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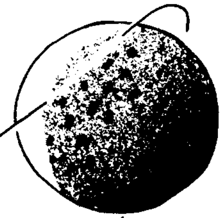

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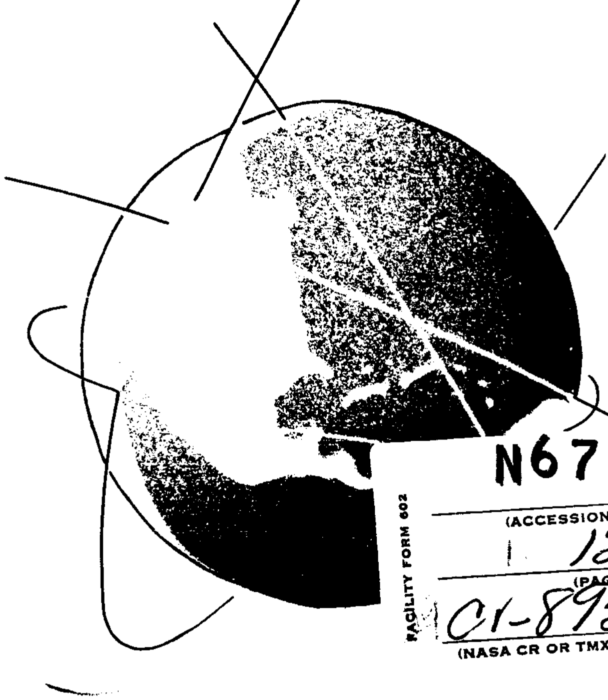

N67-39037

(ACCESSION NUMBER) (THRU) 1

12 (PAGES) (CODE) 30

CR-89329 (NASA CR OR TMX OR AD NUMBER) (CATEGORY)

FACILITY FORM 602

ENGINEERING MECHANICS RESEARCH LABORATORY

THE UNIVERSITY OF TEXAS · AUSTIN, TEXAS

ON THE EQUIVALENCE OF THREE TRAJECTORY OPTIMIZATION SCHEMES

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EMRL RM 1020 February 1967

This report was prepared under

Grant NsG-551

for the

National Aeronautics and Space Administration

by the

Engineering Mechanics Research Laboratory

The University of Texas

Austin, Texas

ON THE EQUIVALENCE OF THREE TRAJECTORY OPTIMIZATION SCHEMES

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A perturbation method and an adjoint-perturbation method for the solution of two-point boundary value problems, which arise in the variational treatment of optimal control problems, are shown to be equivalent to a procedure based on the classical Newton's Method for m simultaneous equations in m unknowns.

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1. Introduction.

Recently three computational methods [Refs. 1, 2, 3] have been proposed for the solution of two-point boundary value problems arising from a variational calculus treatment of optimal control problems. Computational experience [Ref. 4] has indicated that two of these methods, those of Refs. 1 and 2, are essentially the same. The conclusions of Ref. 4 are verified by the results of this note; that is, all three methods are mathematically equivalent. A similar type of analysis was given by Roberts and Shipman [Ref. 5] for the method of Goodman and Lance [Ref. 6].

In Ref. 3 it is shown that a variational two-point boundary value problem can be expressed in the following form:

$$\dot{Z} = \frac{dZ}{dt} = F(Z,t) , \qquad 0 \leq t \leq Q \tag{1}$$

$$Z(0) = K(C) \tag{2}$$

$$L(Z(Q),Q) = 0 \tag{3}$$

where Z, F and K are 2n-vectors, L is an m-vector, C is an (m-1)-vector, and Q is a scalar. In the analysis of the three methods, Q will be assumed to be unknown; thus $m = n + 1$.

In order to solve the boundary value problem, C and Q must be known. In each of the three methods to be discussed, values of C and Q are estimated and Eq. (1) is integrated from $t = 0$ to $t = Q$ with initial conditions given by Eq. (2). In general the terminal conditions, Eq. (3), will not be satisfied. (A solution satisfying Eqs. (1) and (2) but not necessarily Eq. (3), is known as a nominal trajectory.) In each method a procedure is given for determining corrections to C and Q so that a new nominal trajectory will better satisfy Eq. (3). These correction-procedures are shown to give identical results for the same nominal trajectory.

2. Newton's Method.

Newton's Method for the solution of the two-point boundary value problem, Eqs. (1) - (3), is discussed in Ref. 3. This method is based on the classical Newton Method for the solution of m simultaneous equations in m unknowns. The iteration scheme is

$$B^{(k+1)} = B^{(k)} - A^{-1}(B^{(k)})L^{(k)} \,, \quad k = 0,1,\ldots \qquad (4)$$

where $B^T = [C^T \; Q]$, (superscript T denotes transpose)

$$L^{(k)} = L(B^{(k)})$$

$$A(B) = \left[\frac{\partial L}{\partial Z}\,\frac{\partial Z}{\partial C}\middle|\frac{dL}{dQ}\right]_{t=Q} \qquad (5)$$

$$\frac{dL}{dQ} = \frac{\partial L}{\partial Z}\,\frac{\partial Z}{\partial Q} + \frac{\partial L}{\partial Q} = \frac{\partial L}{\partial Z}\,F(Z(Q),Q) + \frac{\partial L}{\partial Q} \,.$$

From Eq. (1),

$$Z(t) = Z(0) + \int_0^t F(Z,s)ds \tag{6}$$

and hence

$$\frac{\partial Z(t)}{\partial C} = \frac{\partial K}{\partial C} + \int_0^t \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial C}\,ds \ . \tag{7}$$

Differentiating Eq. (7) with respect to t gives

$$\frac{d}{dt}\frac{\partial Z}{\partial C} = \frac{\partial F}{\partial Z}\frac{\partial Z}{\partial C} \tag{8}$$

with

$$\frac{\partial Z(0)}{\partial C} = \frac{\partial K}{\partial C} \ . \tag{9}$$

The matrix $\partial F/\partial Z$ is evaluated on the nominal trajectory. For computational purposes the iteration scheme given by Eq. (4) is modified as follows:

$$b^{(k)} = -A^{-1}(B^{(k)})L^{(k)} \tag{10}$$

$$B^{(k+1)} = B^{(k)} + \alpha_k b^{(k)} \tag{11}$$

where $0 < \alpha_k \leq 1$. The factor α_k is chosen so that

$$||L^{(k+1)}|| < ||L^{(k)}||.$$

3. Perturbation Method.

In the perturbation method [Ref. 1] a perturbed trajectory $(Z+z, C+c, Q+q)$ is considered. Equations (1) and (2) are written in terms of the perturbed trajectory; F and K are expanded in a Taylor series (at each t) about the nominal trajectory. Assuming that (z,c,q) are "small" only first order terms are retained in the Taylor series. The linear perturbation equations are

$$\dot{z} = \frac{\partial F}{\partial Z} z \tag{12}$$

$$z(0) = \frac{\partial K}{\partial C} c \ . \tag{13}$$

Because Eq. (3) is not satisfied on the nominal trajectory, the change in the terminal conditions, due to the perturbation (z,c,q), is given by

$$\Delta L = \frac{\partial L}{\partial Z} z + \frac{dL}{dQ} q = -L \ . \tag{14}$$

Let $M(t)$ be a fundamental matrix [Ref. 7] for Eq. (12), with $M(0) = I$, the $2n \times 2n$ identity matrix. Then a solution of Eq. (12) is

$$z(t) = M(t)z(0) = M(t)\frac{\partial K}{\partial C} c \ . \tag{15}$$

On setting $t = Q$ in Eq. (15) and substituting this expression into Eq. (14), the equation becomes

$$\frac{\partial L}{\partial Z} M(Q) \frac{\partial K}{\partial C} c + \frac{dL}{dQ} q = -L .$$

Let

$$D(B) = \left[\frac{\partial L}{\partial Z} M(Q) \frac{\partial K}{\partial C} \mid \frac{dL}{dQ}\right]_{t=Q} \tag{16}$$

$$B^T = [C^T \; Q]$$

$$b^{(k)} = -D^{-1}(B^{(k)})L^{(k)} . \tag{17}$$

The iterative procedure is therefore

$$B^{(k+1)} = B^{(k)} + \alpha_k b^{(k)} . \tag{18}$$

Equation (18) is the same as Eq. (11) if and only if $D(B^{(k)}) = A(B^{(k)})$. The matrices D and A are identical if and only if

$$\frac{\partial Z(Q)}{\partial C} = M(Q) \frac{\partial K}{\partial C} .$$

The matrices $\frac{\partial Z(t)}{\partial C}$ and $M(t)\frac{\partial K}{\partial C}$ satisfy the same differential equation $\dot{Y} = \frac{\partial F}{\partial Z} Y$ and at $t = 0$, $\frac{\partial Z(0)}{\partial C} = \frac{\partial K}{\partial C} = M(0)\frac{\partial K}{\partial C}$, since $M(0) = I$, $2n\times 2n$ identity matrix. Since the matrices are equal at $t = 0$ and they satisfy the same differential equation, they are equal for each t, $0 \leq t \leq Q$. Then $D(B^{(k)}) = A(B^{(k)})$ and therefore the perturbation method is equivalent to Newton's Method.

4. Adjoint-Perturbation Method.

The adjoint-perturbation method [Ref.2] uses the adjoint equation to Eq. (12),

$$\dot{y} = -\left(\frac{\partial F}{\partial Z}\right)^T y \tag{19}$$

to determine the corrections for C and Q. If $N(t)$ is a matrix whose rows are solutions of Eq. (19), then the solution of Eq. (12) can be written as

$$N(t)z(t) = N(0)z(0) = N(0)\frac{\partial K}{\partial C}\, c$$

since $$\frac{d}{dt}(Nz) = \dot{N}z + N\dot{z} = -N\frac{\partial F}{\partial Z}z + N\frac{\partial F}{\partial Z}z = 0 .$$

At $t = Q$, choose $N(Q)$ as

$$N(Q) = \left[\frac{\partial L}{\partial Z}\right]_{t=Q} . \tag{20}$$

The matrix $N(t)$ is obtained by integrating Eq. (19) from $t = Q$ to $t = 0$ with initial conditions given by Eq. (20).

Thus,

$$N(Q)z(Q) = N(0)\frac{\partial K}{\partial C}\, c$$

and so Eq. (14) becomes

$$N(0)\frac{\partial K}{\partial C}\, c + \frac{dL}{dQ}\, q = -L \ .$$

Once again, let

$$B^T = [C^T \; Q]$$

$$E(B) = [N(0)\frac{\partial K}{\partial C} \mid \frac{dL}{dQ}]_{t=Q} \qquad (21)$$

$$b^{(k)} = -E^{-1}(B^{(k)})L^{(k)} \ . \qquad (22)$$

The iteration scheme

$$B^{(k+1)} = B^{(k)} + \alpha_k b^{(k)} \qquad (23)$$

is identical to Eq. (11) provided $E(B^{(k)}) = A(B^{(k)})$. The matrices E and A are equal if and only if $N(0)\frac{\partial K}{\partial C} = [\frac{\partial L}{\partial Z}\frac{\partial Z}{\partial C}]_{t=Q}$. To show that these matrices are equal consider a fundamental matrix $R(t)$ for Eq. (19), with $R(Q) = I$, the

$2n \times 2n$ identity matrix. The matrix $N(t)$ can be expressed as

$$N^T(t) = R(t)N^T(Q) \ .$$

It is known [Ref. 7] that the fundamental matrices $M(t)$ and $R(t)$, for Eqs. (12) and (19) respectively, are related by the equation

$$R^T(t_1)M(t_1) = R^T(t_2)M(t_2) \ .$$

With $t_1 = Q$ and $t_2 = 0$ we have $M(Q) = R^T(0)$ and therefore $N(0) = N(Q)M(Q)$. It was shown in Section 3 that

$$\frac{\partial Z(Q)}{\partial C} = M(Q)\frac{\partial K}{\partial C}$$

so that

$$N(0)\frac{\partial K}{\partial C} = N(Q)M(Q)\frac{\partial K}{\partial C} = \left[\frac{\partial L}{\partial Z}\ \frac{\partial Z}{\partial C}\right]_{t=Q}$$

where $N(Q)$ is given by Eq. (20). Therefore, the adjoint-perturbation method is also equivalent to Newton's Method.

5. Conclusions.

The analysis in Sections 2, 3, and 4 has shown that the three methods

(1) the perturbation method [Ref. 1],

(2) the adjoint-perturbation method [Ref. 2], and

(3) the Newton method [Ref. 3]

are mathematically equivalent. The three methods are just variants of the classical Newton Method for the solution of m simultaneous equations in m unknowns.

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