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"Hydromagnetic Waves - Theory and Applications"

by

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ABSTRACT

In this work we investigate the influence of Magnetohydrodynamic waves on various physical phenomena such as magnetosphere, heating of the solar corona, controlled thermonuclear reaction, propagation of the whirl rings from the sun's core to its surface and the cosmic rays. The article begins with a survey of general theory and results of MHD waves.

Hydromagnetic Waves - Theory and Applications

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I. Introduction

A conducting fluid moving in a magnetic field produces electric currents in the medium. The intersection of these electric currents and the magnetic field gives rise to the mechanical forces which change the hydrodynamic motion of the fluid. In the process the magnetic field is also modified. Magnetohydrodynamics is the combined study of these interactions. In the theoretical study of the subject there appear results which are very different from those known in hydrodynamics and electromagnetism when studied separately. The most fascinating of these results are the ones related to the wave propagation. We present here a brief study of the magnetohydrodynamic sound as well as shock waves and point out their applications and occurrences in various physical phenomena.

In compressible fluids the only possible waves are longitudinal waves travelling at a finite speed. For incompressible fluids this speed becomes infinite. However, when the fluid is magnetized Alfvén discovered transverse waves which propagate at a finite speed even in the incompressible fluids. In fluids which are both compressible as well as magnetized there emerge various kinds of magnetosonic waves which, in general, are neither transverse nor longitudinal. There are the fast, Alfvén, and slow waves; the classical sound speed and Alfvén speed have values which are intermediate between the fast and the slow speeds. Unlike the classical sound wave, the magnetosonic waves generate vorticity and electric currents and are responsible for the stretching and shearing of the fluid elements. Moreover, these waves are anisotropic. The fast wave is slightly anisotropic and somewhat resembles the isotropic gasdynamic sound wave. The Alfvén wave is highly degenerate and propagates along the lines of force. It is forever contained in a fixed tube. The slow wave is even more singular; its locus consists of two cusped regions.

The discontinuities characterized by the condition that at least one of the flow and field quantities should have a jump across them are called shock waves. Non-magnetized and non-dissipative gas-dynamical motions involving a compression are known to break down at some point. From that point on the shock discontinuities appear. The flow quantities on both sides of the shock front are governed by the laws of conservation. The same effects appear in the hydromagnetic motion as well. When we analyze the jump conditions, we are led to various solutions, but all these solutions do not give rise to physically admissible shocks. For a solution to qualify for an admissible shock it must satisfy at least one of the following restrictions. The solution must correspond to the non-decrease of entropy across the wave. Secondly, it should be the limit of the dissipative solution as the dissipation becomes small. Finally, the shock must be the limit of steepening compression waves. In the gas-dynamical case all these restrictions are equivalent but not so in the present general case of magnetohydrodynamics. Alfvén discontinuity does not satisfy these requirements and is therefore not called a shock. Across this discontinuity there is no jump in any of the thermodynamic variables or the normal velocity, while the tangential components of the magnetic field as well as velocity maintain their magnitude and merely rotate in the plane of the wave front. The fast and slow wave solutions involve an entropy jump and are called shocks.

Alfvén pointed out the connection of magnetohydrodynamic waves with the theory of sunspots and other solar phenomena. Since then, continued interest inspired by the astrophysics, solar physics, thermonuclear power-generation, magnetosphere, cosmic rays and other relevant physical theories have led to finding their relationship to these new waves. For example, the heating of the solar corona can be explained by the generation and propagation of hydromagnetic shock waves. The solar wind emanating from these regions carries some of these waves with it. Moreover, as this supersonic solar wind impinges on the earth's magnetosphere, it creates a bow shock wave in front of the magnetosphere. The boundary of this magnetosphere itself can be calculated by the theory of magnetohydrodynamic contact discontinuities. A contact discontinuity is a surface of discontinuity across which there is no mass flow and this surface emerges as one of the solutions of the jump relations mentioned above. Because of the rapid inside and outside motion of the magnetospheric boundaries, various small amplitude disturbances are created, giving rise to all kinds of magnetosonic waves. In this connection we shall discuss both the particle model of Chapman and Ferraro, and the fluid model of Spreiter, Summers and Alsksne. As for the other applications, we point out the important role these waves play in the generation of whirls inside the sun, in the controlled thermonuclear reactions and in the acceleration of the cosmic rays.

II. The Equations of Motion

We consider first the non-relativistic case and adopt the usual notation of writing $\underline{H}$ for the magnetic field, $\underline{v}$ for the velocity field, p for the pressure, ρ for the density, S for the entropy and μ for the magnetic permeability. The definitive system of equations, describing the motion of an electrically conducting fluid which is devoid of the dissipative mechanism of viscosity, heat conduction and electrical resistance, is

$$\text{div}\,\underline{H} = 0, \tag{1}$$

$$\frac{d\,\underline{H}}{d\,t} - \underline{H}\cdot\text{grad}\,\underline{v} + \underline{H}\,\text{div}\,\underline{v} = 0, \tag{2}$$

$$\rho\,\frac{d\,\underline{v}}{d\,t} + \text{grad}\,p + \mu\,\underline{H}\times\text{curl}\,\underline{H} = 0, \tag{3}$$

$$\frac{d\rho}{dt} + \rho \operatorname{div} \underline{v} = 0, \tag{4}$$

$$\frac{dS}{dt} = 0, \tag{5}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \underline{v} \cdot \text{grad}$$

and there is an equation of state relating p, ρ, and S. The equation (1) is really a consequence of equation (2). The rest of the equations form a system of eight quasi-linear partial differential equations of first order for the eight quantities $\underline{H}$, $\underline{v}$, ρ, and S.

Magnetosonic waves can be studied by treating them as singular surfaces of order one such that all the field and fluid parameters are continuous across them; however, there is a discontinuity in the first derivative of at least one of these quantities. Thus, a sonic wave front—$f(x,t) = 0$, $|\text{grad } f| \neq 0$, is a surface in space-time across which can exist discontinuities of derivatives of the dependent variables. The normal vector $\underline{n}$ to such a surface is grad f/ $|\text{grad f}|$ and the normal speed of advance G (also called the phase speed) is $-\ \partial f/\partial t/ |\text{grad } f|$. Let us denote the jump of the derivatives of a function F(x, t) as δF. The dispersion relation for the various normal speeds G can then be obtained from the system of equations (1) to (5) by setting:

$$\left.\begin{aligned} \left[\frac{\partial F}{\partial t}\right] &= -G\,\delta \underline{F}, \quad \left[\nabla \underline{F}\right] = \underline{n}\,\delta \underline{F}, \\ \left[\nabla \cdot \underline{F}\right] &= \underline{n} \cdot \delta \underline{F}, \quad \left[\nabla \times \underline{F}\right] = \underline{n} \times \delta \underline{F}, \end{aligned}\right\} \qquad (6)$$

where [ ] stands for the jump in the required quantity across the wave front. This leads to the following systems of difference relations [10] .

$$-U\,\delta \underline{H} - H n\,\delta \underline{v} + \underline{H}\,(\delta \underline{v}) \cdot \underline{n} = 0, \ldots \qquad (7)$$

$$-\rho U\,\delta \underline{v} + a^2 \underline{n}\,\delta \rho + B\,\underline{n}\,\delta S + \mu\,\underline{n}\,(\underline{H} \cdot \delta \underline{H}) - \mu H_n\,\delta \underline{H} = 0, \qquad (8)$$

$$-U\,\delta \rho + \delta \underline{v} \cdot \underline{n} = 0, \qquad (9)$$

$$-U\,\delta S = 0, \qquad (10)$$

where U is the phase speed of a magnetosonic wave relative to the fluid and a is the speed of the sound:

$$U = G - \underline{v} \cdot \underline{n}, \quad a^2 = (\partial p / \partial \rho)_S, \qquad (11)$$

and by H_n, etc. we mean $\underline{H} \cdot \underline{n}$. Furthermore, we have used the relation between ρ, p and S as given by the equation of state.

The system of algebraic equations (7) to (10) is a definite system of eight homogeneous algebraic equations in the eight quantities

$\delta \underline{v}$, $\delta \underline{H}$, $\delta \rho$, and δS. By putting the discriminant of this algebraic system equal to zero we get the required equation for U:

$$\rho U^2 (\rho U^2 - \mu H_n^2) \left\{ \rho U^4 - (\rho a^2 + \mu H^2) U^2 + a^2 \mu H_n^2 \right\} = 0, \quad (12)$$

III. Magnetosonic Waves

From equation (10) we observe that either $U = 0$, $\delta S \neq 0$; or $U \neq 0$, $\delta S = 0$. In the first case the wave front is a material surface (contact discontinuity). This corresponds to the root $U = 0$, of the discriminant relation (12). The second case, i.e., $\delta S = 0$, gives three non-zero values of U. One of these values arises from the second root of the equation (12):

$$U^2 = \mu H_n^2 / \rho = A^2. \quad (13)$$

The quantity A gives the Alfvén speed. Alfvén waves are transverse waves and propagate in incompressible as well as compressible fluids. Now in compressible non-magnetized fluids for which $p = p(\rho, S)$, the only possible waves are longitudinal waves travelling at the speed, a, which is given by the relation (11). In fluids which are both compressible as well as magnetized, there emerge waves which, in general, are neither transverse nor longitudinal. These speeds are given by the last biquadratic factor in the equation (12). This factor can be written in a more suggestive form:

$$(U^2 - a^2)(U^2 - A^2) = U^2 \mu \frac{(H^2 - H_n^{\;2})}{\rho} . \qquad (14)$$

The larger and smaller roots $U > 0$, of this equation are denoted by U_{fast} and U_{slow} (written U_f and U_s in the sequel). From the equation (14) it follows that

$$U_s \leq a \leq U_f; \; U_s \leq A \leq U_f. \qquad (15)$$

In view of this relation, A is also called the intermediate wave speed.

In order to construct the surfaces of these normal speeds, we define two dimensionless parameters:

$$r^2 = U^2 \rho / \mu H^2, \qquad (16)$$

$$b^2 = a^2 \rho / \mu H^2, \qquad (17)$$

and let Θ denote the angle between $\underline{n}$ and $\underline{H}$. In this notation the non-zero roots of the equation (12) become

$$r_{slow} = \left[\frac{1}{2} (1 + b) - \frac{1}{2} \left\{ (1 + b)^2 - 4 b \cos^2 \Theta \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}}, \qquad (18)$$

$$r_{Alfven} = \cos \Theta , \qquad (19)$$

$$r_{fast} = \left[\frac{1}{2} (1 + b) + \frac{1}{2} \left\{ (1 + b)^2 - 4 b \cos^2 \Theta \right\}^{\frac{1}{2}} \right]^{\frac{1}{2}}, \qquad (20)$$

***

Figure 1 is a $(r - \Theta)$ plot of the magnetosonic normal speeds for the cases (i) $b < 1$, (ii) $b = 1$, (iii) $b > 1$.

(*** Figure 1 inserted here)

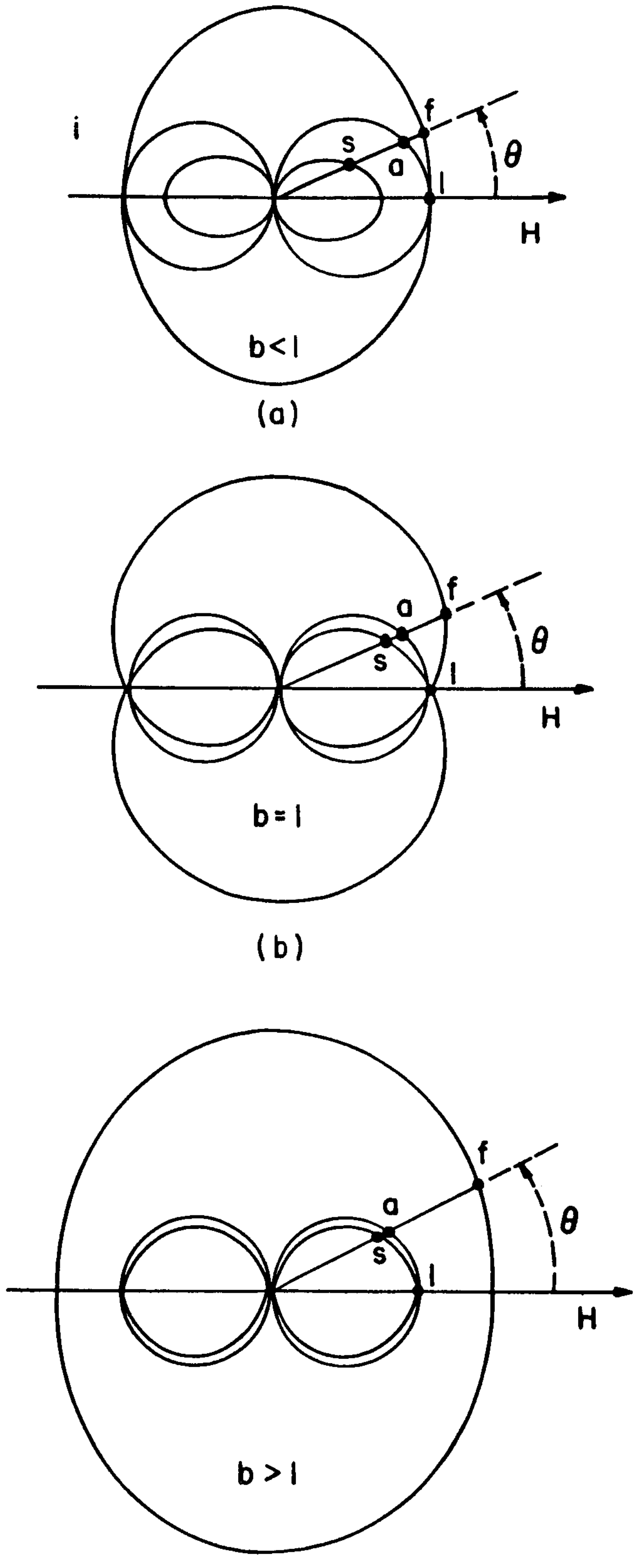

Fig. 1. Surfaces of normal speeds.

These figures have not, however, brought out clearly the anistropy of the wave fronts. The wave fronts themselves can be plotted if we observe that the length of the vector from the origin to a point on the curve of the figure (1) represents the distance travelled by the plane wave which is normal to that vector, and the wave front due to an isotropic pulse can be obtained by constructing the envelop of these normal planes. These wave front diagrams can be obtained by rotating the curves of the Figure (2) about the $\underline{H}$ - axis. The surfaces so obtained are also called the ray surfaces for the following reason: Each factor of Equation (12) when put equal to zero, is itself a partial differential equation of the first order because the unit normal vector $\underline{n}$, involved in that equation, is equal to grad $f/|\text{grad } f|$. As such the surfaces $f(x, t)$ given by this first order differential equation can be spanned by its own characteristic curves, known as bicharacteristics or rays for the present case. *****

From Figure (2) we observe that the outer fast locus is somewhat anisotropic but resembles the sonic wave front. The locus for the intermediate wave is extremely degenerate and consists of two points on the H-axis together with the included line segment. This degenerate locus represents strictly one-dimensional propagation along magnetic lines. Thus, the Alfvén wave front is forever contained within a fixed tube. The slow locus is even more singular. It consists of two disconnected cusped regions together with the intercepted line segment. The qualitative significance is easily appreciated by the example in Figure (3). This figure depicts the evolution of an originally spherical slow wave front in a uniform medium. It overtakes the Alfvén wave front after the singularities develop in its shape. +++ +++

(***** Figure 2 inserted here.), (Figure 3 inserted here.)

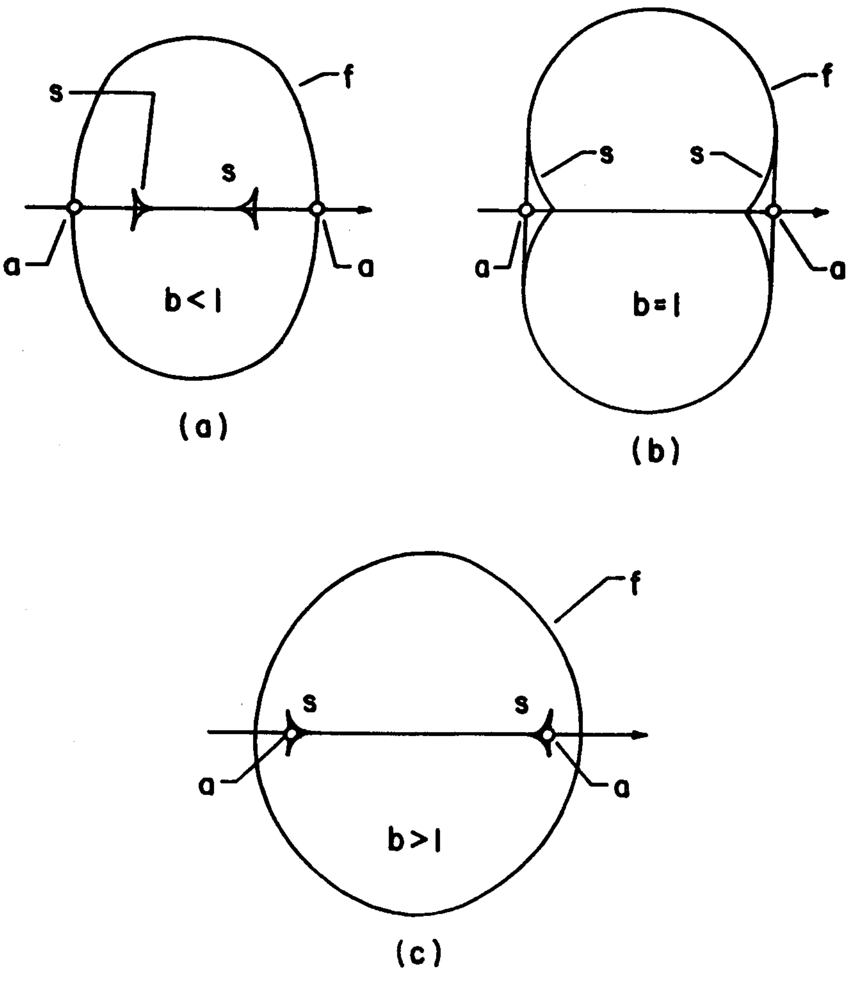

Fig. 2. Ray surface.

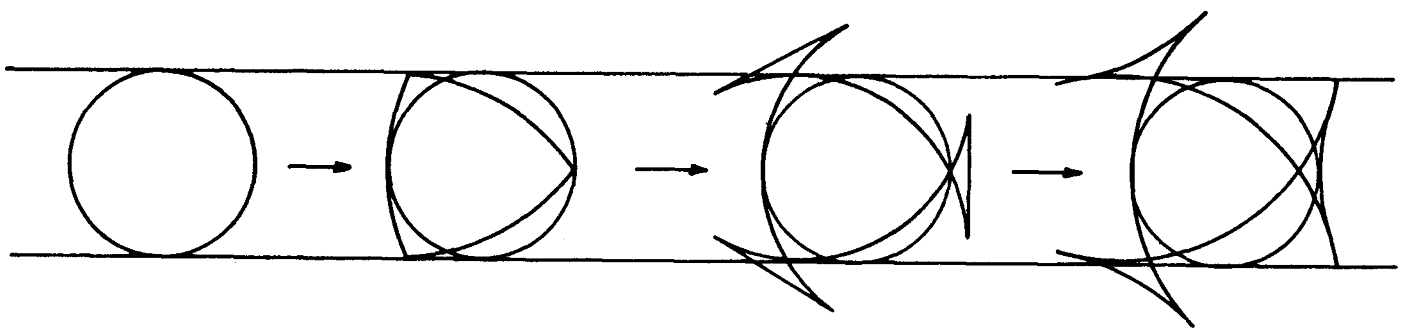

Fig. 3. It illustrates the evolution of a slow wave front from a spherical initial manifold.

IV. Shock Waves

Shock waves appear when non-dissipative gasdynamical motions involving compression waves steepen. Across these waves the fluid properties undergo jumps. However, the flow equations on both sides of the shock wave are governed by the conservation laws and other conditions such as the non-decrease of the entropy. Similar phenomenon occurs for the hydromagnetic flow as well, except that in the present case, the variety of discontinuities extends beyond those in gas-dynamics. The jump relations follow directly from the integral form of the conservation laws and form a determinate system of difference equations. Every non-trivial solution of these difference equations gives rise to a possible discontinuous flow, but physically not all of these solutions are observable or admissible. Identification of the physically relevant solutions can not be made merely on the basis of entropy consideration, which is sufficient for the gas-dynamical case. Different kinds of restrictions are placed in identifying the magnetohydrodynamic shocks. First of all, of course, is the entropy consideration. The discontinuous solution must correspond to the non-decrease of entropy. Secondly, the MHD shocks must be the limit of dissipative solutions as dissipation becomes small. Another restriction is that these shocks should be the limit of a steepening compression wave. In pure gas-dynamical shocks all these restrictions are equivalent, but in the present case they are different. When screened by these restrictions, all the extraneous solutions which do not occur in nature are eliminated.

The physically irrelevant solutions were originally referred to as unstable. This term has now been generally abandoned since there does not arise the ordinary instability which grows with time, but what happens is a rather sudden disintegration of the wave. This disintegration phenomenon can be determined by the steepening condition. This is referred to as the evolutionary criterion. The evolutionarity of shock wave stipulates that the outgoing plane-disturbance waves from the shock front be linearly independent. This condition places more stringent restrictions on the gas velocities than those imposed by the jump relations.

In the sequel we shall have terms such as switch-on and switch-off shocks. The term switch-on stands for the shock if a tangential component of the magnetic field is produced through the wave if it was absent ahead of it. Similarly, for a switch-off shock, the tangential component of the field present upstream of the shock is wiped out by it.

V. Jump Relations

The jump conditions across a shock wave can be derived from the set of equations (1) to (5) by setting $\partial \underline{F} / \partial t$ equal to $-G[\underline{F}]$, and $\nabla \underline{F}$ equal to $\underline{n}[\underline{F}]$, where $[\underline{F}] = (F_2 - F_1)$, and the subscript 1 stands for the upstream quantity and the subscript 2 stands for the downstream one. The normal speed of the shock is G and the relative normal speed is $(G - v_n)$. It is convenient to use a coordinate system fixed to the discontinuity surface. This transforms the problem to that of steady flow and the jump relations take the simple form:

$$[H_n] = 0, \qquad (21)$$

$$[\rho v_n] = 0, \tag{22}$$

$$[v_n \underline{H} - H_n \underline{v}] = 0, \tag{23}$$

$$[\rho v_n \underline{v} + (p + \mu \frac{H^2}{2}) \underline{n} - \mu H_n \underline{H}] = 0, \tag{24}$$

$$[v_n (\frac{1}{2} \rho v^2 + \rho E + \mu \frac{H^2}{2}) + v_n (p + \mu \frac{H^2}{2}) - \mu H_n (\underline{H} \cdot \underline{v})] = 0, \tag{25}$$

where we have, as before, set F_n, the component of $\underline{F}$, in the direction of the unit normal $\underline{n}$, and $\underline{n}$ points downstream.

To have a quick survey of the different modes of shock propagation it is convenient to set

$$m = \rho v_n, \quad \tau = \frac{1}{\rho}, \tag{26}$$

and make use of the notation $\tilde{F} = \frac{1}{2}(F_1 + F_2)$. Moreover, the equation (21) merely implies that H_n is continuous across a shock. The rest of the equations take the following form:

$$m[\tau] - [v_n] = 0, \tag{27}$$

$$m\tilde{\tau}[\underline{H}] + \tilde{\underline{H}}[v_n] - H_n[\underline{v}] = 0, \tag{28}$$

$$m[\underline{v}] + [p]\underline{n} + \mu \bar{\tilde{\underline{H}}}[\underline{H}] = 0, \tag{29}$$

$$m\left\{[\underline{E} + \frac{H^2}{2}\tau] + [\tau](\tilde{p} + \frac{\tilde{H}_t^2}{2} - \frac{H_n^2}{2}) - [\tau \underline{H}_t] \cdot \tilde{H}_t\right\} = 0, \tag{30}$$

where $\underline{H}_t$ is the tangential component of the magnetic field along the surface of shock. These equations form a determinate system of eight equations in the eight unknowns $[\ \underline{v}\ ]$, $[\ \underline{H}\ ]$, $[\ \tau\ ]$, and $[\ E\ ]$. Since E appears only in the relation (30), we can solve the rest of the determinate system of seven equations in seven unknowns. A shock front has to bear a discontinuity in at least one of these seven quantities; therefore, the discriminant of this algebraic system must be zero.

$$\tilde{\tau}^2 m\,(\tilde{\tau} m^2 - \mu H_n^2)\quad \tilde{\tau} m^4 + \left((\tilde{\tau}\,[\ \tau\ ]^{-1})\, p - \mu H_n^2 \right) m^2 - [\ p\ ]\ H_n^2\,[\ \tau\ ]^{-1} = 0. \qquad (31)$$

This is an equation for the flux m, but it may as well be considered as an equation for the shock speed G, because of the relation (in the original coordinate system).

$$G = \tilde{v}_n - m\,\tilde{\tau}. \qquad (32)$$

From (31) it follows that there are contact, Alfvén, slow and fast discontinuities, and the relationship between their magnitudes is the same as in the magnetosonic case. As we shall soon see, the Alfvén discontinuities do not qualify to be true shocks while the slow and fast discontinuities do. Let us classify and discuss the roots of the equation (31) in the following way.

VI. Classification of Shock Waves

Case 1, $m = 0$, $[\ \tau\ ] \neq 0$: Contact discontinuities.

They are material surfaces which convect with the gas. Since m = 0 but $[\,\rho\,] \neq 0$, this means that $[\,v_n\,] = 0$. There are two distinct cases:

(a) $H_n = 0$, in this case in addition to the normal component of the velocity, the total pressure $p + \mu \frac{H^2}{2}$, is also continuous. All the other quantities have an arbitrary jump. Thus, the velocity and magnetic fields are transversal and can have arbitrary jumps, both in magnitude and direction. For that reason, this kind of material surface is also called a tangential discontinuity.

(b) $H_n \neq 0$, then $[\,v\,] = [\,\underline{H}\,] = [\,p\,] = 0$, and the entropy may have an arbitrary jump.

<u>Case 2</u>. $[\,\tau\,] = 0,\ m \neq 0$: Alfvén discontinuities.

From the jump relations and the second factor of the equation (31), it follows that

$$m = (\mu/\tau)^{1/2} H_n, \quad [\,\underline{v}_t\,] = (\mu\,\tau)^{1/2}\,[\,\underline{H}_t\,], \tag{33}$$
$$[\,E\,] = 0, \quad [\,H_t^2\,] = 0.$$

Since the density and the internal energy are continuous, all the thermodynamic quantities are continuous. Furthermore, the normal velocity and the normal magnetic field are continuous. Thus, the transverse magnetic field and the transverse velocity maintain their magnitude. They merely rotate in the plane of the shock. For this reason, these waves are called rotational discontinuities.

The phase velocity of these waves i.e., $H_n\,(\mu\,\tau)^{1/2}$ is independent of the strength of the discontinuity; as such they can not have arisen by steepening of corresponding simple waves. So we find that Alfvén discontinuities have neither an entropy increase across them nor do they satisfy the steepening criterion.

Case 3. $m \neq 0$, $[\tau] \neq 0$: MHD shocks.

These discontinuities follow from the solution of the last factor of the equation (31). They involve an entropy rise accompanied by an increase in density and pressure. Furthermore, these shocks are determined completely by the upstream state. Their nature is analyzed if we write their equation as

$$(m^2 + [p] \; [\tau]^{-1}) (\tilde{\tau} m^2 - H_n{}^2) = m^2 \tilde{H}_t{}^2, \qquad (34)$$

and subdivide their discussion in the following way:

(i) Gasdynamic shocks. If the magnetic field is normal on both sides and thus continuous, we recover the gasdynamic shocks and the Alfvén wave as factors of the above equation. The gasdynamic shock is uniquely determined by the upstream state or a downstream one. The normal velocity upstream is supersonic; downstream it is subsonic. In case the quantity b, the ratio of the sound speed to Alfven speed as defined by the relation (17), is less than or equal to unity, there is an additional restriction that $a_1 < V_{n1} \leq A_1$, when the upstream state is given; $a_2 < V_{n2} \leq A_2$, if the downstream is given. If these conditions are not satisfied then the switch-on shock occurs in the first case and the switch-off one in the second case.

(ii) Fast and slow shocks. They arise as solutions of the equation (34) in the same way as the fast and slow magnetosonic waves arose from their corresponding equation. For a fast shock $U_{f1} < v_{n1}$ and for a slow shock $U_{s1} < v_{n1} \leq A_1$. From the jump conditions it follows that the magnetic field strength increases across a fast shock and decreases across a slow one. In view of the continuity of the normal component of the magnetic field, it follows that for a fast shock the magnetic field bends away from the normal and for a slow shock it bends toward the normal.

The evolutionary condition implies that for a physically admissible shock the small disturbances incident on it produce small, uniquely determined disturbances in the flow pattern. This results in severe restrictions on the gas velocities [3, 9]. In fact, for a fast shock, the condition is that $v_{n1} \geq U_{f1}$; $U_A < v_{n2} \leq U_{f2}$; and for a slow shock $U_{s1} \leq v_{n1} < U_{A1}$; $v_{n2} \leq U_{s2}$. Thus, a fast shock represents a jump of the fluid velocity from the super-fast to the subfast as well as super-Alfvén; a slow shock from the super-slow but sub-Alfvén to sub-slow. All the other solutions of the jump relations which require the shock to cross the Alfvén speed are non-evolutionary and without steady structure. Switch-on and switch-off shocks are also ruled out by the evolutionary condition. Recently, a more physical description of the criterion for the admissibility of the MHD shocks has been advanced by Kantrowitz and Pelschek [11] .

There is one special case of fast shocks known as the perpendicular shocks. In this case the magnetic field lies in the surface of the shock and is thus perpendicular to the direction F propagation of the shock. This is, of course, meaningful for plane shocks. The relation (34), when H_n is put equal to zero in it, becomes

$$(m^2 + [\,p\,] \quad [\,\tau\,]^{-1} \tilde{\tau} = \tilde{H}_t^{\,2}. \tag{35}$$

By an appropriate adjustment of the equation of state (7) this case can be reduced to that of a gas-dynamical shock. When we analyze the jump relations for this case, it emerges that the tangential magnetic field increases across the perpendicular shock as it does for a fast shock. The normal velocity decreases across it while the tangential velocity is preserved.

VII. Relativistic Effects

In our previous study of the coupling of hydrodynamics with electromagnetism, we have adopted only the quasi-equilibrium approximation to the electrodynamic equations in which the displacement currents and charge accumulations are neglected. Such an analysis merely allows the possibility that the velocities, both random and ordered, of fluid elements are comparable with the classical velocity of sound. In order to couple the full equations of electrodynamics, which give a covariant description of the electromagnetic phenomena, with those of hydrodynamics, we must first give the covariant formulation for the flow description. The subsequent coupling of these two systems of equations then yields magneto-sonic waves whose speeds are of the order of the speed of light. We briefly point out the salient features; the detailed analysis is available elsewhere [10] .

Let g_{AB}, (A, B = 0, 1, 2, 3) denote the metric components of an Einstein-Riemann space and let W_A and h_A be the velocity and magnetic four-vectors. The definitive system of relativistic MHD equations then take the form

$$W^A W_{A;B} = 0, \tag{36}$$

$$W^A h_{A;B} + W^A_{;B} h_A = 0, \tag{37}$$

$$W^B h^A_{;B} + h^A W_{;B}{}^B - h^B W_{;B}{}^A - W^A h_{;B}{}^B = 0, \tag{38}$$

$$(c^2 \rho_{,B} + p_{,B}) W^A W^B + (c^2 \rho + p + \mu |h|^2)(W^A W_{;B}{}^A$$

$$+ W^B W_{;B}{}^A) - p_{,B} g^{AB} - 2\mu (W^A W^B - \tfrac{1}{2} g^{AB}) h_c h^c_{;B}$$

$$-\mu (h_{;B}{}^A h_B + h^A h_{B;B}) = 0, \tag{39}$$

and

$$p_{,A} W^A - a^2 \rho_{,A} W^A = 0, \tag{40}$$

where the semicolon denotes the covariant and the comma the ordinary differentiation. The quantity c is the speed of light. When we subject these equations to the same mathematical treatment for finding the various modes of wave propagation as we did to the non-relativistic equations, we get qualitatively the same picture. In fact, we have the intermediate waves which have the speed V.

$$V^2 = \frac{\mu h_N^2 c^2}{c^2 \rho + \rho + \mu |h|^2}, \tag{41}$$

where $h_N = h_A N_A$ and N_A stands for the four-normal vector to the sonic hypersurface. The slow and fast sonic speeds are derived from the equation

$$(1 - \frac{a^2}{c^2})(\rho + \frac{p}{c^2}) V^4 - \left\{ \mu \mid h \mid^2 + a^2 (\rho + \frac{p}{c^2}) \right\} V^2 + \mu a^2 h_N^2 = 0. \qquad (42)$$

In the non-relativistic limit these relations reduce to their corresponding values, as given earlier.

The geometry of the wave fronts is also similar. When we plot the hyper-surfaces of the wave normals, we find that the required surface for the intermediate wave consists of two planes. The corresponding surfaces for the slow and fast waves form two distinct nappes of a conoid; the surface of the wave normals for the intermediate wave always touches the conoid. Thus, the relativistic MHD waves have the same configuration as do their non-relativistic counterparts.

VIII. MHD Waves and Solar Phenomena

The solar electromagnetic fields are responsible for various aspects of solar physics. Alfvén [1, 2] has developed a MHD wave theory of sunspots and discussed the propagation of the sunspot zones from higher altitudes towards the equator according to this theory which is compatible with a dipole field. The solar matter, which is a good conductor, embedded in a magnetic field can be described by the equations studied in this paper. There are various kinds of disturbances and pulses arising in the core of the sun,and they give rise to all kinds of MHD waves. These waves travel outward towards the surface of the sun. The Alfvén waves propagate along the magnetic lines of force until they reach the photosphere, giving rise to the sunspots. The magnetic fields associated with the waves are identified with the strong magnetic fields always observed in the sunspots. The bipolarity of the sunspots also follows from the forward and backward propagation of Alfvén waves.

The granulation in the sun's photosphere has also been explained by MHD waves. Alfvén points out that the instability of the highly ionized hydrogen inside the sun is propagated outwards as Alfvén waves.

Another phenomenon which can be explained by the MHD waves is the propagation of the whirl rings from the sun's core to its surface. This was first discussed by Alfvén in connection with the sunspot theory where they provide a reasonable explanation of the sunspot progression curves. Recently, these whirl rings have attracted fresh interest because of a discovery that indicates a time variation in the solar differential rotation. This time, variation may be intimately connected with the redistribution of the angular momentum due to the propagation of the whirl rings. We advance here a MHD wave theory for these processes by proving that these waves are intrinsically responsible for the generation of the rings.

Let us go back to the relations (7) to (10) and use the fact that the jumps in the vorticity vector and the electric current density vector can be obtained by using the relations

$$[\,\underline{\omega}\,] = \underline{n} \times \delta \underline{v}\,; \quad [\,\underline{J}\,] = \underline{n} \times \delta \underline{H}\,. \qquad (43)$$

It readily follows [10] that

$$U [\underline{\omega}] + H_n [\underline{J}] = 0. \qquad (44)$$

This means that MHD sound waves can generate vorticity as well as electric current as they propagate. Now in the pure gas-dynamical case, only the shock waves have such a property. The MHD shock also, of course, retains that property. Thus, all kinds of MHD waves, as they propagate in the interior of the sun, give rise to whirl rings and should account for the effects mentioned in the preceding paragraph.

IX. Solar Wind

The space surrounding the earth and the other planets is far from empty. A fast wind of hydrogen emanates from the sun and blows continuously through the solar system. Its speed past the earth is between 300 and 700 kilometers per second. It has a volume density of .3 to $10/cm^3$. Despite the fact that the material density of this wind is less than the best vacuum thus far created in a laboratory, the consequences of this exceedingly tenuous matter are rather crucial to a surprising variety of disciplines. In fact, like a broom, it sweeps away the gases evaporated from planets and comets, fine particles of meteorite dust, and even cosmic rays. It is responsible for the variations which occur in the strength and direction of the solar magnetic field lines as they are carried across the space by it. It is also responsible for the outer portions of the Van Allen radiation belts around the earth, for auroras in the earth's atmosphere and for triggering the magnetic storms on the earth.

The existence of the solar wind was first asserted by Birkeland in the year 1896. He observed the similarity between the electric discharge in the newly invented fluorescent tubes and the aurora borealis. He surmised that the aurora was caused by the electrically charged particles coming from the sun. The mathematical details of Birkland's ideas were given by Sormer. Many years later and with the advent of radio and telephone communications, came a further evidence of the existence of the solar wind. This related to the magnetic storms which are associated with the disruption of these communications. Chapman and Ferraro suggested that these storms are caused by the corpuscular emissions from the sun. Their theoretical picture of such a disturbance of the earth's magnetic field so closely resembled the observed field fluctuations that their theory was widely accepted.

Another manifestation of the solar wind was noted in the late 1940's in connection with the fluctuations in the bombardment of the earth by cosmic rays. It was observed that the intensity of the cosmic radiation reaching the earth was low during the height of solar activity in the sunspot cycle. It could be accounted for by the fact that a stream of charged particles shooting from the sun, with a magnetic field embedded in them, would tend to sweep cosmic ray particles out of the interplanetary space. However, the decisive proof for the existence of the solar wind came from the behavior of the gaseous tails of the comets. No matter where the comet may be in its orbit through the solar system, its head is always toward the sun and its tail streams away from the sun. In 1950 Bierman pointed out that the pressure of the sunlight is not sufficient enough to account for the violence with which a comet's gases are blown away from its head. Furthermore, from this evidence, it became plain that the corpuscular radiation could not be coming merely in bursts or isolated beams. This radiation is, in fact, blowing continuously. The streaming of the particles might intensify during the solar activity but the solar wind is present all the time [12, 16, 17] .

The magnetohydrodynamic waves start playing an important role right at the source of the solar wind. When a large flare occurs on the sun, the resultant sudden local increase in temperature and pressure within the ionized gases of the corona overlying the sun shoots a burst of gas across space. This more dense, higher velocity gas compresses the relatively dilute plasma already in space, thus creating a MHD shock front. This wave hereafter precedes the plasma cloud. This highly conducting gas behind the shock front stretches out the sun's lines of magnetic field. Thus, the magnetic field strength is higher behind the shock than its value ahead of the shock. This shock phenomena explains the rise of temperature in the corona [19] .

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The earth is implanted in the supersonic solar wind which, when it impinges on the geomagnetic field, generates a magnetohydrodynamic shock front ahead of it. The interaction of this solar wind with the earth's magnetic field creates the phenomenon of magnetosphere, which forms the next topic.

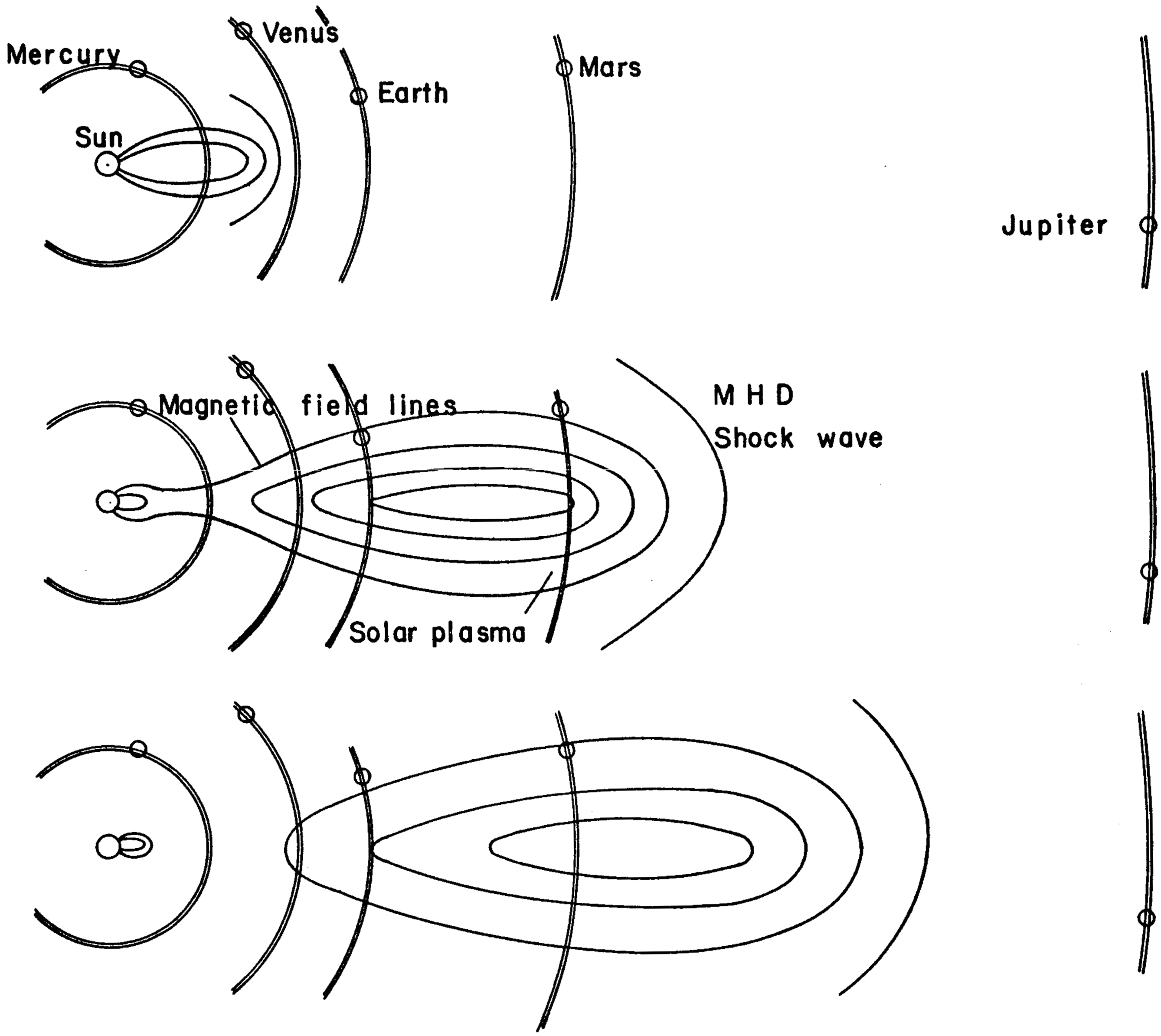

Fig. 4 The solar wind compresses the relatively dilute plasma already in space, thus creating a MHD shock front.

X. Magnetosphere

(a) Chapman-Ferraro Model.

This is the particle model of the magnetosphere [6] . They proposed that the streams of ionized particles were compressing the magnetic field of the earth. The magnetic field of the earth is equivalent to a magnetic dipole. Chapman and Ferraro based their analysis on the diamagnetic relation

$$p + \mu \frac{H^2}{2} = \text{constant.} \tag{45}$$

They assumed that the incident plasma is free of magnetic field and the outer magnetosphere is free of plasma. In fact, if the perfectly conducting stream of plasma leaves the sun with no magnetic field in it then no field can exist in the stream at any subsequent time, as follows from the frozen-in-field property. Under that hypothesis, they recognized that the diamagnetic behavior of the plasma would shut out the geomagnetic field resulting in a sharp boundary between the plasma and the geomagnetic field. The above relation implies that the pressure communicated to the front surface of the magnetosphere is ; $(\underline{n} \cdot \frac{\underline{v}}{v})$, where $\underline{n}$ is the unit normal to the surface and $\underline{v}/v$ is the unit vector in the direction of the stream velocity. The above equation finally leads to the relation

$$\mu \underline{H} = (2\, p_o)^{1/2}\, \underline{n} \cdot \underline{v}/v. \tag{46}$$

The boundary conditions imply that the normal component of $\underline{H}$

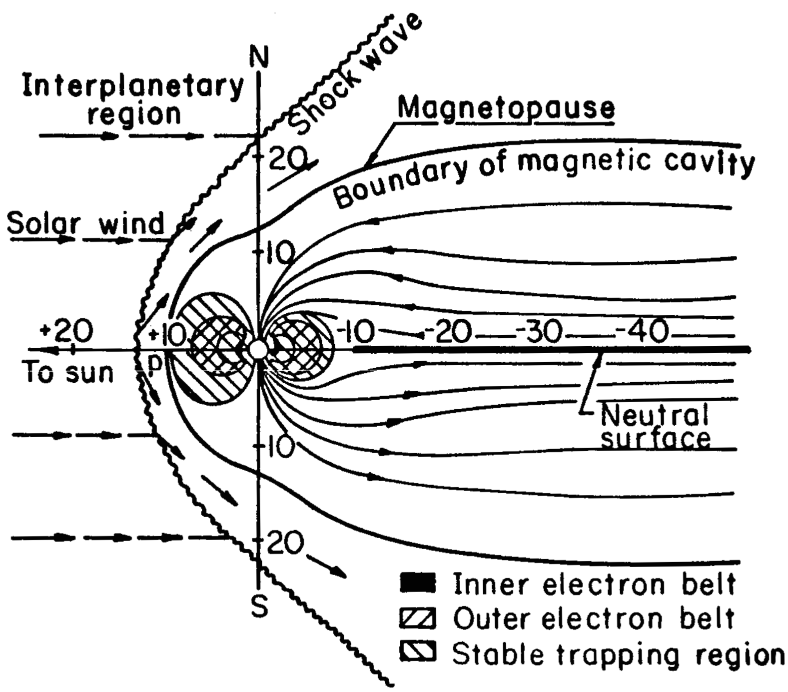

Fig. 5. The noon meridian cross-section of the magnetosphere (after Ness [15]).

is continuous across the surface. Since this component is zero outside the surface, it must be zero inside as well. Thus, the magnetic field inside and close to the surface is everywhere tangential to the surface. We can therefore write the above relation as

$$\mu\,(\underline{n} \times \underline{H}) = -\,(2\,p_o)^{1/2}\,\underline{n} \cdot \underline{v} \,/\, v, \qquad (47)$$

where the negative sign has been taken since $\underline{n} \cdot \underline{v} / v$ must be negative everywhere on the surface for the outside of the surface to be exposed to solar wind. If one takes the surface to be $f(x, y, z) = \text{constant}$, then $\underline{n} = \text{grad } f / \left| \text{grad } f \right|$. Substitution in (47) then leads to the required equation for the surface.

The picture which emerges from this analysis is that the magnetic pressure of the earth's magnetic field opposes the advance of the solar wind. If the dipole is roughly parallel to the magnetospheric surface (see Figure 5), the pressure is greatest on the part of the surface near P. The surface is distorted by the magnetic pressure making it concave towards the dipole. This surface is eventually stopped as soon as the stream pressure is balanced by the magnetic pressure. Far from the dipole the field ceases to have any effect on the flow and the stream flows around all sides of the dipole. The surface is thus folded around the dipole to form a cavity in the stream. The surface currents keep the magnetic field confined to the inside of the cavity [5] .

A large number of authors have contributed to the particle-based theory of the magnetosphere [4, 8, 14, 17] . These reviews contain exhaustive references to other contributions in the last three decades. The studies based on the most recent observations and space-craft experiments have considerably modified the Chapman-Ferraro model. It has been found that the supersonic motion of the solar wind creates a bow shock in front of the magnetosphere. The magnetosphere is shaped like a tear drop with a tremendously elongated tail. It is further surrounded by the fluctuating but discrete boundary layer called the magnetopause. The solar wind pinches off huge portions of the earth's wagging, multi-million mile long tail and hurtles it outward into space, sometimes it pinches the magnetic tail so tightly that the magnetic bubbles break off and become accelerated by the solar plasma.

(b) Magnetohydrodynamic model

As pointed out in the previous section, the molecular flow model given by Chapman and Ferraro neglected the interplanetary magnetic field in the upstream solar wind. When the existence of an interplanetary magnetic field was observed, the free molecular flow model was realized to be invalid. Dungey [8] and various other workers in the field modified the model to take into account the interplanetary field. However, pure magnetohydrodynamic model was suggested by Spreiter, Summers and Alskine [18] . They discussed the powerful role played by magnetohydrodynamic waves in the magnetospheric geometry. The fundamental assumption is that the flow can be described adequately by the mathematical analysis based on the equations of magnetohydrodynamics. From their analysis it emerges that the Chapman-Ferraro model is compatible with the tangential discontinuities of case 1 a (i.e., $H_n = 0$), although not identical. As pointed out in the previous section, Chapman-Ferraro model is based on the assumption that the incident field is free of magnetic field and the cavity containing the geomagnetic field is free of plasma. In the present notation it amounts to putting $H_1 = 0$, $p_2 = 0$, and $p_1 = \mu H_2^2/2$, in the jump relations for the tangential discontinuities. Magnetohydrdynamic model includes H_1 and p_2. Furthermore, the contact discontinuities of the case 1 b ($H_n \neq 0$) are appropriate for the boundary of the distant wake far downstream from the earth. At the far tail of the magnetosphere the external magnetic field penetrates the discontinuity while the thermodynamic properties of the wake remain different from those of the surrounding flow because of the different previous histories of the gases.

The bow shock wave on the sunlit side of the magnetosphere fits in neatly with the description of the fast shock wave. In fact, the magnetic field is stronger on the magnetospheric side than on the interplanetary side. In the downstream flow we might expect the existence of slow shocks as well as additional fast shocks.

XI. Magnetosonic Waves Inside the Magnetosphere

The pressure of the ionized gas behind the bow shock distorts the lines of force inside the magnetosphere. Since the geomagnetic field is weakest immediately inside the magnetopause, the distortion is largest there. This gives rise to magnetosonic waves. The most important source of the generation of the MHD sound waves comes from the rapid inside and outside motion of the magnetopause. This motion causes oscillations in the outermost lines of force and consequently in the gas particles bound to these lines. The fast as well as Alfvén waves have been observed by the space-probes. The waves observed are of long period waves. It is not surprising if one considers the immense size of the magnetosphere, These MHD waves, apparently have a considerable influence on the trapped particles in the magnetosphere. These waves possibly accelerate the particles from the lower to higher energies. Sometimes, they change the trajectories of the particles in such a way that they are lost in the atmosphere.

XII. MHD Waves in the Inter-stellar and Intergalactic Spaces

Recently there have been indications that the collisions between the gas clouds of two galaxies produce MHD shock waves. There is a possible connection between these shocks and the mysterious celestial emission of radio energy. Furthermore, it has been surmised by astronomers that certain lines of luminosity found in the heavens are manifestations of shocks in the highly ionized gases. Such luminous lines have been observed in the shock tubes when argon is accelerated to Mach 10.

It has been pointed out that MHD waves offer a plausable explanation for the great energy of the cosmic rays. In fact it is now well established that the interstellar space is not absolutely void but consists of ionized gases. Although the matter in it is very thin, it adds up to an enormous amount in the vastness of the space. Besides, there are weak magnetic fields pervading throughout this space. We therefore have an enormous amount of ionized gas flowing in a magnetic field. In this hydromagnetic flow there arise disturbances of all kinds, for example, due to the collisions of various galaxies. These disturbances, in turn, generate magnetohydrodynamic wave fronts of all kinds. In the neighborhood of the stars the charged atomic nuclei pass through these waves and get accelerated thus acquiring tremendous energies. Furthermore, since the MHD waves create electric currents, these currents could also be responsible for the acceleration of the cosmic rays.

XIII. MHD Waves and Controlled Thermonuclear Reactions

The first efforts to produce controlled thermonuclear reactions relied on the pinch effect of the ionized gases. This effect can be easily understood as follows. Suppose a bank of electrical condensers is discharged abruptly through electrodes to a plasma, with the result that the current flows as a sheet at the boundary between the gas and its container. This hollow cylinder of current--the current sheet--collapses inward by a radial implosion subjecting the plasma to a sudden pinch. There are two kinds of pinch discharges. In what is called the Z pinch the current flows along the axis of the electrode gap and produces a magnetic field circumferentially around the cylinder. The second type is called the theta pinch. In this case the current flows circumferentially around the cylinder and the magnetic field is axial. In either case the Lorentz force acts inwards in the radial direction on the current sheet, causing it to collapse. The gas enclosed by the current is swept up in the implosion and compressed at the axis of the container. By discharging the condenser bank sufficiently rapidly, the speed of the collapse can be made supersonic. We then have the shock tube effect with the current sheet acting as a piston, producing shock ahead of itself.

The experiments with the pinch effect unfortunately resulted in an unstable magnetic force that distorts the plasma column, eventually causing it to break. It was found that the plasma can be stabilized by creating a magnetic field inside and along the axis of the column--the so called backbone field. This field is in place when the condenser bank is discharged and the shock wave implodes radially into the preionized gas. The inward motion of the conducting gas in the presence of the magnetic field sets up a magnetohydrodynamic phenomena. The shock becomes a magnetohydrodynamic shock [13] .

Figure 6

Acknowledgement

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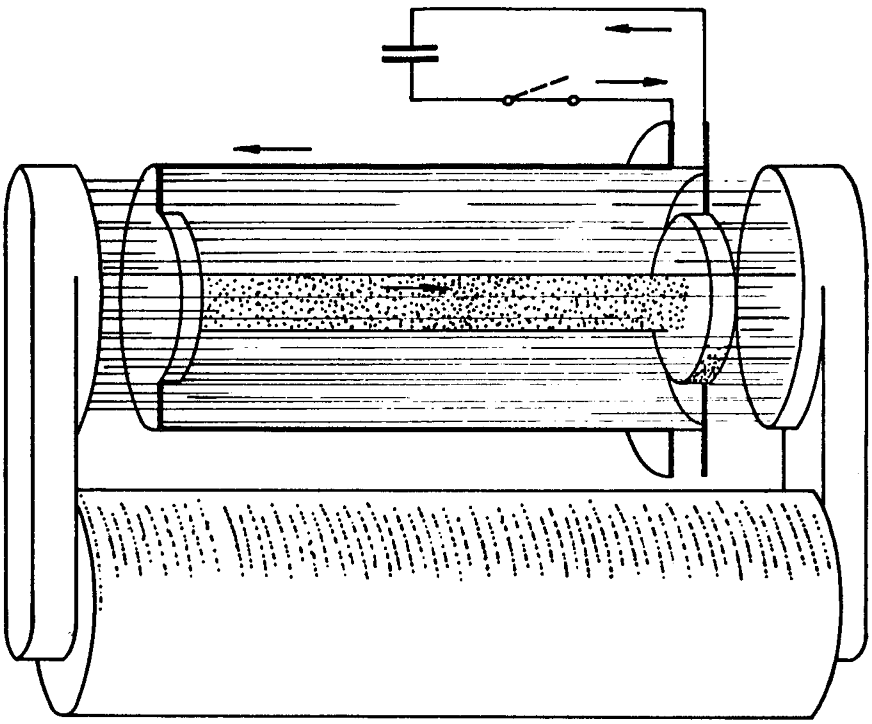

Fig. 6. M H D SHOCK WAVE may be produced when a Z-pinch discharge occurs in an already existing longitudinal magnetic field (straight black lines). Although the discharge flow is indicated (black arrows), the conditions illustrated are those before discharge.

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