WHIRLING OF THE SINGLE MASS ROTOR

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ABSTRACT

The general equations of motion for a single mass, unbalanced rotor on a massless elastic shaft with damping are presented and analyzed for various conditions of synchronous and nonsynchronous precession or whirling. The analysis shows that the introduction of the external damping considerably influences the rotor characteristics and yields results different from those previously reported for the case of no damping. External damping in general, suppresses certain motions and permits only forward synchronous precession. With light rotor damping, a whirl ratio of 1/ 3 is predicted when the rotor is operating at three times the rotor critical speed.

BACKGROUND AND INTRODUCTION

The first recorded article on "whirling" or precessing of a shaft was presented in 1869 by Rankine (1) who introduced the concept of indifferent rotor equilibrium. Because he neglected the influence of the Coriolis force he concluded that; motion is stable below the first critical speed, is neutral or in "indifferent" equilibrium at the critical speed, and unstable above the critical speed. The neglect of the Coriolis term has caused several writers to deduce a fictitious critical condition at $1/\sqrt{2}$ times the critical speed.

During the next half century, this analysis led engineers to believe that operation above the first critical speed was impossible. It was not until 1895 that De Laval demonstrated experimentally that a steam turbine was capable of sustained operation above the first critical speed.

Although both Dunkerly (2) in 1894 and Chree (3) in 1904 did extensive studies on the natural lateral vibrations of shafts, it was not until 1919 that H. H. Jeffcott (4) explained the motion of the single mass rotor (see Fig. 1). Jeffcott demonstrated that a rotor could operate at the critical speed if sufficient damping on the rotor is present. Assuming that the angular velocity of the rotor is constant, Jeffcott arrived at the conclusion that the center of the rotor revolves or precesses at the same angular velocity as the disc. This condition of the rotor centerline moving with the same angular velocity as the

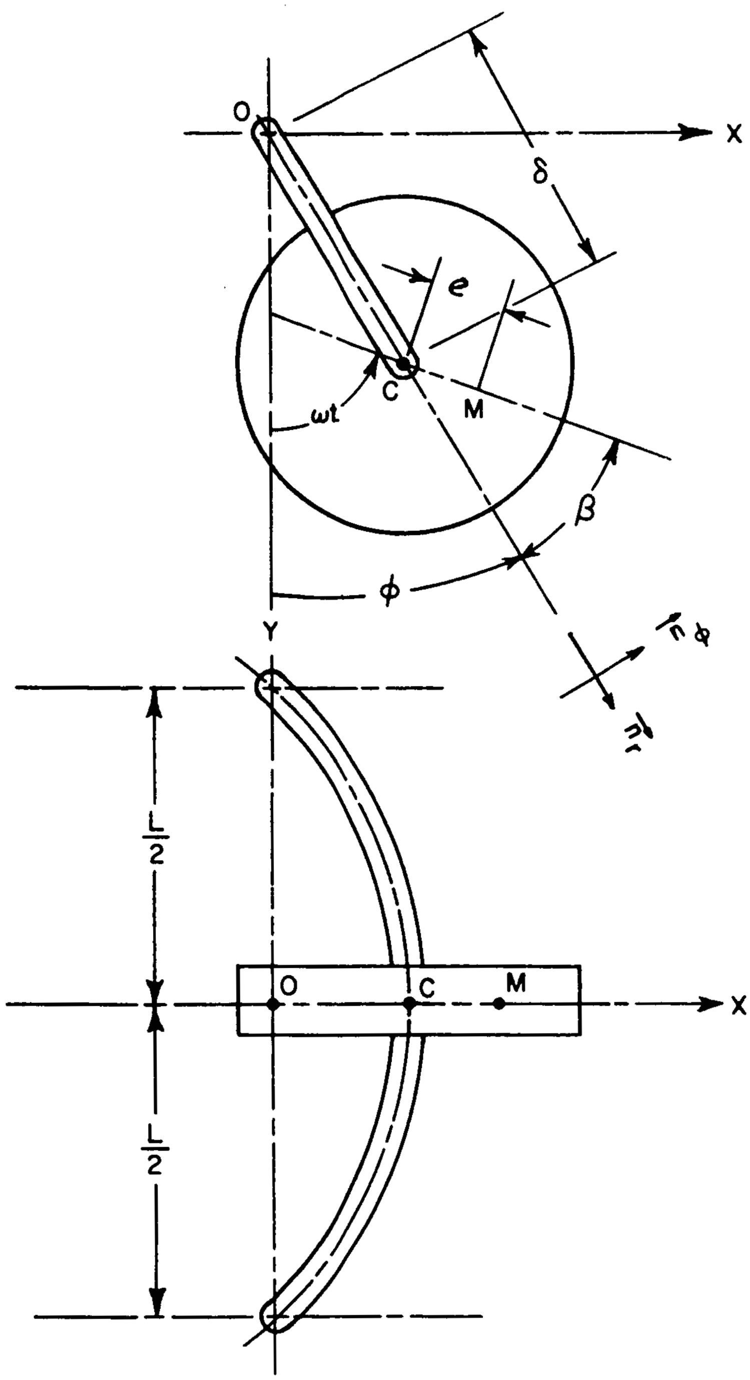

FIGURE I

SINGLE MASS FLEXIBLE ROTOR

mass center is defined as synchronous precession.

Newkirk (5, 6), in 1924, discovered several instances of rotor whirling or nonsynchronous precession in which the plane of the bent shaft rotated at the shaft critical speed while the shaft itself rotates at a higher speed. The cause of this whirl motion was later identified to be due to influences such as internal rotor friction and fluid film bearings, and represents a self-excited vibration (7, 8). Works of Hagg (9), Hori (10), Poritsky (11) and others furnish considerable insight into this phenomenon.

Robertson (12) in 1935 conducted an experimental and theoretical investigation on the transient whirling of the single mass Jeffcott rotor. Robertson observed that the rotor elastic centerline could possess both forward and backward precessive motion depending upon the initial conditions. The influence of external damping causes the transient motion to die out until only the steady-state synchronous component caused by unbalance remained. He observed that only in the case where the deflection of the rotor was sufficient to cause it to strike the guard ring was it possible to develop a sustained transient motion.

Although there is considerable material in the literature on rotor dynamics and whirling, there is still a lack of understanding of the whirl behavior of even fundamental systems such as the Jeffcott model. Kane (13), in his recent paper on rotor whirling, has attempted to analyze the various whirl motions possible with the single mass Jeffcott model. Since Kane assumed a conservative system, he predicted various modes of forward and backward precessive motion which are not generally observed in practice.

In this analysis, the influence of external damping and gravity is considered on the general whirling motion of the shaft. The results show that there is a considerable difference in the behavior of the conservative and dissipative systems.

Equations of Motion

Fig. 1 describes the system under consideration in which the Z-axis in a conventional right-handed coordinate system coincides with the axis of the rotor in the undeformed or undeflected position (i. e., position of axis with no dynamic or gravity forces acting) "C" describes the geometric center of the rotor; "M", the mass center of the rotor, the two being separated by a distance, "e", the eccentricity. It is assumed for the system that the coordinates X, Y, Z, are fixed in space. The angular position of the rotor is given by two quantities, the precession angle, ϕ, and the phase angle, β. The total angular velocity of the system, then, is given by $\omega = \dot{\beta} + \dot{\phi}$, in which the dot over a quantity indicates the derivative of the quantity with respect to time. In Fig. 1, the direction of deformation of the shaft is given by the line OC, noted by $\vec{n}_r$ ($\vec{n}_r$ and $\vec{n}_\phi$ are orthogonal unit vectors moving in space).

The system under consideration possesses three degrees of freedom. The two sets of generalized coordinates which may be employed to describe this system are:

(a) δ - deflection of rotor center from origin 0.

ϕ - precession angle

β - phase angle

or

(b) X, Y - Cartesian coordinates of the displaced rotor center

β - phase angle

In this analysis, the first coordinate system will be used. Jeffcott (4) in his analysis used the Cartesian coordinate representation, which does not readily allow evaluation of whirling in general. Since the system has three degrees of freedom, there will be three equations of motion; one for each generalized coordinate.

Lagrange's Equations of Motion

The three required equations of motion will be derived by means of Lagranges Equations which state:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) - \frac{\partial L}{\partial q_r} = F_{qr} \qquad [1]$$

Where:

L = Lagrangian = T-V

F_{qr} = Generalized force for the q_r coordinate.

Kinetic Energy

The kinetic energy of the system is given by

$$T = 1/2\, MV^2_m + 1/2\, \omega_i \omega_j \phi_{ij} .$$

A position vector to the rotor mass center M from the fixed point 0 is given by:

$$\vec{P}^{M/0} = [\delta + e \cos\beta]\, \vec{n}_r + e \sin\beta\, \vec{n}_\phi . \qquad [2]$$

The velocity of the mass center M is given by:

$${}^R\vec{V}^{M/0} = \frac{{}^R d}{dt} [\vec{P}^{M/0}] = \frac{{}^{R'} d}{dt} [\vec{P}^{M/0}] + {}^R\vec{\omega}^{R'} \times \vec{P}^{M/0} . \qquad [3]$$

Expanding Eq. 3 results in

$${}^R\vec{V}^{M/0} = [\dot{\delta} - e \sin\beta\, \omega]\, \vec{n}_R + [\delta \dot{\phi} + e \cos\beta\, \omega]\, \vec{n}_\phi . \qquad [4]$$

If the disc is constrained to move in the X-Y plane only (no gyroscopic forces), then the total kinetic energy of the system is given by

$$T = \frac{1}{2} m [{}^R\vec{V}^{M/0} \cdot {}^R\vec{V}^{M/0}] + \frac{1}{2} \omega \cdot \omega I$$

$$= \frac{1}{2} \{m[\dot{\delta} - e\omega \sin\beta]^2 + m[\delta \dot{\phi} + e\omega \cos\beta]^2 + I \omega^2\} \qquad [5]$$

Potential Energy - V

The potential energy of the rotor is composed of the strain energy of deformation of the rotor and the vertical position of the rotor mass center.

$$V = \frac{1}{2} K \delta^2 + mgh \qquad [6]$$

where

K = rotor stiffness coefficient

$$h = \vec{P}^{M/0} \cdot \vec{n}_y = \delta \sin\phi + e \sin(\beta + \phi)$$

$$\therefore \quad V = \frac{1}{2} K \delta^2 + mg[\delta \sin\phi + e \sin(\beta + \phi)] \; . \qquad [7]$$

Generalized Forces

The external forces and torques acting on the system which have not been taken into consideration is the rotor damping force acting at C and the rotor drive torque T. The damping force acting at C is given by:

$$\vec{F}^{C}_{ext.} = -[C \dot{\delta} \vec{n}_r + C \dot{\delta} \phi \vec{n}_\phi] \qquad [8]$$

The generalized forces for each of the coordinates are given by

$$F_{qr} = \sum_{i=1}^{N} \vec{F}^{i}_{ext.} \cdot \frac{\partial \vec{V}}{\partial \dot{q}_r} \; ; \quad \vec{V}^{C} = \dot{\delta} \vec{n}_r + \dot{\delta} \phi \vec{n}_\phi \qquad [9]$$

$$\text{(a)}\quad \delta_j;\ F_\delta = -[C\dot{\delta}\vec{n}_r + C\delta\dot{\phi}\vec{n}_\phi]\cdot\vec{n}_R = -C\dot{\delta}$$

$$\text{(b)}\quad \beta;\ F_\beta = -\vec{T}\cdot\frac{\partial\vec{\omega}}{\partial\beta} = -T$$

$$\text{(c)}\quad \phi;\ F_\phi = -[C\dot{\delta}\vec{n}_r + C\delta\dot{\phi}\vec{n}_\phi]\cdot\delta\vec{n}_\phi - \vec{T}\cdot\frac{\partial\vec{\omega}}{\partial\phi} \qquad [10]$$

$$= -C\delta^2\dot{\phi} - T\ .$$

The Lagrangian of the system is given by

$$L_{1,2} = \frac{m}{2}\{[\dot{\delta} - e\,\omega\sin\beta]^2 + [\delta\dot{\phi} + e\,\omega\cos\beta]^2 + k^2\omega^2 +$$

$$-2g\,(\delta\sin\phi + e\sin[\beta+\phi])\} - \frac{1}{2}K\delta^2\ . \qquad [11]$$

The equations of motion of the system are thus given by the following:

$$\text{(a)}\quad \delta;\ \frac{d}{dt}\{m[\dot{\delta} - e\sin\beta\omega]\} + m[g\sin\phi - \delta\dot{\phi}^2 - e\omega\cos\beta] +$$

$$+ K\delta = -C\dot{\delta}$$

$$[12]$$

$$\text{(b)}\quad \beta;\ \frac{d}{dt}\{me[\omega e[1+(\frac{k}{e})^2] - \dot{\delta}\sin\beta + \delta\dot{\phi}\cos\beta]\} +$$

$$me[\omega(\dot{\delta}\cos\beta + \delta\dot{\phi}\sin\beta) + g\cos(\beta+\phi)] = -T$$

$$(c)\quad \phi;\ m\,\frac{d}{dt}\left[\delta^2\dot{\phi} - e\dot{\delta}\ \sin\beta + e^2\omega + e\,\delta\ \cos\beta\,(\omega+\dot{\phi}) + K^2\omega\right] +$$

$$mg\left[\delta\cos\phi + e\cos(\beta+\phi)\right] = -\ C\,\delta^2\,\dot{\phi} - T$$

If the total angular velocity ω of the system is assumed to be constant

$$\omega = \dot{\phi} + \dot{\beta} = \text{constant}$$

$$\dot{\omega} = 0\ ;\ \text{and}\ \ddot{\beta} = -\ \ddot{\phi}\ .$$

Hence, the equations of motion reduce to

$$(a)\quad \delta;\ \ddot{\delta} + \frac{C}{m}\dot{\delta} + (\omega^2_{cr} - \phi^2)\,\delta = e\,\omega^2\cos\beta - g\sin\phi$$

$$(b)\quad \beta;\ e\left\{\left[\delta\ddot{\phi} + 2\,\dot{\delta}\,\dot{\phi}\right]\ \cos\beta - \left[\ddot{\delta} - \delta\,\dot{\phi}^2\right]\ \sin\beta\right\}$$

$$= -\ eg\ \cos\,(\beta+\phi) - \frac{T}{m} \qquad [13]$$

$$(c)\quad \phi;\ \delta^2\ddot{\phi} + \left[\frac{C}{m}\,\delta + 2\,\dot{\delta}\right]\delta\,\dot{\phi} + e\left[(\delta\ddot{\phi} + 2\,\dot{\delta}\,\dot{\phi})\cos\beta\ +\right.$$

$$\left. -\ \sin\beta\ (\ddot{\delta} + \delta[\omega^2 - \dot{\phi}^2])\right] = -\ g\left[\delta\cos\phi + e\cos(\beta+\phi)\right] - \frac{T}{m}$$

The torque T will be eliminated between [12b and c] to yield the system:

$$(a)\quad \ddot{\delta} + K_s\,\dot{\delta} + (\omega^2_{cr} - \dot{\phi}^2)\,\delta = e\omega^2\cos\beta - g\sin\phi$$

$$(b)\quad \delta\,\ddot{\phi} + (K_s\,\delta + 2\,\delta)\,\phi = e\omega^2\sin\beta - g\cos\phi\ . \qquad [14]$$

ANALYSIS OF ROTOR MOTION

Specific cases for the governing equations of motion will be considered:

<u>Case I - Synchronous Precession</u>

Synchronous precession implies that the precession rate ϕ of the rotor is equal to the total angular velocity of the system ω. The equations of motion are:

$$\text{(a)} \quad \ddot{\delta} + K_s \dot{\delta} + (\omega^2_{cr} - \omega^2)\,\delta = e\omega^2 \cos\beta - g \sin\phi \qquad [15]$$

$$\text{(b)} \quad \delta\,\ddot{\phi} + (K_s\,\delta + 2\delta\,)\,\phi = e\omega^2 \sin\beta - g \cos\phi\,. \qquad [16]$$

Assume a condition of steady-state whirling of a vertical rotor. This condition implies:

δ	=	constant	$\dot{\phi} = \omega$ = constant
β	=	constant	$g = 0$.

The governing equations reduce to:

$$\text{(a)} \quad (\omega^2_{CR} - \omega^2)\,\delta = me\,\omega^2 \cos\beta$$

$$[17]$$

$$\text{(b)} \quad K_s\,\delta\,\omega = e\omega^2 \sin\beta\,.$$

Solving for the phase angle β:

$$\tan \beta = \frac{K_s \omega}{\omega_{cr}^2 - \omega^2} \qquad [18]$$

Solving for the rotor deflection δ :

$$\delta = \frac{e\omega^2 \sin\beta}{K_s \omega} = \frac{e}{\sqrt{\left\{\left(\frac{\omega_{cr}}{\omega}\right)^2 - 1\right\}^2 + \left(\frac{K_s}{\omega}\right)^2}} \qquad [19]$$

The force transmitted to each bearing is given by:

$$F = \left(\frac{1}{2} Ke\right) A$$

where A = amplitude factor $= \dfrac{1}{\sqrt{\left\{\left(\frac{\omega_{cr}}{\omega}\right)^2 - 1\right\}^2 + \left(\frac{K_s}{\omega}\right)^2}}$ [20]

The amplification factor A is in agreement with the results obtained by Jeffcott. Fig. 2 represents a plot of the rotor amplification factor for various values of damping.

The critical speed is defined as the speed at which

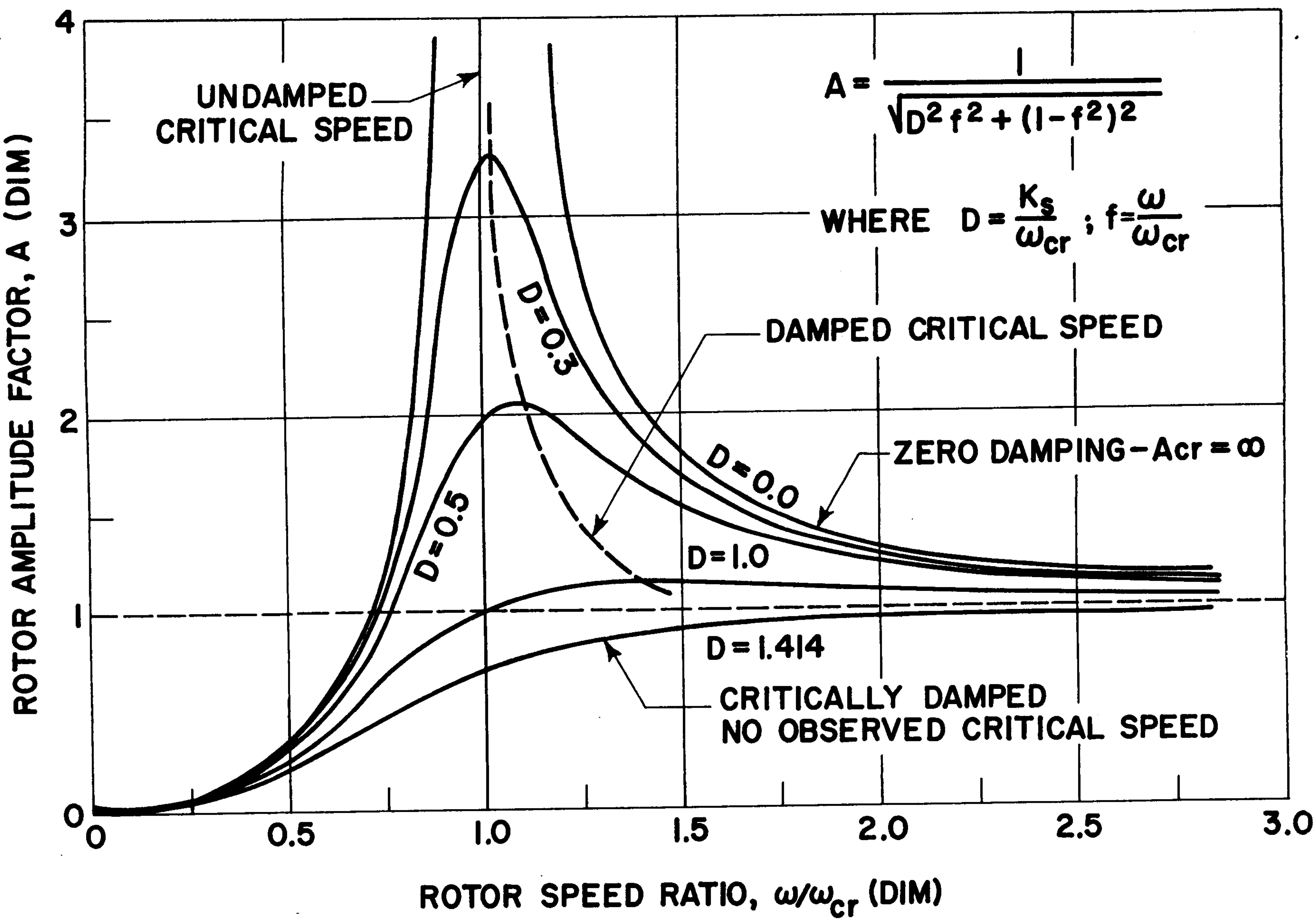

FIGURE 2

ROTOR AMPLITUDE RESPONSE FOR VARIOUS DAMPING VALUES

$$\frac{dF}{d\omega} = 0 = \frac{Ke}{2}\left\{\frac{2\omega}{\sqrt{K_s^2\,\omega^2 + (\omega_{cr}^2 - \omega^2)^2}} - \frac{\omega^2[2\omega K_s^2 - 4\omega\,(\omega_{cr}^2 - \omega^2)]}{2[K_s^2\,\omega^2 + (\omega_{cr}^2 - \omega^2)^2]^{3/2}}\right\}$$

$$= \frac{Ke}{2[K_s^2\,\omega^2 + (\omega_{cr}^2 - \omega^2)^2]^{3/2}}\{K_s\omega^2 + 2(\omega_{cr}^2 - \omega^2)\,\omega_{cr}^2\} = 0 \;. \qquad [21]$$

If we assume that the denominator will be non-zero, the actual system resonance frequency is given by

$$\omega_s = \omega_{cr}\sqrt{\frac{1}{1 - \frac{1}{2}\left(\frac{K_s}{\omega_{cr}}\right)^2}} \;. \qquad [22]$$

From the above equation, it can be seen that only for the case of zero damping ($K_s = 0$) will the system resonance frequency (critical speed) correspond to the natural lateral frequency ω_{cr}. In general, the effects of damping will increase the system resonance frequency as can be seen from Fig. 2.

The maximum force transmitted to the bearings during the system resonance is given by:

$$F_{max} = \frac{Ke\omega_{cr}}{2K_s\sqrt{1 - \frac{1}{2}\left(\frac{K_s}{\omega_{cr}}\right)^2}} \;. \qquad [23]$$

In general, the ratio $\omega_{cr}/K_s >> 1.0$. In this case the maximum force transmitted may be expressed by:

$$F_{max} = \frac{e\omega_{cr}}{2} \left(\frac{K}{K_s}\right) \qquad [24]$$

Thus, it is seen that with the perfectly balanced rotor ($e = 0$), the force transmitted to the bearings will be zero. In actuality a finite value will exist for e depending upon the rotor balancing equipment used.

Case II - Zero Precession

The condition of zero precession implies that the rotor vibrates in a plane. This is given by the precessional angular velocity $\dot{\phi} = 0$

$$\text{(a)} \quad \ddot{\delta} + K_s \dot{\delta} + \omega_{cr}^2 \delta = e\omega^2 \cos\beta - g \sin\phi$$

$$\text{(b)} \quad 0 = e\omega^2 \sin\beta - g \cos\phi \; . \qquad [25]$$

The above condition is possible only if the eccentricity e (unbalance) or the total angular velocity ω is zero. In either case the resulting equation of motion is

$$\ddot{\delta} + K_s \dot{\delta} + \omega_{cr}^2 \delta = 0 \qquad [26]$$

which is the equation of free, damped lateral vibrations. It is important to note that the majority of present methods for calculating critical speeds are based upon finding the natural lateral frequencies of undamped motion. From this simple model it is seen that the normal unbalanced rotor doesn't vibrate in a plane but revolves or precesses to form an orbit.

Case III - Secondary Critical Speed (Effect of Gravity)

Assume rotor synchronous precession

$\dot{\phi} = \omega$

$\phi = \omega t - \beta$

$$\text{(a)} \quad \ddot{\delta} + K_s \dot{\delta} + (\omega_{cr}^2 - \omega^2)\delta = e\omega^2 \cos\beta - g \sin(\omega t - \beta)$$

$$\text{(b)} \quad 2\omega\dot{\delta} + K_s \omega \delta = e\omega^2 \sin\beta - g\cos(\omega t - \beta) . \qquad [27]$$

Solving [27b] for a particular solution we obtain

$$\delta = \frac{e}{2D}\sin\beta - \frac{g}{2\omega^2[1 + D^2]}[\sin(\omega t - \beta) + D\cos(\omega t - \beta)] \qquad [28]$$

where

$$D = \frac{K_s}{2\omega} .$$

[28] must also represent a particular solution of [27a]. Substitution of the above into the first equation of motion results in the following conditions

to be met in order that δ be a valid solution.

$$\text{(a)} \quad \sin\beta - \frac{2D}{\left(\frac{\omega_{cr}}{\omega}\right)^2 - 1} \cos\beta = 0$$

$$\text{(b)} \quad \frac{g}{2[1+D^2]} \left[3 + \frac{5}{2} D^2 + \left(\frac{\omega^2 - \omega^2_{cr}}{\omega^2}\right)\right] \sin(\omega t - \beta) = 0 \qquad [29]$$

$$\text{(c)} \quad \frac{gD}{2[1+D^2]} \left[\frac{1}{2} + \left(\frac{\omega^2 - \omega^2_{cr}}{\omega^2}\right)\right] \cos(\omega t - \beta) = 0$$

The first condition, Equation [29] is satisfied by the requirement that the rotor phase angle β be given by

$$\beta = \tan^{-1} \frac{K_s \omega}{\omega^2_{cr} - \omega^2} \qquad [30]$$

the above is identical to Eq. [18] obtained for synchronous precession in general. The second relationship requires that

$$3 + \frac{5}{2} D^2 + \frac{\omega^2 - \omega^2_{cr}}{\omega^2} = 0$$

or

$$\omega = \frac{1}{2} \omega_{cr} \sqrt{1 - \frac{5}{8}\left(\frac{K_s}{\omega_{cr}}\right)^2} \; . \qquad [31]$$

The above condition represents the system secondary critical speed. Note that the last two conditions are identically satisfied if $g = 0$ and ω may be any speed.

Eq. [29c] requires that either

$$\text{(a)} \quad \frac{1}{2} + \frac{\omega^2 - \omega_{cr}^2}{\omega^2} = 0$$

or [32]

$$\text{(b)} \quad \frac{gD}{2[1 + D^2]} = 0 \ .$$

The first condition leads to the contradictory statement that

$$\omega = \sqrt{\frac{2}{3}}\ \omega_{cr}$$

which is in conflict with Eq. [31]. Thus it is necessary that

$$\frac{gD}{2[1 + D^2]} \rightarrow 0 \ .$$

Substitute δ_{st} = rotor static deflection = $\frac{Mg}{K}$

and $D = \frac{K_s}{2\omega} = \frac{C}{2M\omega}$

$$\therefore \left(\frac{\omega_r}{2\omega}\right) \frac{\omega K_s \delta_{st}}{[1 + D^2]} \approx \frac{C \omega \delta_{st}}{m} \qquad [33]$$

The third condition implies that the rotor damping force $C \cdot \omega \delta$ divided by the bearing mass M must be a small quantity or

$$\frac{F_{damping}}{m} \rightarrow 0$$

in order to observe a secondary system critical speed. This criterion may help to explain why secondary critical speeds have sometimes been observed with heavy, massive low speed turborotors, but seldom with light-weight high speed rotors. If the system damping characteristics are too high, this phenomenon is completely suppressed.

The rotor deflection at the secondary critical speed ($\omega = \frac{\omega_{cr}}{2}$ for light damping) is given by

$$\delta_{[28]} = [30] \frac{e}{\sqrt{1 + (\frac{2}{3} D)^2}} - \frac{2\,\delta_{st}}{[1 + D^2]} \left[\sin\left(\frac{\omega_{cr}}{2} t - \beta\right) + D \cos\left(\frac{\omega_{cr}}{2} t - \beta\right)\right] \qquad [34]$$

Hence we conclude that when the rotor angular velocity is equal to one half the first critical speed, a horizontal rotor is capable of processing a secondary critical speed. The radius of the whirl orbit is equal to twice the static deflection (or initial rotor sag). Note that gravity is not the only cause of secondary critical speeds. Rotors with unsymmetric shaft properties can cause excessive rotor deflection (14).

The investigation of the possible occurrence of subcritical resonance vibrations has been discussed by several authors. Rankine, (1) in his early publications on vibrations of rotors, stated that a resonance vibration at $\sqrt{1/2}\ \omega_{cr}$ was possible. This value was later shown to be erroneous since Rankine neglected the Coriolis acceleration term in his equations of motion. Stodola (15) was the first to demonstrate that the disc weights of a horizontal shaft can create disturbing forces which at a certain speed can produce considerable shaft vibration. Timoshenko gives a simplified explanation of the secondary critical speed effect, developed along the lines of Stodola, in his text Vibration Problems in Engineering (16). The actual observation of the secondary critical speed phenomenon was reported as early as 1919 by FÖPPL (17).

An extensive article on the subcritical speeds of a rotating shaft was presented by Soderberg (18) in the past decade. Soderberg examines and compares the resonance amplitudes at the critical speed to the rotor subcritical vibrations caused by gravity and by variable rotor elasticity for an undamped rotor. In his investigation of the secondary critical speed due to gravity, he arrives at the following equation

$$\frac{d^2r}{dt^2} + (\omega^2_{cr} - \omega^2 + 2\mu\,\omega^2 \sin \omega t)\; r = \omega^2_{cr}\; e \qquad [35]$$

where r is the displacement of the rotor mass center from the steady-state position.

The Eq. [29] of Ref. (18) is a nonhomogeneous Mathieu equation of the form

$$\frac{d^2W}{dz^2} + [\,\delta + \epsilon \;\cos z\,]\; W = C$$

and its solutions and regions of stability are discussed in detail in Stoker (19). Soderberg approximates the solution by solving the equation considering the term $(2\mu r\,\omega^2) \sin \omega t$ as a forcing function independent of r, which results in

$$r = \frac{e\,\omega^2_{cr}}{\omega^2_{cr} - \omega^2}\left[1 + \frac{2\epsilon\;\omega^2\,(\omega^2_{cr} - \omega^2)}{\omega^2_{cr}(\omega^2_{cr} - 4\,\omega^2)}\;\sin \omega t\right]. \qquad [36]$$

He then concluded that since r becomes unbounded when the rotor speed is exactly one-half the rotor critical speed, then the rotor precession angle must be of the form

$$\theta = \omega t + \chi\;\omega t \sin \omega t \qquad [37]$$

which leads to a higher order Mathieu Equation. The solution he obtains when $\omega = \omega_{cr}/2$ is given by

$$r = \frac{4}{3} e \left[1 - 3/8 \, \epsilon \, (2 \sin \omega t + \omega t \cos \omega t)\right] \qquad [38]$$

where

$$\epsilon = e y_o \left(\frac{\omega_c}{\rho\omega}\right)^2 \; ; \; \rho = \text{radius of gyration} \; .$$

Even though the term ϵ is a small quantity, Soderberg predicts that the vibration amplitudes of an undamped rotor will become unbounded if operated continuously at one-half the rotor critical speed. This finding is in contrast to Eq. (34) which shows that the subcritical vibration amplitude of an undamped rotor is bounded and also that the inclusion of sufficient rotor damping will suppress this phenomenon.

Case IV - General Whirling (Non-Synchronous Precession)

Let

$$\dot{\beta} = n\omega$$

$$\beta = n\omega t + \beta_o$$

$$\dot{\beta} + \dot{\phi} = \omega \; .$$

The equations of motion (neglecting gravity) are,

$$\text{(a)} \quad \ddot{\delta} + K_s \dot{\delta} + [\omega^2_{cr} + (1 - n)^2 \omega^2] \; \delta = e\omega^2 \cos (n\omega t + \beta_o) . \qquad [39]$$

$$\text{(b)} \quad [2 \dot{\delta} + K_s \delta] \; \omega \, [1 - n] = e\omega^2 \sin (n\omega t + \beta_o) \; .$$

Solving for δ

$$\delta = A e^{-\frac{K_s}{2}t} - \frac{e\omega \cos(\beta - \alpha)}{2[1-n]\left[\left(\frac{K_s}{2}\right)^2 + n^2\omega^2\right]^{1/2}} \qquad [40]$$

where $\beta - \alpha = n\omega t + \beta_o - \tan^{-1}\left(\frac{K_s}{2n\omega}\right)$.

Applying the initial condition of

$$\delta(0) = \delta_o$$

$$\delta_o = A - R\cos(\beta_o - \alpha) \qquad [41]$$

where

$$R = \frac{e\omega}{2[1-n]\left[\left(\frac{K_s}{2}\right) + n^2\omega^2\right]^{1/2}} .$$

Hence

$$e^{-\frac{K_s}{2}t}\left\{\left[\omega_{cr}^2 - (1-n)^2\omega^2 - \frac{K_s}{4}\right][\delta_o + R\cos(\beta_o - \alpha)]\right\} + R\cos(\beta - \alpha)[n^2\omega^2 + - \omega_{cr}^2 + (1-n)^2\omega^2] + n\omega K_s R\sin(\beta - \alpha) - e\omega^2\cos\beta = 0 \qquad [42]$$

(The above equation represents an extension of the work of Kane (13) who neglected the effects of damping in his equations. It will be seen that even for the case of light damping, the nature of the solutions is considerably altered.)

Problem:

Do any values of n exist such that the above equation is satisfied for all time t ?

If we consider light damping then

$$\frac{K_s}{2n\omega} \rightarrow 0 ; \quad R \rightarrow \frac{e}{2n[1-n]}$$

$$e^{-\frac{K_{st}}{2}} [\omega^2_{cr} - (1-n)^2\omega^2] \left[\delta_o + \frac{e}{2n[1-n]} \cos(\beta_o - \alpha) \right] +$$

$$\frac{e}{2n[1-n]} [(2n-1)^2 \omega^2 - \omega^2_{cr}] \cos\beta = 0 . \qquad [43]$$

Consider values of n (other than 0 or 1) which will make the above equation identically vanish.

Let

$$\omega^2_{cr} - (1-n)^2\omega^2 = 0 \qquad [44]$$

and

$$[2n - 1]^2 \omega^2 - \omega^2 = 0 \ .$$

Solving for n

$$n = 2/3 \ .$$

Hence

$$\omega = 3 \omega_{cr} \quad \text{and} \quad \phi = \omega_{cr} \ . \qquad [45]$$

The above condition implies that if the rotor angular velocity ω of the system is three times the natural lateral critical frequency ω , one possible motion is for the system to precess at a rate equal to the critical speed. This has been reported to occur with an externally pressurized gas bearing rotor (20) and has been referred to as "Fractional frequency whirl." (Although the single mass Jeffcott model is physically unlike a rigid rotor or externally pressurized bearings, the equations of cylindrical precession are similar.)

As a second case consider the less stringent condition that the transient whirl dies out. The steady state equation ($t \rightarrow \infty$) is

$$\frac{e}{2n(1 - n)} \cos \beta \left[(2n - 1)^2 - \left(\frac{\omega_{cr}}{\omega}\right)\right] \omega^2 = 0 \ . \qquad [46]$$

Consider the case where $\omega \gg \omega_{cr}$ or the angular velocity is much higher than the first critical speed. In this case [46] reduces to

$$(2n - 1)^2 = 0 \qquad [47]$$

or $\qquad n = 1/2$.

Hence

$$\dot{\phi} = \omega/2 .$$

Thus we have demonstrated that half-frequency whirling is possible only in the limiting case as the rotor approaches speeds considerably greater than the first critical. Note that it is impossible to obtain this conclusion unless damping is retained in the equations of motion.

Half-frequency whirling is usually associated with hydrodynamic fluid film bearings. At least two bearing coefficients are required to represent the bearing stiffness characteristics; a radial "spring" rate and a tangential spring rate. It is the presence of the tangential or out-of-phase bearing force which causes self excited half-frequency whirl to occur at approximately twice the rotor critical speed (11). In the absence of this force half-frequency whirling cannot occur.

SUMMARY AND CONCLUSIONS

In Table I are presented a summary of the various forms of rotor whirling. For each particular case there are three subsections which represent various degrees of rotor damping. Line A which represents the rotor behavior with zero damping, was obtained from Ref. (13). Line B represents the rotor performance with non-zero damping forces.

It is important to note the influence of even small damping on the rotor characteristics. For example, in the first two cases which represent synchronous rotor precession, the introduction of damping eliminates the possibility of backward synchronous motion and also causes the rotor phase relationship to be single valued. In case 1A and 1B, we see that if the rotor is running at the critical speed or resonance frequency, then the rotor amplitude will increase continuously with time. If the rotor damping is non-zero (10) then the rotor amplitude will be bounded. The rotor deflection at the critical speed will be some multiple of the rotor unbalance e. This amplitude factor is referred to as the rotor critical amplification factor, $Acr = \omega_{cr} / K_s$ for a simple system, and is an important parameter in the study of rotor stability. Case 2 represents rotor synchronous precession in general. The rotor deflection given in 2C is identical to the results stated by Jeffcott (4) and Fig. 2 represents a plot of this function. Notice in 2C that damping causes the rotor phase angle to be zero at low speed and increase smoothly with speed to a maximum value of π. For a single mass rotor in which the

TABLE I

DESCRIPTION OF VARIOUS MODES OF ROTOR MOTION

	CASE	ROTOR SPEED	DAMPING	PRECESSION RATE - ϕ^o	ROTOR DEFLECTION, δ	ROTOR PHASE ANGLE, β
	A		$K_s=0^{(1)}$	$\phi^o = \pm\,\omega_{cr}$	$\delta = \delta_0 \pm e\ \omega_{cr}/2\ t$	$\pm\,\pi/2$
1	B	$\omega = \omega_{cr}$	$K_s \to 0$	$\phi^o = \omega_{cr}$	$\delta = \delta_0 + e\ \omega_{cr}/2\ t$	$+\,\pi/2$
	C		$K_s \neq 0$	$\phi^o = \omega_{cr}$	$\delta = e\ \omega_{cr}/K_s$	$+\,\pi/2$
	A		$K_s=0^{(1)}$	$\pm\omega$	$\pm\, e\omega^2/(\omega^2-\omega_{cr}^2)$	$(1 \pm 1)\,\pi/2$
2	B	$\omega \neq \omega_{cr}$	$K_s \to 0$	ω	$e\omega^2/\sqrt{(\omega^2-\omega_{cr}^2)^2}$	0 if $\omega < \omega_{cr}$ π if $\omega > \omega_{cr}$
	C		$K_s \neq 0$	ω	$e\omega^2/\sqrt{(\omega^2-\omega_{cr}^2)^2+(K_s\omega)^2}$	$\mathrm{TAN}^{-1}\left[K_s\,\omega/(\omega_{cr}^2-\omega^2)\right]$
	A		$K_s=0^{(1)}$	$\pm\,\omega$	$\delta_0 + 9/4\ e\left[\mathrm{COS}\,\beta_0 - \mathrm{COS}(\beta_0 \pm 2\,\omega_{cr}\,t)\right]$	UNRESTRICTED
	B	$\omega = 3\omega_{cr}$	$K_s \to 0$	ω_{cr}	$\delta_0 + 9/4\, e\left[\mathrm{COS}(\beta_0 + 2\,\omega_{cr}\,t) - \mathrm{COS}\,\beta_0\right]$	UNRESTRICTED
3	C		$K_s \neq 0^{(2)}$	ω_{cr}	$\delta_0 e^{-K_s/2t} + e\omega/2(1-n)\left((K_s/2)^2+n^2\omega^2\right)^{-1/2}$ $\times\left[\mathrm{COS}\left(\beta_0+2\omega_{cr}t+\mathrm{TAN}^{-1}(K_s/2\omega_{cr})\right)\right.$ $\left.-e^{-K_s/2\,t}\left(\mathrm{COS}(\beta_0+\mathrm{TAN}^{-1}(K_s/2\omega_{cr}))\right)\right]^{(2)}$	UNRESTRICTED
	A		$K_s=0^{(1)}$	$(\omega+\omega_{cr})/2$	$2e\omega^2/(\omega_{cr}^2-\omega^2)\ \mathrm{COS}\left[\beta_0 + 1/2\ (\omega \pm \omega_{cr})\,t\right]$	$\mathrm{COS}^{-1}(\delta_0(\omega_{cr}^2-\omega^2)/2e\omega^2)$
4	B	$\omega \neq \pm\,\omega_{cr}$	$K_s \to 0$	$(\omega+\omega_{cr})/2=\omega$	SAME AS 1B	SAME AS 1B
	C		$K_s \neq 0$	$(\omega+\omega_{cr})/2=\omega$	SAME AS 1C	SAME AS 1C
	A		$K_s=0^{(1)}$	$\pm\,\omega_{cr}$	$-9/4\ e\ \mathrm{COS}(\beta_0 \pm 2\,\omega_{cr}\,t)$	$\mathrm{COS}^{-1}\ (-4\delta_0/9e)$
5	B	$\omega = 3\omega_{cr}$	$K_s \to 0$	———	SUPPRESSED TRANSIENT	———
	C		$K_s \neq 0$	———	SUPPRESSED TRANSIENT	———
6	A	$\omega >> \omega_{cr}$	$K_s = 0$	NON-EXISTENT	NON-EXISTENT	———
	B		$K_s = 0^{(2)}$	$t \to \infty$, $\phi^o = \omega/2$	$-2e\ \mathrm{COS}\ \omega/2t$	UNRESTRICTED
	A	$\omega = 1/2\ \omega_{cr}$	$K_s = 0$	ω	$-2\ \delta_{ST}\ \mathrm{SIN}\ (\omega_{cr}\,t/2) + e/3^{(3)}$	0
7	B	$\omega = 1/2\ \times$ $\sqrt{\omega_{cr}^2 - 5/8\ K_s^2}$	$K_s \neq 0^{(2)}$	ω	$-2\ \delta_{ST}/(1+D^2)\left[\mathrm{SIN}(\omega_{cr}t/2-\beta)\right.$ $\left.+D\ \mathrm{COS}(\omega_{cr}t/2-\beta)\right] + e/3\sqrt{1+(2/3\,D)^2}$	SAME AS 2C

(1) REF. (13) (2) $K_s/2\omega << 1$ (3) $\delta_{ST} = MG/K$

motion is confined in a plane, there is only one phase angle. At the rotor critical speed of this system the rotor phase angle is 90° and the eccentricity vector is orthogonal to the rotor deflection. If additional degrees of freedom such as conical modes or multi-masses are introduced into the system, there will be additional rotor phase angles corresponding to each mode.

From the examination of Cases 1, 2, and 7, the following characteristics concerning rotor synchronous precession are summarized as follows:

1. For small values of the damping parameter and (or) $\omega \ll \omega_{cr}$, the phase angle β is zero. Thus, for small damping and speeds below the first critical speed, the unbalance is in phase with the maximum deflection and the mass center rotates about the volume center.

2. As the rotor speed ω approaches the critical speed ω_{cr}, the phase angle β approaches $\pi/2$. At this speed, if no damping is present, amplitudes of vibration of dangerous proportions can result.

3. For the condition where $\omega \gg \omega_{cr}$ and low damping, the phase angle approaches π as a limit. In this situation the volume center is revolving around the mass center and the force transmitted to the bearings reaches an asymptote equal to $Ke/2$.

4. If large amounts of damping are present in the system, a peak vibration is not observed at the system critical speed. The rotor deflection increases smoothly from o to e as the rotor speed ω increases from 0 to $\omega >> \omega_{cr}$.

5. The system critical speed increases slightly with an increase in viscous damping. The system critical corresponds to the natural lateral frequency of vibration ω_{cr} only for the case when the damping is zero or the damping forces are proportionate to the velocity squared (21).

6. The rotor phase angle is a single valued and continuous function in a damped system.

7. Synchronous backward precession is not possible even in a lightly damped rotor.

8. A lightly damped horizontal rotor may exhibit a secondary critical speed effect when operating at one-half the rotor first critical speed. The rotor whirl orbit will be approximately twice the rotor static deflection (Case 7).

The cases 3 through 6 represent various modes of whirling or non-synchronous precession. For example Case 3 shows that when the rotor speed reaches three times the rotor critical speed, the rotor is capable of forward or backward precession equal to the rotor critical speed ω_{cr}. The inclusion of damping, however minute, eliminates the possibility of backward precession. If finite damping is considered, this motion is possible only if the system damping is light, i.e., if $Ks/2\ \omega_{cr} << 1.0$.

In all of the above cases of whirling in general, it was found that the inclusion of sufficient damping will suppress all whirl tendency and permit only synchronous forward rotor precession. The inclusion of damping into the equations of motion considerably changes the fundamental nature of the motion as described in Ref. (13). For example, Case 4 reduces to Case 1 and Case 5 vanishes altogether when damping is considered. Thus the only two distinct cases of whirling are 3 and 6. Case 6 states that the rotor is capable of half-frequency whirling ($\phi^{\circ} = \omega/2$) when the rotor speed becomes infinitely high for a lightly damped system. Note that this conclusion, although unrealistic, cannot be obtained from a system in which the damping is excluded.

In conclusion we find that it is impossible to examine or explain the occurrence of rotor whirling by means of a conservative system. It is impossible with this system to completely explain the rotor whirling as observed by Newkirk, Stodola, Pinkus and others. Whirling or nonsynchronous precession can occur only in non-conservative systems in which the system dissipation function possesses special characteristics (8).

NOMENCLATURE

A	Rotor amplification factor
c	Rotor volume center
C	Damping coefficient
e	Eccentricity of rotor unbalance mass
F	Force
F_{qr}	Generalized force
g	Gravity
I	Rotor polar moment of inertia about C. G.
K	Rotor spring rate
k	Radius of gyration
K_s	Damping factor = C/m, Rad/ sec
L	Lagrangian
m	Rotor mass
n	Coefficient
o	Undeflected rotor position
$\vec{n}_r, \vec{n}_\phi$	Unit vector set fixed in reference frame R'
P	Position vector
qr	Generalized coordinate
R	Fixed reference frame
R'	Relative reference frame moving with angular velocity $\phi°$
T	Kinetic energy
t	Time

V	Potential energy
V_m	Velocity of rotor mass center
β	Rotor phase angle
δ	Rotor deflection
ϕ°	Rotor precession rate or whirl speed
ω	Rotor angular velocity
${}^R\omega^{R'}$	Angular velocity vector of relative reference frame R' in R = $\phi^\circ n_z$
ω_{cr}	Rotor natural lateral frequency $= \sqrt{\frac{K}{m}}$
ω_s	Rotor actual critical speed

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