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It is shown that over a time interval $\tau = \frac{4\pi mc}{e|\mathcal{E}|}$ the phenomenon of Landau damping, for electron mass small compared to κT, can induce a cooling mechanism in a relativistic plasma by an "amplification" of longitudinal as well as of transverse waves.

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Introduction

In a plasma, energy losses by a beam of particles traversing it can be put into two categories: (i) Incoherent losses or microscopic losses arising from close collision of ions and electrons. Such losses occur inside a cylinder (of Debye radius $r_D = \left(\frac{\kappa T}{4\pi ne^2}\right)^{1/2}$ whose axis of symmetry coincides with the path of the particle; (ii) coherent or macroscopic losses arising from an interplay of collective oscillations and particles outside the Debye cylinder.

For a relativistic (rarefied) plasma, the cross-section for the direct Coulomb interaction of charged particles decreases and coherent energy losses eventually take over. For a non-magnetized plasma the excitation of longitudinal plasma waves constitutes the main mechanism for almost all coherent losses suffered by a beam of particles.

The phase velocity of a longitudinal plasma wave, with a first order relativistic correction, is

$$\upsilon = \frac{\omega}{k} = \left[3\,\frac{\kappa T}{m} + \frac{\omega_p^2}{k^2} + \frac{\omega_p^2}{c^2 k^2}\,\frac{\kappa T}{m}\right]^{1/2} \quad , \qquad \text{(I.1)}$$

where

$$k = \frac{2\pi}{\lambda}\ ,\ \omega_p = \sqrt{\left(\frac{4\pi ne^2}{m}\right)},\ \sqrt{\left(\frac{\kappa T}{m}\right)} = V_T = \text{mean thermal velocity}$$

of the plasma electrons.

The basic restriction on the emission of a plasma wave is contained in the "plasma propagator" defined by

$$\delta_+ \left(\frac{\omega E}{c^2} - \underset{\sim}{k}\cdot\underset{\sim}{p}\right) = P\left[\frac{1}{\frac{\omega E}{c^2} - \underset{\sim}{k}\cdot\underset{\sim}{p}}\right] - i\,\pi\,\delta\left(\frac{\omega E}{c^2} - \underset{\sim}{k}\cdot\underset{\sim}{p}\right) \tag{I.2}$$

where

$$E = \gamma\, mc^2, \quad \underset{\sim}{p} = \gamma\, m\underset{\sim}{v}, \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} .$$

The second term generates the Landau damping of the plasma waves when they are emitted, viz.,

$$\frac{\omega E}{c^2} - \underset{\sim}{k}\cdot\underset{\sim}{p} = 0 \quad ,$$

which can be written as

$$\omega = k\ v\ \cos\theta \quad . \tag{I.3}$$

Thus the necessary condition for the emission of plasma waves is

$$v > \upsilon \ , \quad \upsilon = \frac{\omega}{k} \quad . \tag{I.4}$$

If the particle velocity **v** is less than $\sqrt{3}\ V_T$ then the necessary condition (I.4) cannot be satisfied, and in this case most of the energy losses experienced by the particles are due to close collisions (incoherent losses). If particle velocity v exceeds $\sqrt{3}\ V_T$ the condition (I.4) is satisfied for certain range of wave lengths $\frac{2\pi}{k}$, and coherent losses can occur. However, we shall see presently that the nature of these losses, i.e. from particles to waves or from waves to particles will, for relativistic plasmas, depend on the energy of the particles.

II. Relativistic Landau Damping

Landau damping is a property of the collisionless plasma and to that extent it is rather sensitive to the form of velocity-distribution function. More importantly the damping seems to assume a basic role in differentiating between short and long time processes in the plasma. A discussion of plasma phenomena in terms of time does, naturally, necessitate a careful analysis of the linear and nonlinear nature of plasma oscillations. A reasonable limit on the time interval, during which a plasma behaves linearly, can be obtained. Let us consider the nonlinear integral equation for a relativistic plasma[1] (in k space),

$$\rho(k,p) = -\frac{e}{8\pi c}\frac{1}{p_\alpha k^\alpha}\int f^{\mu\nu}(q)\ L_{\mu\nu}\ \rho(k-q,p)\ d^4q \quad , \qquad \text{(II.1)}$$

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[1] B. Kursunoglu, Nuclear Fusion 1, 213-223 (1961).

where

$$f_{\mu\nu}(k) = - \frac{4\pi ie}{mc} \int \frac{k_\mu p_\nu - k_\nu p_\mu}{k_\alpha k^\alpha} \rho(k,p)\ d^4p \qquad \text{(II.2)}$$

is the Fourier transform of the electromagnetic field in the plasma and

$$L_{\mu\nu} = i \left(p_\mu \frac{\partial}{\partial p_\nu} - p_\nu \frac{\partial}{\partial p_\mu}\right) \qquad \text{(II.3)}$$

is the relativistic angular momentum operator.

We are using relativistic 4-vectors according to

$$k^\alpha = g^{\alpha\mu} k_\mu,\ k_\alpha k^\alpha = k_4^2 - k_1^2 - k_2^2 - k_3^2 ,$$

and arguments of f and ρ are 4-dimensional vectors.

In (II.1) a definite direction of time flow is chosen by assuming that $E = cp_4 = c\sqrt{(p^2+m^2c^2)}$ has a small positive imaginary part (see I.2). However, in the case of nonlinear theory, in view of an interaction between plasma oscillations, the relationship of particle velocities to phase velocity of plasma oscillations is not well defined. The two cannot be expected to be in phase and hence flow of energy from one to the other has a random behaviour in a nonlinear state. Therefore, on this basis Landau damping cannot occur. However for shorter time intervals, namely just before the appearance of nonlinear effects, the plasma as a linear system can experience a Landau damping.

The equation (II.1) can be written as

$$\rho(k,p) = \frac{1}{4\pi}\frac{e}{\omega p_4 - c\underset{\sim}{k}\cdot\underset{\sim}{p}} \int (\underset{\sim}{\mathcal{E}}\cdot\underset{\sim}{N} - \underset{\sim}{\mathcal{H}}\cdot\underset{\sim}{M})\rho(k-q,p)\ d^4q\ , \qquad \text{(II.4)}$$

where $\underset{\sim}{M}$ and $\underset{\sim}{N}$ are angular momentum vector operators corresponding to (II.3), and $\omega = ck_4$. One can apply perturbation theory to (II.4) (or the linearized relativistic collisionless Boltzmann equation) provided in the rest frame of the particle (i.e. $\underset{\sim}{p} = 0$, $\underset{\sim}{\mathcal{H}} = 0$) we impose the condition

$$\frac{1}{4\pi}\ \frac{e|\mathcal{E}|}{\omega p_4} < 1\ . \qquad \text{(II.5)}$$

Putting $p_4 = mc$ we may write for the time interval $\tau = \frac{1}{\omega}$, over which the linear behaviour can be assumed, the expression

$$\tau \leqq \frac{4\pi mc}{e|\mathcal{E}|}\ . \qquad \text{(II.6)}$$

This provides, for a relativistic plasma, a reasonable criterion for the existence of favourable states in which Landau damping can occur. The condition (II.6) does not, of course, determine the direction of the damping, i.e. energy absorption by the particle or by the wave.

For large electric fields the electrons have available a much smaller time interval than ions to absorb from or to lose energy to the wave.

Now to calculate the actual damping we resort to linearized theory (as was already done in reference 1) and obtain the dispersion relations

$$1 = \frac{\omega_p^2}{\omega^2 - k^2c^2} - \frac{\omega_p^2}{c^2} I_1^1 \quad , \tag{II.7}$$

$$\frac{\omega_p^2 + k^2c^2}{\omega^2 - k^2c^2} = \omega_p^2 \frac{\partial I}{\partial \omega} + \frac{2\omega\, \omega_p^2}{\omega^2 - k^2c^2} I_o \quad , \tag{II.8}$$

refering to transverse and longitudinal waves, respectively. The dispersion integrals are given by

$$I_o = I_p + i\, I_L \; ,$$

where

$$I_p = \frac{\sqrt{(1+\eta^2)}}{ck\, K_2(\alpha)} \int_o^\eta \left[Z\, K_1(Z) + K_2(Z) \right] du, \quad Z = \alpha \sqrt{(1+\eta^2-u^2)},$$

$$\eta = \frac{\upsilon/c}{\sqrt{(1 - \frac{\upsilon^2}{c^2})}} \quad ,$$

$\upsilon = \frac{\omega}{k}$ = phase velocity of the wave, $\alpha = \frac{mc^2}{\kappa T}$,

$$I_L = - \frac{\alpha}{ck\, K_2(\alpha)} \frac{d^2}{d\alpha^2} \left[\frac{\pi}{2\alpha} \exp\left(-\alpha \sqrt{(1+\eta^2)}\right) \right] \quad . \tag{II.9}$$

The term I_L is the cause of Landau damping. The integral I_1^1 in (II.7) is given by

$$I_1^1 = - \frac{1}{2k^2} + \frac{c^2}{2} \frac{\partial}{\partial\omega} \left[(\mu^2-1)I_o\right] + \frac{\alpha c}{2k\, K_2(\alpha)} \frac{\partial\Lambda}{\partial\omega} ,$$

where

$$\Lambda = \Lambda_p + i\Lambda_L ,$$

$$\Lambda_p = \sqrt{(1+\eta^2)} \int_o^{\eta} du \frac{K_1\left[\alpha \sqrt{(1+\eta^2-u^2)}\right]}{\sqrt{(1+\eta^2-u^2)}} du ,$$

$$\Lambda_L = \frac{\pi}{2\alpha} \exp\left[- \alpha \sqrt{(1+\eta^2)}\right] .$$

The damping of a plasma wave of velocity $\upsilon = \frac{\omega}{k}$ by a relativistic electron with velocity v depends on the form of I_L given by (II.9). It can be written as

$$I_L = - \frac{\pi}{ck\, K_2(\alpha)} \left[\frac{1}{\alpha^2} + \frac{1}{\alpha} \gamma_W + \frac{1}{2} \gamma_W^2\right] \exp\left(- \frac{E\beta}{\kappa T}\right) , \qquad \text{(II.10)}$$

where

$$\beta = \frac{\gamma_W}{\gamma_e} , \quad \gamma_W = \left(1 - \frac{\upsilon^2}{c^2}\right)^{-\frac{1}{2}} , \quad \gamma_e = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} ,$$

$$E = \gamma_e mc^2 .$$

Thus the damping of a plasma wave is proportional to the density of plasma electrons, possessing at $\upsilon = v$ (or $\beta = 1$), according to a relativistic Maxwell distribution, an energy $E = \gamma_e mc^2$ and velocity equal to the velocity of the wave. This is in exact correspondence with the absorption mechanisms pointed out above. However, it is important to note that absorption of energy by the electron from a plasma wave decreases with an increase of electron energy.

As an illustration, let us consider the extreme case where the rest mass of the electron is small compared to κT. In this case the relativistic dispersion integral (see ref. 1) for longitudinal waves becomes

$$I_o = \frac{1}{ck} \left[\log \left(\frac{kc+\omega}{kc-\omega}\right) - i\pi \right] \tag{II.11}$$

where the imaginary part refers to Landau damping in this extreme case. Using (II.11) in the dispersion relation (II.8) we obtain the result

$$\frac{\upsilon}{c} = \coth \left[\frac{c}{\upsilon} \left(1 + \frac{c^2}{\upsilon^2} \frac{\omega^2}{2\omega_p^2}\right)\right] \quad . \tag{II.12}$$

From (II.12) it follows that longitudinal wave velocity exceeds the velocity of light. Therefore, in this extreme case, the reverse of Landau damping occurs: the wave absorbs energy from the electron and reaches phase velocities exceeding that of light. This means that Landau damping in the extreme relativistic state provides a mechanism for cooling the plasma as contrasted with the heating effect that it provides in a warm plasma. If the imaginary part of (II.11) were omitted then one would obtain the relation (II.12), with tanh instead of coth. In this case υ is always less than c.

On the basis of this extreme relativistic case the dispersion relation for transverse waves has a similar structure to that of longitudinal waves. The transverse dispersion integral I_1^1 together with transverse dispersion relation leads to

$$\frac{\upsilon}{c} = \coth \left[\frac{c}{\upsilon}\left(\frac{\upsilon^2}{\upsilon^2 - c^2} - \frac{\omega^2 c^2}{\upsilon^2\ \omega_p^2}\right)\right] \quad , \qquad \text{(II.13)}$$

one of its solutions being $\upsilon = c$. For all other solutions, $\upsilon > c$.

The energy absorbed by the waves may be dissipated amongst the waves themselves, enhancing the mechanism of waves scattering from one another. The plasma thus assumes its nonlinear behaviour which does not allow Landau damping[(2)] after the time τ.

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[2] For a detailed discussion of Landau damping in nonrelativistic plasma, see Chapter 7 of THE THEORY OF PLASMA WAVES, by Thomas H. Stix, 1962, McGraw-Hill.