GENERALIZED RATIONAL APPROXIMATIONS WITH APPLICATIONS TO PROBLEMS IN CONTROL THEORY

INTERIM TECHNICAL REPORT
14 August 1967

Contract No. NAS8-20403

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For

National Aeronautics and Space Administration
George C. Marshall Space Flight Center
Huntsville, Alabama 35812

Attn: Mr. Mario Rheinfurth, R-AERO-D

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PREFACE

This report covers research initiated by National Aeronautics and Space Administration, George C. Marshall Space Flight Center, on Contract No. NAS8-20403, "Analysis of the Coupled Frequencies and Mode Shapes of Vehicle Control Systems." The research work upon which this report is based was accomplished at Midwest Research Institute with Mr. Mario Rheinfurth as project monitor.

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by

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I. INTRODUCTION

In [1] Kalman shows that the solution of many important problems in control theory, for example the problem of prediction theory and that of perturbation guidance, depends directly upon the solution of the matrix Ricatti equation. Although the solution to this equation can be obtained by getting the solution to an associated system of linear differential equations (whose order is twice the order of the Ricatti system), this is of little help since one must resort to numerical integration (except in the constant coefficient case). Thus there are no simple direct methods of gaining information about the solution to the matrix Ricatti equation, except possibly numerical integration. As is well known, the latter technique is ineffective if there are singularities in the range of interest, and singularities are commonplace in the case of nonlinear equations.

We have developed a new technique for obtaining formal rational approximations to the solution of the matrix Ricatti equation that should be a valuable tool for describing the behavior of the solution both locally and globally. Note that the matrix Ricatti equation includes systems of linear differential equations as a special case.

In our previous work [2,3], we indicated the versatility and power of rational approximations when applied to solutions of some important scalar nonlinear differential equations. The results obtained thus far show that the formal rational approximations are as effective as those in the scalar case. It calls for a remark that formal rational approximations have already been employed in the integration of large systems of linear differential equations, see the paper by Legras [4]. Thus the search for rational approximations to solutions of nonlinear systems is a natural course of investigation to follow. Preliminary work in this direction has been initiated and the results are included in this report.

Section II contains the development of the rational approximations to the solution of the matrix Ricatti equation. The problem of convergence of these approximations is discussed in Section III. The examples of Section IV illustrate the wide applicability and utility of the approximations. Section V completes the report with a discussion of the scope of the technique and indicates some important areas of future research.

II. DEVELOPMENT OF RATIONAL APPROXIMATIONS

In this section, all matrices considered are $m \times m$ and all the required inverses are assumed to exist. By means of a sequence of linear fractional transformations we obtain the formal continued fraction representation of the solution to the matrix Ricatti equation. Truncation of this continued fraction yields a rational approximation to the solution.

Consider the equation

$$(-1)^n tW_n' + A_n W_n + W_n B_n + W_n C_n W_n + t^{r_n} D_n = 0 \ , \ W_n(0) = 0 \ , \qquad (2.1)$$

where the coefficients in (2.1) are power series in the variable t. Also r_n is a positive integer. A preliminary transformation may be needed to cast a given equation into the form (2.1). We set

$$W_n = t^{r_n}(I + W_{n+1})^{-1}E_n \ , \qquad (2.2)$$

and put (2.2) into (2.1) and require that

$$(-1)^n r_n I + A_n(0) + E_n B_n(0) E_n^{-1} + D_n(0) E_n^{-1} = 0 \ , \qquad (2.3)$$

in which case we get (after dividing the resulting equation by t^{r_n}) Eq. (2.1) with all the subscripts increased by unity. The new coefficients are

$$A_{n+1} = (-1)^n r_n I + E_n B_n E_n^{-1} + D_n E_n^{-1} \ ,$$

$$B_{n+1} = A_n + D_n E_n^{-1} \ ,$$

$$C_{n+1} = D_n E_n^{-1} \ ,$$

and

$$t^{r_{n+1}} D_{n+1} = t^{r_n} E_n C_n + F_n \ ,$$

where

$$F_n = A_n + E_n B_n E_n^{-1} + D_n E_n^{-1} + (-1)^n r_n I \ , \ F_n(0) = 0 \ . \tag{2.4}$$

Thus, if $W = W_o$, we have

$$W = \cfrac{t^{r_o} E_o}{I + \cfrac{t^{r_1} E_1}{I + \ddots}} \ . \tag{2.5}$$

The n^{th} approximant of the continued fraction furnishes a rational approximation to W ,

$$W_n^* = Q_n^{-1} P_n \ , \tag{2.6}$$

where P_n and Q_n satisfy the recurrence relations

$$Q_{n+1} = Q_n + t^{r_{n+1}} E_{n+1} Q_{n-1} \ ,$$

$$P_o = t^{r_o} E_o \ , \ Q_o = I \ , \ P_1 = t^{r_o} E_o \ , \text{ and } \ Q_1 = I + t^{r_1} E_1 \ . \tag{2.7}$$

III. CONVERGENCE THEORY

In order to discuss the convergence of the formal continued fraction (2.4) we must define a suitable norm for matrices. For the present, if A is an $m \times m$ real matrix and x is an $m \times 1$ real vector, we define the norm of A by

$$||A|| = \sup_{||x|| = 1} \left\{ ||Ax|| \right\} \ ,$$

where

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2} \quad . \tag{3.1}$$

Thus, the following properties obtain,

$$\|A+B\| \leq \|A\| + \|B\| \quad ,$$

$$\|cA\| = |c| \, \|A\| \quad ,$$

and

$$\|AB\| \leq \|A\| \, \|B\| \quad , \tag{3.2}$$

where c is a scalar and A and B are $m \times m$ real matrices. We say that the sequence of $m \times m$ matrices $\{A_k\}$ converges, if there is an $m \times m$ matrix A such that

$$\lim_{k \to \infty} \|A_k - A\| = 0 \quad . \tag{3.3}$$

Also, the continued fraction (2.4) is said to converge if the sequence of approximants $\left\{ Q_k^{-1} P_k \right\}$ converges.

At this point in the development of formal continued fractions, little is known about convergence. There is, however, a criterion for convergence stated without proof in a paper by Wynn [5]. It reads as follows:

Theorem: If $\sum_{n=0}^{\infty} \|A_n\| < 1$, then the continued fraction

$$\cfrac{A_0}{I + \cfrac{A_1}{I + \ddots}} \tag{3.4}$$

converges. Here, the A_i are $m \times m$ real matrices and I is the $m \times m$ unit matrix.

Much work needs to be done in the theory of convergence of formal continued fractions and this will be part of our future investigations.

IV. EXAMPLES AND APPLICATIONS

Here we apply the results of the previous development to four examples which illustrate the utility and diverse applications of generalized rational approximations.

Example 1: We obtain the solution to the "variance equation" associated with the following problem from linear filtering and prediction theory, see [1]. A particle leaves the origin at time $t_o = 0$ with a fixed but unknown velocity of zero mean and known variance. The position of the particle is continually observed in the presence of additive white noise. The best estimator of position and velocity is desired. The associated variance equation is

$$P' - FP - PF^T + PGP = 0 ,$$

$$P = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} , \; F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \; G = \begin{bmatrix} r & 0 \\ 0 & 0 \end{bmatrix} ,$$

$$p_1(0) = p_2(0) = 0 \; , \; p_3(0) = q > 0 \; , \; r > 0 \; , \qquad (4.1)$$

and F^T denotes the transpose of the matrix F. It is convenient to introduce the transformations

$$p_1 = t^2 v_1 \; , \; p_2 = tv_2 \; , \; p_3 = q + v_3 \text{ and } x = t^3 \; , \qquad (4.2)$$

so that (4.1) becomes

$$3xV' - AV - VA^T + rxVBV + C = 0 ,$$

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & -q \\ -q & 0 \end{bmatrix} \text{ and } V(0) = \begin{bmatrix} q & q \\ q & 0 \end{bmatrix}. \tag{4.3}$$

Now $V(0) \neq 0$, so our first transformation is

$$V = (I + V_1)^{-1}E_1 \; , \; V_1(0) = 0 \; . \tag{4.4}$$

Proceeding as in Section II, we find that the first approximation to V is

$$V_1^* = \frac{3q}{3+rqx} \begin{bmatrix} 1 & 1 \\ 1 & -rqx \end{bmatrix} . \tag{4.5}$$

The corresponding first approximation to P is

$$P_1^* = \frac{3q}{3+rqt^3} \begin{bmatrix} t^2 & t \\ t & 1 \end{bmatrix} , \tag{4.6}$$

which is exact. Thus it appears that if the solution to the matrix Ricatti equation is a rational function, then it is possible to get the exact solution.

Example 2: The following matrix Riccati equation arises in the investigation of the oscillatory behavior of solutions to a fourth-order (linear) boundary value problem, see [6].

$$Z' - ZCZ + B = 0 \; , \; Z(0) = 0 \; ,$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & q \end{bmatrix}, \; C = \begin{bmatrix} p & 0 \\ 0 & r \end{bmatrix}, \tag{4.7}$$

where p, q and r are continuous non-negative functions. For our example, we consider them to be constant. Applying the sequence of transformations of Section II, we obtain the expansion

$$Z = \cfrac{-Bt}{I + \cfrac{t^2CB/1\cdot 3}{I + \cfrac{t^2BC/3\cdot 5}{I + \cfrac{t^2CB/5\cdot 7}{I + \ddots}}}} \quad . \tag{4.8}$$

For t small, (4.8) satisfies the hypothesis of the theorem of Section III.

Example 3: It is of interest to examine (2.1) in case W is a diagonal matrix. Consider the equation

$$xW' + CW - xW^2 - xI = 0 \ , \ W(0) = 0 \ ,$$

$$W = \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} . \tag{4.9}$$

Then $w_1 = \tan x$ and $w_2 = J_1(x)/J_0(x)$. The results of Sections II and III yield

$$W = \cfrac{xE_0}{I - \cfrac{x^2E_1}{I - \cfrac{x^2E_2}{I - \ddots}}} \ ,$$

$$E_n = \begin{bmatrix} \dfrac{1}{(2n+1)(2n-1)} & 0 \\ 0 & \dfrac{1}{4(n+1)(n+2)} \end{bmatrix}, \ E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} . \tag{4.10}$$

Again, we can show that (4.10) converges for restricted x. We can also show that the $n^{\underline{th}}$ approximant of the formal continued fraction (4.10) is just the matrix whose diagonal elements are, respectively, the $n^{\underline{th}}$ approximants of the usual continued fraction representations of $\tan x$ and $J_1(x)/J_0(x)$. Consequently we have convergence in the finite complex plane, except at the poles of the functions.

Example 4: The formal continued fraction for the square root of a matrix is employed here to obtain $A^{\frac{1}{2}}$ where

$$A = \begin{bmatrix} 18 & 9 \\ 18 & 27 \end{bmatrix} . \tag{4.11}$$

Truncation of the continued fraction to obtain the second order main diagonal Padé approximation yields

$$A^{\frac{1}{2}} \cong \begin{bmatrix} 4.006 & 0.997 \\ 1.995 & 5.003 \end{bmatrix} , \tag{4.12}$$

which compares favorably with

$$A^{\frac{1}{2}} = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix} . \tag{4.13}$$

Although the square root of a matrix can be gotten in closed form, this example is instructive in two ways. First it affords a single example for investigation of convergence and second, the computation via this technique can be much faster than obtaining the closed form expression.

V. CONCLUSIONS AND RECOMMENDATIONS

The results of this report show that formal rational approximations can be successfully applied in a great number of diverse problems. In particular they provide a convenient algorithm for analyzing solutions of the matrix Riccati equation which arises in many areas of control theory.

Although the theory of formal rational approximations is still in the infant stage, the approximations are extremely useful. The value of the approximations would be greatly enhanced by the development of convergence criteria. The areas below will be investigated in the future.

1. Development of general convergence theory of formal continued fractions.

2. Application of the sequence of linear fractional transformations to the matrix Riccati equation to obtain approximate solutions.

3. Utilization of the approximations in the analysis of important problems in control theory including numerical results and their physical interpretation.

REFERENCES

1. Kalman, R. E., and R. S. Bucy, "New Results in Linear Filtering and Prediction Theory," Journal of Basic Engineering, Trans. ASME, Vol. 93D (1961), pp. 95-108.

2. Fair, W. "Padé Approximation to the Solution of the Riccati Equation," Math. Comp., Vol. 18 (1964), pp. 627-634.

3. Fair, W., and Y. L. Luke, "Rational Approximations to the Solution of The Second Order Riccati Equation," Math. Comp., Vol. 20 (1966), pp. 602-606.

4. Legras, J., "Résolution Numérique des Grands Systèmes Différentials Linéaires," Numerische Math, Vol. 8 (1966), pp. 14-28.

5. Wynn, P., "On Some Recent Developments in the Theory and Application of Continued Fractions," SIAM, J. Numer. Anal., Series B, Vol. 1 (1964) pp. 177-197.

6. Atkinson, F. V., Discrete and Continuous Boundary Problems, Academic Press, (1964).