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Bounds for the Zeros of Polynomials

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The following results are derived from an application of a theorem of A. Brauer to the companion matrix of a polynomial and application of the theorem of Rouche. The pattern follows that of the theorems mentioned in the contract proposal and are in some sense a rounding out of those results derived earlier by the principal investigator. No negative results (and there were many) are reported.

It was suggested in the first progress report that the results of this study, and more likely the study itself, may have applications in obtaining analytic criteria for irreducibility of polynomials with integer coefficients. This has not been pursued under this grant, but will be undertaken shortly by the principal investigator. Any results published will be appropriately credited to this grant.

The principal investigator wishes to take this opportunity to thank the National Aeronautics and Space Administration for the opportunity afforded by this grant to pursue these studies.

In [1], Brauer proves the following

Theorem. Suppose $A = (a_{\kappa\lambda})$ is a square matrix of order n with complex entries. For each κ and λ with $\kappa \neq \lambda$ one sets

$$Q_{\kappa\lambda} = \sum_{\nu}\{|a_{\kappa\lambda}a_{\lambda\nu}| + |a_{\lambda\kappa}a_{\kappa\nu}| + |a_{\kappa\nu}a_{\lambda\nu}|\}$$

$$+ \sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu} + a_{\lambda\nu}a_{\kappa\mu}|,$$

where μ and ν run from 1 to n but $\mu \neq \kappa,\lambda$ and $\nu \neq \kappa,\lambda$. Moreover, let $D_{\kappa\lambda}$ denote the principal subdeterminant of the characteristic determinant of A which contains the rows κ and λ. Then each characteristic root of A lies in the union of the Cassini ovals

$$|D_{\kappa\lambda}| \leqq Q_{\kappa\lambda} \qquad (\kappa,\lambda=1,2,\ldots,n).$$

We wish to apply this theorem to the companion matrix

$$\begin{pmatrix} -a_1 & -a_2 & -a_3 & \ldots & -a_{n-1} & -a_n \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

of the polynomial $f = z^n + a_1z^{n-1} + \ldots + a_n$. We first note that

$$D_{\kappa\lambda} = (w-a_{\kappa\kappa})(w-a_{\lambda\lambda}) - a_{\kappa\lambda}a_{\lambda\kappa}$$

and observe that for the above companion matrix all diagonal elements are zero except in the 1,1 position. Thus if $\kappa = 1$, then $D_{1\lambda} = (w-a_{11})w - a_{1\lambda}a_{\lambda 1}$. Further $a_{11} = -a_1$ so that this becomes $D_{1\lambda} = (w+a_1)w - a_{1\lambda}a_{\lambda 1}$. Moreover, since $\kappa \neq \lambda$, then $\lambda \geqq 2$ and in the first column all entries are 0 except the first two. Thus we have two situations: $\lambda = 2$, and $\lambda > 2$.

$$D_{12} = (w+a_1)w + a_2 \qquad \text{for } \lambda = 2$$

$$D_{1\lambda} = (w+a_1)w \qquad \text{for } \lambda > 2.$$

For $\kappa > 1$ either $a_{\kappa\lambda} = 0$ or $a_{\lambda\kappa} = 0$ since in the companion matrix the only non-zero elements not in the first row occur in the $\kappa,\kappa-1$ position. Hence for $\kappa > 1$ we have only one form for $D_{\kappa\lambda}$, viz.,

$$D_{\kappa\lambda} = w^2.$$

We now wish to calculate the $Q_{\kappa\lambda}$ values for our distinct ovals, Q_{12}, $Q_{1\lambda}(\lambda>2)$, and $Q_{\kappa\lambda}(\kappa>1)$.

$$Q_{12} = \sum_{\nu}\{|a_{12}a_{2\nu}| + |a_{21}a_{1\nu}| + |a_{1\nu}a_{2\nu}|\} + \sum_{\mu<\nu}|a_{1\nu}a_{2\mu} + a_{2\nu}a_{1\mu}|$$

Now $\sum_{\nu}|a_{12}a_{2\nu}| = |a_{12}|\sum_{\nu}|a_{2\nu}| = |-a_2|\sum_{\nu=3}^{n}|a_{2\nu}|$. But all elements of the 2nd row of the companion matrix are zero except in the first column so that $\sum_{\nu}|a_{12}a_{2\nu}| = |-a_2|\sum_{\nu=3}|a_{2\nu}| = 0$. In a similar manner we see that $\sum_{\nu}|a_{1\nu}a_{2\nu}| = 0$, and that $\sum_{\nu}|a_{21}a_{1\nu}| = |a_{21}|\sum_{\nu}|a_{1\nu}| = \sum_{\nu=3}^{n}|a_{1\nu}| = |a_3| + |a_4| + |a_5| + \ldots + |a_n|$. Now

$$\sum_{\nu<\mu}|a_{1\nu}a_{2\mu} + a_{2\nu}a_{1\mu}| = \sum_{3\leqq\nu<\mu\leqq n}|a_{1\nu}a_{2\mu} + a_{2\nu}a_{1\mu}|.$$

The term $a_{1\nu}a_{2\mu}$ must always be zero since $a_{2\mu} = 0$ for all $\mu > 1$, and 1 is excluded from the range of μ. The term $a_{2\nu}a_{1\mu}$ is not necessarily zero unless $a_{2\nu}$ is zero and $a_{2\nu} = 0$ for all $\lambda > 1$. Since 1 is not in the range for λ we have that $\sum_{\nu<\mu}|a_{1\nu}a_{2\mu} + a_{2\nu}a_{1\mu}| = 0$. Hence $Q_{12} = |a_3| + |a_4| + |a_5| + \ldots + |a_n|$.

Consider now $Q_{1\lambda}$ for $\lambda > 2$.

$$Q_{1\lambda} = \sum_{\nu}|a_{1\lambda}a_{\lambda\nu}| + \sum_{\nu}|a_{\lambda 1}a_{1\nu}| + \sum_{\nu}|a_{1\nu}a_{\lambda\nu}| +$$

$$\sum_{\nu<\mu}|a_{1\nu}a_{\lambda\mu} + a_{\lambda\nu}a_{1\mu}|, \qquad \text{where } \nu \neq 1,\lambda, \mu \neq 1,\lambda.$$

Now by examining the companion matrix we easily see that $\sum_{\nu}|a_{1\lambda}a_{\lambda\nu}| = |a_{\lambda}|$, while $\sum_{\nu}|a_{\lambda 1}a_{1\nu}| = 0$, and $\sum_{\nu}|a_{1\nu}a_{\lambda\nu}| = |a_{\lambda-1}|$. Moreover,

$$\begin{aligned}
\sum_{\nu<\mu}|a_{1\nu}a_{\lambda\mu} + a_{\lambda\nu}a_{1\mu}| = &\sum_{\nu<\mu_1}|a_{1\nu}a_{\lambda\mu_1} + a_{\lambda\nu}a_{1\mu_1}| + \\
&\sum_{\nu<\mu_2}|a_{1\nu}a_{\lambda\mu_2} + a_{\lambda\nu}a_{1\mu_2}| + \qquad\qquad (A)\\
&\cdots\cdots\cdots \\
&\qquad\qquad + \\
&\sum_{\nu<\lambda-1}|a_{1\nu}a_{\lambda,\lambda-1} + a_{\lambda\nu}a_{1,\lambda-1}| + \qquad\qquad (B)\\
&\sum_{\nu<\lambda+1}|a_{1\nu}a_{\lambda,\lambda+1} + a_{\lambda\nu}a_{1,\lambda+1}| + \\
&\sum_{\nu<\lambda+2}|a_{1\nu}a_{\lambda,\lambda+2} + a_{\lambda\nu}a_{1,\lambda+2}| + \qquad\qquad (C)\\
&\cdots\cdots\cdots \\
&\qquad\qquad + \\
&\sum_{\nu<n}|a_{1\nu}a_{\lambda n} + a_{\lambda\nu}a_{1n}|,
\end{aligned}$$

where $\mu_1, \mu_2 \ldots$ run through the admissible values for μ. Now each sum in (A) is zero, since $a_{\lambda\mu_i} = 0$ for $\mu_i < \lambda - 1$ and $a_{\lambda\nu} = 0$ for $\nu < \mu_i < \lambda - 1$. In the sum (B) we have $a_{\lambda,\lambda-1} = 1$ but $a_{\lambda\nu} = 0$ for all $\nu < \lambda - 1$ so that the sum (B) is $\sum_{i=2}^{\lambda-2}|a_i|$. In the sums in (C) we have $a_{\lambda,\lambda+j} = 0$ and $a_{\lambda\nu} = 0$ except for $\nu = \lambda - 1$, so that each sum collapses to a single term

$|a_{1,\lambda+j}|$. Hence the sum of all sums in (C) is $\sum_{i=\lambda+1}^{n} |a_i|$. Therefore

$$\sum_{\nu<\mu} |a_{1\nu}a_{\lambda\mu} + a_{\lambda\nu}a_{1\mu}| = \sum_{\substack{i=2 \\ i\neq\lambda-1,\lambda}}^{n} |a_i|.$$

Hence $Q_{1\lambda} = |a_2| + |a_3| + \ldots + |a_n|$.

We now calculate $Q_{\kappa\lambda}$ for $\kappa > 1$.

$$Q_{\kappa\lambda} = \sum_{\nu} |a_{\kappa\lambda}a_{\lambda\nu}| + \sum_{\nu} |a_{\lambda\kappa}a_{\kappa\nu}| + \sum_{\nu} |a_{\kappa\nu}a_{\lambda\nu}|$$

$$+ \sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu} + a_{\lambda\nu}a_{\kappa\mu}|$$

First consider the special case when $\lambda = 1$, and examine the sums one at a time.

$$\sum_{\nu} |a_{\kappa 1}a_{1\nu}| = |a_{\kappa 1}| \sum_{\nu} |a_{1\nu}|$$

$$= \begin{cases} \sum_{i=3}^{n} |a_i| & \text{for } \kappa=2 \\ 0 & \text{for } \kappa>2. \end{cases}$$

$$\sum_{\nu} |a_{1\kappa}a_{\kappa\nu}| = |a_{1\kappa}| \sum_{\nu} |a_{\kappa\nu}|$$

$$= \begin{cases} 0 & \text{for } \kappa=2 \\ |a_\kappa| & \text{for } \kappa>2. \end{cases}$$

$$\sum_{\nu} |a_{\kappa\nu}a_{1\nu}| = \begin{cases} 0 & \text{for } \kappa=2 \\ |a_{\kappa-1}| & \text{for } \kappa>2. \end{cases}$$

Moreover

$$\sum_{\nu<\mu} |a_{\kappa\nu}a_{\lambda\mu} + a_{\lambda\nu}a_{\kappa\mu}| = \begin{cases} 0 & \text{for } \kappa=2 \\ \sum_{\substack{i \\ i\neq 1,\kappa-1,\kappa}} |a_i| & \text{for } \kappa>2. \end{cases}$$

Hence $Q_{21} = \sum_{i=3}^{n} |a_i|$ and $Q_{\kappa 1} = \sum_{i=2}^{n} |a_i|$ for $\kappa > 2$.

Now we calculate $Q_{\kappa\lambda}$ for $\kappa, \lambda > 1$. We have

$$\sum_{\nu} |a_{\kappa\lambda} a_{\lambda\nu}| = \begin{cases} 0 & \text{for } \lambda = \kappa - 1 \\ 1 & \text{for } \lambda \neq \kappa - 1. \end{cases}$$

$$\sum_{\nu} |a_{\lambda\kappa} a_{\kappa\nu}| = \begin{cases} 1 & \text{for } \lambda = \kappa + 1 \\ 0 & \text{for } \lambda \neq \kappa + 1. \end{cases}$$

$$\sum_{\nu} |a_{\kappa\nu} a_{\lambda\nu}| = 0$$

$$\sum_{\nu < \mu} |a_{\kappa\nu} a_{\lambda\mu} + a_{\lambda\nu} a_{\kappa\mu}| = \begin{cases} 0 & \text{for } \lambda = \kappa - 1 \\ 0 & \text{for } \lambda = \kappa + 1 \\ 2 & \text{otherwise} \end{cases}$$

Hence for $\kappa, \lambda > 1$ we have

$$Q_{\kappa\lambda} = \begin{cases} 1 & \text{for } \lambda = \kappa - 1 \\ 1 & \text{for } \lambda = \kappa + 1 \\ 2 & \text{otherwise} \end{cases}$$

<u>SYNPOSIS</u>: The distinct ovals then are

(A) $|w(w+a_1) + a_2| \leqq |a_3| + |a_4| + \ldots + |a_n|$

(B) $|w(w+a_1)| \leqq |a_2| + |a_3| + \ldots + |a_n|$

(C) $|w|^2 \leqq Q_{\kappa\lambda}$,

where $Q_{21} = \sum_{i=3}^{n} |a_i|$, $Q_{\kappa 1} = \sum_{i=2}^{n} |a_i|$ for $\kappa > 2$, and for $\lambda > 1$ we have

$Q_{\kappa\lambda} = 1$ or 2 according as $\lambda = \kappa \pm 1$ or not.

In the contract proposal the sum $\sum_{i=2}^{n} |a_i|$ was denoted by S and there we outlined the earlier work of M. Parodi and of the principal investigator in dealing with the circle

$$|w| \leqq \sqrt{S}$$

and with the oval $|w(w+a_1)| \leqq S$. In the earlier work simple bounds were obtained by subjecting the companion matrix to certain similarity transformations. But in the present situation the oval (B) is contained in the oval (A), since

$$\sum_{i=3}^{n} |a_i| \geqq |w(w+a_1) + a_2| \geqq |w(w+a_1)| - |a_2| \rightarrow$$

$$|w(w+a_1)| \leqq \sum_{i=2}^{n} |a_i|,$$

and the earlier techniques seem not to yield fruitful results here. One may, however, ask under what circumstances the oval (A) may be used for localization of the zeros of f, and this is the subject of the following remarks.

Consider the Cassini oval

$$\Omega : \quad |w^2 + a_1 w + a_2| \leqq K$$

whose foci f_1 and f_2 are given by

$$f_1 = \tfrac{1}{2}\{-a_1 + \sqrt{a_1^2 - 4a_2}\} \qquad \text{and}$$

$$f_2 = \tfrac{1}{2}\{-a_1 - \sqrt{a_1^2 - 4a_2} \; .$$

Thus Ω is

$$|w - f_1||w - f_2| \leqq K.$$

Let us consider the restricted situation where the coefficients a_1 and a_2 are real and $a_1^2 \geqq 4a_2$. Now the oval splits into two branches provided the midpoint of the line segment joining the foci does not belong to Ω; that is, provided

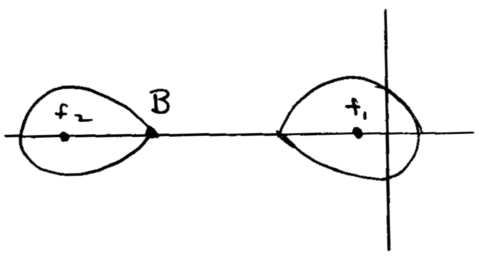

fig. 1a

$a_1>0$, $a_2>0$

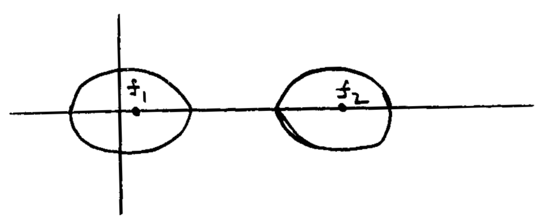

fig. 1b

$a_1<0$ $a_2>0$

$$\left|\frac{f_1+f_2}{2} - f_1\right|\left|\frac{f_1+f_2}{2} - f_2\right| > K \leftrightarrow$$

$$|f_1-f_2|^2 > 4K \leftrightarrow$$

$$\left|\frac{1}{2}\{-a_1 + \sqrt{a_1^2-4a_2}\} - \frac{1}{2}\{-a_1 - \sqrt{a_1^2-4a_2}\}\right|^2 > 4K \leftrightarrow$$

$$|a_1^2 - 4a_2| > 16K.$$

Moreover, if the oval Ω is disconnected, then the point B with minimum modulus on the branch containing f_2 in the case of fig. 1a (or containing f_1 in the case illustrated in Figure 1b) has modulus (see 2 or 3)

$$|B| = \frac{1}{2}\{|a_1| + \sqrt{a_1^2-4a_2-4K}\}$$

Hence $|B| \geqq 1$ if and only if

$$|a_1| + \sqrt{a_1^2-4a_2-4K} \geqq 2.$$

If we let S' denote the sum $|a_3| + |a_4| + \ldots + |a_n|$ and Ω_{f_2} denote the branch of Ω containing f_2 then we may formulate the following

Theorem 1. Suppose $f = z^n + a_1z^{n-1} + \ldots + a_n$ has the property that $a_1 > 0$, a_2 is real, $a_1^2 \geqq 4a_2$ and K is any real number such that

i) $a_1^2 - 4a_2 \geqq 16K \geqq 16S'$

ii) $|a_1| + \sqrt{a_1^2-4a_2-4K} \geqq 2.$

Then the oval Ω: $|w(w+a_1) + a_2| \leqq K$ is disconnected and Ω_{f_2} contains exactly one zero of f.

Proof: The preceding discussion and the hypothesis assure that for each w on the boundary of Ω_{f_2} we have $|w| \geqq 1$. The hypothesis guarantees $K > S'$, so that for each w on the boundary of Ω_{f_2}

$$|w^n + a_1 w^{n-1} + a_2 w^{n-2}| = |w^2 + a_1 w + a_2||w^{n-2}| \geq |w^2 + a_1 w + a_2||w^{n-3}|$$

$$\geq K\ |w^{n-3}| > S'\ |w^{n-3}| = \sum_{i=3}^{n} |a_i| w^{n-3}|$$

$$\geq \sum_{i=3}^{n} |a_i||w^{n-i}| \geq |\sum_{i=3}^{n} a_i w^{n-i}|.$$

Hence f and $w^n + a_1 w^{n-1} + a_2 w^{n-2}$ have the same number of zeros interior to Ω_{f_2} by the theorem of Rouché. But the midpoint of the line segment joining the foci is $-a_1/2$ so that in the case under consideration the origin is not in Ω_{f_2}. Hence f_2 is the only zero of $w^n + a_1 w^{n-2}$ in Ω_{f_2} and the theorem follows.

A similar theorem holds in the case where $a_1 < 0$ (see fig. 1b) for the branch Ω_{f_1} which contains f_1.

We continue to assume $a_1 > 0$ and consider the situation for $a_2 > 0$. Suppose that the oval Ω is disconnected and that the branch Ω_{f_1} containing f_1 also contains the origin as in figure 2.

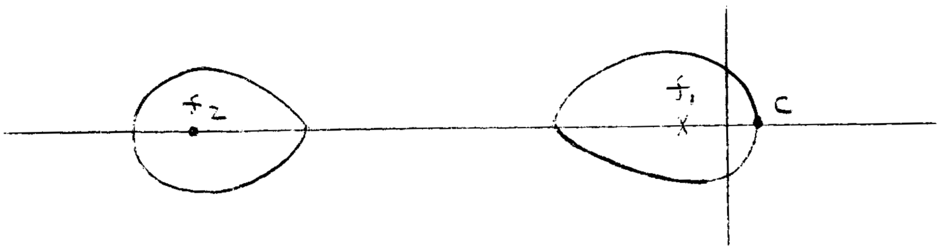

fig. 2

It is known (see [2] or [3]) that the circle with center f_1 and radius $|f_1 - c|$ is contained in Ω_{f_1}; hence if $|c| \geq |$ then the unit circle also lies in Ω_{f_1}.

Now

$$|c| = |f_1 - c| - |f_1| = \frac{1}{2}\{\sqrt{a_1^2-4a_2+4k} - \sqrt{a_1^2-4a_2}\} - |\tfrac{1}{2}\{-a_1 + \sqrt{a_1^2-4a_2}\}|$$

$$= \frac{1}{2}\{\sqrt{a_1^2-4a_2+4k} - a_1\},$$

so that $|c| \geqq 1$ if and only if $\sqrt{a_1^2-4a_2+4k} - a_1 \geqq 2$; that is, if and only if $K \geqq a_1 + a_2 + 1$.

We are now able to formulate the following

Theorem 2. Suppose that $a_1, a_2 > 0$, that $a_1^2 - 4a_2 \geqq 0$ and that K is any number such that

i) $a_1^2 - 4a_2 > 16K \geqq 16S'$

ii) $K \geqq a_1 + a_2 + 1$.

Then the oval Ω is disconnected and the branch Ω_{f_1} contains exactly n - 1 zeros of $f = z^n + a_1 z^{n-1} + \ldots + a_n$.

Proof: The hypothesis assures that Ω is disconnected, that the branch Ω_{f_1} contains the unit circle, and that f_1 and C are separated by the origin on the real axis. Finally for each w on the boundary of Ω_{f_1}

$$|w^n + a_1 w^{n-1} + a_2 w^{n-2}| = |w^2 + a_1 w + a_2||w^{n-2}|$$

$$\geqq |w^2 + a_1 w + a_2||w^{n-3}| = K\,|w^{n-3}|$$

$$\geqq S'|w^{n-3}| \geqq \sum_{i=3}^{n} |a_i||w^{n-1}|$$

$$\geqq |\sum_{i=3}^{n} a_i w^{n-i}|.$$

Hence $w^n + a_1 w^{n-1} + a_2 w^{n-2}$ and f have, by Rouché's theorem the same number of zeros interior to Ω_{f_1}, namely n - 1 zeros.

Again, an analogous result is available to us in the case that $a_1 < 0$ and $a_2 > 0$ (see fig. 1b).

In the following we let β' denote the maximum of $|a_3|, |a_4|, \ldots, |a_n|$.

Theorem 3. Suppose that $a_1, a_2 > 0$ and $a_1^2 \geq 4a_2$ and further that K is any number such that

$$\text{i)} \qquad a_1^2 > 16K$$

$$\text{ii)} \qquad a_1 + \sqrt{a_1^2 - 4a_2 - 4K} > 2$$

$$\text{iii)} \qquad \frac{\beta'}{\frac{1}{2}\{a_1^2 + \sqrt{a_1^2 - 4a_2 - 4K}\} - 1} < K.$$

Then the Cassini oval

$$\Omega: \quad |w - f_1|\,|w - f_2| \leqq K$$

is disconnected and the branch Ω_{f_2} containing f_2 contains exactly one zero of $f = z^n + a_1 z^{n-1} + \ldots + a_n$.

Proof: As in Theorem 1, the hypothesis (i) is sufficient for the oval to be disconnected; (ii) assures, as in figure 1a, that the minimum modulus on the boundary of Ω_{f_2} is greater than 1, so that with (iii) we are assured that for each w on the boundary of Ω_{f_2}

$$\frac{\beta'}{|w| - 1} < K.$$

Thus for each w on the boundary of Ω_{f_2}, we have

$$\left|\sum_{i=3}^{n} a_i w^{n-i}\right| \leqq \sum_{i=3}^{n} |a_i|\,|w|^{n-i} \leqq \beta' \sum_{i=3}^{n} |w|^{n-i}$$

$$= \frac{\beta'}{|w| - 1}\left(|w|^{n-2} - 1\right) < |w^2 + a_1 w + a_2|\,|w|^{n-2}$$

$$= |w^n + a_1 w^{n-1} + a_2 w^{n-2}|,$$

and the theorem follows from Rouché's theorem.

Theorem 4. Suppose that $\alpha_1, \alpha_2 > 0$ and $a_1^2 \geqq 4a_2$ and further that K is any number such that

i) $a_1^2 > 16K$

ii) $K > a_1 + a_2 + 1$

iii) $$\frac{\beta'}{\frac{1}{2}\{a_1 + \sqrt{a_1^2-4a_2-4K}\} - 1} < K.$$

Then the oval Ω is disconnected and the branch Ω_{f_1} contains exactly n - 1 zeros of f.

Proof: The hypothesis (i) assures us that Ω is disconnected; (ii) assures us (see Theorem 2) that the unit circle lies inside Ω_{f_1} and (iii) assures us that the inequality appearing in the proof of Theorem 3 is valid for each w on the boundary of Ω_{f_1}. Hence the theorem follows from Rouché's theorem.

Literature Cited

1. Brauer, A. "Uber die Lage der charakteristischen Wurzeln einer Matrix," J. fur die Reine und Angew. Math., Band 192, Heft 2, pp. 113-116 (1953).

2. Ganguli, S. The Theory of Higher Plane Curves, University of Calcutta, 1926.

3. Painter, R. Extensions of Theorems of Ostrowski on the Zeros of Certain Classes of Polynomials, doctoral dissertation, Univ. of North Carolina, Chapel Hill, N. C., 1963.