SOME EXTENSIONS OF THE KUHN-TUCKER RESULTS IN CONCAVE PROGRAMMING

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I. INTRODUCTION, NOTATION

In their seminal paper [11], Kuhn and Tucker proved an equivalence between the existence of a saddle point and the maximization of a concave function f subject to $x \geqq 0$, and $g(x) \geqq 0$, where g is a vector of concave functions.[2] Uzawa later provided a somewhat simpler proof of this result [14], as well as extending the basic theorem to the case where the function f and the functions g_i are not necessarily differentiable nor even continuous.[3] In a fundamental article in the same volume, Hurwicz [8] generalized the Kuhn-Tucker results to the case where the functions involved map a (real) linear space into linear topological spaces; as well as providing interesting and important extensions of the saddle point notion to cases involving more general orderings.[4]

The purpose of this paper is two-fold:

1.) We shall provide a fairly systematic treatment of the theory of the constrained maximization of nondifferentiable vector-valued functions defined on a finite-dimensional Euclidean space (Sec. II). In some respects, much of this portion of the paper is not new. In fact, some of the theorems presented are special cases of Professor Hurwicz's results for the non-differentiable case. However, where the results presented here are special cases of Professor Hurwicz's work, we have generally been able to take advantage of the more elementary spaces with which we are concerned here to develop somewhat simpler proofs. Moreover, some of the results presented in Section II are at least mild generalizations of the heretofore published work on the constrained maximization of nondifferentiable vector-valued functions.

2.) In Section III, we undertake a systematic exploration of the nature of the constraint qualifications which have been used in this type of maximization problem. We there examine both the geometric role the constraint qualification plays in the problem, and the relationships among the various constraint qualifications which have been used.

In order to more clearly define the kind of problem with which we shall be dealing, suppose we first introduce the following notation.

Let E_n denote n-dimensional Euclidean space. We shall use x, y, z, etc., to denote points in this space, which we think of (where the distinction is important) as column vectors. If x is the vector with elements $x_1, x_2, \ldots, x_n$, we write

$$x = \langle x_1, x_2, \ldots, x_n \rangle.$$

We shall denote the set of unit (Cartesian) coordinate vectors in E_n by $\{e^1, \ldots, e^n\}$, i.e.,

$$e^i = \langle \delta_{i1}, \delta_{i2}, \ldots, \delta_{in} \rangle \text{ for } i=1, \ldots, n;$$

where δ_{ij} is the Kronecker delta.

We shall use what seems to be a standard notation for vector inequalities:

$$x \geqq y \text{ iff } x_i \geqq y_i \text{ for } i=1, \ldots, n;$$

$$x > y \text{ iff } x \geqq y \text{ and } x \neq y;$$

$$x >> y \text{ iff } x_i > y_i \text{ for } i=1, \ldots, n.$$

Using these definitions, we define the non-negative orthant in E_n, E_n^+, by:

$$E_n^+ = \{x \varepsilon E_n | \; x \geq \theta_n\};$$

where θ_n denotes the origin in E_n. In line with this notation, E_1 will denote the real line, and E_1^+ the set of non-negative real numbers.

If x, $y \varepsilon E_n$, we shall denote the:

1.) <u>INNER PRODUCT OF x AND y</u> by $x \cdot y$,

    $$x \cdot y = \Sigma_{i=1}^n x_i y_i.$$

2.) <u>NORM OF x</u> by $||x||$, <u>i.e.</u>,

    $$||x|| = x \cdot x.$$

3.) <u>DISTANCE BETWEEN x AND y</u> (the metric on E_n) by

    $$d(x, y) = ||x-y|| = [\Sigma_{i=1}^n (x_i - y_i)^2]^{1/2}$$

4.) <u>SPHERICAL NEIGHBORHOOD OF x WITH RADIUS $\varepsilon > 0$</u> by

    $$N(x, \varepsilon) = \{y \varepsilon E_n | \; d(x, y) < \varepsilon\}.$$

    Where the radius is unimportant, we use N(x) to denote an arbitrary (non-empty) spherical neighborhood of x.

If $X \subseteq E_n$, we denote the closure of X by $\overline{X}$, and the interior of X by int(X), <u>i.e.</u>,

$$\text{int}(X) = \{x \varepsilon X | \; (\exists N(x)) \; N(x) \subseteq X\}.$$

If A and B are subsets of E_m and E_n, respectively, we denote the Cartesian Product of A and B by

$$A \times B = \{\langle a, b\rangle \varepsilon E_{m+n} | \; a \varepsilon A, \; b \varepsilon B\}.$$

Extending the above notation, we shall frequently partition vectors in, say, E_m, writing, <u>e.g.</u>, $x = \langle x^1, x^2\rangle$. Where we write

$$\langle x^1, x^2\rangle \varepsilon E_{n+p},$$

we shall understand that

$$x^1 \varepsilon E_n, \; x^2 \varepsilon E_p.$$

We say that a set $X \subseteq E_n$ is:

1.) a CONE if ($x \varepsilon X$ and $\lambda \varepsilon E_1^+$): $\lambda x \varepsilon X$.

2.) CONVEX if (x^1, $x^2 \varepsilon X$ and $\lambda \varepsilon [0, 1]$): $\lambda x^1 + (1-\lambda) x^2 \varepsilon X$

3.) a CONVEX CONE if (x^1, $x^2 \varepsilon X$ and λ_1, $\lambda_2 \varepsilon E_1^+$) $\lambda_1 x^1 + \lambda_2 x^2 \varepsilon X$;

while if $X \subseteq E_n$, we define:

1.) the CONJUGATE CONE OF X, denoted X*, by

$$X^* = \{y \varepsilon E_n \mid (x \varepsilon X) x \cdot y \geqq 0\},$$

2.) $X^\perp = \{y \varepsilon E_n \mid (x \varepsilon X) x \cdot y = 0\}$

3.) $-X = \{y \varepsilon E_n \mid (-1) y \varepsilon X\}$

4.) (for $Y \subseteq E_n$):

$$X+Y = \{z \varepsilon E_n \mid (\exists x \varepsilon X, y \varepsilon Y)\ z = x+y\}.$$

Finally, we shall make frequent use of the following definitions.

DEFINITION 1: Let g: $E_m \to E_n$. We shall say that g is AFFINE[5] if g is of the form:

$$g(x) = Gx+b,$$

where G is an $n \times m$ matrix of constants, and b is an $n \times 1$ column vector of constants.

DEFINITION 2: Let $D \subseteq E_n$ be convex, let g: $D \to E_m$, and let $Y \subseteq E_m$ be a convex cone. We shall say that g is Y-CONCAVE ON D if for every x^1, $x^2 \varepsilon D$ and $\lambda \varepsilon$ [0, 1], we have

$$g[\lambda x^1 + (1-\lambda) x^2] - [\lambda g(x^1) + (1-\lambda) g(x^2)] \varepsilon Y.$$

This second definition is equivalent (for the case with which we're dealing here) to the definition of concavity introduced by Professor Hurwicz in [8] (p. 68). Note that if g: $D \to E_1$, the usual definition of concavity is equivalent to the statement that g is E_1^+-concave on D; while if g is an m-vector of functions, each of which is concave by the usual definition, then g is E_m^+-concave on D.

We now set out the maximization problem with which we shall deal in this paper as follows:

DEFINITION 3: Let $D \subseteq E_m$, and suppose that:

$$f\colon D \to E_n, \quad g\colon D \to E_p,$$

and that $X \subseteq D$ is non-empty, $Y \subseteq E_p$ is a non-empty convex cone. We shall then say that $\langle f, g, X, Y\rangle$ defines a MAXIMIZATION PROBLEM, π, and that $\bar{x}$ is a SOLUTION of π provided that:

(1) $\quad \bar{x} \in X, \; g(\bar{x}) \in Y,$

and

(2) $\quad \nexists \hat{x} \in X \ni g(\hat{x}) \in Y \text{ and } f(\hat{x}) > f(\bar{x}).$[6]

Notice that if n=1, so that f is real-valued, we have, as a special case, the maximization (in the usual sense) of a real-valued function subject to the constraints $x \in X$ and $g(x) \in Y$. Moreover, in the very special case where n=1, $X=E_m^+$, and $Y=E_p^+$; our maximization problem reduces to the much more familiar problem of maximizing f subject to $x \geq \theta_m$, $g(x) \geq \theta_p$. We note also that, since $\{\theta_p\}$ is a convex cone, the general maximization problem formulated in Definition 3 includes as a special case the classical Lagrangian problem of maximizing a real-valued function f subject to the constraint $g(x)=\theta$ (in the case where n=1, X=D, and $Y=\{\theta_p\}$)[7]. As a final example, suppose we wish to maximize (in the sense of definition 3)[8] some vector-valued function f subject to $x \in X$ and

$$h_i(x) = b_i \text{ for } i=1, \ldots, q;$$
$$(3) \qquad h_i(x) \geq b_i \text{ for } i=q+1, \ldots, r$$
$$h_i(x) \leq b_i \text{ for } i=r+1, \ldots, p.$$

Define

$$g^{(1)}(x) = \langle h_1(x)+(-b_1), \ . \ . \ . \ , h_q(x)+(-b_q)\rangle$$
$$g^{(2)}(x) = \langle h_{q+1}(x)-b_{q+1}, \ . \ . \ . \ , h_p(x)-b_p\rangle$$
$$g(x) = \langle g^{(1)}(x), g^{(2)}(x)\rangle;$$

and

$$Y_1 = \{\theta_q\},$$
$$Y_2 = E^+_{r-q}\times(-E^+_{p-r}),$$
$$Y = Y_1\times Y_2.$$

Then Y, being the Cartesian Product of convex cones, is itself a convex cone; and the constraints in (3) can equivalently be expressed by the requirement $g(x)\varepsilon Y$.[9] Hence this example is also a special case of the type of general maximization problem formulated in Definition 3.

One further aspect of this definition deserves some discussion. The reader will note that in Definition 3, we have not required D, the domain of definition of the functions f and g, to coincide with E_m (and in fact, our definition and the theorems of the next section apply to the case where D=X). The extent of the domain of definition is important in this kind of maximization problem for at least two reasons.

First of all, the saddle point theorems of the next section do not require the functions f and g to be continuous. However, the necessity theorems (Theorems 3-5 of Section II) do require f and g to be concave (in the usual applications); and a function which is defined and concave on an open convex set in E_m is continuous on this set (see Berge [6], p. 193). Hence if we assume $D=E_m$, we would implicitly be assuming that f and g were continuous.

A more important consideration stems from the following reasoning. One might conjecture [10] that if a function is defined and concave on a convex set in E_m, it is always extendible to a function which is defined and concave over all of E_m. However, in spite of the apparent plausibility of this statement, it is incorrect; as we can easily see from the following counterexample: let

$$f(x) = \begin{cases} 0 \text{ if } x=0 \\ 1 \text{ if } x>0. \end{cases}$$

It is clear that f is defined and concave on E_1^+, but it is obvious that there is no way of extending it to a function which is defined and concave over all of E_1.[11] Hence if we have, say, a function f which is defined and concave over the non-negative orthant in E_m (<u>e.g.</u>, a production function), a saddle point theorem which requires the domain of definition to be E_m (and the functions f and g to be concave on this domain) is not applicable without enough additional specifications on the nature of the function f to guarantee that it is extendible. The formulation of our Definition 3, which is followed in the theorems of the next section, is applicable to this sort of situation without the additional specifications.

<u>DEFINITION 4</u>: Let π be the maximization problem defined by $\langle f, g, X\ Y\rangle$. We define the <u>GENERALIZED LAGRANGIAN EXPRESSION ASSOCIATED WITH π</u>, Φ_π, on $D\times E_n\times E_p$ by:

$$\Phi_\pi(x, v, w) = v\cdot f(x)+w\cdot g(x).$$ [12]

In the next section, we shall be concerned with the investigation of the relationship between the existence of the solution of a maximization problem, π, and the existence of a Saddle Point, of one of the following types, for Φ_π.

DEFINITIONS :[13] Let $\langle f, g\ X, Y\rangle$ define a maximization problem, π, and let Φ_π denote the Generalized Lagrangian Expression associated with π.

Then we shall say that:

5.) a point $\langle\bar{x}, \bar{v}, \bar{w}\rangle \varepsilon E_{m+n+p}$ is a GENERALIZED SADDLE POINT (GSP) for Φ_π, or that Φ_π has a GSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$, if:

(4) $\bar{x}\varepsilon X,\ \bar{v}\varepsilon E_n^+,\ \bar{w}\varepsilon Y^*,\ \langle\bar{v}, \bar{w}\rangle \neq \theta_{n+p}$,

and

(5) $\Phi_\pi(x, \bar{v}, \bar{w}) \leqq \Phi_\pi(\bar{x}, \bar{v}, \bar{w}) \leqq \Phi_\pi(\bar{x}, \bar{v}, w)$ for all $x\varepsilon X$, $w\varepsilon Y^*$.

Extending this terminology somewhat, we shall sometimes say that Φ_π has a GSP at $\bar{x}\varepsilon X$, or that a GSP exists for Φ_π at $\bar{x}$, if $\exists \langle\bar{v}, \bar{w}\rangle \varepsilon E_n^+ \times Y^* \ni \langle\bar{x}, \bar{v}, \bar{w}\rangle$ is a Generalized Saddle Point for Φ_π. (Similar conventions will be followed for the types of saddle points defined in the following.)

6.) a point $\langle\bar{x}, \bar{v}, \bar{w}\rangle \varepsilon E_{m+n+p}$ is a GENERALIZED NON-DEGENERATE SADDLE POINT (GNSP) for Φ_π, or that Φ_π has a GNSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$, if:

$\bar{x}\varepsilon X,\ \bar{v}\varepsilon E_n^+ \setminus \{\theta_n\}$[14], $\bar{w}\varepsilon Y^*$,

and (5) holds. Equivalently, Φ_π has a GNSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$ if Φ_π has a GSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$ and $\bar{v}\neq\theta_n$.

7.) a point $\langle\bar{x}, \bar{v}, \bar{w}\rangle \varepsilon E_{m+n+p}$ is a GENERALIZED PROPER SADDLE POINT (GPSP) for Φ_π if $\bar{x}\varepsilon X$, $\bar{v} >> \theta_n$, $\bar{w}\varepsilon Y^*$,

and (5) holds.

8.) a point $\langle\bar{x}, \bar{w}\rangle \varepsilon E_{m+p}$ is a SADDLE POINT for Φ_π (in the special case where n=1, i.e., $f\colon D\to E_1$) if Φ_π has a GNSP at $\langle\bar{x}, 1, \bar{w}\rangle \varepsilon E_{m+1+p}$, that is, if:

(6) $\bar{x}\varepsilon X,\ \bar{w}\varepsilon Y^*$,

and

(7) $f(x)+\bar{w}\cdot g(x) \leqq f(\bar{x})+\bar{w}\cdot g(\bar{x}) \leqq f(\bar{x})+w\cdot g(\bar{x})$ for all $x\varepsilon X$, $w\varepsilon Y^*$.

Note that for the special case where n=1 (i.e., where $f: D \to E_1$), the distinction between a GNSP and a GPSP disappears (the distinction is of some importance when n>1, however, as we shall see). Moreover, in the case where n=1, the existence of a GNSP is logically equivalent to the existence of a Saddle Point. To see this, we first note that a Saddle Point in this situation is a special case of a GNSP (having $\bar{v}=1$). Moreover, if Φ_π has a GNSP at $<\bar{x}, \bar{v}, \bar{w}> \varepsilon E_{m+1+p}$; then, as we can easily verify, Φ_π has a Saddle Point at $<\bar{x}, (1/\bar{v})\bar{w}>$.

II. THE PRINCIPAL THEOREMS

The following theorem is a special case of theorem V.1, p. 86, in Hurwicz [8]. It deals with a sufficient condition for a constrained maximum; and, it should be noted, holds with no restrictive assumptions (e.g., concavity) on f and g whatever. It is also perhaps worth emphasizing that X can be any point set in E_m (even a finite point set), while Y can be any closed convex cone in E_p (and we may have, for instance $Y \cap E_p^+ = \{\theta_p\}$).

THEOREM 1 (HURWICZ):

If:

1.) $<f, g, X, Y>$ defines a maximization problem, π (see Definition 3), where Y is a closed convex cone;

2.) $<\bar{x}, \bar{v}, \bar{w}> \varepsilon E_{m+n+p}$ is a GPSP for Φ_π; then $\bar{x}$ is a solution of π.

PROOF (HURWICZ):

By hypothesis (2), we have

(1) $\bar{x} \varepsilon X, \; \bar{v} >> \theta_n, \; \bar{w} \varepsilon Y^*,$

and

(2) $\bar{v} \cdot f(x) + \bar{w} \cdot g(x) \leqq \bar{v} \cdot f(\bar{x}) + \bar{w} \cdot g(\bar{x}) \leqq \bar{v} \cdot f(\bar{x}) + w \cdot g(\bar{x})$ for all $x \varepsilon X$, $w \varepsilon Y^*$.

From the r.h.s. of (2), we have:

(3) $\bar{w}\cdot g(\bar{x}) \leqq w\cdot g(\bar{x})$ for all $w\varepsilon Y^*$.

However, since $\bar{w}\varepsilon Y^*$, we have $w+\bar{w}\varepsilon Y^*$ for all $w\varepsilon Y^*$ (since Y a convex cone implies Y* a convex cone). Hence, from (3), we have:

$$\bar{w}\cdot g(\bar{x}) \leqq (w+\bar{w})\cdot g(\bar{x}) \text{ for all } w\varepsilon Y^*,$$

or

(4) $w\cdot g(\bar{x}) \geqq 0$ for all $w\varepsilon Y^*$.

Therefore $g(\bar{x})\varepsilon Y^{**}$. However, since Y is a closed convex cone, we have (see Karlin [10], p. 403) $Y=Y^{**}$. Hence

(5) $g(\bar{x})\varepsilon Y$.

Moreover, it follows immediately from (1), (3), and (5), that

(6) $\bar{w}\cdot g(\bar{x})=0$

Suppose now that $x\varepsilon X$ and $g(x)\varepsilon Y$. Then by (1), (6), and l.h.s. of (2), we have:

$$\bar{v}\cdot f(x) \leqq \bar{v}\cdot f(x)+\bar{w}\cdot g(x) \leqq \bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x})=\bar{v}\cdot f(\bar{x}).$$

Hence

(7) $\bar{v}\cdot f(x) \leqq \bar{v}\cdot f(\bar{x})$ for all $x\varepsilon X \ni g(x)\varepsilon Y$.

Therefore, since $\bar{v} \gg \theta_n$, (7) implies that:

$$\nexists \hat{x}\varepsilon X \text{ with } g(\hat{x})\varepsilon Y \ni f(\hat{x})>f(\bar{x}),$$

and it follows from (1) and (5) that $\bar{x}$ is a solution of π.

Q.E.D.

Our next theorem deals with necessary conditions for a constrained maximum, and is a generalization of a theorem by Berge (Cf., Berge [6], p. 227).[15] The theorem stated here is implicit in Hurwicz's treatment in [8], although it is not stated explicitly. It is a fairly natural extension of the approach to the classical Lagrangian problem developed by Bliss in [7].

THEOREM 2:

If:

1.) $\langle f, g, X, Y\rangle$ defines a maximization problem, π, where:

a.) X is convex,

b.) f is concave (i.e., E_n^+-concave) on X,

c.) g is Y-concave on X,

2.) $\bar{x}\varepsilon X$ is a solution for π;

then

$\exists\ \bar{v}\varepsilon E_n^+,\ \bar{w}\varepsilon Y^* \ni \Phi_\pi$ has a GSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$.

The method of proof used in the following is an adaptation of that originated by Hurwicz in [8] and Uzawa in [14]. It depends heavily on two convex and disjoint sets, A and B, which are (in our case) subsets of E_{n+p}. In order to define these sets, we first define:

$$(8)\quad \begin{cases} h(x) = \langle f(x), g(x)\rangle \text{ for } x\varepsilon X, \\ Z = E_n^+ \times Y. \end{cases}$$

We note that h is Z-concave on X, and that Z is a convex cone (since it is the Cartesian Product of two convex cones).

For each $x\varepsilon X$, define:

$$(9)\qquad A(x) = \{a\varepsilon E_{n+p} \mid h(x)-a\varepsilon Z\}.$$

We then define:

$$(10)\qquad A = \{a\varepsilon E_{n+p} \mid (\exists x\varepsilon X)\, a\varepsilon A(x)\} = \bigcup_{x\varepsilon X} A(x)$$

$$= \{a = \langle s, t\rangle\varepsilon E_{n+p} \mid (\exists x\varepsilon X)\, f(x)\geq s,\ g(x)-t\varepsilon Y\},$$

and

$$(11)\quad \begin{cases} B = \{b=\langle z, y\rangle\varepsilon E_{n+p} \mid z>f(\bar{x}),\ y\varepsilon Y\} \\ \quad = \{b=\langle z, y\rangle\varepsilon E_{n+p} \mid \langle z, y\rangle\varepsilon[f(\bar{x})+(E_n^+\setminus\{\theta\})] \times Y\}. \end{cases}$$

LEMMA 1: Under the hypotheses of Theorem 2, the sets A and B defined in (10) and (11) are disjoint, convex, and non-empty. Moreover, for every $x \varepsilon X$, we have $h(x)=<f(x), g(x)> \varepsilon A$.

PROOF OF LEMMA 1:

i.) Since $\theta_{n+p} \varepsilon Z$, it is clear that

$(x \varepsilon X): h(x) \varepsilon A$.

Since this is the case, it is obvious that if $X \neq \emptyset$, then $A \neq \emptyset$. It is also obvious that if $Y \neq \emptyset$, then $B \neq \emptyset$; and it is clear that B is convex, since it is the Cartesian Product of two convex sets.

ii.) In order to prove that A is convex, suppose that

(12) $\hat{a} = <\hat{s}, \hat{t}>, \tilde{a}=<\tilde{s}, \tilde{t}> \varepsilon A$.

Then $\exists \hat{x}, \tilde{x} \varepsilon X \ni$

(13) $h(\hat{x})-\hat{a} \varepsilon Z$,

(14) $h(\tilde{x})-\tilde{a} \varepsilon Z$.

Let $\lambda \varepsilon [0, 1] \subseteq E_1^+$, and define

(15) $a(\lambda)=\lambda\hat{a}+(1-\lambda)\tilde{a}$,

(16) $x(\lambda)=\lambda\hat{x}+(1-\lambda)\tilde{x}$.

Since Z is convex, we have by (13) and (14):

(17) $\lambda[h(\hat{x})-\hat{a}]+(1-\lambda)[h(\tilde{x})-\tilde{a}]=\lambda h(\hat{x})+(1-\lambda)h(\tilde{x})-a(\lambda) \varepsilon Z$.

Moreover, since X is convex, and h is Z-concave on X:

(18) $h[x(\lambda)]-[\lambda h(\hat{x})+(1-\lambda)h(\tilde{x})] \varepsilon Z$.

Hence, since Z is a convex cone, we have by (17) and (18):

$$\{h[x(\lambda)]-[\lambda h(\hat{x})+(1-\lambda)h(\tilde{x})]\}+\{\lambda h(\hat{x})+(1-\lambda)h(\tilde{x})-a(\lambda)\}=h[x(\lambda)]-a(\lambda) \varepsilon Z.$$

Therefore

$a(\lambda) \varepsilon A[x(\lambda)] \subseteq A$;

and we conclude that A is convex.

iii.) In order to show that $A\cap B=\emptyset$, suppose b.w.o.c. that $\exists\langle\hat{s}, \hat{t}\rangle\varepsilon A\cap B$. Then, since $\langle\hat{s}, \hat{t}\rangle\varepsilon A$,

$$\exists\hat{x}\varepsilon X \ni h(\hat{x})-\langle\hat{s}, \hat{t}\rangle\varepsilon\ Z.$$

But then, since $\langle\hat{s}, \hat{t}\rangle\varepsilon B$, we have

(19) $f(\hat{x})\geq\hat{s}>f(\overline{x})$,

$$\hat{t}\varepsilon Y,\ g(\hat{x})-\hat{t}\varepsilon Y,$$

and therefore, since Y is a convex cone:

(20) $(g(\hat{x})-\hat{t})+\hat{t} = g(\hat{x})\varepsilon Y$.

However, (19) and (20) together contradict the assumption that $\overline{x}$ is a solution of π. Hence $A\cap B=\emptyset$.

Q.E.D.

<u>LEMMA 2</u>: Under the hypotheses of Theorem 2, and with A and B defined as in (10) and (11),

$$\exists\ \langle\overline{v}, \overline{w}\rangle\varepsilon E_{n+p} \ni$$

i.) $\langle\overline{v}, \overline{w}\rangle\neq\theta$;

ii.) $\overline{v}\cdot s+\overline{w}\cdot t\leq\overline{v}\cdot z+\overline{w}\cdot y$ for all $\langle s, t\rangle\varepsilon A$, $\langle z, y\rangle\varepsilon B$;

iii.) $\overline{v}\varepsilon E_n^+$, $\overline{w}\varepsilon Y^*$,

iv.) $\overline{v}\cdot f(x)+\overline{w}\cdot g(x)\leq\overline{v}\cdot f(\overline{x})$ for all $x\varepsilon X$,

v.) $\overline{w}\cdot g(\overline{x})=0$.

<u>PROOF OF LEMMA 2</u>:

By Lemma 1 and the "separating hyperplane theorem" (<u>Cf</u>., Berge [6], p. 163):

$$\exists\langle\overline{v}, \overline{w}\rangle\varepsilon E_{n+p}$$

satisfying (i) and (ii).

By the conclusion of Lemma 1, $\langle f(x), g(x)\rangle\varepsilon A$ for every $x\varepsilon X$. Hence it follows from (ii) that we must have, in particular:

(21) $\quad \bar{v}\cdot[z-f(\bar{x})]+\bar{w}\cdot[y-g(\bar{x})]\geqq 0$ for all $z>f(\bar{x})$, $y\varepsilon Y$;

from which it follows immediately that:

(22) $\quad \bar{v}\varepsilon E_n^+$, $\bar{w}\varepsilon Y^*$,

which verifies (iii).

Since $<f(\bar{x}),\ \theta_p>$ is on the boundary of B and by Lemma 1,

$$(x\varepsilon X):\ <f(x),\ g(x)>\varepsilon A,$$

it also follows from (ii) that we must have:

(23) $\quad \bar{v}\cdot f(x)+\bar{w}\cdot g(x)\leqq\bar{v}\cdot f(\bar{x})$ for all $x\varepsilon X$,

which verifies (iv).

Finally, letting $x=\bar{x}$ on the l.h.s. of (23), and using (22) and the fact that $g(\bar{x})\varepsilon Y$,

we have

$$\bar{w}\cdot g(\bar{x})=0,$$

which verifies (v).

Q.E.D.

We are at last ready to prove Theorem 2.

PROOF OF THEOREM 2:

Combining (iii)-(v) of the conclusion of Lemma 2, we have:

$$\bar{v}\cdot f(x)+\bar{w}\cdot g(x)\leqq\bar{v}\cdot f(\bar{x})=\bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x})\leqq\bar{v}\cdot f(\bar{x})+w\cdot g(\bar{x}) \text{ for all } x\varepsilon X,\ w\varepsilon Y^*.$$

Combining this result with (iii) of Lemma 2 and the definition of $\bar{x}$, we see that $<\bar{x},\ \bar{v},\ \bar{w}>$ is a GSP for Φ_π.

Q.E.D.

Under certain assumptions, one obtains in the classical theory of constrained extrema (with equality constraints, and where all the functions involved are differentiable):

(24) $\exists <\lambda_0,\lambda>\varepsilon E_{1+p} \ni \lambda_0 f_x(\overline{x})+\lambda\cdot g_x(\overline{x}) = 0,$

where $\overline{x}$ maximizes f subject to $g(x)=\theta_p$, f_x represents the gradient vector of f, and g_x the matrix of partial derivatives $[\partial g_i/\partial x_j]$. Theorem 2 is the analogue of this result in the case where our maximand function is vector-valued and nondifferentiable (more specifically, where our maximization problem is of the form specified in Definition 3). If we add the rank condition to the hypotheses implying (24), we can conclude $\lambda_1 \neq 0$, and obtain:

$$f_x(\overline{x})+\overline{\lambda}\cdot g_x(\overline{x})=0,$$

where $\overline{\lambda}=(1/\lambda_0)\lambda$. Similarly, if we add a constraint qualification (together with some assumptions about the dimensions of Y and X) to the hypotheses of Theorem 2, we can conclude that $\overline{v}\neq\theta$ in our GSP, and hence that a GNSP exists at $<\overline{x}, \overline{w}, \overline{v}>$ (and if n=1 obtain a Saddle Point at $<\overline{x}, (1/\overline{v})\overline{w}>$ as we noted in our earlier discussion).[16] This is essentially the content of Theorem 3, to which we now turn. We shall, however, have need for the following lemmas in our proof. The result in Lemma 3 is quite well known, and a proof is included here only for the sake of providing a convenient reference.[17]

<u>LEMMA 3</u>: Let $X\subseteq E_n$, and suppose that $\overline{x}\varepsilon\text{int}(X)$, $y\varepsilon E_n$. If $y\cdot\overline{x}\geqq y\cdot x$ [resp., $y\cdot\overline{x}\leqq y\cdot x$] for every $x\varepsilon X$, then $y=\theta_n$.

<u>PROOF</u>:

If $\overline{x}\varepsilon\text{int}(X)$, $\exists\, \overline{\lambda}>0 \ni \overline{x}+\overline{\lambda}y\varepsilon X$, and we have

$$y\cdot[\overline{x}+\overline{\lambda}y] = y\cdot\overline{x}+\overline{\lambda}y\cdot y.$$

Hence, if $y\neq\theta_n$,

$$y\cdot[\overline{x}+\overline{\lambda}y]>y\cdot\overline{x}.$$

The result with the reversed inequality follows immediately from this.

Q.E.D.

LEMMA 4:

If:

1.) $g: E_m \to E_n$, where $m \geq n$,

2.) g is affine, i.e., $g(x)=Gx+b$, and rank $(G)=n$,

3.) $X \subseteq E_m$, $\bar{x} \varepsilon \text{int}(X)$;

then there exist open neighborhoods $N_1(\bar{x}) \subseteq X$, $N_2[g(\bar{x})] \subseteq E_n \ni$

$$N_2[g(\bar{x})] \subseteq g[N_1(\bar{x})].$$

PROOF:

Partition the matrix G by

$$G=[G_1 \;\; G_2],$$

where G_1 is $n \times n$, and we assume w.l.o.g. that rank $(G_1)=n$. By assumption, $\exists N_1(\bar{x}) \ni N_1(\bar{x}) \subseteq X$.

Write

$$\bar{x}=<\bar{x}^1 \;\; \bar{x}^2>,$$

where $\bar{x}^1 \varepsilon \; E_n$, $\bar{x}^2 \varepsilon E_{m-n}$. Then $\exists N_3(\bar{x}^1) \ni z \varepsilon N_3(\bar{x}^1) \Rightarrow <z,\bar{x}^2> \varepsilon N_1(\bar{x})$.

Define h on E_n by

$$h(z)=G_1 z.$$

It then follows by Theorems 7-3 and 7-4, pp. 141 and 143, respectively, in Apostol [1], that:

$$\exists N_4[h(\bar{x}^1)] \ni$$

$$N_4[h(\bar{x}^1)] \subseteq h[N_3(\bar{x}^1)].$$

But, it is clear that

$$\bar{y} \equiv g(\bar{x}) \varepsilon N_4[h(\bar{x}^1)] + G_2\bar{x}^2 + b \subseteq g[N_1(\bar{x})].$$

Hence, noting that if M is an open sphere containing $\bar{y}$, $M+\hat{y}$ is an open sphere containing $\bar{y}+\hat{y}$; we see that if we define:

$$N_2[g(\bar{x})]=N_3[h(\bar{x}^1)]+G_2\bar{x}^2+b,$$

N_2 is an open sphere containing $g(\bar{x})$and

$$N_2[g(\bar{x})]\subseteq g[N_1(\bar{x})].$$

Q.E.D.

THEOREM 3:

If:

1.) $\langle f, g, X, Y\rangle$ defines a maximization problem, π , where:

a.) X is convex, $int(X)\neq\emptyset$,

b.) f is concave (i.e., E_n^+-concave) on X,

c.) Y is of the form $Y=Y_1\times Y_2$, where

i.) $Y_1\subseteq E_q$, $Y_2\subseteq E_r$ $(q+r=p)$

ii.) $int(Y_2)\neq\emptyset$ (in E_r)

d.) g is of the form $g(x)=\langle g^{(1)}(x), g^{(2)}(x)\rangle$, where

i.) $g^{(1)}: D\to E_q$, $g^{(2)}: D\to E_r$

ii.) $g^{(1)}$ is affine[18] $(g^{(1)}(x)=Gx+b)$, and we assume w.l.o.g. that rank $(G)=q$

iii.) $g^{(2)}$ is Y_2-concave on X,

2.) g satisfies:

CQ$_1$: i.) $\exists x^{\dagger}\varepsilon int(X)\ni g^{(1)}(x^{\dagger})\varepsilon Y_1$

ii.) $\exists x^*\varepsilon X\ni g^{(1)}(x^*)\varepsilon Y_1$, $g^{(2)}(x^*)\varepsilon int(Y_2)$

3.) $\bar{x}\varepsilon X$ is a solution of π;

then

$$\bar{v}\varepsilon[E_n^+\backslash\{\theta_n\}],\ \bar{w}\varepsilon Y^*\ni\Phi_\pi \text{ has a GNSP at } \langle\bar{x}, \bar{v}, \bar{w}\rangle.$$

PROOF:

It is clear that g is Y-concave on X. Hence, we can readily verify that the hypotheses of Theorem 2 are satisfied. Therefore, by Theorem 2:

$\exists \langle\bar{v}, \bar{w}\rangle \varepsilon E_{n+p} \ni$

(25) $\langle\bar{v}, \bar{w}\rangle \neq \theta$

(26) $\bar{v} \varepsilon E_n^+$, $\bar{w} \varepsilon Y^*$

and Φ_π has a GSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$. Moreover, by Lemma 2:

(27) $\bar{v}\cdot s+\bar{w}\cdot t \leqq \bar{v}\cdot z+\bar{w}\cdot y$ for all $\langle s, t\rangle \varepsilon A$, $\langle z, y\rangle \varepsilon B$;

where A and B are defined in (10) and (11), above.

Writing $\bar{w}=\langle\bar{w}^1, \bar{w}^2\rangle$, where $\bar{w}^1 \varepsilon E_q$, $\bar{w}^2 \varepsilon E_r$,

we see that we have from (27):

(28) $\bar{v}\cdot f(x)+\bar{w}^1\cdot g^{(1)}(x)+\bar{w}^2\cdot g^{(2)}(x) \leqq \bar{v}\cdot z+\bar{w}^1\cdot y^1+\bar{w}^2\cdot y^2$ for all $x \varepsilon X$, $y^1 \varepsilon Y_1$, $y^2 \varepsilon Y_2$.

Suppose now that $\bar{v}=\theta$. Then by (28) and (ii) of CQ_1, we have:

$$\bar{w}^1\cdot g^{(1)}(x^*)+\bar{w}^2\cdot g^{(2)}(x^*) \leqq \bar{w}^1\cdot g^{(1)}(x^*)+\bar{w}^2\cdot y^2 \text{ for all } y^2 \varepsilon Y_2,$$

or

(29) $\bar{w}^2\cdot g^{(2)}(x^*) \leqq \bar{w}^2\cdot y^2$ for all $y^2 \varepsilon Y_2$.

Hence by Lemma 3:

(30) $\bar{w}^2=\theta_r$ (if $\bar{v}=\theta_n$)

We then have from (28) (if $\bar{v}=\theta_n$):

(31) $\bar{w}^1\cdot g^{(1)}(x) \leqq \bar{w}^1\cdot y^1$ for all $y^1 \varepsilon Y_1$, $x \varepsilon X$.

By (i) of CQ_1, $\exists x^\dagger \varepsilon \operatorname{int}(X) \ni g^{(1)}(x^\dagger) \varepsilon Y_1$. Since $x^\dagger \varepsilon \operatorname{int}(X)$,

$\exists N(x^\dagger) \subseteq X$.

Define

$$y^\dagger=g^{(1)}(x^\dagger).$$

We then have by Lemma 4:

$$\exists N_1(y^\dagger) \subseteq E_q \ni N_1(y^\dagger) \subseteq g^{(1)}[N(x^\dagger)].$$

Hence by (31), we have:

$$\bar{w}^1\cdot y^1 \leqq \bar{w}^1\cdot y^\dagger \text{ for all } y^1 \varepsilon N_1(y^\dagger).$$

It then follows from Lemma 3 that:

(32) $\overline{w}^1 = \theta_q$ (if $\overline{v} = \theta_n$).

Combining (30) and (32), we see that if $\overline{v} = \theta_n$, we have:

$$\langle \overline{v}, \overline{w} \rangle = \theta_{n+p},$$

contradicting (25). Therefore $\overline{v} \neq \theta$, and Φ_π has a GNSP at $\langle \overline{x}, \overline{v}, \overline{w} \rangle$.

Q.E.D.

The following result is almost a special case of Theorem 3[19]; and is, moreover, essentially a special case of Professor Hurwicz's Theorem V.3.1 in [8] (p. 91). It is included here for the sake of completeness.

THEOREM 4 - COROLLARY (HURWICZ):

If:

1.) $\langle f, g, X, Y \rangle$ defines a maximization problem, π, where:

   a.) X is convex,

   b.) f is concave (*i.e.*, E_n^+-concave) on X,

   c.) $\text{int}(Y) \neq \emptyset$, g is Y-concave on X

2.) g satisfies:

   CQ_s: $\exists x^* \varepsilon X \ni g(x^*) \varepsilon \text{int}(Y)$,[20]

3.) $\overline{x} \varepsilon X$ is a solution of π;

then

$$\exists\, \overline{v} \varepsilon [E_n^+ \setminus \{\theta_n\}],\ \overline{w} \varepsilon Y^* \ni \Phi_\pi \text{ has a GNSP at } \langle \overline{x}, \overline{v}, \overline{w} \rangle.$$

PROOF:

Re-examining the proof of Theorem 3, we see that the only steps in the argument which used the assumption $\text{int}(X) \neq \emptyset$ were in the proof that $\overline{v} = \theta_n$ implies $\overline{w}^1 = \theta_q$. Hence Theorem 4 follows as a corollary of the proof of Theorem 3.

Q.E.D.

The following result is an immediate corollary of the proof of Theorem 3:

THEOREM 5 - COROLLARY:

If:

1.) $\langle f, g, X, Y\rangle$ defines a maximization problem, π, where:

a.) X is convex, $\mathrm{int}(X)\neq\emptyset$,

b.) f is concave (i.e., E_n^+-concave) on X,

c.) g is affine,

2.) g satisfies:

CQ$_3$: $\exists x^\dagger \varepsilon \mathrm{int}(X) \ni g(x^\dagger)\varepsilon Y$,

3.) $\bar{x}\varepsilon X$ is a solution of π;

then:

$\exists \bar{v}\varepsilon[E_n^+ \setminus \{\theta_n\}]$, $\bar{w}\varepsilon Y^* \ni \Phi_\pi$ has a GNSP at $\langle\bar{x}, \bar{v}, \bar{w}\rangle$.

Theorem 3 is a generalization and slight correction of Theorem 3 in Uzawa [14], p. 36.[21] An example of a situation wherein Theorem 3, but not Theorem 4, is applicable is given by the last example on p. 5 ; if the functions h_i appearing there are assumed to be affine for $i=1, \ldots,$ q, concave for $i=q+1, \ldots, r$, and convex for $i=r+1, \ldots, p$ (and we suppose that X is convex, and f is E_n^+-concave on X). To see this, suppose we define

$$Y_1 = \{\theta_q\},\ Y_2=E_{r-q}^+ \times[-E_{p-r}^+], \text{ and } Y=Y_1\times Y_2.$$

We note that, under the current assumptions, $\langle f, g, X, Y\rangle$ defines a maximization problem, and g is Y-concave on X. However, it is impossible for g to satisfy CQ$_2$ in this case, since $\mathrm{int}(Y)=\emptyset$.[22] We can, however, apply Theorem 3 if g satisfies CQ$_1$.

III. CONSTRAINT QUALIFICATIONS AND THE GEOMETRY OF GENERALIZED SADDLE POINTS

We shall begin our discussion of considering some facets of the geometric nature of a GSP. Suppose we have a maximization problem, π, defined by $\langle f, g, X, Y\rangle$, and suppose Φ_π has a GSP at $\langle \bar{x}, \bar{v}, \bar{w}\rangle \varepsilon E_{m+n+p}$. Then

(1) $\bar{v} \varepsilon E_n^+$, $\bar{w} \varepsilon Y^*$,

and

(2) $\bar{v}\cdot f(x)+\bar{w}\cdot g(x) \leqq \bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x}) \leqq \bar{v}\cdot f(\bar{x})+w\cdot g(\bar{x})$ for all $x \varepsilon X$, $w \varepsilon Y^*$.

It is clear, then, that the existence of a GSP at $\langle \bar{x}, \bar{v}, \bar{w}\rangle$ implies:

(3) $\bar{w}\cdot g(\bar{x})=0$.

Therefore, if $b=\langle z, y\rangle \varepsilon E_{n+p}$ is such that:

(4) $z \geq f(\bar{x})$, $y \varepsilon Y$;

we have:

$$\bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x}) \leqq \bar{v}\cdot z+\bar{w}\cdot y.$$

Recalling the definition of the set B used in Section II:

(5) $B(\pi, \bar{x}) = \{b=\langle z, y\rangle \varepsilon E_{n+p} \mid z>f(\bar{x}), y \varepsilon Y\}$,

we see that:

(6) $\bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x}) \leqq \bar{v}\cdot z+\bar{w}\cdot y$ for all $\langle z, y\rangle \varepsilon B(\pi, \bar{x})$.

Moreover, by (1) and (2), we see that if $\hat{x} \varepsilon X$ and $a=\langle s, t\rangle \varepsilon E_{n+p}$ are such that:

$$f(\hat{x}) \geq s, \; g(\hat{x})-t \varepsilon Y,$$

then

$$\bar{v}\cdot[f(\hat{x})-s] \geqq 0, \; \bar{w}\cdot[g(\hat{x})-t] \geqq 0;$$

and therefore

(7) $\bar{v}\cdot s+\bar{w}\cdot t \leqq \bar{v}\cdot f(\hat{x})+\bar{w}\cdot g(\hat{x}) \leqq \bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x})$.

Recalling our definition of the set A given in Section II:

(8) $A(\pi) = \{a=\langle s, t\rangle \varepsilon E_{n+p} \mid (\exists x \varepsilon X) f(x) \geq s,\ g(x)-t \varepsilon Y\}$;

we see that:

(9) $\bar{v}\cdot s+\bar{w}\cdot t \leqq \bar{v}\cdot f(\bar{x})+\bar{w}\cdot g(\bar{x})$ for all $\langle s, t\rangle \varepsilon A(\pi)$.

If as in Section II we write

(10) $h_\pi(x) = \langle f(x), g(x)\rangle$,

and

(11) $\bar{u} = \langle \bar{v}, \bar{w}\rangle \ \varepsilon \ E_n^+ \times Y^*$,

we have by (6) and (9):

(12) $\bar{u}\cdot a \leqq \bar{u}\cdot h_\pi(\bar{x}) \leqq \bar{u}\cdot b$ for all $a \varepsilon A(\pi)$, $b \varepsilon B(\pi, \bar{x})$.

From (12) we see, therefore, that a necessary condition for the existence of a GSP for Φ_π at $\bar{x}$ is that there exist a vector $\bar{u}=\langle \bar{v}, \bar{w}\rangle \varepsilon E_n^+ \times Y^*$ such that $\bar{u}$ separates the set $A(\pi)$ and $B(\pi, \bar{x})$ (Clearly this is also a sufficient condition, as we showed in the proof of Theorem 2). The question of whether Φ_π also has a GNSP at $\bar{x}$ boils down to whether there exists such a vector $\bar{u}$ which has $\bar{v} \neq \theta_n$. We shall now examine the function of the constraint qualification in guaranteeing that such a $\bar{u}$ does exist.

Suppose we begin by examining an illustrative situation in which no GNSP exists. In his very important 1950 article [13], Slater presents an example to show that the constraint qualification he'd introduced could not be dispensed with if one was concerned with the existence of a GNSP (in our terminology). Slater's example deals with the maximization problem, $\bar{\pi}$, defined by $\langle f, g, E_1, E_1^+\rangle$, where

$$f(x) = 1-x$$

$$g(x) = -(x-1).^2$$

Clearly the only solution of this problem is at $\bar{x}=1$. The image of the function

$h_{\overline{\pi}}(x)$ and the sets $A(\overline{\pi})$ and $B(\overline{\pi}, 1)$ for this case are graphed in Figure 1, below.

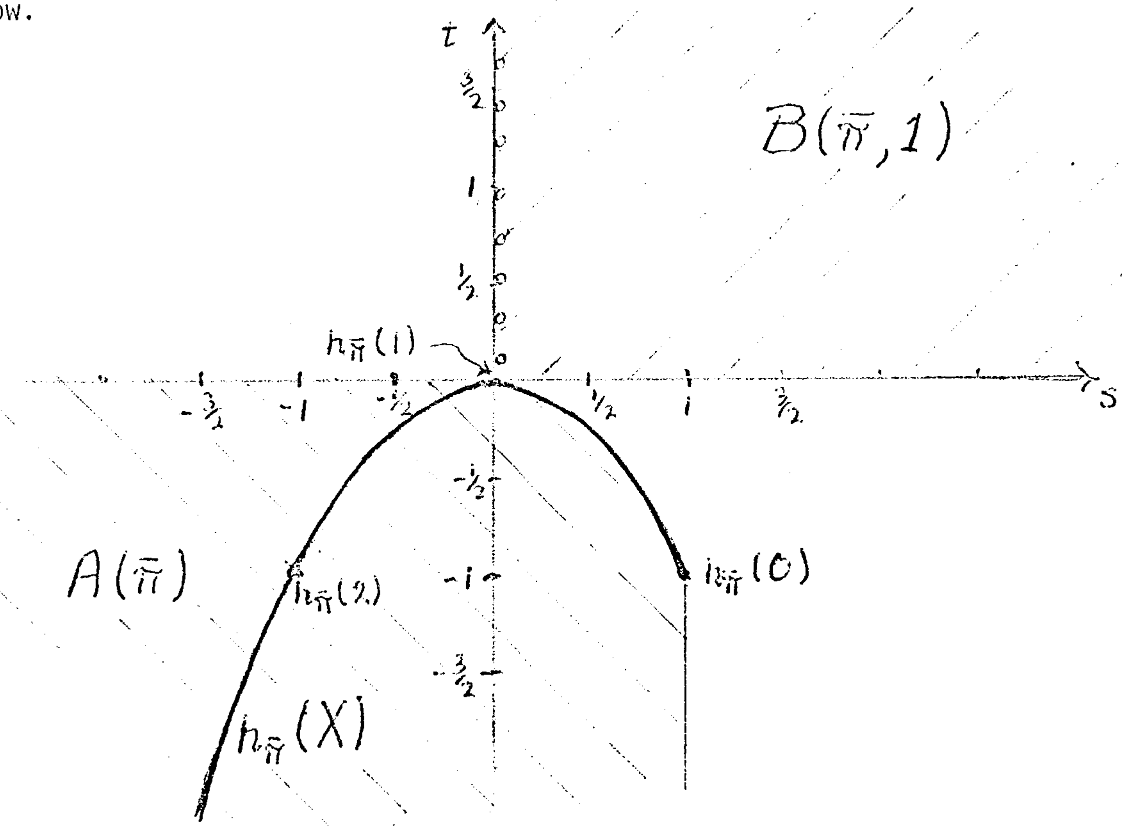

Figure 1

In this case, it is apparent from Figure 1 that no vector $\overline{u}\varepsilon E_1^+ \times E_1^+ = E_2^+$ exists which has a non vanishing first coordinate and which separates the sets $A(\overline{\pi})$ and $B(\overline{\pi}, 1)$. Hence no GNSP exists for Φ_{π} at $\overline{x}=1$. Notice, however, that $\Phi_{\overline{\pi}}$ does have a GSP at $\overline{x}$ (as we would expect, since all the hypotheses of Theorem 2 are satisfied here); in fact, Φ_{π} has a GSP at <1, 0, 1>.

In order to examine the workings of the Slater Constraint Qualification (which we shall hereafter refer to as the Slater CQ) in a little greater detail, suppose we consider the class of maximization problems, P, defined by <f, g, X, E_1^+>, where:

$$(13) \quad \begin{cases} X \subseteq D \subseteq E_1, \\ f: D \to E_1, \\ g: D \to E_1, \end{cases}$$

and f and g are concave on X. Let $\pi \varepsilon P$ and suppose $\overline{x}$ is a solution of π. If the Slater CQ holds, we can distinguish two cases, as follows.

CASE 1: $g(\overline{x})>0$.

In this case we'll have the sort of situation shown in Figure 2, below. While we don't have enough information to graph the set $A(\pi)$, we know that the set A^1 shown in Fig. 2 will be a subset of $A(\pi)$. Clearly, then, any vector $\overline{u}$ separating $A(\pi)$ and $B(\pi, \overline{x})$ must have $\overline{u}_1=\overline{v}\neq 0$ (in fact, any separating vector u must be a scalar multiple of $\overline{u}=\langle 1, 0\rangle$).[23]

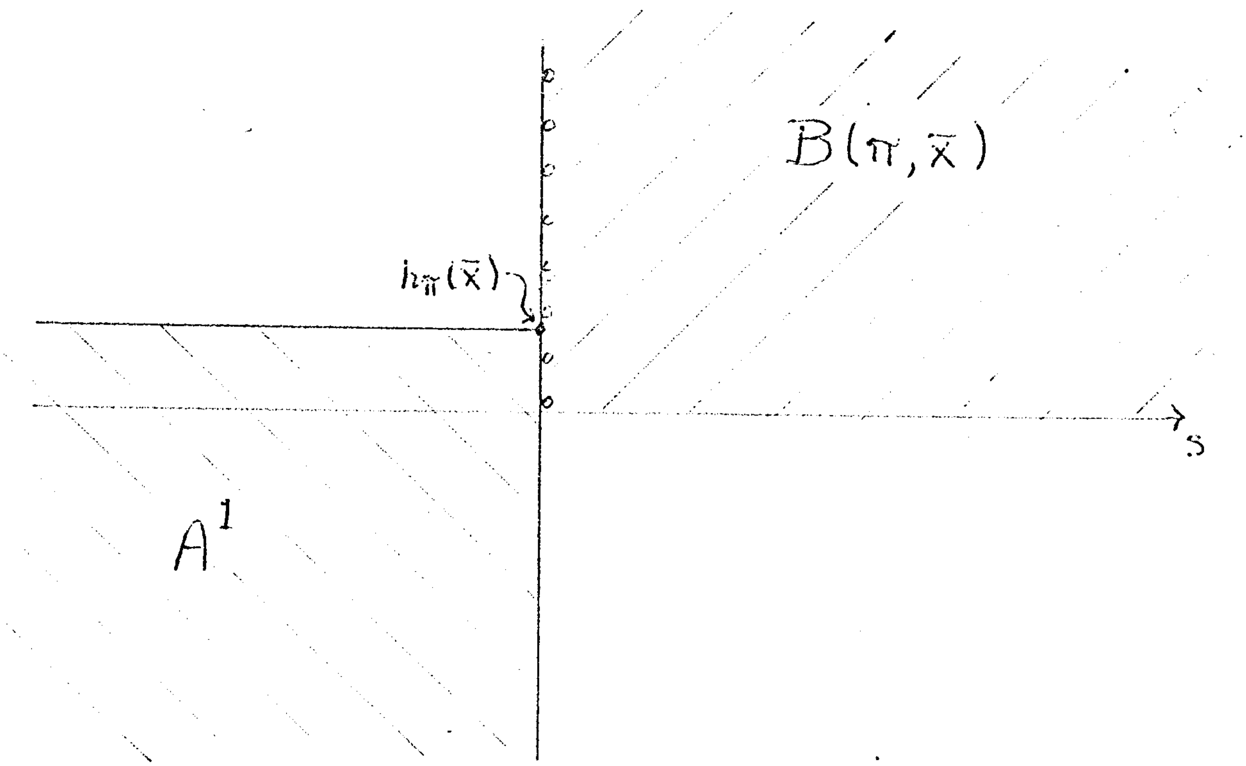

Figure 2

CASE 2: $g(\bar{x})=0$.

In this situation, according to the Slater CQ, there exists $x^* \varepsilon X \ni g(x^*)>0$. Hence $h_\pi(x^*)$ must stand in something like the relationship to $h_\pi(\bar{x})$ shown in Figure 3, below. While once again we do not have sufficient information to graph $A(\pi)$, we know that the set A^1 shown in Fig. 3 will be a subset of $A(\pi)$. Hence, it is clear from our diagram that any vector $\bar{u}=<\bar{v}, \bar{w}> \varepsilon E_2^+$ which separates $A(\pi)$ and $B(\pi, \bar{x})$ must have $\bar{v}\neq 0$.

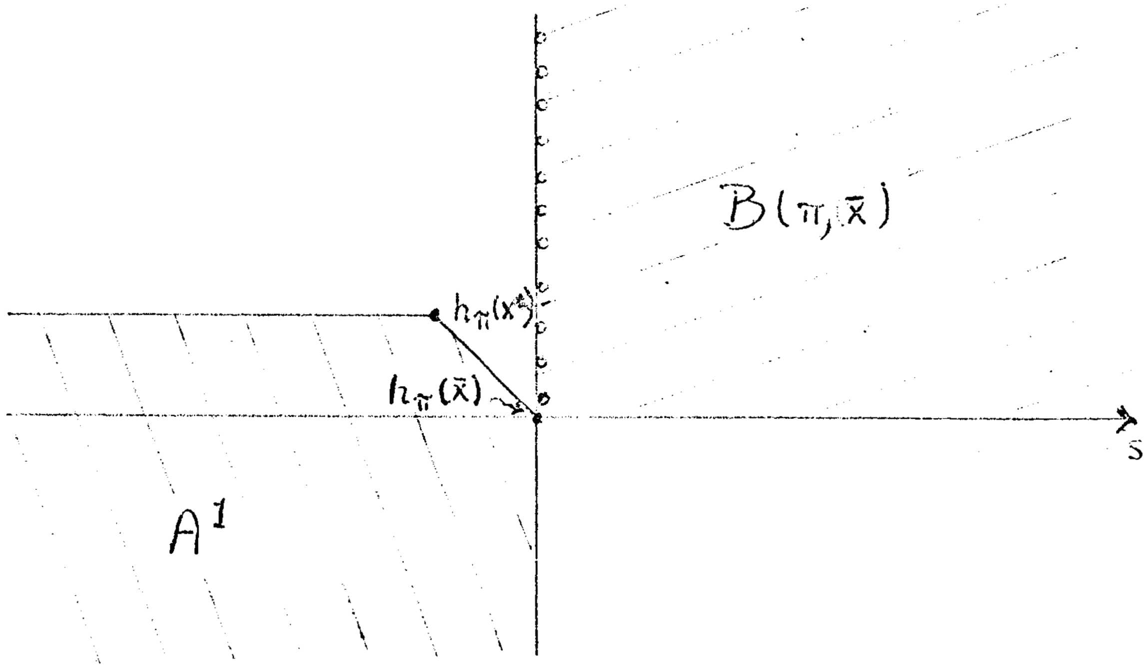

Figure 3

It is apparent from our discussion of the above two cases (and from a careful reading of the proof of Theorem 3) that if $\pi \varepsilon P$, the Slater CQ does more than guarantee the existence of a GNSP for Φ_π at a solution of π. It actually guarantees that if Φ_π has a GSP at $<\bar{x}, \bar{v}, \bar{w}>$, it is necessarily a GNSP. Consequently, it might appear that if one wished to concentrate on developing a set of conditions sufficient only to ensure the existence of a GNSP at $\bar{x}$, a solution of π, for some $\bar{u}\varepsilon E_n^+\times Y^*$ (and allowing for the possibility of the existence of other vectors $u=\langle v, w\rangle \ni \langle x, v, w\rangle$ is a GSP for Φ_π, but $v=\theta_n$); one could weaken the Slater CQ for this purpose. However, if the Slater CQ does not hold, the possibility arises of obtaining the sort of tangency solution depicted in Fig. 1; and it is difficult to see how any weaker condition could be developed which would have anything like the "nice" operational properties of the Slater CQ, and which would guarantee that this sort of tangency could not occur.

In the development of a theorem analogous to our Theorem 4, Karlin introduced ([10], p. 201) an interesting constraint qualification of a form different from the Slater CQ. Hurwicz and Uzawa proved in [9] that in very general spaces these two constraint qualifications were actually equivalent. Our next lemma is a special case of the Hurwicz-Uzawa result. It is presented here for both the sake of completeness and because it seems reasonable to take advantage of the more elementary spaces with which we're dealing to present a proof involving more elementary mathematics than that used by Professors Hurwicz and Uzawa.

<u>LEMMA 5</u> (HURWICZ-UZAWA):

If:

1.) $X\subseteq D\subseteq E_m$, $g\colon D\to E_p$,

2.) X is convex and non-empty,

3.) $Y\subseteq E_p$ is a convex cone, $\mathrm{int}(Y)\neq\emptyset$,

4.) g is Y-concave on X;

then the following are equivalent:

CQ_S: $\exists x^*\varepsilon X \ni g(x^*)\varepsilon\,\mathrm{int}(Y)$,

CQ_K: $(z\varepsilon[Y^*\setminus\{\theta_p\}])(\exists x\varepsilon X)\colon z\cdot g(x)>0.$

PROOF:

i.) $CQ_S \Rightarrow CQ_K$.

If $\bar{z}\varepsilon Y^*$ is $\ni$ $\bar{z}\cdot g(x^*)=0$, we have

$$\bar{z}\cdot g(x^*) \leqq \bar{z}\cdot y \text{ for all } y\varepsilon Y.$$

Hence by Lemma 3, $\bar{z}=\theta_p$. Therefore:

$$(z\varepsilon[Y^*\setminus\{\theta_p\}])\ z\cdot g(x^*)>0.$$

ii.) $CQ_K \Rightarrow CQ_S$.

Suppose CQ_K holds, but that:

(13) $\nexists x^*\varepsilon X \ni g(x^*)\varepsilon \text{int}(Y)$.

Define

$$A = \{t\varepsilon E_p \mid (\exists x\varepsilon X) g(x)-t\varepsilon Y\}.$$

Clearly A is convex (see the statement and proof of Lemma 1, above), and non-empty. Moreover, if there existed a vector $\bar{t}$ such that $\bar{t}\varepsilon A\cap\text{int}(Y)$, then we would have:

$$\exists \bar{x} \ni g(\bar{x})-\bar{t}\varepsilon Y.$$

But then, since $\bar{t}\varepsilon\text{int}(Y)$, it would follow that:

$$[g(\bar{x})-\bar{t}]+\bar{t} = g(\bar{x})\varepsilon\text{int}(Y)$$

(since if $y\varepsilon Y$, $\bar{y}\varepsilon\text{int}(Y)$, $y+\bar{y}\varepsilon\text{int}(Y)$; for Y a convex cone), which contradicts (13). Therefore:

(14) $A\cap\text{int}(Y)=\emptyset$.

Hence, since the convexity of int(Y) follows from the convexity of Y; we have by the "Separating Hyperplane Theorem"(see Berge [6], p. 163):

$\exists w\varepsilon E_p \ni$

(15) $w\neq\theta_p$, and

(16) $w\cdot t \leqq w\cdot y$ for all $t\varepsilon A$, $y\varepsilon\text{int}(Y)$.

However, it is clear from (16) that:

(17) $w \varepsilon Y^*$,

and from (16) and the definition of A that:

(18) $(x \varepsilon X): w \cdot g(x) \leqq 0$.

But (17) and (18) together contradict the assumption that the Karlin CQ (CQ_K) holds. Therefore (13) is false, that is, $\exists x^* \varepsilon X \ni g(x^*) \varepsilon \operatorname{int}(Y)$.

Q.E.D.

In reading the literature on saddle point theorems for the non-differentiable case, one is likely to get the feeling that a constraint qualification is not needed for the existence of a GNSP in the case where the constraint function g is affine.[24] More precisely, one might speculate that Theorem 5 of the previous section would remain correct if hypothesis 2 (CQ_3) were omitted. The constraint qualification cannot be dispensed with in this case, however, as the following example shows. Let the maximization problem $\overline{\pi}$ be defined by $<f, g, E_1^+, E_1^+>$, where $X=D=E_1^+$,

$$f(x)=\begin{cases} 0 \text{ for } x=0 \\ 1 \text{ for } x>0 \end{cases}$$

$$g(x)=-x.$$

Clearly $\overline{x}=0$ is the only solution for $\overline{\pi}$, and all the hypotheses of Theorem 4 are satisfied except hypothesis 2 (CQ_3).[25] The sets $A(\overline{\pi})$ and $B(\overline{\pi}, \overline{x})$, and the image of the function $h_{\overline{\pi}}$ are shown in Figure 4, below. It is apparent that any vector $\overline{u}=<\overline{v}, \overline{w}> \varepsilon E_2$ which separates $A(\overline{\pi})$ and $B(\overline{\pi}, \overline{x})$

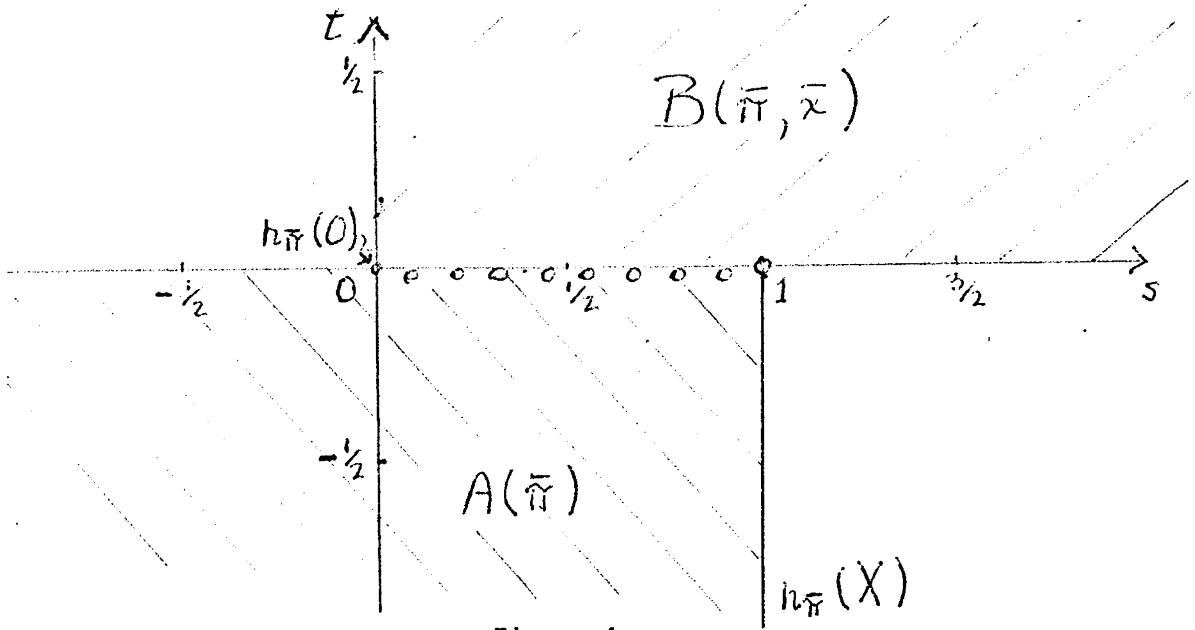

Figure 4

must have $\bar{v}=0$. Hence $\Phi_{\bar{\pi}}$ does not have a GNSP at $\bar{x}$. It should be noted that the need for the constraint qualification is not eliminated by requiring the maximimand function, f, to be continuous. The reader can easily verify that the maximization problem defined by <f, g, X, Y>, where

$$D=X=[0,\ 1],$$
$$f(x) = {}^{+}\!\sqrt{1-x^2},$$
$$g(x) = x-1,$$
$$Y = E_1^+ \text{ or } Y=\{0\};$$

does not have a GNSP at its solution, $\bar{x}=1$.

We have shown in the above examples that the constraint qualifications used in Theorems 3-5 of Section II cannot be dispensed with. Theorem 3 can be proved, however, with any one of several constraint qualifications; which, at least at first glance, appear to be non-equivalent. Hence the following result may be of some interest.

LEMMA 6:

If:

1.) $X\subseteq D\subseteq E_m$, $g(x)=\langle g^{(1)}(x),\ g^{(2)}(x)\rangle$, where $g^{(1)}$: $D\to E_q$, $g^{(2)}$: $D\to E_r$, 2.) X is convex, $\text{int}(X)\neq\emptyset$,

3.) $Y_1\subseteq E_q$ and $Y_2\subseteq E_r$ are convex cones,

4.) $\text{int}(Y_2)\neq\emptyset$ (in E_r),

5.) $g^{(1)}$ is affine,

6.) $g^{(2)}$ is Y_2-concave on X;

then the following are all equivalent:

CQ_1: i.) $\exists x^{\dagger}\varepsilon\,\text{int}(X)\ni g^{(1)}(x^{\dagger})\varepsilon Y_1$

ii.) $\exists x^{*}\varepsilon X\ni g^{(1)}(x^{*})\varepsilon Y_1,\ g^{(2)}(x^{*})\varepsilon\,\text{int}(Y_2)$,

CQ_4: $\exists\ \tilde{x}\varepsilon\,\text{int}(X)\ni g^{(1)}(\tilde{x})\varepsilon Y_1,\ g^{(2)}(\tilde{x})\varepsilon\,\text{int}(Y_2)$,

CQ_5: i.) $\exists\hat{x}\varepsilon\,\text{int}(X)\ni g^{(1)}(\hat{x})\varepsilon Y_1,\ g^{(2)}(\hat{x})\varepsilon Y_2$

ii.) $\exists x^{**}\varepsilon D\ni g^{(1)}(x^{**})\varepsilon Y_1,\ g^{(2)}(x^{**})\varepsilon\,\text{int}(Y_2)$.

PROOF:

i.) Obviously $CQ_4\Rightarrow CQ_1$. To prove the converse, define

$y^{\dagger} = g^{(2)}(x^{\dagger}),\ y^{*} = g^{(2)}(x^{*})$.

Since $y^{*}\varepsilon\,\text{int}(Y_2),\ \exists\,\bar{\lambda}\varepsilon(0,\ 1)\ni$

(18) $\bar{\lambda}y^{\dagger}+(1-\bar{\lambda})y^{*}\varepsilon\,\text{int}(Y_2)$.

Define

$$\tilde{x} = \bar{\lambda}x^{\dagger}+(1-\bar{\lambda})x^{*}.$$

Then, since $x^{\dagger}\varepsilon\,\text{int}(X)$, $\bar{\lambda}>0$, and X is convex:

(19) $\tilde{x}\varepsilon\,\text{int}(X)$.

Moreover, since $g^{(1)}$ is affine, and Y_1 is convex:

(20) $g^{(1)}(\tilde{x}) = \bar{\lambda}g^{(1)}(x^{\dagger})+(1-\bar{\lambda})g^{(1)}(x^{*})\varepsilon Y_1$;

while, since $g^{(2)}$ is Y_2-concave:

(21) $g^{(2)}(\tilde{x})-[\bar{\lambda}y^{\dagger}+(1-\bar{\lambda})y^{*}]\varepsilon Y_2$.

By (18) and (21), it follows that:

(22) $\{g^{(2)}(\tilde{x})-[\bar{\lambda}y^{\dagger}+(1-\bar{\lambda})y^{*}]\}+\{\bar{\lambda}y^{\dagger}+(1-\bar{\lambda})y^{*}\}=g^{(2)}(\tilde{x})\varepsilon\,\text{int}(Y_2)$.

Therefore, by (19), (20), and (22), we see that $\tilde{x}$ satisfies the requirements of CQ_4. Hence,

$$CQ_4 \Leftrightarrow CQ_1.$$

ii.) Obviously $CQ_4 \Rightarrow CQ_5$. The proof of the converse proceeds in a fashion very similar to that developed in (i), above. This time we can choose $\bar{\lambda}\varepsilon(0, 1)$ small enough so that, letting

$$\tilde{x} = \bar{\lambda}x^{**}+(1-\bar{\lambda})\hat{x},$$

we have:

$$\tilde{x}\varepsilon\, \mathrm{int}(X).$$

We then can easily show, in the same way as in (i), that:

$$g^{(1)}(\tilde{x})\varepsilon Y_1,\ g^{(2)}(\tilde{x})\varepsilon\, \mathrm{int}(Y_2).$$

Hence $CQ_5 \Leftrightarrow CQ_4$.

Q.E.D.

By way of concluding our discussion, suppose we consider a problem tangentially related to the material of this section. In many applications of saddle point theorems to problems in Economics, one may be interested in conditions sufficient to guarantee that $\bar{w}$ does not vanish. For instance, the vector $\bar{w}$ may lend itself to a "shadow price" interpretation; and in these circumstances it is clearly of some importance to determine whether or not $\bar{w}=\theta$. After our discussion of the function of the constraint qualification in guaranteeing that $\bar{v}\neq\theta$, however, the following result is fairly obvious.

THEOREM 6:

If:

1.) $\langle f, g, X, Y \rangle$ defines a maximization problem, π, 2.) Φ_π has a GSP at $<\overline{x}, \overline{v}, \overline{w}>\varepsilon E_{m+n+p}$,

3.) $\exists \hat{x}\varepsilon X \ni f(\hat{x})>>f(\overline{x})$

then

$\overline{w}\neq\theta$.

PROOF:

If $\overline{w}=\theta$, then by the existence of a GSP at $<\overline{x}, \overline{v}, \overline{w}>$, we have:

$$\overline{v}\cdot f(x)\leqq\overline{v}\cdot f(\overline{x}) \text{ for all } x\varepsilon X,$$

where $\overline{v}>\theta_n$. But this would mean

$$\overline{v}\cdot f(\hat{x})\leqq\overline{v}\cdot f(\overline{x});$$

which is impossible; since

$$\overline{v}\cdot[f(\hat{x})-f(\overline{x})]>0.$$

Hence $\overline{w}\neq\theta$.

Q.E.D.

COROLLARY:

If:

1.) $<f, g, X, Y>$ defines a maximization problem, π, where:

a.) X is convex,

b.) f is concave (i.e., E_n^+-concave) on X,

c.) g is Y-concave on X,

2.) $\overline{x}\varepsilon X$ is a solution for π,

3.) $\exists \hat{x}\varepsilon X \ni f(\hat{x})>>f(\overline{x})$,

then

$\exists \overline{v}\varepsilon E_n^+$, $\overline{w}\varepsilon[Y^*\setminus\{\theta_p\}] \ni \pi$ has a GSP at $<\overline{x}, \overline{v}, \overline{w}>$.

PROOF:

This result follows immediately from Theorems 2 and 6.

Notice that the conditions sufficient for the non-vanishing of $\bar{w}$ stated in the above results require that the constraint $g(x) \varepsilon Y$ be effective in the sense that the solution $\bar{x}$ is <u>not</u> a solution of the problem:

maximize $f(x)$ subject to $x \varepsilon X$.[26]

Moreover, in the special case where $n=1$, (<u>i.e.</u>, where $f: D \to E_1$), we see that the effectiveness of the constraint $g(x) \varepsilon Y$ (in the sense just stated) is sufficient to guarantee the non-vanishing of $\bar{w}$.[27]

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APPENDIX

1.) In this section, we shall show that if f is defined on E_1^+ by

$$f(x) = +\sqrt{x},$$

then $\nexists h \ni$

$$h: E_1 \to E_1,\ h(x)=f(x) \text{ for } x \in E_1^+,$$

and h is concave on all of E_1.

PROOF:

Suppose b.w.o.c. that there exists such a function, h. Then clearly we must have:

$$h(x)<0 \text{ for } x<0.$$

Suppose, then, that we are given $\bar{x}<0$, and

(1) $\bar{y}=h(\bar{x})<0.$

Consider the point x defined by

(2) $x = \dfrac{\bar{x}^2}{4\bar{y}^2}$

Clearly $x>0$, and therefore

(3) $h(x) = +\sqrt{x} = \dfrac{\bar{x}}{2\bar{y}} > 0.$

Define

(4) $\bar{\lambda} = \dfrac{4\bar{y}^2}{4\bar{y}^2-\bar{x}}$

Then, since $\bar{x}<0$, we have:

$$0<\bar{\lambda}<1.$$

Therefore, if h is concave, it must be the case that

$$h[\bar{\lambda}x+(1-\bar{\lambda})\bar{x}] \geqq \bar{\lambda}h(x)+(1-\bar{\lambda})h(\bar{x})$$

(5) $$= \left[\frac{4\bar{y}^2}{4\bar{y}^2-\bar{x}}\right] \frac{\bar{x}}{2\bar{y}} + \left[1-\frac{4\bar{y}^2}{4\bar{y}^2-\bar{x}}\right]\bar{y} = \frac{\overline{yx}}{4\bar{y}^2-\bar{x}} > 0,$$

However, we have:

$$\overline{\lambda}x + (1-\overline{\lambda})\overline{x} = 0,$$

and therefore

$$h[\overline{\lambda}x + (1-\overline{\lambda})\overline{x}] = f(0) = 0,$$

which contradicts (5). Consequently if $h\colon E_1 \to E_1$ and $h(x) = f(x)$ for $x \in E_1^+$, h is not concave.

2.) PROFESSOR UZAWA'S THEOREM 3.

As indicated previously (n. 21), there appear to be some misprints in the statement and proof of Professor Uzawa's Theorem 3, pp. 35-37, in [14]. Because of the importance of the Uzawa article, it would seem that a brief discussion of these apparent misprints would be of some value; especially since the published version of Professor Uzawa's Theorem 3 (with misprints) makes it appear that some of the hypotheses of our Theorem 3 could be weakened in a fashion which would, in fact, make the theorem incorrect.

Using the notation developed in this paper, the necessity portion of Professor Uzawa's Theorem 3 can be stated as follows:

If:

1.) $\langle f, g, E_m^+, Y\rangle$ defines a maximization problem, π, where

   a.) $D=E_m$,

      i.) $f\colon E_m \to E_1$

      ii.) $g = \langle g^{(1)}, g^{(2)}\rangle$, where

         $g^{(1)}\colon E_m \to E_q$

         $g^{(2)}\colon E_m \to E_r$, $(q+r=p)$,

   b.) f is concave on E_m,

   c.) Y is of the form $\{\theta_q\} \times E_r^+$

   d.) $g^{(1)}$ is affine, $g^{(1)}(x) = Gx+b$, and we assume w.l.o.g. that rank $(G) = q$,

   e.) $g^{(2)}$ is concave (<u>i.e.</u>, E_r^+-concave) on E_m, 2.) g satisfies:

$\overline{CQ}_u$: for each i, i=1, . . ., m, $\exists x^i > \theta_m \ni x_i^i > 0,\ g^{(1)}(x^i) = \theta_q,\ g^{(2)}(x^i) \geqq \theta_r$,

3.) $\bar{x} \varepsilon X$ is a solution of π;

then

$\exists\ \bar{w} \varepsilon Y^*(=E_q \times E_r^+) \ni \Phi_\pi$ has a Saddle Point at $<\bar{x}, \bar{w}>$.

This statement is incorrect, the problem being a misprint in the Constraint Qualification, apparently. We can verify this as follows:

First of all, we note that $\overline{CQ}_u$ is equivalent to:

$$\overline{CQ}\text{:}\ \exists x^\dagger >> \theta_m \ni g^{(1)}(x^\dagger) = \theta_q,\ g^{(2)}(x^\dagger) \geqq \theta_r.$$

(To show that $\overline{CQ}_u \Rightarrow \overline{CQ}$, let

$x^\dagger = \Sigma_{i=1}^m \lambda_i x^i$, where $\lambda_i > 0$ for i=1, . . ., m; and $\Sigma_{i=1}^m \lambda_i = 1$)

The following then provides a counterexample (it was, of course, developed from the counterexample presented by Slater in [13]) to the above statement. Let

$$f(x) = x_1 + 2x_2$$

$$g(x) = <g^{(1)}(x),\ g^{(2)}(x)> = <x_1 + x_2 - 1,\ -(2x_1 + x_2 - 3/2)^2>.$$

Let π be defined by $<f,\ g,\ E_2^+,\ \{0\} \times E_1^+>$. We can readily verify that π satisfies hypothesis 1, and that $\bar{x} = <1/2,\ 1/2>$ satisfies $\overline{CQ}$ and is the solution of π. We can readily show, however, that if there existed a $\bar{w} \varepsilon E_1 \times E_1^+ \ni \Phi_\pi$ had a Saddle Point at $<\bar{x}, \bar{w}>$, it would be necessary that

$$\bar{w}_2 \geqq - \frac{x_1 - 1/2}{(x_1 - 1/2)^2} = \frac{1}{1/2 - x_1} \quad \text{for } x_1 \varepsilon [0,\ 1/2),\ x_1 + x_2 = 1.$$

But this is impossible, since the expression on the right approaches $+\infty$ as $x_1 \to 1/2^-$. Hence $\nexists\, \bar{w} \varepsilon E_1 \times E_1^+ \ni \Phi_\pi$ has a Saddle Point at $<\bar{x}, \bar{w}>$.

Professor Uzawa's Theorem 3 becomes correct if we substitute:

CQ_u: for each i, i=1, . . ., m, $\exists x^i > \theta_m \ni$
$x_i^i > 0$, $g^{(1)}(x^i) = \theta$, $g^{(2)}(x^i) >> \theta_m$;

which is apparently the constraint qualification which would have appeared in his Theorem 3 but for the misprint. Equivalently, we could use:

$$CQ_4^+ : \exists x^\dagger >> \theta_m \ni g^{(1)}(x^\dagger) = \theta,\ g^{(2)}(x^\dagger) >> \theta.$$

It should also be pointed out, however, that there is a misprint of some significance in Professor Uzawa's proof. The set B used in the proof should be defined by (in Prof. Uzawa's notation):

$$B = \{\langle z_0, z, y\rangle \mid z_0 > f(\overline{x}),\ z^I = 0,\ z^{II} > 0,\ y \geqq 0\}.$$

FOOTNOTES

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[2]If $g: D \to E_n$, where $D \subseteq E_m$ is convex, we say that g is <u>CONCAVE</u> on D if for all $x^1, x^2 \varepsilon D$, and for all scalars $\lambda \varepsilon [0, 1]$, we have

$$g[\lambda x^1 + (1-\lambda)x^2] \geqq \lambda g(x^1) + (1-\lambda)g(x^2).$$

A function, g, is <u>CONVEX</u> if -g is concave.

[3]A line of development first explored by Slater [13].

[4]Kuhn and Tucker, in [11], had begun this investigation with their consideration of the "vector maximum problem."

[5]The terminology used here is not quite consistent with normal mathematical terminology, since the term "affine" is normally used for a mapping of a space into itself. "Affine" seems to be a better term than "linear," however, since "linear" is normally taken to mean (in the Euclidean case) that g is of the form $g(x)=Gx$.

[6]Our notation here is an adaptation of that introduced by Hurwicz in [8]. Note that we're using π generically to denote maximization problems of the type defined in Definition 3. It should also be noted that our treatment here is somewhat asymmetric. If $Y \subseteq E_p$ is a convex cone, the ordering defined by $x \geq_Y y$ iff $x - y \varepsilon Y$ is what is known as a vector ordering. Moreover, the ordering, $\geq$, of E_n which we've defined in the text, is a special case of a vector ordering. Hence in many ways a more natural approach would be to deal with maximization problems, π, defined (given the situation of Definition 3) by $\langle f, g, X, \geq_1, \geq_2 \rangle$, where $\geq_1$ is a vector ordering of E_n and $\geq_2$ is a vector ordering of E_p. We would then say that $\bar{x}$ is a solution of π iff:

(1[1]) $\bar{x}\varepsilon X$, $g(\bar{x}) \underset{2}{\geq} \theta_p$,
and
(2[1]) $\nexists \hat{x}\varepsilon X \ni g(\hat{x}) \underset{2}{\geq} \theta_p$ and $f(\hat{x}) >_1 f(\bar{x})$.

This is the kind of approach taken by Prof. Hurwicz in [8] (a more symmetric approach is also followed by the present author in [12]). For the purposes of this paper, however, it seemed that the problem under discussion here was sufficiently simpler than this more symmetric treatment, and sufficiently general, to justify our Definition 3.

[7]Recall, however, that we intend to treat only the case where the functions f and g are not necessarily differentiable.

[8]A case of this sort which is familiar to Economists occurs when f is a vector of utility functions, and we are seeking a Pareto-Optimal point.

[9]Moreover, if $g^{(1)}$ is affine, h_i is concave for i=q+1, . . ., r, and convex for r+1, . . . , p; g will be Y-concave, so that the necessity results of Section II will apply. Note in particular Theorems 2 and 3.

[10]As is apparently the case in the parenthetical remark on p. 780 of the valuable work by Arrow and Enthoven [2]. It may be that Professors Arrow and Enthoven did not mean to imply that a function defined and concave on a convex subset of E_m could be extended to a function defined and concave on all of E_m; but rather that, under these circumstances, it could be extended to a function defined and quasi-concave over the whole space. This latter statement also appears to be incorrect, however, as the following example shows. Let f be defined on $E_1^+ \setminus \{0\}$ by:

$$f(x) = \log x.$$

Then f is defined and concave on $E_1^+ \setminus \{0\}$, which is a convex set, but it is clear that there is no way of extending f to a real-valued function defined and quasi-concave over the whole space. It should be emphasized, however, that the statement in question is in the nature of an aside, and in no way affects the text of the Arrow and Enthoven article.

[11]An example of a function which is concave and continuous on E_1^+, has continuous derivatives of all orders on the interior of E_1^+, but which is nonetheless not extendible is provided by:

$$f(x) = +\sqrt{x} \quad \text{for } x \geqq 0.$$

For a proof that this function is not extendible see Appendix (1), p. 36.

[12]This terminology is an adaptation of that introduced by Hurwicz in [8]. Note that if n=1 (<u>i.e.</u>, if f: $D \to E_1$), we have

$$\Phi_\pi(x, 1, w) = f(x) + w \cdot g(x);$$

which is the usual form of the Lagrangian expression.

[13]The types of saddle points introduced here are given somewhat different definitions by the author in [12]. The concepts developed there reduce to the definitions presented here, however, for the type of maximization problem with which we are concerned in this paper. Once again the notation is an adaptation of that introduced by Hurwicz in [8].

[14]We denote the set-theoretic difference of A and B by $A \setminus B$, i.e.,

$$A \setminus B = \{a \in A \mid a \notin B\}.$$

[15]The author is grateful to Dr. Mohamed El-Hodiri for bringing both the book and this particular theorem to his attention. The theorem stated by Berge is also implicit in Uzawa's proof of his Theorem 2 in [14].

[16]The reader might argue, however, that in view of Theorem 1, the more basic question is whether a GPSP exists. We shall not examine this question in the present paper, but the author has essayed such an investigation in [12]. Note, however, that if π has a GNSP at $\langle \bar{x}, \bar{v}, \bar{w} \rangle$, and we define $F(x) = \bar{v} \cdot f(x)$, and consider the maximization problem, π^1, defined by $\langle F, g, X, Y \rangle$, π^1 has a Saddle Point at $\langle \bar{x}, \bar{w} \rangle$. Therefore, by Theorem 1, $\bar{x}$ is a solution of π^1. This property is often useful in applications.

[17]Lemma 4 is more or less a standard result of Functional Analysis, and in fact is usually proved for spaces of greater generality than those with which we are dealing here. The author has been unable, however, to locate a reference presenting the special case of Lemma 4 (which makes possible a more elementary proof than that usually provided in the texts on Functional Analysis). This is why a proof is included here.

[18]Note that we can generalize this theorem by substituting

ii') $g^{(1)}$ is open and Y_1-concave on X, for hypothesis 1-d-ii of the text.

[19]It is not quite a special case of Theorem 3, for the reader will note that the assumption

$$\mathrm{int}(X) \neq \emptyset$$

is not included in the hypotheses of Theorem 4.

[20]The reader will recognize this as a generalization of the constraint qualification introduced by Slater in [13]. It was first used in this general form by Hurwicz in [8].

[21]There is an apparent misprint in the statement of the constraint qualification in Professor Uzawa's theorem. We shall discuss this result in Appendix (2).

[22] Defining $g^{(3)} = -g^{(1)}$, and $Y=E_q^+ \times Y_2 \times E_q^+$ does not, of course, solve the problem either; since, defining $g = \langle g^{(1)}, g^{(2)}, g^{(3)} \rangle$, the existence of an $x^* \varepsilon X$ satisfying CQ_S would then involve a contradiction.

[23] Notice that in this case the constraint $g(\underline{x}) \varepsilon E_1^+$ is not effective in the sense that, if f and g are concave on X, $\bar{x}$ maximizes f subject to $x \varepsilon X$. We can show that this must be the case by supposing b.w.o.c. that $\exists \hat{x} \varepsilon X \ni f(\hat{x}) > f(\bar{x})$. Define

$$\bar{\lambda} = \frac{-g(\hat{x})}{[g(\bar{x})-g(\hat{x})]}, \text{ and } \tilde{x} = \bar{\lambda}\bar{x} + (1-\bar{\lambda})\hat{x}.$$

Then

$$0<\bar{\lambda}<1, \text{ so that } \tilde{x} \varepsilon X;$$

but:

$$g(\tilde{x}) \geqq \bar{\lambda} g(\bar{x}) + (1-\bar{\lambda}) g(\hat{x}) = 0, \text{ and}$$

$$f(\tilde{x}) \geqq \bar{\lambda} f(\bar{x}) + (1-\bar{\lambda}) f(\hat{x}) > f(\bar{x}),$$

which contradicts the assumption that $\bar{x}$ is a solution of π. Alternatively, we can show the same result by the following reasoning. It follows by Theorem 2 that a GSP exists for π at $\bar{x}$; and therefore we have by the parenthetical remark in the text:

$$f(x) \leq f(\bar{x}) \text{ for all } x \varepsilon X.$$

[24] See, e.g., Karlin [10], Theorem 7.1.2, p. 203. Note, however, that the example in the text is not a counterexample to Professor Karlin's theorem, which requires (in the context of our example) f to be defined and concave over all of E_1.

[25] Note, moreover, that all the hypotheses of Theorem 4 are satisfied except CQ_S. Hence this example also shows that Theorem 4 does not remain correct if the assumption that g is affine is substituted for CQ_S.

[26] See p.24 and n. 23, above.

[27] See [12] for applications of these results to Activity Analysis.