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THE PRECESSION AND NUTATION OF DEFORMABLE BODIES IV: DEFORMATION OF SELF-GRAVITATING ELASTIC SOLIDS

by

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ABSTRACT

In two preceding reports of this series (Boeing Documents D1-82-0611 and D1-82-0618, hereafter referred to as Reports II and III, respectively) differential equations have been set up which govern the deformations of self-gravitating globes of viscous fluids in an external field of force, and solved in a closed form for the case of an incompressible homogeneous liquid. The aim of the present report will be to derive the explicit form of equivalent equations which govern the deformations of self-gravitating *elastic* bodies of non-uniform internal temperature, whose dynamical behavior is characterized by variable Lamé parameters λ and μ.

Special respect will be paid to small deformations (or oscillations) about the state of hydrostatic equilibrium, characterized by spheroidal or toroidal symmetry. The former possess a counterpart in the case of fluidity treated previously in Reports II and III. Analytical similarities as well as differences between these two cases will be pointed out (inviscid fluid case obtaining as a limit when $\mu = 0$); and closed solutions constructed for homogeneous configurations distorted by rotational or tidal forces. On the other hand, toroidal deformations are characteristic of elastic solids alone, and possess no counterpart in our previous work.

The sections as well as equations of the present report will be numbered consecutively to those of Reports I - III (Boeing Documents D1-82-590, D1-82-611, and D1-82-618.

XIV. DEFORMATION OF SELF-GRAVITATING ELEASTIC SOLIDS IN EXTERNAL FIELD OF FORCE: FUNDAMENTAL EQUATIONS

Consider a self-gravitating solid configuration of density ρ, the elastic properties which are characterized by the Lamé parameters λ and μ, and the coefficient of volume thermal expansion α - none of which need to be constant. As is well known, the nine stress components σ_{ik} of such a body at a temperature T can be expressed in polar coordinates r, θ, ϕ, (cf. e.g., Boley and Weimer, 1960)

$$\sigma_{rr} = \lambda\Delta + 2\mu\varepsilon_{rr} - (\lambda + \frac{2}{3}\mu)\alpha T, \tag{14-1}$$

$$\sigma_{\theta\theta} = \lambda\Delta + 2\mu\varepsilon_{\theta\theta} - (\lambda + \frac{2}{3}\mu)\alpha T, \tag{14-2}$$

$$\sigma_{\phi\phi} = \lambda\Delta + 2\mu\varepsilon_{\phi\phi} - (\lambda + \frac{2}{3}\mu)\alpha T, \tag{14-3}$$

$$\sigma_{r\theta} = 2\mu\varepsilon_{r\theta} = \sigma_{\theta r} \tag{14-4}$$

$$\sigma_{r\phi} = 2\mu\varepsilon_{r\phi} = \sigma_{\phi r} \tag{14-5}$$

$$\sigma_{\theta\phi} = 2\mu\varepsilon_{\theta\phi} = \sigma_{\phi\theta} \tag{14-6}$$

in terms of the six strain components

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r}, \tag{14-7}$$

$$\varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \tag{14-8}$$

$$\varepsilon_{\phi\phi} = \frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot\theta}{r}, \qquad (14\text{-}9)$$

$$2\varepsilon_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}, \qquad (14\text{-}10)$$

$$2\varepsilon_{r\phi} = \frac{1}{r \sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \qquad (14\text{-}11)$$

$$2\varepsilon_{\theta\phi} = \frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{r}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\phi \cot\theta}{r}, \qquad (14\text{-}12)$$

where u_r, u_θ, u_ϕ are the components of the displacement vector $\vec{u}$, and

$$\Delta \equiv \operatorname{div} \vec{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{2u_r}{r} + \frac{u_\theta \cot\theta}{r}\,. \qquad (14\text{-}13)$$

If so, the equations governing the deformation of self-gravitating elastic bodies, expressed in terms of the stress components σ_{ij} (cf., e.g., Love, 1927, p. 91), will assume the forms

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{2\sigma_{rr} - \sigma_{\theta\theta} - \sigma_{\phi\phi} + \sigma_{r\theta}\cot\theta}{r} = \rho\left\{\frac{\partial^2 u_r}{\partial t^2} - \frac{\partial V}{\partial r}\right\}, \qquad (14\text{-}14)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{\theta\phi}}{\partial \phi} + \frac{(\sigma_{\theta\theta} - \sigma_{\phi\phi})\cot\theta + 3\sigma_{r\theta}}{r} = \rho\left\{\frac{\partial^2 u_\theta}{\partial t^2} - \frac{1}{r}\frac{\partial V}{\partial \theta}\right\}, \qquad (14\text{-}15)$$

$$\frac{\partial \sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\phi}}{\partial \theta} + \frac{1}{r \sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{3\sigma_{r\phi} + 2\sigma_{\theta\phi}\cot\theta}{r} = \rho\left\{\frac{\partial^2 u_\theta}{\partial t^2} - \frac{1}{r \sin\theta}\frac{\partial V}{\partial \phi}\right\}, \qquad (14\text{-}16)$$

where t denotes the time and V, the total potential of forces acting on our configuration - including the disturbing potential of exterior forces (if any).

If we insert in (14-14) - (14-16) the expressions (14-1) - (14-6) for the actual stress components in terms of the strains (14-7) - (14-12), the foregoing equations of motion can be rewritten, more explicitly, as

$$(\lambda + 2\mu)\frac{\partial\Delta}{\partial r} - \frac{2\mu}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(\tilde{\omega}_\phi \sin\theta) - \frac{\partial\tilde{\omega}_\theta}{\partial\phi}\right]$$

$$- (\lambda + \frac{2}{3}\mu)\alpha\frac{\partial T}{\partial r} + \mathcal{E} = \rho\left\{\frac{\partial^2 u_r}{\partial t^2} - \frac{\partial V}{\partial r}\right\}, \qquad (14\text{-}17)$$

$$\frac{\lambda + 2\mu}{r}\frac{\partial\Delta}{\partial\theta} - \frac{2\mu}{r \sin\theta}\left[\frac{\partial\tilde{\omega}_r}{\partial\phi} - \frac{\partial}{\partial r}(r\tilde{\omega}_\phi \sin\theta)\right]$$

$$- \frac{3\lambda + 2\mu}{3r}\frac{\partial T}{\partial\theta} + \mathcal{F} = \rho\left\{\frac{\partial^2 u_\theta}{\partial t^2} - \frac{1}{r}\frac{\partial V}{\partial\theta}\right\}, \qquad (14\text{-}18)$$

and

$$\frac{\lambda + 2\mu}{r \sin\theta}\frac{\partial\Delta}{\partial\phi} - \frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\tilde{\omega}_\theta) - \frac{\partial\tilde{\omega}_r}{\partial\theta}\right]$$

$$- \frac{3\lambda + 2\mu}{3r \sin\theta}\frac{\partial T}{\partial\phi} + \mathcal{G} = \rho\left\{\frac{\partial^2 u_\phi}{\partial t^2} - \frac{1}{r \sin\theta}\frac{\partial V}{\partial\phi}\right\} \qquad (14\text{-}19)$$

where

$$\tilde{\omega}_r = \frac{1}{2r^2 \sin\theta}\left\{\frac{\partial}{\partial\theta}(r u_\phi \sin\theta) - \frac{\partial}{\partial\phi}(r u_\theta)\right\} \qquad (14\text{-}20)$$

$$\tilde{\omega}_\theta = \frac{1}{2r \sin\theta}\left\{\frac{\partial u_r}{\partial\phi} - \frac{\partial}{\partial r}(r u_\phi \sin\theta)\right\} \qquad (14\text{-}21)$$

$$\tilde{\omega}_\phi = \frac{1}{2r}\left\{\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right\} \qquad (14\text{-}22)$$

are the respective components of curl $\vec{u}$, and

$$\mathcal{E} = \Delta \frac{\partial\lambda}{\partial r} + 2\left\{\varepsilon_{rr}\frac{\partial\mu}{\partial r} + \frac{\varepsilon_{r\theta}}{r}\frac{\partial\mu}{\partial\theta} + \frac{\varepsilon_{r\phi}}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\right\} - T\frac{\partial}{\partial r}\left[\alpha(\lambda + \frac{2}{3}\mu)\right], \tag{14-23}$$

$$\mathcal{J} = \frac{\Delta}{r}\frac{\partial\lambda}{\partial\theta} + 2\left\{\varepsilon_{r\theta}\frac{\partial\mu}{\partial r} + \frac{\varepsilon_{\theta\theta}}{r}\frac{\partial\mu}{\partial\theta} + \frac{\varepsilon_{\theta\phi}}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\right\} - \frac{T}{r}\frac{\partial}{\partial\theta}\left[\alpha(\lambda + \frac{2}{3}\mu)\right], \tag{14-24}$$

$$\mathcal{G} = \frac{\Delta}{r\sin\theta}\frac{\partial\lambda}{\partial\phi} + 2\left\{\varepsilon_{r\phi}\frac{\partial\mu}{\partial r} + \frac{\varepsilon_{\theta\phi}}{r}\frac{\partial\mu}{\partial\theta} + \frac{\varepsilon_{\phi\phi}}{r\sin\theta}\frac{\partial\mu}{\partial\phi}\right\} - \frac{T}{r\sin\theta}\frac{\partial}{\partial\phi}\left[\alpha(\lambda + \frac{2}{3}\mu)\right] \tag{14-25}$$

are terms depending on the spatial derivatives of the elastic parameters α, λ and μ (and vanishing identically if all these quantities were constant).

Equations (14-17) - (14-19) together with (14-20) - (14-25) constitute a simultaneous system of sixth order in the displacement components u_r, u_θ, u_ϕ ; but their determination requires a knowledge of elastic parameters α, λ, μ as well as of the density ρ, temperature T, and potential V as functions of the independent variables. The latter is, however, constrained to satisfy the Poisson equation

$$\nabla^2 V = -4\pi G\rho, \tag{14-26}$$

where G denotes the constant of gravitation; while the distribution of internal temperature in solid bodies sill be generally governed by the equation

$$\rho C_v \frac{\partial T}{\partial t} = \mathrm{div}(\kappa \text{ grad } T) + Q \qquad (14\text{-}27)$$

of heat conduction, where C_v and κ denote the specific heat at constant volume and the coefficient of heat conduction (not necessarily constant); while $Q(r,\theta,\phi,t)$ stands for a function representing the action of internal heat sources (if any).

Consistent with the basic premises underlying the theory of elasticity, the left-hand sides of equations (14-17) - (14-19) are linear in the displacement components u_r, u_θ, and u_ϕ; but this is not necessarily the case with their right-hand sides. In what follows we shall, however, assume the displacement to be small enough for their effects on the density ρ and the potential V to represent small quantities whose squares and cross-products can be ignored --an assumption which ensures the linearity of the entire system. Furthermore, let us assume that the mass of our self-gravitating configuration is sufficiently large to be in hydrostatic equilibrium*

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* This will be true whenever the forces of self-attraction (which grow proportionally with the mass) will exceed the molecular forces of solid state which depend on the kind of the material, but not on its total mass). For self-gravitating globes consisting of silicate materials of density ρ this is expected (cf. Wildt, 1963) to occur when their radius exceeds approximately $580\rho^{-\frac{1}{2}}$ kms. The Moon (of mean radius of 1738 km and of density $3.34 g/cm^3$) exceeds this limit by so wide a margin that it should assume essentially spherical form even if it were (as it may be) solid throughout its interior; and may deviate from it only to an extent maintained by external forces.

in its undisturbed state, in which it assumes the form of a sphere (the zero-order gravity being balanced by internal pressure). Moreover, the equilibrium values of the state parameters α, λ, μ as well as of the density ρ and temperature T can then be regarded as functions of r only; and under stress can differ from them only by amounts of the order of magnitude of the displacements. This means that, within the framework of a linear theory, their equilibrium values may be used whenever multiplied by u_r, u_θ, or u_ϕ.

In particular, let $\rho_0(r)$ represent the density distribution of our configuration in its equilibrium (undisturbed) state. If so, the conservation of mass requires that the difference $\rho - \rho_0$ in density brought about by stress be expressible as

$$\rho - \rho_0 = \rho' = - \operatorname{div}(\rho_0 \vec{u}) \; ; \tag{14-28}$$

which (since ρ_0 depends on r only) can be rewritten as

$$\rho' = - \rho_0 \Delta - u_r \frac{\partial \rho_0}{\partial r} , \tag{14-29}$$

representing the "equation of continuity" of our present problem, and equivalent to equation (5-19) of Report II in fluid mechanics.

Similarly, if $V_0(r)$ denotes the gravitational potential of our configuration in its state of equilibrium (giving rise to acceleration balanced up by hydrostatic pressure), its change under stress can be expressed

$$V = V' + u_r \frac{\partial V_0}{\partial r} , \tag{14-30}$$

where the increment V' in potential due to deformation (and including that of the external forces which caused it) must satisfy the differential equation

$$\nabla^2 V' = - 4\pi G \rho' , \tag{14-31}$$

following from (14-26), in which ρ' can be inserted from (14-29).

In consequence, the linearized right-hand sides of equation (14-17) - (14-19) are, therefore, found to assume the forms

$$\rho_0 \left\{ \frac{\partial^2 u_r}{\partial t^2} + \Delta \frac{\partial V_0}{\partial r} - \frac{\partial}{\partial r} \left(V' + u_r \frac{\partial V_0}{\partial r} \right) \right\},$$

$$\rho_0 \left\{ \frac{\partial^2 u_\theta}{\partial t^2} - \frac{1}{r} \frac{\partial}{\partial \theta} \left(V' + u_r \frac{\partial V_0}{\partial r} \right) \right\}$$

$$\rho_0 \left\{ \frac{\partial^2 u_\phi}{\partial t^2} - \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left(V' + u_r \frac{\partial V_0}{\partial r} \right) \right\},$$

where the term $\Delta(\partial V_0/\partial r)$ in the first one of these expressions (but in that only) represents the increment in gravitational force per unit volume due to a change of $-\rho_0\Delta$ in density on the right-hand side of equation (14-29); it is absent from others because V_0 does not depend on the angular variables. If so, however, it becomes also unnecessary hereafter to characterize the equilibrium density ρ_0 and potential V_0 by zero subscript; and these will henceforward be dropped.

Equation (14-29) renders ρ' a linear and homogeneous function of the components of displacement; and so will be V', by virtue of (14-31), as long as the disturbance of the potential arises solely from displacement of its own mass. In such a case our system of equations for u_r, u_θ, and u_ϕ becomes linear and homogeneous in these variables, and governs free oscillations of the respective configurations. The same will, however, not be true if V' contains a component of external force; in such a case our system of equations becomes nonhomogeneous, and the displacement governed by it of the nature of forced deformation which may, but need not, be oscillatory in nature.

The role of the terms involving the temperature T in our equation of motion remains yet to be specified. If, in conformity with our previous process of linearization, we express the latter in the form

$$T = T_0(r) + T'(r,\theta,\phi;t), \tag{14-32}$$

where $T_0(r)$ denotes the steady-state equilibrium temperature and $T'(r,\theta,\phi;t)$ its changes arising from the deformation (or whatever other cause). The effects of $T_0(r)$ will, in hydrostatic equilibrium, again be balanced up by the pressure; so that its changes T' alone need to be considered in our linearized equations of motion. These changes may be invoked by an expansion or contraction of our solid (in which case they can be expressed in terms of the respective components of displacement); or be due to the internal heat sources represented in equation (14-27) by the Q-term on its right-hand side; or again by secular cooling by the escape of internal heat into space (governed by the same equation but without the Q-term).

In the latter two cases the function $T'(r,\theta,\phi;t)$ can be independent of the displacement. If, on the other hand, the stresses σ_{ij} inside our configuration perform work in the course of each displacement which is convertible into heat, equation (14-27) - representing as it does the conservation of energy - must take this into account. If unrestricted expansion were possible at constant pressure, equation (14-27) would continue to hold good as it stands, provided only that the coefficient C_v on its left-hand side were replaced by C_p, the specific heat at constant pressure (cf. Carslaw and Jaeger, 1959; p. 13). If, however, the displacement is not piezotropic (as will generally be the case), additional terms will appear in the energy equation (cf. Jeffreys, 1930; Biot, 1959), which in the present case will be equivalent (cf. e.g., Nowacki, 1962; pp. 38 ff) to a source function

$$Q = -(\lambda + \frac{2}{3}\mu)(\alpha T_0)\,\mathrm{div}\,\dot{\vec{u}}, \tag{14-33}$$

where the dot over $\vec{u}$ denotes the differentiation of the displacement vector with respect to the time*. In such a case, the energy and momentum equations of the underlying thermoelastic problem would be coupled through the term (14-33), and thus constitute a simultaneous system which - in the absence of external forces - would be homogeneous in its dependent variables; and in the next section we shall proceed to reduce it to a form more amenable to actual solution, subject to appropriate boundary conditions.

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* It may also be noted that, by virtue of the thermo-dynamical definition of the respective parameters, $(\lambda + \frac{2}{3}\mu)\alpha T_0 = \rho C_v(\gamma - 1)/\alpha$, where γ stands for the ratio of specific heats C_p/C_v.

Some of the boundary conditions which such solutions must obey are trivially simple. Thus we shall require that, at the center of our configuration $(r = 0)$, the displacements must be zero - i.e., that

$$u_r(0) = u_\theta(0) = u_\phi(0) = 0. \qquad (14\text{-}34)$$

Next, we require the vanishing, at the outer boundary $r = a$, of the radial components of the stress tensor

$$\sigma_{rr}(a) = \sigma_{r\theta}(a) = \sigma_{r\phi}(a) = 0. \qquad (14\text{-}35)$$

In addition, we have to ensure that, for self-gravitating configurations, the gravitational potential and its normal component (i.e., acceleration) remain continuous at $r = a$ for any displacement; but an explicit formulation of the constraint which this imposes on our solution is more involved, and will be postponed for the next section.

XV. SPHEROIDAL DEFORMATIONS

In the preceding section of this report we have reduced the linearized equations of motion which govern the deformation of a self-gravitating elastic globe from a state of hydrostatic equilibrium to the form of a simultaneous system

$$(\lambda + 2\mu)\frac{\partial \Delta}{\partial r} - \frac{2\mu}{r \sin\theta}\left[\frac{\partial}{\partial\theta}(\tilde{\omega}_\phi \sin\theta) - \frac{\partial\omega_\theta}{\partial\phi}\right] \tag{15-1}$$

$$- (\lambda + \tfrac{2}{3}\mu)\,\alpha\,\frac{\partial T'}{\partial r} + \mathcal{E} = \rho\left\{\frac{\partial^2 u_r}{\partial t^2} + \Delta\frac{\partial V}{\partial r} - \frac{\partial}{\partial r}\left(V' + u_r\frac{\partial V}{\partial r}\right)\right\},$$

$$\frac{\lambda + 2\mu}{r}\frac{\partial\Delta}{\partial\theta} - \frac{2\mu}{r\sin\theta}\left[\frac{\partial\tilde{\omega}_r}{\partial\phi} - \frac{\partial}{\partial r}(r\tilde{\omega}_\phi \sin\theta)\right] \tag{15-2}$$

$$- \frac{3\lambda + 2\mu}{3r}\,\alpha\,\frac{\partial T'}{\partial\theta} + \mathcal{F} = \rho\left\{\frac{\partial^2 u}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(V' + u_r\frac{\partial V}{\partial r}\right)\right\},$$

$$\frac{\lambda + 2\mu}{r\sin\theta}\frac{\partial\Delta}{\partial\phi} - \frac{2\mu}{r}\left[\frac{\partial}{\partial r}(r\tilde{\omega}_\theta) - \frac{\partial\tilde{\omega}_r}{\partial\theta}\right] - \frac{3\lambda + 2\mu}{3r\sin\theta}\,\alpha\,\frac{\partial T'}{\partial\phi} \tag{15-3}$$

$$+ \mathcal{G} = \rho\left\{\frac{\partial^2 u}{\partial t^2} - \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(V' + u_r\frac{\partial V}{\partial r}\right)\right\},$$

where $\tilde{\omega}_r$, $\tilde{\omega}_\theta$, ω_ϕ continue to be given in terms of the displacement components u_r, u_θ, u_ϕ by means of equations (14-20) - (14-22); Δ by (14-13); the function $\mathcal{F}, \mathcal{G}, \mathcal{J}$ by (14-23) - (14-25); and the potentials V, V' are to be obtained by a solution of the equations (14-26) and (14-31), respectively; while T' follows from (14-27).

Let us assume now - similarly as in Section VI of Report II of this series - that the deformations of our self-gravitating configuration from the state of hydrostatic equilibrium are *spheroidal* - an assumption which constrains the displacement components u_r, u_θ, and u_ϕ to be expressible in the form

$$u_r(r,\theta,\phi;t) = u(r,t)Y_j^i(\theta,\phi), \tag{15-4}$$

$$u_\theta(r,\theta,\phi;t) = v(r,t)\frac{\partial Y_j^i}{\partial\theta}, \tag{15-5}$$

$$u_\phi(r,\theta,\phi;t) = \frac{v(r,t)}{\sin\theta}\frac{\partial Y_j^i}{\partial\phi}, \tag{15-6}$$

where the $Y_j^i(\theta,\phi)$'s are surface harmonics of index i and order j, which satisfy the differential equation

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y}{\partial\theta}) + j(j+1)\,Y = 0, \tag{15-7}$$

and $u(r,t)$, $v(r,t)$ are new functions of r and t only which remain to be determined.

If, as in Section VI, we abbreviate

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) - j(j+1)\frac{v}{r} = y \tag{15-8}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) - \frac{u}{r} = z, \tag{15-9}$$

it follows by insertion of (15-8) - (15-9) in (14-13) again that

$$\Delta = yY_j^i , \tag{15-10}$$

while the curl components (14-20) - (14-22) reduce likewise to

$$\tilde{\omega}_r = 0 , \tag{15-11}$$

$$\tilde{\omega}_\theta = - \frac{z}{2 \sin \theta} \frac{\partial Y_j^i}{\partial \phi} , \tag{15-12}$$

$$\tilde{\omega}_\phi = + \frac{z}{2} \frac{\partial Y_j^i}{\partial \theta} . \tag{15-13}$$

Let us, furthermore, assume that the changes in temperature and the potential of the strained body are expansible in the form

$$T'(r,\theta,\phi;t) = \sum_{i,j} \tau_{i,j}(r,t) Y_j^i(\theta,\phi) \tag{15-14}$$

and

$$V'(r,\theta,\phi;t) = \sum_{i,j} R_{i,j}(r,t) Y_j^i(\theta,\phi) . \tag{15-15}$$

An insertion of this latter expansion together with (14-29) in (14-31) discloses that the function $R(r,t)$ must satisfy the differential equation

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r} \frac{\partial R}{\partial r} - j(j+1) \frac{R}{r} = 4\pi G\rho \left\{ y + \frac{u}{\rho} \frac{\partial \rho}{\partial r} \right\} , \tag{15-16}$$

the solution of which can be expressed (cf. equation 7-17 of Report II) in the integral form

$$R_j = -\frac{4\pi G}{2j+1}\left\{\frac{1}{r^{j+1}}\int_0^r \left(\rho y + u\frac{\partial\rho}{\partial r}\right) r^{j+2}\,dr + r^j\int_r^\infty \left(\rho y + u\frac{\partial\rho}{\partial r}\right) r^{1-j}\,dr\right\} + c_{i,j}\,r^j \qquad (15\text{-}17)$$

or, on partial integration,

$$R_j = \frac{4\pi G}{2j+1}\left\{\frac{}{r^{j+1}}\int_0^r \rho\left[\frac{\partial}{\partial r}(ur^{j+2}) - yr^{j+2}\right]dr + r^j\int_r^\infty \rho\left[\frac{\partial}{\partial r}(ur^{1-j}) - yr^{1-j}\right]dr\right\} + c_{i,j}r^j\,, \qquad (15\text{-}18)$$

when the particular integral (in curly brackets) represents the perturbation in potential arising from the distortion of our configuration, and the complementary function $c_{i,j}r^j$ where the c's are constants (or arbitrary functions of the time) specifies the force (of rotational or tidal origin) which is responsible for distortion. Moreover, if the density distribution ρ is such that $\rho(r) = 0$ for $r > a$ (where a denotes the radius of our configuration in its equilibrium state) the infinite upper limit of integration on the right-hand side of (15-17) or (15-18) can be replaced by a.

Furthermore, since in the undistorted state

$$V = \frac{4\pi G}{r}\int_0^r \rho r^2 dr + 4\pi G\int_r^a \rho r\,dr, \qquad (15\text{-}19)$$

it follows that

$$u_r\frac{\partial V}{\partial r} = -\frac{4\,GuY_j^i}{r^2}\int_0^r \rho r^2 dr = -\,guY_j^i \qquad (15\text{-}20)$$

where

$$g = \frac{4\pi G}{r^2}\int_0^r \rho r^2 dr \tag{15-21}$$

denotes the gravitational acceleration, it follows that

$$V' + u_r\left(\frac{\partial V}{\partial r}\right) = -\, 4\pi G\, R_j(r,t) Y_j^i(\theta,\phi), \tag{15-22}$$

where

$$R_j = \frac{u}{r^2}\int_0^r \rho r^2\, dr - \frac{1}{2j+1}\left\{\frac{1}{r^{j+1}}\int_0^r \rho\left[\frac{\partial}{\partial r}(ur^{j+2}) - yr^{j+2}\right] dr + r^j \int_r^a \rho\left[\frac{\partial}{\partial r}(ur^{1-j}) - yr^{1-j}\right] dr\right\} - \frac{c_{i,j} r^j}{4\pi G} \tag{15-23}$$

Lastly, it follows likewise from (15-10) and (15-19) that

$$\Delta \frac{\partial V}{\partial r} = -\frac{4\pi G y}{r^2}\int_0^r \rho r^2\, dr = -\, g y\, Y_j^i\ . \tag{15-24}$$

If we insert now all the foregoing results in equation (15-1) we find the latter to reduce to

$$(\lambda + 2\mu)\frac{\partial y}{\partial r} + \left\{\frac{\partial \lambda}{\partial r} + g\rho\right\} y + 2\left(\frac{\partial \mu}{\partial r}\right)\frac{\partial u}{\partial r} + j(j+1)\frac{\mu z}{r} - \frac{\partial}{\partial r}(\lambda + \frac{2}{3}\mu)\alpha\tau = \rho\left\{\frac{\partial^2 u}{\partial t^2} + 4\pi G\frac{\partial R}{\partial r}\right\}; \tag{15-25}$$

while equations (15-2) and (15-3) assume the identical form

$$(\lambda + 2\mu)y + \frac{\partial}{\partial r}(\mu r z) + 2(u-v)\frac{\partial \mu}{\partial r} -$$

$$- (\lambda + \frac{2}{3}\mu)\alpha\tau = \rho\left\{r\frac{\partial^2 v}{\partial t^2} + 4\pi GR\right\}. \tag{15-26}$$

The foregoing equations (15-25) and (15-26) represent the fundamental set of equations governing spheroidal oscillations of self-gravitating elastic globes; but they are not sufficiently explicit inasmuch as they involve functions u and v which specify displacement not only through their derivatives, but also behind the integral sign in the expression (15-23) for R; and to remove this an elimination of R between (15-25) and (15-26) appears desirable. In order to do so we can proceed similarly as in Section VI of Report II. First, divide both sides of equation (15-26) by ρ, differentiate with respect to r, and then eliminate $\partial R/\partial r$ between the outcome of this operation and equation (15-25): the result assumes the form

$$\rho\frac{\partial^2}{\partial t^2}(rz) + g\rho y + \frac{1}{\rho}\frac{\partial\rho}{\partial r}\left\{(\lambda + 2\mu)y + \frac{\partial}{\partial r}(\mu rz) + 2(u-v)\frac{\partial\mu}{\partial r} - (\lambda + \frac{2}{3}\mu)\alpha\tau\right\} = \tag{15-27}$$

$$= \left\{\frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{r^2}\right\}(\mu rz) + 2\frac{\partial}{\partial r}\left[(u-v)\frac{\partial\mu}{\partial r}\right] + \left[y - \frac{\partial u}{\partial r}\right]\frac{\partial\mu}{\partial r} ;$$

or, on insertion for y and z from (15-8) and (15-9),

$$\mu r\frac{\partial^3 v}{\partial r^3} + \left(3\mu + 2r\frac{\partial\mu}{\partial r}\right)\frac{\partial^2 v}{\partial r^2} + r\left(\frac{\partial^2\mu}{\partial r^2} + \frac{2}{r}\frac{\partial\mu}{\partial r} - \frac{j(j+1)\mu}{r^2}\right)\frac{\partial v}{\partial r}$$

$$- \left(\frac{\partial^2\mu}{\partial r^2} + \frac{2j(j+1)}{r}\frac{\partial\mu}{\partial r} + \frac{j(j+1)}{r^2}\mu\right)v - \mu\frac{\partial^2 u}{\partial r^2}$$

$$+\left(\frac{\partial^2\mu}{\partial r^2} + \frac{4}{r}\frac{\partial\mu}{\partial r} + \frac{j(j+1)}{r^2}\mu\right) u - \rho r \frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial r} + \frac{v-u}{r}\right)$$

$$= \Bigg\{\mu r \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}(r^2\mu)\frac{\partial v}{\partial r} + (\lambda + \mu)\frac{\partial u}{\partial r}$$

$$+ \left[\frac{\partial\mu}{\partial r} + \frac{2(\lambda + 2\mu)}{r}\right] u - \left[\frac{\partial\mu}{\partial r} + \frac{j(j+1)(\lambda+2\mu)}{r}\right] v$$

$$- (\lambda + \frac{2}{3}\mu)\alpha\tau\Bigg\}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial r}\right) + g\rho\left\{\frac{\partial u}{\partial r} + \frac{2u}{r} - \frac{j(j+1)v}{r}\right\}. \qquad (15\text{-}28)$$

In order to obtain the second independent relation between $u(r,t)$ and $v(r,t)$, let us note that the function $R_j(r,t)$ as defined by equation (15-23) satisfies the differential equation

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} - \frac{j(j+1)}{r^2} R = \rho\left\{2\left(\frac{\partial u}{\partial r} - y\right)\right\} + \frac{\bar{\rho}}{3}\left\{r^2\frac{\partial^2}{\partial r^2} - j(j+1)\right\}\left(\frac{u}{r}\right), \qquad (15\text{-}29)$$

where

$$\bar{\rho} = \frac{3}{r^3}\int_0^r \rho r^2\, dr \qquad (15\text{-}30)$$

denotes the mean density of a sphere of radius r. Let us differentiate now (15-25) with respect to r and solve the outcome for $\partial^2 R/\partial r^2$ which we insert in (15-19). If, moreover, we insert similarly for $\partial R/\partial r$ and R from (15-25) and (15-26) as they stand, equation (15-29) can be reduced to the form

$$(\lambda + 2\mu)\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2}\right\} y - \rho\frac{\partial^2 y}{\partial t^2}$$

$$+ 2 \frac{\partial}{\partial r}(\lambda + \mu) \frac{\partial y}{\partial r} + \frac{y}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \lambda}{\partial r}\right) + \frac{2}{r^2} \frac{\partial u}{\partial r} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \mu}{\partial r}\right)$$

$$+ 2 \frac{\partial \mu}{\partial r} \left\{\frac{\partial^2 u}{\partial r^2} - \frac{j(j+1)}{r^2} (u-v)\right\}$$

$$- \left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{j(j+1)}{r^2}\right\} (\lambda + \frac{2}{3} \mu)\alpha\tau =$$

$$= g\rho\left\{j(j+1) \left(z - \frac{2v}{r}\right) - 4\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) + 6 \frac{\rho}{\rho}\left(\frac{\partial u}{\partial r} - y\right)\right\}$$

$$+ \frac{1}{\rho} \frac{\partial \rho}{\partial r} \left\{\frac{\partial}{\partial r} [(\lambda + 2\mu)y] + 2\left[\frac{\partial u}{\partial r} - y\right]\frac{\partial \mu}{\partial r} + j(j+1)\frac{\mu z}{r}\right.$$

$$\left. - \frac{\partial}{\partial r}\left[(\lambda + \frac{2}{3})\alpha\tau\right]\right\} ; \qquad (15\text{-}31)$$

or, on insertion for y and z from (15-8) and (15-9),

$$(\lambda + 2\mu)\frac{\partial^3 u}{\partial r^3} + \frac{2}{r^2} \frac{\partial}{\partial r}\left[r^2 (\lambda + 2\mu)\right] \frac{\partial^2 u}{\partial r^2} +$$

$$+ \left\{\frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r} (\lambda + 2\mu)\right] - j(j+1)\frac{\lambda+2\mu}{r^2} + \frac{4}{r} \frac{\partial}{\partial r}(\lambda + \mu)\right\}\frac{\partial u}{\partial r}$$

$$+ 2\left\{\frac{\partial^2 \lambda}{\partial r^2} - [j(j+1) + 2] \frac{1}{r} \frac{\partial \mu}{\partial r} - \frac{j(j+1)}{r^2} (\lambda + 2\mu)\right\}\frac{u}{r}$$

$$- \frac{j(j+1)(\lambda+2\mu)}{r} \frac{\partial^2 v}{\partial r^2} - \frac{2}{r} \left\{j(j+1) \frac{\partial}{\partial r}(\lambda + \mu)\right\} \frac{\partial v}{\partial r}$$

$$- j(j+1) \left\{\frac{\partial^2 \lambda}{\partial r^2} + \frac{2}{r}[j(j+1) - 1] \frac{\partial \mu}{\partial r} - \frac{j(j+1)}{r^2}(\lambda + 2\mu)\right\}\frac{v}{r}$$

$$- \left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} (\lambda + \frac{2}{3}\mu)\alpha\tau$$

$$- \rho \frac{\partial^2}{\partial t^2} \left\{ \frac{\partial u}{\partial r} + \frac{2u}{r} - \frac{j(j+1)}{r} v \right\}$$

$$= g\rho \left\{ j(j+1)\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) + 6\frac{\bar{\rho}}{\rho}[j(j+1)v - 2u] \right.$$

$$\left. - 4\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) - \frac{j(j+1)}{r} u \right\}$$

$$+ \frac{1}{\rho}\frac{\partial \rho}{\partial r} \left\{ (\lambda + 2\mu)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\lambda + 2\mu)\right]\frac{\partial u}{\partial r} \right.$$

$$- j(j+1)\frac{\lambda+\mu}{r}\frac{\partial v}{\partial r} + \frac{1}{r}\left[2\frac{\partial \lambda}{\partial r} - \frac{2(\lambda+2\mu)}{r} - \frac{j(j+1)\mu}{r}\right] u - \frac{j(j+1)}{r}\left[\frac{\partial \lambda}{\partial r} - \frac{\lambda+3\mu}{r}\right] v$$

$$\left. - \frac{\partial}{\partial r}\left(\lambda + \frac{2}{3}\mu\right)\alpha\tau \right\}. \qquad (15\text{-}32)$$

The foregoing pairs of equations (15-27) and (15-32) or (15-28) and (15-32) constitute each a simultaneous system of sixth order for u and v (their reduction to a single sixth-order equation for either u or v is possible, but too cumbersome to be really worth while). Their solution is subject to the boundary conditions (14-34) - (14-35) which, in the case of spheroidal symmetry characterized by (15-4) - (15-6) require that, at the center,

$$u(0,t) = 0, \qquad (15\text{-}33)$$

$$v(0,t) = 0; \qquad (15\text{-}34)$$

while, on the boundary, the vanishing of the spheroidal stress components

$$\sigma_{rr} = \left\{\lambda y + 2\mu \frac{\partial u}{\partial r} - (\lambda + \frac{2}{3}\mu)\alpha\tau\right\} Y_j^i \tag{15-35}$$

$$\sigma_{r\theta} = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{u}{r}\right) \frac{\partial Y_j^i}{\partial \theta}, \tag{15-36}$$

$$\sigma_{r\phi} = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{u}{r}\right) \frac{1}{\sin\theta} \frac{\partial Y_j^i}{\partial \phi} \tag{15-37}$$

reduces to the additional two conditions

$$\lambda y + 2\mu \frac{\partial u}{\partial r} = (\lambda + \frac{2}{3}\mu)\alpha\tau(a,t) \tag{15-38}$$

and

$$\frac{\partial v}{\partial r} - \frac{v}{a} + \frac{u}{a} = 0 \tag{15-39}$$

for $r = a$. If the temperature over the free surface could be regarded as constant, the right-hand side of (15-38) would vanish, and both (15-38) and (15-39) would then be homogeneous in the components of displacement.

The form boundary conditions (15-33) - (15-34) and (15-38) - (15-39) are sufficient to specify particular solutions of the fourth-order system of differential equations (15-25) - (15-26) for a given function R. In actual fact, however, R itself is a function of the displacement components u and v as specified by equation (15-23). A single differention of the product $r^{j+1}R$ reveals, however, that

$$\frac{\partial R}{\partial r} + \frac{j+1}{r} R = - r^{j-1} \int_r^a \rho \left[\frac{\partial}{\partial r}(ur^{1-j}) - yr^{1-j}\right] dr$$

$$+ \frac{g}{4\pi G}\left\{r \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + j \frac{u}{r}\right\} - \frac{2j+1}{4\pi G} c_{i,j} r^{j-1}, \tag{15-40}$$

which on insertion for $\partial R/\partial r$ and R on the left-hand side from (15-35)-(15-36) for $r = a$ assumes the form

$$- \rho_1 \frac{\partial^2}{\partial t^2} \left\{ u + (j+1)v \right\} - \left\{ \frac{\partial}{\partial r} + \frac{j+1}{r} \right\} (\lambda + \frac{2}{3}\mu)\alpha\tau$$

$$+ (\lambda + 2\mu) \left\{ \frac{\partial}{\partial r} + \frac{j+1}{r} \right\} y + \frac{j+1}{r} \left\{ \frac{\partial}{\partial r} + \frac{j}{r} \right\} (\mu r z)$$

$$+ \left\{ \frac{\partial \lambda}{\partial r} + g\rho_1 \right\} y + 2 \frac{\partial u}{\partial r} \left\{ \frac{\partial u}{\partial r} + \frac{j+1}{r} (u-v) \right\}$$

$$= g\rho_1 \left\{ \frac{\partial u}{\partial r} + \frac{j-1}{r} u \right\} - (2j+1)\rho_1 c_{i,j} r^{j-1}, \tag{15-41}$$

where ρ_1 denotes the surface density of our configuration (which, unlike for fluids, need not be zero).

Next, let us ascertain the surface values of other quantities involved in equation (15-41). Since, by (15-38), the boundary value of y can be expressed as

$$y = - 2 \frac{\mu}{\lambda} \left(\frac{\partial u}{\partial r} \right) + \left(1 + \frac{2}{3} \frac{\mu}{\lambda} \right) \alpha\tau, \tag{15-42}$$

a solution of this latter equation together with (15-8) (which is valid everywhere including the boundary) for y and $\partial u/\partial r$ discloses that, at $r = a$,

$$(\lambda + 2\mu) y = \frac{2\mu}{r} [2u - j(j+1)v] + (\lambda + \frac{2}{3}\mu)\alpha\tau \tag{15-43}$$

and

$$(\lambda + 2\mu) \frac{\partial u}{\partial r} = - \frac{\lambda}{r} [2u - j(j+1)v] + (\lambda + \frac{2}{3}\mu)\alpha\tau. \tag{15-44}$$

On the other hand, from a solution of (15-39) valid over the same boundary with (15-9) for z and $\partial v/\partial r$ leads to

$$rz = 2(v-u) \tag{15-45}$$

and

$$\frac{\partial v}{\partial r} = \frac{v-u}{r} . \tag{15-46}$$

If we differentiate now the expressions (15-8) and (15-9) for y and z with respect to r and subsequently insert from (15-44) and (15-46) we find that at $r = a$,

$$\frac{\partial y}{\partial r} = \frac{\partial^2 u}{\partial r^2} + \frac{(j+2)(j-1)}{r^2} u$$

$$+ \frac{2}{r^2(\lambda+2\mu)} \left\{\lambda[j(j+1)v - 2u] + (\lambda + \frac{2}{3}\mu)\alpha r\tau\right\} \tag{15-47}$$

and

$$\frac{\partial}{\partial r}(rz) = r\frac{\partial^2 v}{\partial r^2} - \frac{1}{r(\lambda+2\mu)}\left\{4\mu u + [j(j+1)\lambda - 2(\lambda + 2\mu)]v\right.$$

$$\left. + (\lambda + \frac{2}{3}\mu)\alpha r\tau\right\} . \tag{15-48}$$

If we insert now the foregoing expressions (15-43) - (15-48) in (15-41), this latter equation can be reduced to a requirement that, at the boundary $r = a$,

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial r^2} + (j+1)\mu\frac{\partial^2 v}{\partial r^2} - \rho_1\frac{\partial^2}{\partial t^2}\left\{u + (j+1)v\right\}$$

$$+ \left\{(j+3)(j-2)\frac{\lambda}{r} + \frac{4j\mu}{r} - (j-3)g\rho_1 + \right.$$

$$+ \frac{4\mu^2}{\lambda+2\mu}\left[\frac{\partial}{\partial r}\left(\frac{\lambda}{\mu}\right) - \frac{j+1}{r}\right]\Bigg\}\frac{u}{r}$$

$$- (j+1)\left\{\frac{2(j^2-1)}{r}\mu - \frac{2j\lambda}{r} + jg\rho_1\right.$$

$$\left. + \frac{j\mu^2}{\lambda+2\mu}\left[2\frac{\partial}{\partial r}\left(\frac{\lambda}{\mu}\right) + \frac{j+1}{r}\left(\frac{\lambda}{\mu}\right)\right]\right\}\frac{v}{r}$$

$$= \Bigg\{(\lambda + 2\mu)\frac{\partial}{\partial r}\left(\frac{\lambda + \frac{2}{3}\mu}{\lambda + 2\mu}\right) - \frac{2}{r}(\lambda + \frac{2}{3}\mu)$$

$$+ \frac{\lambda + \frac{2}{3}\mu}{\lambda + 2\mu}\left[\frac{j+1}{r}\mu\right]\Bigg\}\alpha\tau - (2j+1)\rho_1 c_{i,j}\, r^{j-1}. \qquad (15\text{-}49)$$

If, moreover, we remember that, quite generally,

$$\mu = \rho c_t^2 \qquad (15\text{-}50)$$

$$\lambda + 2\mu = \rho c_\ell^2 \qquad (15\text{-}51)$$

where c_ℓ and c_t denote the velocities of propagation of the longitudinal and transversal waves in elastic solids, while

$$\lambda + \frac{2}{3}\mu = \rho\left(c_\ell^2 - \frac{4}{3}c_t^2\right) = k \qquad (15\text{-}52)$$

stands for their compression modulus, the foregoing boundary condition (15-49) can be rewritten alternatively as

$$c_\ell^2 r^2\frac{\partial^2 u}{\partial r^2} + (j+1)c_t^2 r^2\frac{\partial^2 v}{\partial r^2} - r^2\frac{\partial^2}{\partial t^2}\{u + (j+1)v\}$$

$$+ \Big\{(j+3)(j-2)c_\ell^2 - 2(j-3)(j+2)c_t^2 - (j-3)gr$$

$$+ 4 \frac{c_t^4}{c_\ell^2} \left[r \frac{\partial}{\partial r} \left(\frac{c_\ell^2}{c_t^2} \right) - (j+1) \right] \Bigg\} u$$

$$- (j+1) \Bigg\{ 2(j^2+2j-1) c_t^2 - 2jc_\ell^2 + jgr$$

$$+ j \frac{c_t^4}{c_\ell^2} \left[2r \frac{\partial}{\partial r} \left(\frac{c_\ell^2}{c_t^2} \right) + (j+1) \left(2 - \frac{c_t^2}{c_\ell^2} \right) \right] \Bigg\} v$$

$$= \Bigg\{ - \frac{4}{3} c_\ell^2 \, r \frac{\partial}{\partial r} \left(\frac{c_t^2}{c_\ell^2} \right) - 2c_\ell^2 + \frac{8}{3} c_t^2$$

$$+ \left(1 - \frac{4}{3} \frac{c_t^2}{c_\ell^2} \right) (j+1) c_t^2 \Bigg\} \alpha\tau - (2j+1) c_{i,j} \, r^j . \qquad (15\text{-}53)$$

The foregoing equations (15-49) or (15-53) valid at $r = a$ ensure that the gravitational potential as well as acceleration remain continuous across the boundary of our distorted configuration, and represent the fifth boundary condition of our problem. The sixth and last one obtains by investigating the behavior of the function R at the origin in the following manner. Let both sides of equation (15-23) defining R be divided by r^j, and differentiated with respect to r : the outcome discloses that

$$r^{j+1} \frac{\partial}{\partial r} \left(\frac{R}{r^j} \right) = \frac{g}{4\pi G} \left\{ r \frac{\partial u}{\partial r} - (j+2) u \right\}$$

$$+ \frac{1}{r^{j+1}} \int_0^r \rho \left[\frac{\partial}{\partial r} (ur^{j+2}) - yr^{j+2} \right] dr, \qquad (15\text{-}54)$$

which for $r = 0$ reduces to

$$R(0,t) = 0, \qquad (15\text{-}55)$$

a result for which the previous conditions (15-33) and (15-34) are sufficient (though not necessary; since in the neighborhood of the origin the gravitational acceleration $g(r)$ varies as r and tends to zero in its own right; the only conditions necessary for the validity of (15-55) are that u and v be bounded at the origin).

Since, moreover, as $r \rightarrow 0$,

$$\lim \frac{u}{r} = \frac{\partial r}{\partial r} \quad \text{and} \quad \lim \frac{v}{r} = \frac{\partial v}{\partial r} \tag{15-56}$$

by virtue of (15-33) and (15-34), so that

$$\lim_{r \rightarrow 0} y = 3 \frac{\partial u}{\partial r} - j(j+1) \frac{\partial v}{\partial r}, \tag{15-57}$$

equations (15-26) together with (15-56) - (15-57) reveal that, at the origin,

$$(3\lambda + 5\mu) \frac{\partial u}{\partial r} + \left[2\mu - j(j+1)(\lambda+2\mu)\right] \frac{\partial v}{\partial r}$$

$$= (\lambda + \frac{2}{3} \mu)\alpha\tau(0,t); \tag{15-58}$$

or, alternately,

$$(3c_\ell^2 - c_t^2) \frac{\partial u}{\partial r} + \left[2c_t^2 - j(j+1)c_\ell^2\right] \frac{\partial v}{\partial r}$$

$$= (c_\ell^2 - \frac{4}{3} c_t^2)\alpha\tau(0,t), \tag{15-59}$$

which become again homogeneous in u and v if the central temperature of our configuration is constrained to remain constant.

Equations (15-33) - (15-34), (15-38) - (15-39), together with (15-59) or (15-53) and (15-58) or (15-59) constitute six boundary conditions requisite for a complete specification of the desired solutions of the sixth-order system of fundamental equations (15-27) and (15-31) or (15-28) and (15-32) of our problem; the conditions (15-33) - (15-34) and (15-58) or (15-59) being valid at the center, and (15-38) - (15-39) and (15-49) or (15-53) at the boundary of our configuration. Before we proceed with the construction of the requisite particular solutions, however, we propose to establish first the explicit form of the equivalent set of equations which govern the deformations of self-gravitating elastic globes characterized by toroidal rather than spheroidal symmetry; and to this task we shall address ourselves in the next section of this report.

XVI. TOROIDAL DEFORMATIONS

Having completed the reduction of the fundamental equations of section XIV governing the deformation of self-gravitating elastic globes to a form appropriate for the spheroidal symmetry defined by equations (15-4) - (15-6), let us turn our attention to another particular case of interest, arising if the deformation of our globe possesses *toroidal* symmetry - such as will obtain if the three components of the displacement vector $\vec{u}$ can be represented by

$$u_r = 0 \tag{16-1}$$

$$u_\theta = \frac{v(r,t)}{\sin\theta}\frac{\partial Y_j^i}{\partial\phi}, \tag{16-2}$$

$$u_\phi = -\,v(r,t)\frac{\partial Y_j^i}{\partial\theta}, \tag{16-3}$$

in place of (15-4) - (15-6), when the $Y_j^i(\theta,\phi)$'s continue to be surface harmonics satisfying equation (15-7).

If so, an insertion of (16-1) - (16-3) in (14-13) reveals that, in the present case,

$$\Delta = 0 ; \tag{16-4}$$

which together with $u_r = 0$ renders, by (14-29) the perturbation ρ' in density - and, therefore, by (14-31) in potential V' - identically zero. The torsional motion characterized by velocity components of the form (16-1) - (16-3) does not, therefore, disturb the gravitational potential of our globe - a fact which will far-reachingly simplify the analysis.

As the reader can easily verify, the assumed form of the velocity components (16-1) - (16-3) reduces the components (14-20) - (14-22) of the curl

$\vec{u}$ to the form

$$2\bar{\omega}_r = \frac{j(j+1)v}{r} Y_j^i, \tag{16-5}$$

$$2\bar{\omega}_\theta = \left\{\frac{\partial v}{\partial r} + \frac{v}{r}\right\} \frac{\partial Y_j^i}{\partial \theta}, \tag{16-6}$$

$$2\bar{\omega}_\phi = \left\{\frac{\partial v}{\partial r} + \frac{v}{r}\right\} \frac{1}{\sin\theta} \frac{\partial Y_j^i}{\partial \phi}. \tag{16-7}$$

The terms $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{G}$ as defined by (14-13) - (14-25) are (for $\Delta = 0$) obviously independent of λ; but they can be made to vary as the requisite single harmonic only if μ is independent of the angular variables; in which case

$$\mathcal{E} = 0,$$

$$\mathcal{F} = \left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\left(\frac{1}{\sin\theta}\frac{\partial Y}{\partial \phi}\right)\frac{\partial \mu}{\partial r}, \tag{16-9}$$

$$\mathcal{G} = -\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\left(\frac{\partial Y}{\partial \theta}\right)\frac{\partial \mu}{\partial r}. \tag{16-10}$$

Lastly, a separation of the physical and angular variables in (15-1) - (15-3) with the aid of (16-1) - (16-3) becomes possible only provided that

$$\operatorname{grad}\left\{(\lambda + \tfrac{2}{3}\mu)\alpha\tau\right\} = 0 \tag{16-11}$$

If so, however, equation (15-1) becomes identically zero; while (15-2) and (15-3) reduce to identical second-order differential equations for v of the form

$$\rho\frac{\partial^2 v}{\partial t^2} - \frac{j(j+1)\mu}{r^2} v = \mu\left\{\frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r}\right\} + \frac{\partial \mu}{\partial r}\left\{\frac{\partial v}{\partial r} - \frac{v}{r}\right\}, \tag{16-12}$$

subject again to the boundary conditions (14-34) and (14-35) which, in the present case, reduce to the requirements that, at the origin,

$$v(0,t) = 0; \tag{16-13}$$

while, at the outer boundary $r = a$,

$$\frac{\partial v}{\partial r} = \frac{v}{a} . \tag{16-14}$$

An inspection of the foregoing results discloses that - unlike the case of spheroidal deformation of section XV whose analytical formulation constitutes a differential problem of sixth order - the toroidal deformations (subject to additional restricting condition represented by equation (16-11)) reduce to a problem of second order only. This reduction goes back partly to the fact that only one radial function $v(r,t)$ is found sufficient to describe toroidal deformations - in place of two such functions $u(r,t)$ and $v(r,t)$ in the spheroidal case - and partly to the fact that the toroidal deformations do not perturb the gravitational potential of the respective body.

It may be added that the fundamental equation (16-12) of the toroidal case was first derived by Alterman, Jarosch and Pekeris (1959) in connection with their investigation of free toroidal oscillations of the Earth; while for its subsequent applications to the Moon cf. Takeuchi, Saito, and Kobayashi (1961) or Carr and Kovach (1962).

XVII. SPHEROIDAL DEFORMATIONS: PARTICULAR CASES

The sixth-order system (15-1) - (15-3) of fundamental equations governing spheroidal deformations of self-gravitating globes of arbitrary equilibrium structure is too complicated to admit of any but numerical solutions in an unrestricted case; and such solutions can be constructed by standard methods for any given functions $\alpha(r)$, $\lambda(r)$, $\mu(r)$ and $\rho(r)$ or $\tau(r)$. Certain particular cases exist, however, in which our problem simplifies sufficiently to admit of analytic solutions expansible in series of well-known elementary or transcendental functions. The aim of the present section will be to point out such cases and to construct their appropriate solutions - with particular attention to the similarities obtaining between our present treatment of elastic solids and viscous fluids investigated in earlier reports of this series.

1. Radial Deformations

As the first instance which we propose to treat in some detail, consider the case of a purely radial deformation (i.e., such as represented by an expansion or contraction of an elastic solid globe) in which, by definition,

$$j = 0 \quad \text{and} \quad v(r,t) = 0, \tag{17-1}$$

so that, by (15-8) and (15-9),

$$y = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) \quad \text{and} \quad z = -\frac{u}{r} . \tag{17-2}$$

Moreover, in such a case equation (15-23) governing the potential disturbance reduces to

$$R = \frac{u}{r^2}\int_0^r \rho r^2 dr + \int_r^a \rho u \, dr - \frac{c_{0,0}}{4\pi G} \tag{17-3}$$

so that

$$\frac{\partial R}{\partial r} = \frac{1}{r^2}\left\{\frac{\partial u}{\partial r} - \frac{2u}{r}\right\} \int_0^r \rho r^2 dr = \frac{g}{4\pi G}\left\{\frac{\partial u}{\partial r} - \frac{2u}{r}\right\} \quad (17\text{-}4)$$

If so,

equations (15-2) - (15-3) or (15-26) become identically zero, while (15-25) will reduce to a second-order differential equation for $u(r,t)$ of the form

$$(\lambda + 2\mu) \frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)\right\} + \left[\frac{\partial \lambda}{\partial r} + g\rho\right]\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u)$$

$$+ 2\frac{\partial \mu}{\partial r}\frac{\partial u}{\partial r} = \rho\frac{\partial^2 u}{\partial t^2} + g\rho\left[\frac{\partial u}{\partial r} - \frac{2u}{r}\right] + \frac{\partial}{\partial r}(\lambda + \frac{2}{3}\mu)\alpha\tau \quad (17\text{-}5)$$

or, more explicitly,

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2(\lambda+2\mu)\right]\frac{\partial u}{\partial r} + \frac{2}{r}\left[2g\rho - \frac{\lambda+2\mu}{r} + \frac{\partial \lambda}{\partial r}\right]u$$

$$= \rho\frac{\partial^2 u}{\partial t^2} + \frac{\partial}{\partial r}(\lambda + \frac{2}{3}\mu)\alpha\tau(r,t). \quad (17\text{-}6)$$

The boundary conditions of our present problem derive from (15-33) and (15-38), which in view (17-2) reduce to

$$u(0,t) = 0 \quad (17\text{-}7)$$

at the center, and

$$(\lambda + 2\mu)\frac{\partial u}{\partial r} + \frac{2\lambda u}{r} = (\lambda + \frac{2}{3}\mu)\alpha\tau(a,t) \quad (17\text{-}8)$$

on the bounding surface $r = a$.

If the motion in question were of oscillatory nature which, for small amplitudes, becomes harmonic with the frequency ν, then by setting

$$\frac{\partial^2}{\partial t^2} = -\,\upsilon^2 \tag{17-9}$$

we can regard (17-6) as an ordinary differential equation for u as a function of r, and solve it as such for given values of α, λ, μ, ρ and τ.

Consider, on the other hand, the steady state case in which none of the quantities involved in equation (17-6) depend on the time. If, moreover, the parameters α, λ, μ as well as ρ can be regarded as constant (i.e., our configuration regarded as a homogeneous elastic solid non-uniformly heated within), equation (17-6) will again reduce to

$$\frac{\partial}{\partial r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)\right\} = \left\{\frac{3\lambda + 2\mu}{3\lambda + 6\mu}\right\}\alpha\frac{\partial\tau}{\partial r} - \left\{\frac{4g\rho}{\lambda+2\mu}\right\}\frac{u}{r}\,, \tag{17-10}$$

where the first term on the right-hand side represents the effects of thermal expansion or contraction; the second, those of compression due to self-attraction.

The foregoing equations can, moreover, be reduced to more symmetrical forms if we introduce a new variable

$$\xi = \frac{u}{r}\,, \tag{17-11}$$

and replace the Lamé constants λ and μ by the coefficient β of isothermal compression (i.e., reciprocal of the bulk modulus k introduced through equation (15-52)) and the Poisson's ratio σ of transverse compression to longitudinal expansion, related with λ and μ by means of the equations

$$\sigma = \frac{\lambda}{2(\lambda+\mu)} \tag{17-12}$$

and

$$\beta = \frac{3}{3 + 2\mu} . \tag{17-13}$$

If we do so and remember that, for homogeneous configurations,

$$g = \frac{4}{3} \pi G \rho r, \tag{17-14}$$

our equation (17-10) can be alternatively rewritten as

$$\frac{\partial^2 \xi}{\partial r^2} + \frac{4}{r} \frac{\partial \xi}{\partial r} = \frac{1 + \sigma}{3(1-\sigma)} \left\{ \frac{\alpha}{r} \frac{\partial \tau}{\partial r} - \frac{16}{3} \pi G \rho^2 \beta \xi \right\} ; \tag{17-15}$$

and its associated boundary condition (17-8) for $r = a$ becomes

$$(1-\sigma) r \frac{\partial \xi}{\partial r} + (1+\sigma) \xi = \frac{1}{3}(1+\sigma) \tau(a) . \tag{17-16}$$

Equations (17-15) - (17-16) were previously used by the present writer (cf. Kopal, 1962, 1963) to study the secular thermal expansion of the Moon radioactively heated within, as well as its contraction due to self-compression; but the more general equation (17-6) has not yet been used to this end.

2. Incompressible Configurations

The aim of the present section will be to study the case of spheroidal deformations of incompressible elastic globes, characterized by the condition

$$\Delta = y = 0, \tag{17-17}$$

which, by (15-8) and (15-9), implies that

$$j(j+1)\frac{v}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) \tag{17-18}$$

and

$$j(j+1)z = \left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{j(j+1)}{r^2}\right\}(ru) \;; \tag{17-19}$$

or, by setting

$$u = \frac{j(j+1)}{r^2}w \tag{17-20}$$

and thus, by (17-18),

$$v = \frac{1}{r}\frac{\partial w}{\partial r}, \tag{17-21}$$

equation (17-19) can be expressed more symmetrically as

$$rz = \frac{\partial^2 w}{\partial r^2} - \frac{j(j+1)}{r^2}w. \tag{17-22}$$

Since, moreover, incompressible configurations cannot be deformed radially, it follows that, in our present case,

$$j > 0. \tag{17-23}$$

Moreover, imcompressibility implies that the velocity c_ℓ of propagation of the longitudinal waves in our solid be infinite - which, by (15-51), implies that the Lamé parameter

$$\lambda = \infty, \tag{17-24}$$

but in such a way that

$$0 < \lambda y < \infty; \tag{17-25}$$

and that, furthermore,

$$0 \leq \mu < \infty, \tag{17-26}$$

which by (17-12) renders the Poisson ratio

$$\sigma = \frac{1}{2} . \tag{17-27}$$

In the limiting case of $\mu = \infty$ equation (17-12) permits σ to assume any value constrained by $0 < \sigma < \frac{1}{2}$; and for $\mu = 0$ (i.e., zero rigidity) we obtain the case of incompressible liquid. All these will be treated in turn.

Let us first take up the harmonic oscillations of a heterogeneous globe of incompressible liquid, for which $\alpha = \mu = 0$ and $\lambda = \infty$. Since, moreover, $y = 0$ (though λy remain finite), the first fundamental equation (15-27) for harmonic motion with a frequency ν introduced by (17-9) will reduce to

$$\frac{\lambda y}{\rho} \frac{\partial \rho}{\partial r} = \nu^2 \rho r z ; \tag{17-28}$$

while the second fundamental equation (15-28) similarly reduces to

$$\left\{ \frac{\partial^2}{\partial r^2} + \left(\frac{2}{r} - \frac{1}{\rho} \frac{\partial \rho}{\partial r} \right) \frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} (\lambda y) =$$

$$= \frac{4}{3}\pi G\rho \left\{ \bar{\rho}\left[j(j+1)\left(\frac{\partial v}{\partial r} - \frac{u+v}{r}\right) - 4\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right)\right] + 6\rho\frac{\partial u}{\partial r}\right\} . \tag{17-29}$$

Eliminating λy between (17-28) and (17-29), and inserting for v in terms of u by means of (17-18), we obtain

$$\nu^2 \left\{ \frac{\partial^2}{\partial r^2} + \left[\frac{2}{r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r}\right]\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} \frac{\rho^2 rz}{\frac{\partial \rho}{\partial r}} =$$

$$= \frac{g\rho}{r}\left\{ r^2 \frac{\partial^2 \xi}{\partial r^2} + 6\frac{\bar{\rho}}{\rho}\left[r\frac{\partial \xi}{\partial r} + \xi\right] - j(j+1)\xi \right\}, \tag{17-30}$$

where rz is expressible in terms of u by (17-19) and where, in accordance with (17-11), ξ continues to stand for the ratio u/r.

Since rz as given by (17-19) is of second order in u, the foregoing equation (17-30) constitutes a differential equation for u of fourth order, which is generally solvable only by quadratures. However, in certain limiting cases it can be reduced to one of second order. Thus, in the equilibrium case (when $\nu = 0$), it can be satisfied only if

$$r^2 \frac{\partial^2 \xi}{\partial r^2} + 6\frac{\bar{\rho}}{\rho}\left\{ r\frac{\partial \xi}{\partial r} + \xi\right\} - j(j+1)\xi = 0 , \tag{17-31}$$

which represents the well-known Clairaut equation of hydrostatic equilibrium (cf.,e.g., Kopal, 1960); or, for homogeneous configurations (ρ = constant), oscillating with a frequency $\nu^2 > 0$ it requires that

$$rz = \frac{\partial^2 w}{\partial r^2} - \frac{j(j+1)}{r^2} w = 0 \tag{17-32}$$

by (17-22).

The particular solution of (17-32) which remains finite at the origin is of the form

$$w(r) = kr^{j+1}, \tag{17-33}$$

so that, by (17-20) and (17-21),

$$u = j(j+1)kr^{j-1} \quad \text{and} \quad v = (j+1)kr^{j-1}, \tag{17-34}$$

where k denotes an arbitrary constant. Moreover, since for homogeneous configurations $\rho/\bar{\rho} = 1$, the expression $\xi = u/r = j(j+1)kr^{j-2}$ satisfies also Clairaut's equation (17-31). Both sides of (17-30) vanish on insertion from (17-34) which represent, therefore, the solutions of the complete equation (17-30) for homogeneous configurations.

In order to determine the frequency ν of the respective oscillations, let us return to our fundamental equation (15-26) in which, for homogeneous configurations, R as defined by (15-23) can be integrated to yield

$$\begin{aligned} R(r) &= \frac{1}{3}\,\rho r u(r) - \left\{ \frac{a^{1-j}u(a)}{2j+1} + \frac{c_{i,j}}{4\pi G\rho} \right\} \rho r^j \\ &= \left\{ \frac{2j(j^2-1)}{3(2j+1)}\,k_j - \frac{c_{i,j}}{4\pi G\rho} \right\} \rho r^j \end{aligned} \tag{17-35}$$

by the first one of equations (17-34). Let us, moreover, particularize now equation (15-26) for $r = 0$. Since, in accordance with the boundary condition (15-38),

$$\lambda y = 0 \quad \text{at} \quad r = a, \tag{17-36}$$

and

$$v(a) = (j+1)k_j a^{j-1} \tag{17-37}$$

by the second one of (17-34), on insertion from (17-34) - (17-37) in (15-26) the latter can be reduced to the algebraic form

$$(j+1)k_j \left(\frac{\nu^2}{4\pi G\rho}\right) = \frac{2}{3}\,\frac{j(j^2-1)}{2j+1}k_j - \frac{c_{i,j}}{4\pi G\rho}\,. \tag{17-38}$$

In the absence of a disturbing force (i.e., when $c_{i,j} = 0$), the foregoing equation yields for the frequency of free harmonic osicllations of self-gravitating globes of incompressible liquids the well-known result

$$\frac{\nu^2}{2\pi G\rho} = \frac{4j(j-1)}{3(2j+1)}\,, \tag{17-39}$$

obtained first by Kelvin (1863) and re-derived subsequently by us, by a different method, in Report III (equation 13-16) of this series. If, on the other hand, $\nu = 0$ in (17-38) - corresponding to the case of hydrostatic equilibrium - equation (17-38) leads to

$$(j+1)k_j = \frac{3(2j+1)}{2j(j-1)}\,\frac{c_{i,j}}{4\pi G\rho} \tag{17-40}$$

and, by (17-34),

$$u = \frac{3(2j+1)}{2(j-1)}\left(\frac{c_{i,j}}{4\pi G\rho}\right) r^{j-1}, \tag{17-41}$$

$$v = \frac{3(2j+1)}{2j(j-1)}\left(\frac{c_{i,j}}{4\pi G\rho}\right) r^{j-1}, \tag{17-42}$$

where the constants $c_{i,j}$ are specified by the nature of the external forces acting on our configuration.

As the next case to be considered in the present section, let us relax the condition $\mu = 0$ and regard this latter quantity as finite but constant such that $0 < \mu < \infty$. If so, equations (15-27) and (15-31) will reduce to

$$\left\{\frac{\partial^2}{\partial r^2} - \left(\frac{1}{\rho}\frac{\partial\rho}{\partial r}\right)\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2}\right\}(\mu rz) = \frac{\lambda y}{\rho}\frac{\partial\rho}{\partial r} - \nu^2\rho rz \tag{17-43}$$

and

$$\left\{\frac{\partial^2}{\partial r^2} + \left[\frac{2}{r} - \frac{1}{\rho}\frac{\partial\rho}{\partial r}\right]\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2}\right\}(\lambda y) =$$

$$= \frac{j(j+1)}{r}\left\{\frac{g\rho}{\bar{\rho}} + \frac{\mu}{\rho^2}\frac{\partial\rho}{\partial r}\right\}\rho z + g\rho r\left(1 - \frac{\rho}{\bar{\rho}}\right)\left\{\frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{r^2}\right\}\frac{u}{r} \tag{17-44}$$

and, for ρ = constant, further to

$$\left\{\frac{\partial^2}{\partial r^2} - \frac{j(j+1)}{r^2}\right\}(\mu rz) = - \nu^2\rho rz, \tag{17-45}$$

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2}\right\}(\lambda y) = \frac{j(j+1)}{r} g\rho z. \tag{17-46}$$

Within the scheme of our approximation, equation (17-45) is independent of (17-46) and can be solved for rz as it stands; whereupon (17-46) can be solved for λy.

If we abbreviate

$$\frac{\nu^2\rho}{\mu} = \kappa^2, \tag{17-47}$$

the particular solution of equation (17-45) which remains finite at the origin can be expressed as

$$rz = B\sqrt{r}\; J_{j+\frac{1}{2}}(\kappa r), \tag{17-48}$$

where B denotes an integration constant. In consequences, the differential equation (17-19) for u assumes now the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{4}{r}\frac{\partial u}{\partial r} - \frac{(j-1)(j+2)}{r^2} = \frac{j(j+1)}{\sqrt{r^3}} BJ_{j+\frac{1}{2}}(\kappa r), \tag{17-49}$$

and its particular solution which is regular at the origin becomes

$$u = Ar^{j-1} + \frac{j(j+1)}{2j+1}\left\{r^{j-1}\int_0^r r^{\frac{1}{2}-j}\, J_{j+\frac{1}{2}}(\kappa r)dr - r^{-j-2}\int_0^r r^{3/2+j} J_{j+\frac{1}{2}}(\kappa r)dr\right\}B, \tag{17-50}$$

where A represents another integration constant.

The integrals on the right-hand side of (17-50) can be evaluated by the same techniques we employed in section XI of Report III. In doing so we find that

$$r^{j-1}\int_0^r r^{\frac{1}{2}-j}\, J_{j+\frac{1}{2}}(\kappa r)dr = -\frac{1}{\kappa\sqrt{r}}\, J_{j-\frac{1}{2}}(\kappa r) \tag{17-51}$$

and

$$r^{-j-2}\int_0^r r^{\frac{3}{2}+j} J_{j+\frac{1}{2}}(\kappa r)dr = \frac{1}{\kappa\sqrt{r}}\, J_{j+3/2}(\kappa r), \tag{17-52}$$

which inserted in (17-50) yields

$$u = Ar^{j-1} - \frac{j(j+1)B}{(2j+1)\kappa\sqrt{r}}\left\{J_{j+3/2}(\kappa r) + J_{j-\frac{1}{2}}(\kappa r)\right\}$$

$$= Ar^{j-1} - \frac{j(j+1)B}{\kappa^2\sqrt{r^3}}\, J_{j+\frac{1}{2}}(\kappa r) \tag{17-53}$$

if advantage is taken of the recursion formula (11-36). Moreover, from (17-18) it follows then that

$$v = \frac{A}{j} r^{j-1} + \frac{B}{\kappa^2 \sqrt{r^3}} \left\{ j\ J_{j+\frac{1}{2}}(\kappa r) - \kappa r\ J_{j-\frac{1}{2}}(\kappa r) \right\} . \qquad (17\text{-}54)$$

In order to proceed further we invoke now the boundary conditions $\sigma_{r\theta} = \sigma_{r\phi} = 0$ which, for $\mu \neq 0$, require that

$$\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{u}{r} = 0 \qquad (17\text{-}55)$$

at $r = a$. Inserting into it for u and v from (17-53) and (17-54) we find that

$$\frac{j-1}{j}(\kappa a)^2 a^{j-2} A + \left\{ \left[\frac{1}{2}(\kappa a)^2 - j(j+2) \right] J_{j+\frac{1}{2}}(\kappa a) + \kappa a\ J_{j-\frac{1}{2}}(\kappa a) \right\} \frac{B}{\sqrt{a}} = 0 \qquad (17\text{-}56)$$

The use of the boundary condition $\sigma_{rr} = 0$ at $r = a$ requires a knowledge of the surface value of the function λy governed by (17-46). On insertion for rz from (17-48), (17-46) assumes the more explicit form

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} \lambda y = j(j+1) g\rho B r^{-3/2} J_{j+\frac{1}{2}}(\kappa r), \qquad (17\text{-}57)$$

and its particular solution which remains finite at the origin assume the form

$$\lambda y = C x^j + \frac{j(j+1)g\rho}{2j+1} \left\{ r^{j-1} \int_0^r r^{\frac{1}{2}-j} J_{j+\frac{1}{2}}(\kappa r)\, dr - r^{-j-2} \int_0^r r^{3/2+j} J_{j+\frac{1}{2}}(\kappa r)\, dr \right\} B, \qquad (17\text{-}58)$$

where C denotes another integration constant; and which on substitution from (17-51) and (17-52) yields

$$\lambda y = Cr^j - \frac{j(j+1)g\rho}{(2j+1)\kappa\sqrt{r}} \left\{ J_{j+3/2}(\kappa r) + J_{j-\frac{1}{2}}(\kappa r) \right\} B$$

$$= Cr^j - \frac{j(j+1)g\rho}{\kappa^2\sqrt{r^3}} BJ_{j+\frac{1}{2}}(\kappa r) \tag{17-59}$$

by (11-36); so that, for $r = a$,

$$\lambda y = Ca^j - \frac{j(j+1)g\rho}{\kappa^2\sqrt{a^3}} BJ_{j+\frac{1}{2}}(\kappa a) \ . \tag{17-60}$$

An insertion from (17-53) and (17-60) in the boundary condition

$$(\lambda y)_a + 2\mu\left(\frac{\partial u}{\partial r}\right)_a = 0 \tag{17-61}$$

then yields

$$2(j-1)\mu a^{j-2}A + \frac{j(j+1)}{(\kappa a)^2}\left\{\left[2(j+2)\mu - g\rho a\right] J_{j+\frac{1}{2}}(\kappa a)\right.$$

$$\left. - 2\mu\kappa a J_{j-\frac{1}{2}}(\kappa a)\right\} \frac{B}{\sqrt{a}} + a^j C = 0 \tag{17-62}$$

Equations (17-56) and (17-62), based on the boundary conditions (17-55) and (17-61) at $r = a$, contain three arbitrary constants A, B, C, and cannot as yet be solved uniquely for them. In order to complete the specification of our problem, a recourse must be had to the fundamental equation (15-26) in which (for constant ρ) R continues to be given by the first part of equation (17-35). Inserting in the latter for u from (17-53) we find that, in the present case,

$$R(a) = \frac{2(j-1)}{2j+1}\left\{Aa^{j-1} - \frac{j(j+1)B}{\kappa^2\sqrt{a^3}} J_{j+\frac{1}{2}}(\kappa a)\right\}\rho a - \frac{c_{i,j}a^j}{4\pi G} \ ; \tag{17-63}$$

and if so, equation (15-36) on insertion for rz, v, λy, and R from (17-48), (17-54), (17-60) and (17-63) will assume the form

$$\left\{\frac{\mu(\kappa a)^2}{j} - \frac{2(j-1)}{2j+1} g\rho a\right\} a^{j-2}A$$

$$- \left\{\frac{3j(j+1)g\rho a}{(2j+1)(\kappa a)^2} J_{j+\frac{1}{2}}(\kappa a)\right\}\frac{B}{\sqrt{a}} + a^j C = - c_{i,j}\rho a^j, \tag{17-64}$$

which represents the third independent relation between our integration constants A, B, and C.

If $c_{i,j} = 0$ on the right-hand side of equation (17-64) - i.e., if our configuration is subject to the action of no external forces, equation (17-56), (17-62) and (17-64) are homogeneous in A, B, C, and can be solved for unrestricted values of these constants only if

$$\begin{vmatrix} \frac{j-1}{j}\alpha^2 & \left[\frac{1}{2}\alpha^2 - j(j+2)\right] J_{j+\frac{1}{2}}(\alpha) + \alpha J_{j-\frac{1}{2}}(\alpha) & 0 \\ 2(j-1) & \frac{j(j+1)}{\alpha^2}\left\{\left[2(j+2) - m\right] J_{j+\frac{1}{2}}(\alpha) - 2\alpha J_{j-\frac{1}{2}}(\alpha)\right\} & 1 \\ \frac{\alpha^2}{j} - \frac{2(j-1)m}{2j+1} & - \frac{3j(j+1)m}{(2j+1)\alpha^2} J_{j+\frac{1}{2}}(\alpha) & 1 \end{vmatrix} = 0 \tag{17-65}$$

i.e., provided that the nondimensional parameter

$$\kappa a = \nu a\sqrt{\rho/\mu} = \alpha \tag{17-66}$$

is a root of the transcendental equation

$$\left\{2j(2j+1)\left[2(j-1)(j+2) - \alpha^2\right] + \alpha^4 + 2j(j-1)\left[2 - \frac{\alpha^2}{2j+1}\right]m\right\} J_{j+\frac{1}{2}}(\alpha)$$

$$= 2\left\{2j(j-1)\left[j+2 + \frac{m}{2j+1}\right] - \alpha^2\right\}\alpha \ J_{j-\frac{1}{2}}(\alpha) ; \qquad (17\text{-}67)$$

or, in a slightly more simplified form,

$$\left\{2(j-1)(2j+1)^2 - (2j+1)\alpha^2 + 2j(j-1)m\right\}\left\{\alpha J_{j+5/2}(\alpha) - J_{j+3/2}(\alpha)\right\}$$

$$+ 4(j+1)(j-1)^2 J_{j+3/2}(\alpha) = 0, \qquad (17\text{-}68)$$

where

$$m = \frac{g\rho a}{\mu} \qquad (17\text{-}69)$$

denotes likewise a nondimensional parameter characterizing the structure of the respective configuration. Once the roots α_i of (17-67) or (17-68) have been determined (numerically or otherwise) for given values of m, the normalized frequency of oscillations itself follows from the equation

$$\frac{\nu^2}{2\pi G\rho} = \frac{2\alpha^2}{3m} \qquad (17\text{-}70)$$

Since Bessel's functions of half-integral argument are expressible as trigonometric polynomials of their arguments, equations (17-67) or (17-68) are in effect trigonometric; and can be solved for the α's corresponding to any given value of m by standard methods applicable to such equations. For small α's, the following expansions may be useful. Since, by definition,

$$J_{j+\frac{1}{2}}(\alpha) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(i+j+3/2)} \left(\frac{\alpha}{2}\right)^{2i+j+\frac{1}{2}} \tag{17-71}$$

and, therefore,

$$\alpha J_{j-\frac{1}{2}}(\alpha) = 2 \sum_{i=0}^{\infty} \frac{(-1)^i (i+j+\frac{1}{2})}{i!\,\Gamma(i+j+3/2)} \left(\frac{\alpha}{2}\right)^{2i+j+\frac{1}{2}} \tag{17-72}$$

an insertion of (17-71) and (17-72) in (17-67) converts the latter into an equation of the form

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(i+j+3/2)} \left\{ -8ij(j-1)(j+2) + 2[2i - (j-1)(2j+1)]\alpha^2 + \alpha^4 - \frac{2j(j-1)}{2j+1}(4i + \alpha^2)m \right\} \left(\frac{\alpha}{2}\right)^{2i} = 0 \,. \tag{17-73}$$

This equation admits obviously of the root $\alpha = 0$, corresponding to the state of equilibrium. For $i = 0$, it is satisfied by

$$\frac{\alpha^2}{m} = \frac{2j(j-1)}{2j+1} + \frac{2(j-1)(2j+1)}{m} \,, \tag{17-74}$$

which on insertion from (17-66) and (17-69) can be rewritten as

$$\frac{\nu^2}{2\pi G\rho} = \frac{4}{3} \left\{ \frac{j(j-1)}{(2j+1)} + (j-1)(2j+1) \frac{\mu}{g\rho a} \right\} ; \tag{17-75}$$

and this for $\mu = 0$ reduces indeed to (17-39).

The foregoing results represent the most general formulation of our results, valid when the surface value of λy in the boundary condition (17-58) is different from zero. Should this not be the case, and (like for fluids) $(\lambda g)_a = 0$, it follows immediately from (17-60) that the constant

$$C = B\left\{ j(j+1)g\rho a^{-j+\frac{1}{2}}\alpha^{-2}J_{j+\frac{1}{2}}(\alpha) \right\} . \tag{17-76}$$

If so, equations (17-56) and (17-62) then become independent of (17-64); and are solvable for arbitrary values of A and B provided that α is the root of the determinantal equation

$$\begin{vmatrix} \frac{j-1}{j}\alpha^2 & \left[\frac{1}{2}\alpha^2 - j(j+2)\right] J_{j+\frac{1}{2}}(\alpha) + \alpha J_{j-\frac{1}{2}}(\alpha) \\ \frac{j-1}{j(j+1)}\alpha^2 & (j+2)J_{j+\frac{1}{2}}(\alpha) - \alpha J_{j-\frac{1}{2}}(\alpha) \end{vmatrix} = 0 \tag{17-77}$$

which can be reduced to

$$\alpha J_{j+5/2}(\alpha) + J_{j+3/2}(\alpha) = 0 ; \tag{17-78}$$

and once these equations have been solved for α, the corresponding value of m can then be obtained from (17-64). The reader may wish to note the similarity of the preceding equations (17-77) - (17-68) with equivalent equations (12-11) - (12-12) valid for the case of incompressible viscous fluids (cf. Report III), which prove to be formally identical with the present results except for the definition of the parameter α (which is, in fact, identical with a geometric mean of the arguments α and β of equations (12-12) and (12-14) in Report III, and related by (11-25)).

The foregoing equations (17-65) - (17-78) apply to a homogeneous system obtaining if the constants $c_{i,j}$ on the right-hand side of equation (17-64) are set equal to zero. If this is not the case by virtue of external forces, unique solutions for the constants A, B, and C can be obtained by a simultaneous solution of (17-56), (17-62) and (17-64) in terms of $c_{i,j}$. Provided that α is not a root of (17-67) or (17-68), the scale constant A which specifies the amplitude of the forced displacements u and v proves readily to be expressible as

$$A = \frac{j\rho a^2}{\mu}\left\{\frac{2\alpha J_{j+3/2} + \left[2(j^2-1)-\alpha^2\right]J_{j+\frac{1}{2}}}{\left[2(j-1)(2j+1)^2 - (2j+1)\alpha^2 + 2j(j-1)m\right]\left[\alpha J_{j+5/2}(\alpha) - J_{j+3/2}(\alpha)\right] + 4(j+1)(j-1)^2 J_{j+3/2}(\alpha)}\right\} \qquad (17\text{-}79)$$

with similar results for B and C. As $\mu \to \infty$, all these constants are bound to tend to zero.

In the *equilibrium case*, for which

$$\nu = 0, \qquad (17\text{-}80)$$

equation (17-45) will possess a particular solution

$$rz = B_0 r^{j+1} \qquad (17\text{-}81)$$

which is well-behaved at the origin; and equation (17-19) for u,

$$\frac{\partial^2 u}{\partial r^2} + \frac{4}{r}\frac{\partial u}{\partial r} - \frac{(j-1)(j+2)}{r^2}u = j(j+1)B_0 r^{j-1}, \qquad (17\text{-}82)$$

will admit of a similar polynomial solution of the form

$$u = A_0 r^{j-1} + \frac{j(j+1)}{2(2j+3)} B_0 r^{j+1}, \qquad (17\text{-}83)$$

corresponding by (17-18) to

$$v = \frac{A_0}{j} r^{j-1} + \frac{j+3}{2(2j+3)} B_0 r^{j+1}. \quad (17\text{-}84)$$

The boundary condition (17-55) than reveals by insertion from (17-83) and (17-84) that

$$\frac{B_0}{A_0} = - \frac{2(j-1)(2j+3)}{j^2(j+2)a^2} , \quad (17\text{-}85)$$

by virtue of which

$$u = A_0 r^{j-1}\left\{1 - \frac{j^2-1}{j(j+2)} \left(\frac{r}{a}\right)^2\right\} , \quad (17\text{-}86)$$

and

$$v = \frac{A_0}{j} r^{j-1}\left\{1 - \frac{(j-1)(j+3)}{j(j+2)} \left(\frac{r}{a}\right)^2\right\} . \quad (17\text{-}87)$$

Since, moreover, in the equilibrium case the boundary condition (17-61) reveals that

$$(\lambda y)_a = - 2\mu\left(\frac{\partial u}{\partial r}\right)_a = \frac{2\mu(j-1)}{j(j+2)} a^{j-2}A_0 , \quad (17\text{-}88)$$

while equation (17-35) together with (17-86) leads to

$$R(a) = \left\{\frac{2(j-1)}{3j(j+2)} \rho A_0 - \frac{c_{i,j}}{4\pi G}\right\} a^j , \quad (17\text{-}89)$$

an insertion of (17-81), (17-85) and (17-88) - (17-89) in (15-26) that

$$A_0 = \frac{3j^2(j+2)}{2(j-1)} \frac{1}{j + (\mu/g\rho a)(2j^2+4j+3)} \left(\frac{c_{i,j}}{4\pi G\rho}\right); \quad (17\text{-}90)$$

so that the displacements u, v caused by an application of an external force specified by the coefficients $c_{i,j}$ of the disturbing potential will ultimately assume the form

$$u(r) = \frac{3j(j+2)}{2(j-1)} \frac{r^{j-1}}{1 + q/m} \left\{ 1 - \frac{j^2-1}{j(j+2)} \left(\frac{r}{a}\right)^2 \right\} \frac{c_{i,j}}{4\pi G\rho} \tag{17-91}$$

and

$$v(r) = \frac{3(j+2)}{2(j-1)} \frac{r^{j-1}}{1 + q/m} \left\{ 1 - \frac{(j-1)(j+3)}{j(j+2)} \left(\frac{r}{a}\right)^2 \right\} \frac{c_{i,j}}{4\pi G\rho} \tag{17-92}$$

where $m \equiv g\rho a/\mu$ continues to be given by equation (17-69) and

$$jq = (2j+1)(2j+3) - 2j(j+2) = 2j^2+4j+3 \ . \tag{17-93}$$

3. Fluid Configurations

If the self-gravitating globe under investigation were fluid (i.e., possessed zero rigidity), we should expect that

$$\alpha = \mu = 0 \; . \tag{17-94}$$

If so, the matter in question will be incapable of transmitting transversal waves (i.e., $c_t = 0$ in accordance with equation (15-50)). However, the velocity c_ℓ of propagation of the longitudinal waves must remain finite - a requirement which implies, by (15-51), that

$$c_\ell^2 = \frac{\lambda}{\rho} < \infty \; . \tag{17-95}$$

Hence, for $\rho \neq 0$, $\lambda < \infty$ in the present case; and equations (15-27) and (15-31) reduce them to

$$\rho \frac{\partial^2}{\partial t^2}(rz) + \left\{ \frac{\lambda}{\rho}\frac{\partial \rho}{\partial r} + g\rho \right\} y = 0 \tag{17-96}$$

and

$$\frac{\partial^2 y}{\partial t^2} - \left\{ \frac{\partial^2}{\partial r^2} + \left[\frac{2}{r} - \frac{1}{\rho}\frac{\partial \rho}{\partial r} \right] \frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} (\lambda y)$$

$$= \left\{ 4y - j(j+1)z + 6\left(1 - \frac{\rho}{\bar{\rho}}\right)\left(\frac{\partial u}{\partial r} - y\right) \right\} \frac{g\rho}{r} \tag{17-97}$$

Let us consider first the equilibrium case, in which neither y nor z depend on the time. If so, equation (17-96) can be satisfied for an arbitrary function $\rho(r)$ only if $y = 0$ as in the incompressible case, for which equations (17-18) and (17-19) continue to hold good. Inserting them on the right-hand side of (17-97) we find this latter equation then to reduce to

$$r^2 \frac{\partial^2 u}{\partial r^2} + 2\left(3\frac{\rho}{\bar{\rho}} - 1\right) r \frac{\partial u}{\partial r} + [2 - j(j+1)]u = 0, \tag{17-98}$$

which on substituting $u = r\xi$ becomes identical with the Clairaut equation (17-31). That this should be so follows indeed directly from equation (15-26), which for $\nu = 0$ and $y = 0$ together with $\alpha = \mu = 0$ reduces (in accordance with 15-23) to

$$R = \frac{u}{r^2} \int_0^r \rho r^2 dr - \frac{1}{2j+1} \left\{ \frac{1}{r^{j+1}} \int_0^r \rho \frac{\partial}{\partial r} (ur^{j+2}) dr \right.$$

$$\left. + r^j \int_r^a \rho \frac{\partial}{\partial r} (ur^{1-j}) dr \right\} - \frac{c_{i,j} r^j}{4\pi G} = 0 , \tag{17-99}$$

which represents Clairaut's equation in its integral form.

Next, let us turn to the non-equilibrium (time-dependent) case, for which $\nu \neq 0$ and

$$\lambda = \rho c^2, \tag{17-100}$$

where c denotes the velocity of propagation of small (longitudinal) disturbances which (like λ and ρ) will hereafter be regarded as a function of r only, and such that

$$c^2 = \gamma \frac{P}{\rho} , \tag{17-101}$$

where P denotes the pressure in the fluid and γ, the ratio of specific heats of the respective material. If so, the reader can easily verify that, for example, equation (17-6) valid for $j = 0$ reduces to the well-known equation governing the harmonic pulsations of self-gravitating globes of inviscid fluid.*

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* The reader may even notice that it continues formally to hold good also for viscous fluids (cf., Kopal, 1964) provided that the Lamé coefficient μ of rigidity if replaced by $i\nu\mu_G$, where μ_G denotes the coefficient of material viscosity; ν, the frequency of oscillations; and i, the imaginary unit.

If, moreover, our fluid at rest is in hydrostatic equilibrium requiring that

$$\frac{\partial P}{\partial r} = - g\rho \ , \tag{17-102}$$

the coefficient of the second term on the left-hand side of equation (17-96) can, by use of (17-100) - (17-102), be expressed as

$$\frac{\lambda}{\rho}\frac{\partial \rho}{\partial r} + g\rho = c^2\left\{\frac{1}{\rho}\frac{\partial \rho}{\partial r} - \frac{1}{\gamma P}\frac{\partial P}{\partial r}\right\} = c^2 A, \tag{17-103}$$

where the term in the curly brackets on the right-hand side becomes identical with the quantity A introduced already by equation (6-19) of Report II - and our present equation (17-96) becomes indeed identical with equation (6-25) if the viscous terms in the latter are omitted (i.e., $\mu = 0$).

The quantity A as defined by (6-19) or (17-103) vanishes if the equilibrium structure of our configuration is adiabatic, so that

$$P = K\rho^{\gamma} \tag{17-104}$$

where K as well as γ are constants; and is different from zero otherwise. If $A = 0$, the validity of equation (17-96) in the time-dependent case requires that

$$z = 0, \tag{17-105}$$

which by (15-8) and (15-9) implies that

$$u = r\frac{\partial v}{\partial r} + v \tag{17-106}$$

and

$$y = \left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} (rv) . \tag{17-107}$$

If we insert now (17-106) - (17-107) in (17-97), the latter will reduce to

$$\left\{ \frac{\rho}{\lambda}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} - \left[\frac{2}{r} - \frac{1}{\rho}\frac{\partial\rho}{\partial r}\right]\frac{\partial}{\partial r} + \frac{j(j+1)}{r^2} \right\} \lambda \left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{j(j+1)}{r^2} \right\} (rv)$$

$$= 2g\rho \left\{ 2\frac{\partial^2}{\partial r^2} + \frac{2}{r}\left[1 + \frac{3\rho}{\bar{\rho}}\right]\frac{\partial}{\partial r} + \left[1 - \frac{3\rho}{\bar{\rho}}\right]\frac{(j-1)(j+2)}{r^2} \right\} v, \tag{17-108}$$

which constitutes a differential equation of fourth order for v.

Therefore, while in the equilibrium case ($\nu = 0$) and for $A \neq 0$ equation (17-96) necessarily requires that $y = 0$, which converts (17-97) into Clairaut's equation (17-98) for u ; in the non-equilibrium case ($\nu \neq 0$) the condition $z = 0$ (leading by (17-106) and (17-107) to (17-108)) becomes necessary only in the adiabatic case (i.e., $A = 0$) and not otherwise ; for $A \neq 0$, $z = 0$ only if $y = 0$; in which case equations

$$\left.\begin{aligned} y &= \frac{\partial u}{\partial r} + \frac{2u}{r} - \frac{j(j+1)}{r} v = 0, \\ z &= \frac{\partial v}{\partial r} + \frac{v}{r} - \frac{u}{r} = 0, \end{aligned}\right\} \tag{17-109}$$

can (for $j > 1$) be satisfied with particular solutions for u and v varying as r^{j-1} which remain finite at the origin. However, a glance at the fundamental equations (17-96) and (17-97) of motion in fluid case reveals that they could also be satisfied by such a solution only if our configuration were homogeneous (i.e., for $\rho = \bar{\rho}$) .

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