditions shown in Equation (14). They are found to be

$$B_j = \sum_{\mu=1}^{\infty} K_{j\mu} \frac{\rho_f(i\omega)\, A_\mu}{D_1(\epsilon_\mu^{\,4} - \lambda_1^{\,4})}\,, \tag{15}$$

where

$$K_{1\mu} = \frac{1 - \cosh\lambda_1 L \cos\lambda_1 L - \sinh\lambda_1 L \sin\lambda_1 L + \cos\epsilon_\mu L\,(\cosh\lambda_1 L - \cos\lambda_1 L)}{2(\cosh\lambda_1 L \cos\lambda_1 L - 1)}$$

$$+ \frac{\frac{\epsilon_\mu}{\lambda_1}\sin\epsilon_\mu L\,(\sinh\lambda_1 L - \sin\lambda_1 L)}{2(\cosh\lambda_1 L \cos\lambda_1 L - 1)}\,, \tag{16}$$

$$K_{2\mu} = \frac{\cosh\lambda_1 L \sin\lambda_1 L + \sinh\lambda_1 L \cos\lambda_1 L - \cos\epsilon_\mu L\,(\sinh\lambda_1 L + \sin\lambda_1 L)}{2(\cosh\lambda_1 L \cos\lambda_1 L - 1)}$$

$$- \frac{\frac{\epsilon_\mu}{\lambda_1}\sin\epsilon_\mu L\,(\cosh\lambda_1 L - \cos\lambda_1 L)}{2(\cosh\lambda_1 L \cos\lambda_1 L - 1)}\,, \tag{17}$$

$$K_{3\mu} = \frac{1 - \cosh\lambda_1 L \cos\lambda_1 L + \sinh\lambda_1 L \sin\lambda_1 L - \cos\epsilon_\mu L\,(\cosh\lambda_1 L - \cos\lambda_1 L)}{2(\cosh\lambda_1 L \cos\lambda_1 L - 1)}$$

$$- \frac{\frac{\epsilon_\mu}{\lambda_1}\sin\epsilon_\mu L\,(\sinh\lambda_1 L - \sin\lambda_1 L)}{2(\cosh\lambda_1 L \cos\lambda_1 L - 1)}\,, \tag{18}$$

$$K_{4\mu} = -K_{2\mu}\,. \tag{19}$$

The equation of motion of the wall along $x = \ell$ is

$$\frac{\partial^4 v}{\partial z^4} + \frac{\rho_2 h_2}{D_2} v = \frac{\rho_f}{D_2} \left.\frac{\partial\varphi}{\partial t}\right|_{x=\ell} \tag{20}$$

The solution of Equation (20) is assumed in the following form:

$$v = V(z)e^{i\omega t} . \tag{21}$$

Substitution of Equations (7) and (21) into Equation (20) gives

$$\frac{d^4V}{dZ^4} - \lambda_2{}^4 V = \frac{\rho_f(i\omega)}{D_2} \sum_{n=1}^{\infty} (A_n \cosh \epsilon_n \ell + B_n \sinh \epsilon_n \ell) \cos \epsilon_n z , \tag{22}$$

where

$$\lambda_2{}^4 = \frac{\rho_2 h_2}{D_2} \omega^2$$

The general solution of the Equation (22) is

$$V = B_1{}^* \cosh \lambda_2 z + B_2{}^* \sinh \lambda_2 z + B_3{}^* \cos \lambda_2 z + B_4{}^* \sin \lambda_2 z$$

$$+ \frac{\rho_f(i\omega)}{D_2} \sum_{n=1}^{\infty} \frac{1}{\epsilon_n{}^4 - \lambda_2{}^4} (A_n \cosh \epsilon_n \ell + B_n \sinh \epsilon_n \ell) \cos \epsilon_n z . \tag{23}$$

The application of the boundary conditions (14) results in the expressions

$$B_j{}^* = \frac{\rho_f(i\omega)}{D_2} \sum_{\mu=1}^{\infty} \frac{K_{j\mu}}{\epsilon_\mu{}^4 - \lambda_2{}^4} (A_\mu \cosh \epsilon_\mu \ell + B_\mu \sinh \epsilon_\mu \ell) \tag{24}$$

for the integration constants $B_j{}^*$.

It is assumed that the liquid remains in contact with the wall throughout the motion. The conditions shown in Equations (4) and (5) must be satisfied. The relationships for the determination of the coefficients $A_n$ and $B_n$ become

$$\sum_{n=1}^{\infty} \left[\frac{\rho_f \omega^2}{D_1(\epsilon_n{}^4 - \lambda_1{}^4)} A_n + \epsilon_n B_n\right] \cos \epsilon_n z + \frac{\rho_f \omega^2}{D_1} \sum_{\mu=1}^{\infty} \frac{A_\mu}{\epsilon_\mu{}^4 - \lambda_1{}^4} [K_{1\mu} \cosh \lambda_1 z$$

$$+ K_{2\mu} \sinh \lambda_1 z + K_{3\mu} \cos \lambda_1 z + K_{4\mu} \sin \lambda_1 z] = 0 \tag{25}$$

and

$$\sum_{n=1}^{\infty}\left\{\left[\epsilon_n \sinh \epsilon_n \ell + \frac{\rho_f \omega^2}{D_2(\epsilon_n^4-\lambda_2^4)} \cosh \epsilon_n \ell\right] A_n + \right.$$

$$\left. + \left[\epsilon_n \cosh \epsilon_n \ell + \frac{\rho_f \omega^2}{D_2(\epsilon_n^4-\lambda_2^4)} \sinh \epsilon_n \ell\right] B_n\right\} \cdot \cos \epsilon_n Z +$$

$$+ \frac{\rho_f \omega^2}{D_2} \sum_{\mu=1}^{\infty} \frac{A_\mu \cosh \epsilon_\mu \ell + B_\mu \sinh \epsilon_\mu \ell}{(\epsilon_\mu^4-\lambda_2^4)} [K_{1\mu} \cosh \lambda_2 z + K_{2\mu} \sinh \lambda_2 z + \tag{26}$$

$$+ K_{3\mu} \cos \lambda_2 z + K_{4\mu} \sin \lambda_2 z] = 0$$

The functions, $\cosh \lambda_k z$, $\sinh \lambda_k z$, $\cos \lambda_k z$ and $\sin \lambda_k z$ expressed in terms of their Fourier series expansions are:

$$\begin{aligned}
\cosh \lambda_k z &= \sum_{n=1}^{\infty} a_n^k \cos \epsilon_n z \, , \\
\sinh \lambda_k z &= \sum_{n=1}^{\infty} b_n^k \cos \epsilon_n z \, , \\
\cos \lambda_k z &= \sum_{n=1}^{\infty} c_n^k \cos \epsilon_n z \, , \\
\sin \lambda_k z &= \sum_{n=1}^{\infty} d_n^k \cos \epsilon_n z \, ,
\end{aligned} \tag{27}$$

where

$$a_n^k = \frac{2}{H^*(\lambda_k^2+\epsilon_n^2)} (\lambda_k \sinh \lambda_k H \cos \epsilon_n H + \epsilon_n \cosh \lambda_k H \sin \epsilon_n H) \, , \tag{28}$$

$$b_n^k = \frac{2}{H^*(\lambda_k^2+\epsilon_n^2)} (\lambda_k \cosh \lambda_k H \cos \epsilon_n H + \epsilon_n \sinh \lambda_k H \sin \epsilon_n H - \lambda_k)$$

$$c_n^k = \frac{2}{H^*(\lambda_k^2-\epsilon_n^2)} (\lambda_k \sin \lambda_k H \cos \epsilon_n H - \epsilon_n \cos \lambda_k H \sin \epsilon_n H) \ , \qquad (28)$$

$$d_n^k = \frac{2}{H^*(\lambda_k^2-\epsilon_n^2)} (\lambda_k - \lambda_k \cos \lambda_k H \cos \epsilon_n H - \epsilon_n \sin \lambda_k H \sin \epsilon_n H) \ ,$$

and

$$H^* = H + \frac{1}{2\epsilon_n} \sin(2\epsilon_n H) \ .$$

Substitution of Equations (27) and (28) into Equations (25) and (26) results in the following algebraic equations:

$$\left\{\frac{\rho_f \omega^2}{D_1(\epsilon_n^4-\lambda_1^4)} A_n + \epsilon_n B_n\right\}\delta_{n\mu} + \frac{\rho_f\omega^2}{D_1} \sum_{\mu=1}^{\infty} \frac{A_\mu}{\epsilon_\mu^4-\lambda_1^4} (K_{1\mu}A_n^{(1)}+K_{2\mu}b_n^{(1)}+K_{3\mu}c_n^{(1)}$$

$$+ K_{4\mu}d_n^{(1)}) = 0 \qquad (29)$$

and

$$\left\{\left[\epsilon_n \sinh \epsilon_n \ell + \frac{\rho_f\omega^2}{D_2(\epsilon_n^4-\lambda_2^4)} \cosh \epsilon_n \ell\right]A_n + \left[\epsilon_n \cosh \epsilon_n \ell + \frac{\rho_f\omega^2}{D_2(\epsilon_n^4-\lambda_2^4)} \sinh \epsilon_n \ell\right]B_n\right\}^{\delta_{n\mu}}$$

$$(30)$$

$$+ \frac{\rho_f\omega^2}{D_2} \sum_{\mu=1}^{\infty} \frac{A_\mu \cosh \epsilon_\mu \ell + B_\mu \sinh \epsilon_\mu \ell}{\epsilon_\mu^4-\lambda_2^4} (K_{1\mu}a_n^{(2)}+K_{2\mu}b_n^{(2)}+K_{3\mu}c_n^{(2)}+K_{4\mu}d_n^{(2)}) = 0 \ .$$

Since n ranges for 1 to infinity this results in a doubly infinite system of equations for the unknown coefficients $A_n$ and $B_n$, i.e.,

$$\begin{bmatrix} \alpha_{n\mu} & \beta_{n\mu} \\ \alpha^*_{n\mu} & \beta^*_{n\mu} \end{bmatrix} \begin{Bmatrix} A_\mu \\ B_\mu \end{Bmatrix} = 0 \ , \qquad (31)$$

where

$$\alpha_{n\mu} = \frac{\rho_f \omega^2}{D_1(\epsilon_\mu{}^4 - \lambda_1{}^4)} \left[(K_{1\mu}a_n^{(1)} + K_{2\mu}b_n^{(1)} + K_{3\mu}c_n^{(1)} + K_{4\mu}d_n^{(1)}) + \delta_{n\mu}\right] ,$$

$$\beta_{n\mu} = \epsilon_n \, \delta_{n\mu} ,$$

$$\alpha^*_{n\mu} = \frac{\rho_f \omega^2 \cosh \epsilon_\mu \ell}{D_2(\epsilon_\mu{}^4 - \lambda_2{}^4)} (K_{1\mu}a_n^{(2)} + K_{2\mu}b_n^{(2)} + K_{3\mu}c_n^{(2)} + K_{4\mu}d_n^{(2)})$$

$$+ \left[\epsilon_n \sinh \epsilon_n \ell + \frac{\rho_f \omega^2 \cosh \epsilon_n \ell}{D_2(\epsilon_n{}^4 - \lambda_2{}^4)}\right] \delta_{n\mu} , \qquad (32)$$

$$\beta^*_{n\mu} = \frac{\rho_f \omega^2 \sinh \epsilon_\mu \ell}{D_2(\epsilon_\mu{}^4 - \lambda_2{}^4)} (K_{1\mu}a_n^{(2)} + K_{2\mu}b_n^{(2)} + K_{3\mu}c_n^{(2)} + K_{4\mu}d_n^{(2)})$$

$$+ \left[\epsilon_n \cosh \epsilon_n \ell + \frac{\rho_f \omega^2 \sinh \epsilon_n \ell}{D_2(\epsilon_n{}^4 - \lambda_2{}^4)}\right] \delta_{n\mu} ,$$

For non-trivial solution, the determinant of the coefficient matrix of the Equation (31) must vanish, or

$$\begin{vmatrix} \alpha_{n\mu} & \beta_{n\mu} \\ \alpha^*_{n\mu} & \beta^*_{n\mu} \end{vmatrix} = 0 , \qquad (33)$$

which represents the frequency equation.

If only one of the two walls, say the left side one, is elastic and the other side is rigid, the following equation may be reduced directly from Equations (29) and (30) by considering $D_2 = \infty$:

$$[\gamma_{n\mu}]\ \{B_{\mu}\} = 0 \tag{34}$$

where

$$\gamma_{n\mu} = \frac{\rho_f \omega^2 \cosh \epsilon_n \ell}{D_1(\epsilon_\mu{}^4 - \lambda^4)} (K_{1\mu} a_n + K_{2\mu} b_n + K_{3\mu} c_n + K_{4\mu} d_n) + \left(\frac{\rho_f \omega^2 \cosh \epsilon_n \ell}{D_1(\epsilon_n{}^4 - \lambda^4)} - \epsilon_n \sinh \epsilon_n \ell\right)\delta_{n\mu} \tag{35}$$

The frequency equation can be obtained immediately by requiring the determinant of the coefficient matrix of Equation (34) to vanish, or

$$|\gamma_{n\mu}| = 0 \tag{36}$$

## Numerical Examples

In order to illustrate the results of the analysis, numerical examples for a tank with one wall elastic and the other one rigid and for a rectangular tank with both wall elastic were worked out on a Burroughs 5500 digital computer. The infinite order determinant of Equations (33) and (34) were truncated to a twentieth order one for the calculation of the first coupled natural frequency.

The results of the numerical examples are given in Tables 2 and 3 where the coupled frequencies are in radians per sec. The variation of frequencies versus liquid height is also plotted in Figure (4). The data common for all cases are

$h = 0.005$ ft., $w = 4$ ft., $L = 5$ ft.

$\rho = 5.217$ lb-sec$^2$/ft.$^4$, $\rho_f = 2.019$ lb-sec$^2$/ft.$^4$

$E = 1.44 \times 10^9$ psf, $\nu = 0.3$

TABLE 2

FUNDAMENTAL CIRCULAR FREQUENCIES FOR A PARTIALLY LIQUID FILLED RECTANGULAR TANK HAVING ONE WALL ELASTIC AND THE OTHER ONE RIGID

| H ft | H/L | $\omega$ rad/sec | $\Omega$ rad/sec* |
|---|---|---|---|
| 5 | 1 | 3.31 | 5.03 |
| 4.6 | 0.92 | 3.70 | 5.03 |
| 4.2 | 0.84 | 4.18 | 5.02 |
| 3.8 | 0.76 | 4.81 | 5.02 |
| 3.4 | 0.68 | 5.73 | 5.00 |
| 3.0 | 0.60 | 7.32 | 4.97 |
| 2.6 | 0.52 | 11.93 | 4.95 |
| 2.2 | 0.44 | 22.49 | 4.87 |
| 1.8 | 0.36 | 22.49 | 4.74 |

*Rigid tank slosh frequencies

TABLE 3

FUNDAMENTAL CIRCULAR FREQUENCIES FOR A PARTIALLY LIQUID FILLED RECTANGULAR TANK WITH BOTH WALLS ELASTIC

| H ft | H/L | $\omega$ rad/sec | $\Omega$ rad/sec* |
|---|---|---|---|
| 5 | 1 | 3.21 | 5.03 |
| 4.6 | 0.92 | 3.61 | 5.03 |
| 4.2 | 0.84 | 4.09 | 5.02 |
| 3.8 | 0.78 | 4.71 | 5.02 |
| 3.4 | 0.68 | 5.62 | 5.00 |
| 3.0 | 0.60 | 7.21 | 4.97 |
| 2.6 | 0.52 | 11.79 | 4.95 |
| 1.8 | 0.36 | 22.01 | 4.74 |

*Rigid tank slosh frequencies

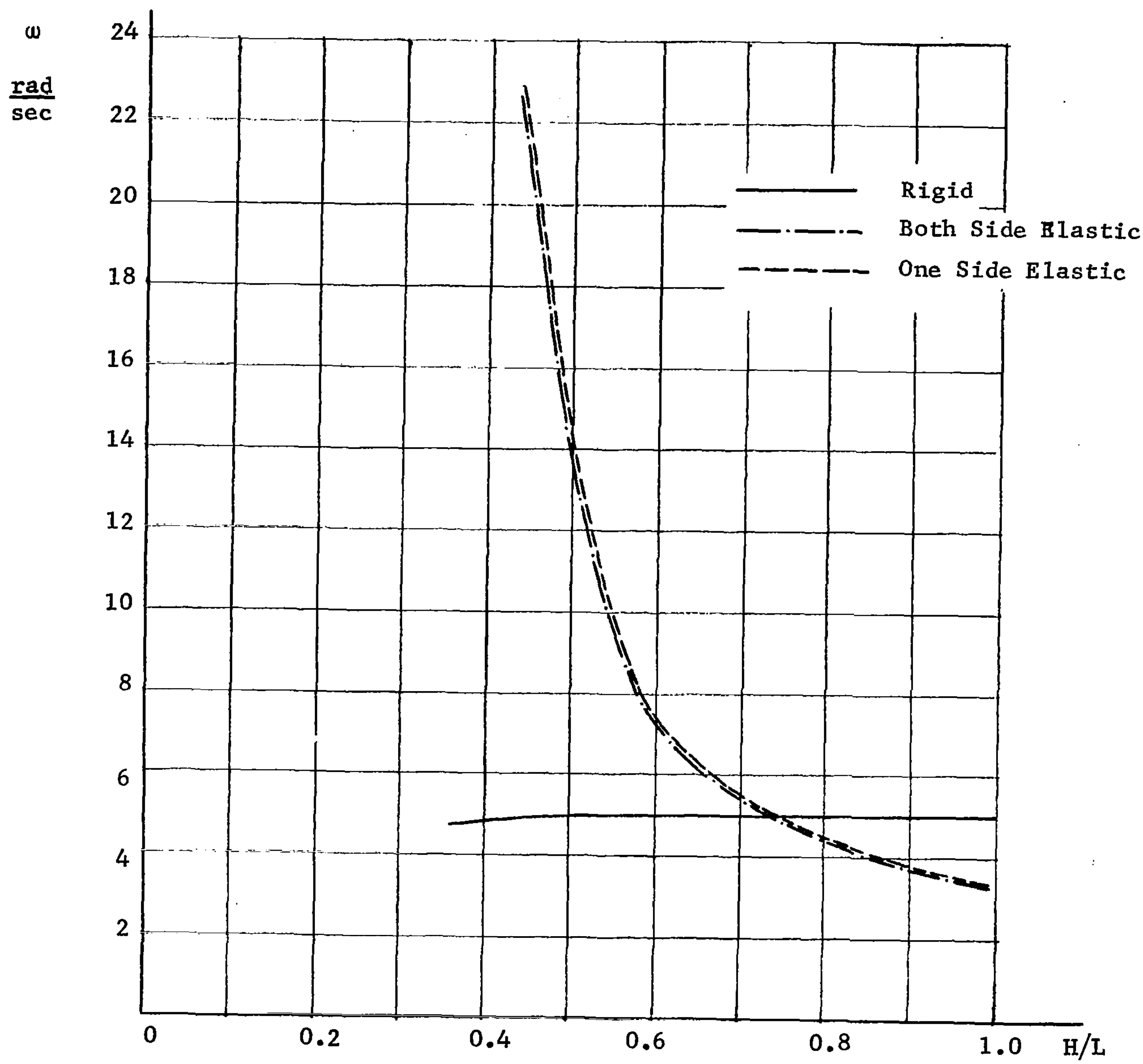


Figure 4. Fundamental Frequencies of Long Rectangular Tank with Elastic Wall.

## CYLINDRICAL TANK WITH ELASTIC WALL

The governing differential equation for the axisymmetrical motion of liquid in cylindrical coordinates is

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{\partial^2 \varphi}{\partial z^2} = 0 \ . \tag{1}$$

The boundary conditions, which must be satisfied at the rigid bottom and at the free surface of the liquid are

$$\left.\frac{\partial \varphi}{\partial r}\right|_{r=0} = 0 \ , \tag{2}$$

$$\left.\frac{\partial \varphi}{\partial z}\right|_{z=0} = 0 \ , \tag{3}$$

$$\left[\frac{\partial^2 \varphi}{\partial t^2} + g\frac{\partial \varphi}{\partial z}\right]_{z=H} = 0 \ . \tag{4}$$

The geometry of the system under consideration is shown in Figure (5).

A separable solution for Equation (1) is assumed and the motion is considered to be harmonic. The velocity potential $\varphi$ thus takes the form

$$\varphi(r,z,t) = R(r)\ z\ (z)e^{i\omega t} \tag{5}$$

where $\omega$ is the natural frequency of the system. The general solution of the Equation (1) with Equations (2), (3), and (4) all satisfied is found to be

$$\varphi(r,z,t) = e^{i\omega t}\sum_{n=1}^{\infty} A_n \cos \epsilon_n z\ I_o(\epsilon_n r) \ , \tag{6}$$

where $\epsilon_n$ are the roots of the Equation

$$\epsilon H \tan \epsilon H = -\frac{\omega^2}{g} H \ . \tag{7}$$

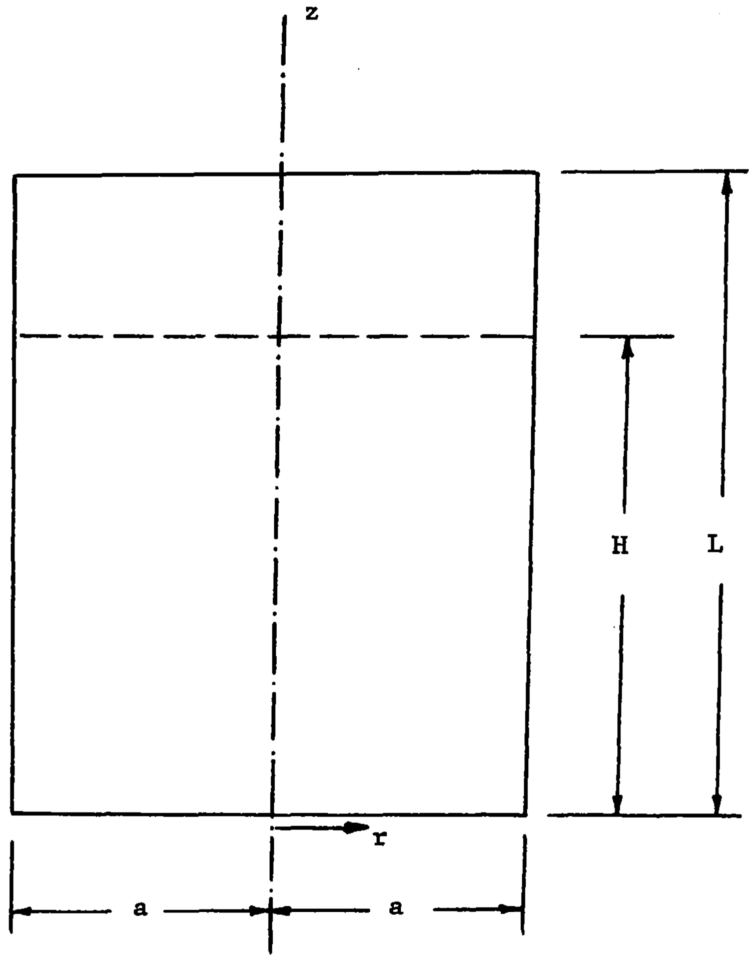


Figure (5)

The pressure at the tank wall is

$$\rho_f \left.\frac{\partial\varphi}{\partial t}\right|_{r=a} = \rho_f(i\omega)e^{i\omega t} \sum_{n=0}^{\infty} A_n \cos \epsilon_n z \; I_o(\epsilon_n a) \; . \tag{8}$$

The following differential equation governing the motion of the cylinder is used:

$$\frac{\partial^4 w}{\partial z^4} + \frac{Eh}{Da^2} w + \frac{\rho h}{D} \frac{\partial^2 w}{\partial t^2} = - \frac{\rho_f}{D} \left.\frac{\partial\varphi}{\partial t}\right|_{r=a} \; . \tag{9}$$

Equation (9) is derived based on the assumption that the longitudinal inertia of the shell has negligible effect on the motion of the shell. The transverse displacement, w, is defined to be positive if it moves along inward normal direction. The motion of the cylinder is assumed to be harmonic and takes the form

$$w(r,t) = W(r)e^{i\omega t} \; . \tag{10}$$

Substitution of Equation (10) into Equation (9) results in the following ordinary differential equation:

$$\frac{d^4 W}{dz^4} + 4\lambda^4 W = \frac{-\rho_f}{D} (i\omega) \sum_{n=0}^{\infty} A_n \cos \epsilon_n z \; I_o(\epsilon_n a) \; , \tag{11}$$

where

$$4\lambda^4 = \frac{1}{D}\left(\frac{Eh}{a^2} - \rho h\omega^2\right) . \tag{12}$$

The boundary conditions for the cylindrical shell are

a) Simply supported

$$w = 0 \,, \tag{13}$$

$$w_{zz} = 0 \,. \tag{14}$$

b) Clamped edge

$$w = 0 \,, \tag{15}$$

$$w_z = 0 \,. \tag{16}$$

The general solution of the differential equation (11) is found to be

$$w = e^{\lambda z}(B_1 \cos \lambda z + B_2 \sin \lambda z) + e^{-\lambda z}(B_3 \cos \lambda z + B_4 \sin \lambda z) - \frac{\rho_f}{D}(i\omega) \sum_{n=1}^{\infty} \frac{I_o(\epsilon_n a)A_n}{\epsilon_n^4 + 4\lambda^4} \cos \epsilon_n z \tag{17}$$

where $B_1$, $B_2$, $B_3$, and $B_4$ are integration constants which may be expressed in terms of Fourier coefficients $A_n$ when the boundary conditions shown in Equations (13), (14), and/or Equations (15) and (16) are applied.

For a cylindrical tank clamped along both ends, boundary conditions shown in Equations (15) and (16) must be satisfied. The integration constants $B_1$, $B_2$, $B_3$, and $B_4$ are found to be

$$B_j = \sum_{\mu=1}^{\infty} \frac{\rho_f(i\omega)I_o(\epsilon_\mu a)A_\mu}{D(\epsilon_\mu^4 + 4\lambda^4)} k_{j\mu}, \; j = 1 \text{ to } 4 \,. \tag{18}$$

where

$$K_{1\mu} = -\frac{1}{c_1}\Big\{e^{-\lambda L}\Big[\cos \epsilon_{\mu L}(\sin \lambda L - \cos \lambda L) - \frac{\epsilon_\mu}{\lambda} \sin \epsilon_{\mu L} \sin \lambda L\Big]$$

$$+ e^{\lambda L}\Big[\cos \epsilon_{\mu L}(\sin \lambda L + \cos \lambda L) + \frac{\epsilon_\mu}{\lambda} \sin \epsilon_{\mu L} \sin \lambda L\Big]$$

$$+ e^{-2\lambda L} - 2 - \sin(2\lambda L) + \cos(2\lambda L)\Big\}$$

$$K_{2\mu} = -\frac{1}{c_2}\left[C_3K_{1\mu} - 2\cos\epsilon_{\mu L}\sin\lambda L + \frac{\epsilon_\mu}{\lambda}\sin\epsilon_{\mu L}(\sin\lambda L + \cos\lambda L)\right]$$

$$K_{3\mu} = +(1 + K_{1\mu})$$

$$K_{4\mu} = +(2K_{1\mu} + K_{2\mu} + 1)$$

and

$$C_1 = 4 - 2\cos 2\lambda L - e^{2\lambda L} - e^{-2\lambda L}$$

$$C_2 = e^{\lambda L}[2 + \sin(2\lambda L) - \cos(2\lambda L)] - e^{-\lambda L}$$

$$C_3 = e^{-\lambda L} - e^{\lambda L}[\sin(2\lambda L) + \cos(2\lambda L)]$$

The functions $e^{\lambda z}\cos\lambda z$, $e^{\lambda z}\sin\lambda z$, $e^{-\lambda z}\cos\lambda z$ and $e^{-\lambda z}\sin\lambda z$ are expanded in Fourier cosine series as

$$e^{\lambda z}\cos\lambda z = \sum_{m=1}^{\infty} a_m \cos\epsilon_m z ,$$

$$e^{\lambda z}\sin\lambda z = \sum_{m=1}^{\infty} b_m \cos\epsilon_m z ,$$

$$e^{-\lambda z}\cos\lambda z = \sum_{m=1}^{\infty} d_m \cos\epsilon_m z ,$$

$$e^{-\lambda z}\sin\lambda z = \sum_{m=1}^{\infty} d_m \cos\epsilon_m z ,$$

where

$$a_m = \frac{1}{\left[H + \frac{1}{2\epsilon_m}\sin(2\epsilon_m H)\right]}\left[\frac{\lambda[e^{\lambda H}\cos(\lambda+\epsilon_m)H-1] + (\lambda+\epsilon_m)e^{\lambda H}\sin(\epsilon_m+\lambda)H}{\lambda^2 + (\lambda+\epsilon_m)^2}\right.$$

$$\left. + \frac{\lambda[e^{\lambda H}\cos(\lambda-\epsilon_m)H-1] + (\lambda-\epsilon_m)e^{\lambda H}\sin(\lambda-\epsilon_m)H}{\lambda^2 + (\lambda-\epsilon_m)^2}\right],$$

$$b_m = \frac{1}{\left[H + \frac{1}{2\epsilon_m}\sin(2\epsilon_m H)\right]}\left[\frac{\lambda e^{\lambda H}\sin(\lambda+\epsilon_m)H - (\lambda+\epsilon_m)[e^{\lambda H}\cos(\lambda+\epsilon_m)H - 1]}{\lambda^2 + (\lambda+\epsilon_m)^2}\right.$$

$$\left. + \frac{\lambda e^{\lambda H}\sin(\lambda-\epsilon_m)H - (\lambda-\epsilon_m)[e^{\lambda H}\cos(\lambda-\epsilon_m)H - 1]}{\lambda^2 + (\lambda-\epsilon_m)^2}\right],$$

$$c_m = \frac{1}{\left[H + \frac{1}{2\epsilon_m}\sin(2\epsilon_m H)\right]}\left[\frac{(\lambda+\epsilon_m)e^{-\lambda H}\sin(\lambda+\epsilon_m)H - \lambda[e^{-\lambda H}\cos(\lambda+\epsilon_m)H - 1]}{\lambda^2 + (\lambda+\epsilon_m)^2}\right.$$

$$\left. + \frac{(\lambda-\epsilon_m)e^{-\lambda H}\sin(\lambda-\epsilon_m)H - \lambda[e^{-\lambda H}\cos(\lambda-\epsilon_m)H - 1]}{\lambda^2 + (\lambda-\epsilon_m)^2}\right],$$

$$d_m = -\frac{1}{\left[H + \frac{1}{2\epsilon_m}\sin(2\epsilon_m H)\right]}\left[\frac{\lambda e^{-\lambda H}\sin(\epsilon_m+\lambda)H + (\lambda+\epsilon_m)[e^{-\lambda H}\cos(\lambda+\epsilon_m)H - 1]}{\lambda^2 + (\lambda+\epsilon_m)^2}\right.$$

$$\left. + \frac{\lambda e^{-\lambda H}\sin(\lambda-\epsilon_m)H + (\lambda-\epsilon_m)[e^{-\lambda H}\cos(\lambda-\epsilon_m)H - 1]}{\lambda^2 + (\lambda-\epsilon_m)^2}\right].$$

For compatible motion of the tank wall and the liquid in the tank, the following condition must be satisfied:

$$\frac{\partial w}{\partial t} = + \left.\frac{\partial \varphi}{\partial r}\right|_{r=a} \tag{26}$$

Substitution of Equations (5) and (10) in conjunction with the Equations (18) through (25), into the Equation (26) yields the doubly infinite system of simultaneous, homogeneous algebraic equations:

$$[\alpha_{n\mu}]\{A_\mu\} = 0 \ , \qquad n,\mu = 1 \text{ to } \infty \tag{27}$$

where

$$\begin{aligned}\alpha_{n\mu} &= \frac{\rho_f\, \omega^2}{D(\epsilon_\mu{}^4+4\lambda^4)} (K_{1\mu} a_n + K_{2\mu} b_n + K_{3\mu} c_n + K_{4\mu} d_n) \\ &+ \left\{\frac{\rho_f\, \omega^2}{D(\epsilon_n{}^4+4\lambda^4)} - \epsilon_n \frac{I_1(\epsilon_n a)}{I_o(\epsilon_n a)}\right\} \delta_{n\mu}\end{aligned} \tag{28}$$

where $\delta_{n\mu}$ is the Kronecker delta. For non-trivial solution, the determinant of the coefficient matrix must vanish. Therefore, the frequency equation becomes

$$|\alpha_{n\mu}| = 0 \tag{29}$$

The fundamental frequencies of a liquid partially filled in an elastic circular cylindrical tank with a rigid bottom are obtained according to the following data:

$\rho g$ = 168 lb/cu. ft. $\qquad \rho_f g$ = 65 lb/cu. ft.

$E = 1.44 \times 10^9$ psf. $\qquad \nu = 0.3$, $h = 0.001$ ft. $\qquad a = 12$ ft.

The results are shown in the Table 4 and Figure (6):

TABLE 4

FUNDAMENTAL CIRCULAR FREQUENCIES FOR A PARTIALLY FILLED CIRCULAR CYLINDRICAL TANK HAVING A FLEXIBLE WALL

| H ft | H/a | $\omega$ rad/sec | $\Omega$ rad/sec* |
|---|---|---|---|
| 2 | 0.166 | 63.28 | 2.40 |
| 3 | 0.250 | 50.21 | 2.76 |
| 4 | 0.333 | 42.48 | 2.96 |
| 5 | 0.417 | 37.17 | 3.08 |
| 6 | 0.500 | 33.16 | 3.14 |
| 7 | 0.583 | 29.98 | 3.17 |
| 8 | 0.667 | 27.37 | 3.19 |
| 9 | 0.750 | 25.15 | 3.19 |
| 10 | 0.833 | 23.26 | 3.20 |

*Rigid tank slosh frequencies

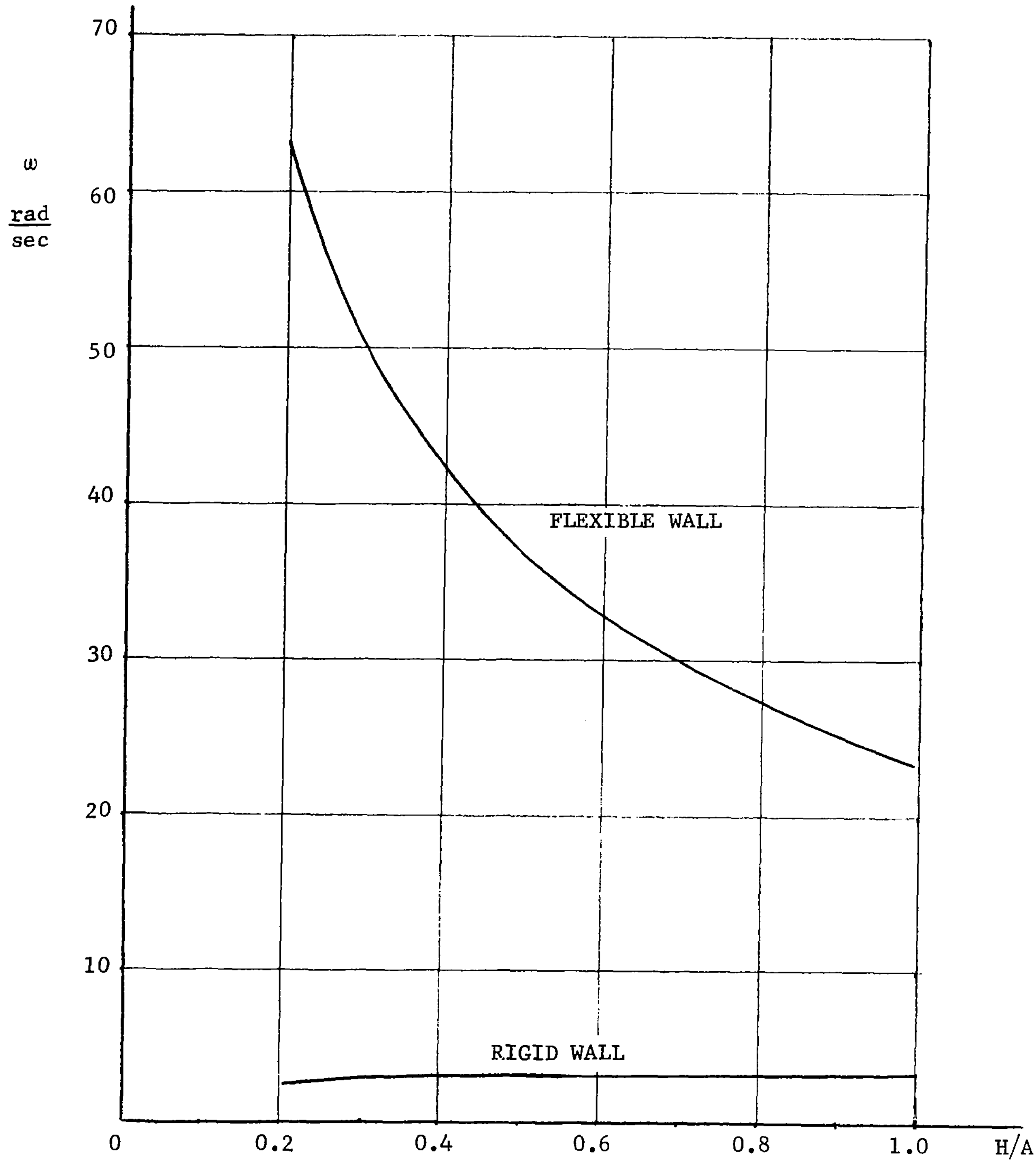


Figure 6. Fundamental Frequencies of Cylindrical Tank with Flexible Wall.

# DISCUSSIONS

The coupled oscillations of a liquid partially filled in a container having an elastic bottom or elastic walls have been studied. Two different tank configurations, namely a rectangular and a circular container both with flat bottoms were considered. Since the effect of the flat elastic bottom of a circular cylindrical tank to the frequencies of the system has been studied by several authors [3], [4], etc. Therefore, no further investigation is attempted in this report. The methods used for both configurations are similar and straight forward. The effects of the flexibility of the container may be summarized according to the results of the numerical examples presented previously.

1. The flexibility of the bottom of a rectangular tank reduces the natural frequencies of the system. The frequency increases as the depth of the liquid increases and gradually approaches to the value corresponding to the case of a rigid container. According to the numerical examples worked, the effect of the flexibility of the bottom becomes negligibly small when the depth of the liquid H is increased to approximately three-fourth of the width of the tank.

2. For the cases where the flexibility of the container walls of a rectangular tank are taken into account, the lowest frequency of the system occurs when the tank is completely filled. The frequencies increase as the depths of the liquid decrease. The frequencies exceed the value corresponding to rigid tank case when the depth of liquid, H, is gradually reduced to approximately three-fifth of the container height.

3. The corresponding natural frequencies for a rectangular tank having one flexible wall are slightly higher, but not appreciable, than the case having both walls flexible.

4. The frequency increases as the depth of liquid decreases in a circular cylindrical container having a flexible wall.

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