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Preprinted for

10TH MIDWEST SYMPOSIUM
ON CIRCUIT THEORY
May 18-19, 1967
Purdue University
Lafayette, Indiana

This research was supported by the
Electronics Division, AFOSR,
SREE
under Contract/Grant AF-AFOSR 766-65

LABORATORIES FOR ELECTRONICS AND
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On the Stability Constraints and Oscillatory Behavior of Coupled Systems

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Abstract:

The system described by the differential equation

$$\ddot{x} + g(\dot{x}) + h(x) = 0$$

has been extensively investigated both from the stability as well as oscillatory points of view. The present paper discusses some properties of a class of higher order systems which are obtained by coupling second order systems of the above form. Sufficient conditions are derived for such a system to be stable and the effects of introducing additional nonlinearities in the feedback path are discussed. The oscillatory behavior is discussed qualitatively and some new results are presented concerning (1) the coupling of an oscillatory system to a stable system, and (2) the coupling of two oscillatory systems.

1. Introduction

Much work has been done on nonlinear second order differential systems from both stability and periodic behavior standpoints because of the ease of applying analytic and topological methods to their solution. An extensive bibliography is found in Cesari [1]. Two of the more general basic second order forms are,

$$\ddot{x} + g(\dot{x}) + h(x) = 0 \quad (\cdot = \frac{d}{dt}) \tag{1}$$

and

$$\ddot{x} + g(x, \dot{x}) + h(x) = 0. \tag{2}$$

Constraints must be placed on the nonlinear functions to assure asymptotic stability in the large, instability, or the existence of a limit cycle. For the sake of completeness two typical results are given below for the system (1).

System (1) is asymptotically stable in the large if

$$\begin{aligned}
&\text{i)} \quad x h(x) > 0, \ x \neq 0; \ h(0) = 0 \\
&\text{ii)} \quad \dot{x} g(\dot{x}) > 0, \ \dot{x} \neq 0; \ g(0) = 0 \qquad (*) \\
&\text{iii)} \quad \int_0^x h(u)du \rightarrow \infty \text{ as } |x| \rightarrow \infty
\end{aligned}$$

This result is proved by using the Liapunov function

$$V(x, \dot{x}) = 1/2 \ \dot{x}^2 + \int_0^x h(u)du \tag{3}$$

and applying LaSalle's Theorem [2] for which (i) and (ii) above assure that the origin is the only solution along which $\dot{V} = 0$.

If $g(\dot{x}) = \dot{x}g_1(\dot{x})$ and $h(x)$ in system (1) are continuous along with their first derivatives, then there exists at least one (nonconstant) periodic solution, $x(t)$, provided there are positive constants a, m, and M such that

$$\begin{aligned}
&\text{i)} \quad g_1(\dot{x}) > m \text{ for } |\dot{x}| \geq a \\
&\text{ii)} \quad g_1(\dot{x}) > -M \text{ for all } \dot{x} \qquad (**)
\end{aligned}$$

iii) $g_1(\dot{x}) < 0$ in some neighborhood of $\dot{x}=0$ excluding $\dot{x}=0$,

iv) $xh(x) > 0$, $x \neq 0$, $h(0) = 0$

v) $|h(x)| \to \infty$ and $h(x)/\int_0^x h(u)du \to 0$ as $|x| \to \infty$.

The proof of this result uses the positive definite auxiliary function (3) to show that all trajectories, except the origin, started within a small neighborhood of the origin leave that neighborhood and all trajectories started sufficiently far from the origin are shown to enter a special curve constructed from V. (For proof see [1]).

For system (2) the Liapunov function (3) gives rise to a stability result similar to (*) and Wax [3] has recently proved the limit cycle theorem for a subclass of this system.

Higher order systems may be constructed by coupling second order subsystems as follows:

$$\ddot{x}_i + g_i(\dot{x}_i) + h_i(x_i) + k_i(\bar{x}) = 0 \qquad (4)$$

$$i = 1,2,\ldots,n$$

$$\bar{x} = (x_1, x_2, \ldots, x_n)$$

The result on asymptotic stability may be extended to include the above higher order systems.

The 2n-order system (4) is asymptotically stable in the large if for $i = 1,2,\ldots,n$

i) $\dot{x}_i g_i(\dot{x}_i) > 0$, $\dot{x}_i \neq 0$; $g_i(0) = 0$

ii) $x_i h_i(x_i) > 0$, $x_i \neq 0$; $h_i(0) = 0$ (***)

iii) $k_i(\bar{x}) = \frac{\partial}{\partial x_i} U(\bar{x})$ where U is a positive semidefinite function

iv) $\int_0^{x_i} h_i(u)du \to \infty$ as $|x_i| \to \infty$, and

v) the set of equations $h_i(x_i) + k_i(\bar{x}) = 0$, $i = 1,2,\ldots,n$, has only the trivial solution $\bar{x} = \bar{0}$.

One can prove this result using LaSalle's Theorem [2] where the Liapunov function is

$$V(\bar{x},\dot{\bar{x}}) = 1/2 \sum_{i=1}^{n} \dot{x}_i^2 + \sum_{i=1}^{n} \int_0^{x_i} h_i(u)du + U(\bar{x}) \tag{5}$$

for which

$$\dot{V}(\bar{x},\dot{\bar{x}}) = - \sum_{i=1}^{n} \dot{x}_i g_i(\dot{x}_i).$$

No results similar to (**) above exist for the system (4). However, very restricted results on amplitude bounds are available.

Aggarwal [4] has shown for the system

$$\begin{aligned} &\ddot{x} + A(\dot{x}^3/3 - \dot{x}) + Bx = -\alpha(x-y) \\ &\ddot{y} + C(\dot{y}^3/3 - \dot{y}) + Dy = \alpha(x-y) \end{aligned} \tag{6}$$

the existence of an "inner" amplitude bound and this result may be generalized to the situation where more than two oscillators are coupled.

This paper discusses the stability constraints and oscillatory behavior of system (4) and its extensions. Also, results on the oscillatory behavior of the coupled systems, when one subsystem is stable and the other oscillatory, or when both subsystems are oscillatory are presented.

2. Stability of Coupled Systems

An extension of the System (4) with a single dissipative element has the form

$$\ddot{x}_i + g_i(\dot{x}_i) + h_i(x_i) + k_i(\bar{x}) + e_i f\left(\sum_{m=1}^{n} e_m \dot{x}_m\right) = 0 \qquad (7)$$

$$i = 1, 2, \ldots, n$$

where

$$e_m = \begin{cases} +1 \\ 0 \\ -1 \end{cases} \quad \text{depending upon the system.}$$

In the above system the coupling terms are

$$k_i(\bar{x}) + e_i f\left(\sum_{m=1}^{n} e_m \dot{x}_m\right)$$

$$i = 1, 2, \ldots, n.$$

System (7) is asymptotically stable in the large if in addition to the conditions (***) the following condition on f holds:

$$uf(u) > 0, \ u \neq 0, \ f(0) = 0.$$

Again this result follows from the use of LaSalle's Theorem [2] where the Liapunov function is (5). This result may easily be extended to systems with several dissipative elements.

Another generalization of the System (4) has the form

$$\ddot{x}_i + f_i(\sigma_i) + k_i(\bar{\sigma}) = 0 \qquad (8)$$

$$i = 1, 2, \ldots, n$$

where

$$\sigma_i = g_i(\dot{x}_i) + h_i(x_i).$$

The 2n-order system (8) is asymptotically stable in the large if for $i = 1, 2, \ldots, n$

i) $\sigma_i f_i(\sigma_i) > 0, \ \sigma_i \neq 0; \ f_i(0) = 0$

ii) $\int_0^{\sigma_i} f_i(u)du \to \infty$ as $|\sigma_i| \to \infty$

iii) $g_i'(\dot{x}_i) > 0$ for all $\dot{x}_i$

iv) $h_i'(x_i) > 0$ for all x_i

v) $g_i(0) = 0, \ h_i(0) = 0$

vi) $k_i(\bar{\sigma}) = \frac{\partial}{\partial \sigma_i} U(\bar{\sigma})$ where U is a positive semidefinite function

vii) the set of equations $f_i(\sigma_i) + k_i(\bar{\sigma}) = 0$ has only the trivial solution $\bar{\sigma} = \bar{0}$.

This theorem is proved by applying the Liapunov function

$$V(\bar{x}, \bar{\dot{x}}) = \sum_{i=1}^{n} \int_0^{\sigma_i} f_i(u)du + \sum_{i=1}^{n} \int_0^{x_i} h_i'(x_i)\dot{x}_i d\dot{x}_i + U(\bar{\sigma})$$

for which

$$\dot{V}(\bar{x}, \bar{\dot{x}}) = -\sum_{i=1}^{n} g_i'(\dot{x}_i) f_i^2(\sigma_i).$$

3. Dissipatively Coupled van der Pol Oscillators

In the case of the conservatively coupled van der Pol oscillators an inner amplitude bound always exists, however, for the case of dissipatively coupled van der Pol oscillators, an inner amplitude bound may be found only for a certain range of resistance. By the inner amplitude bound is meant a closed surface in the phase space such that all trajectories starting within the surface leave the surface and no trajectories enter the surface (except the origin). For the system

$$\ddot{x} + \epsilon_1 \left(\frac{\dot{x}_1^3}{3} - \dot{x}_1\right) + \omega_1^2 x_1 = -C(x_1 - x_2) - R(\dot{x}_1 - \dot{x}_2)$$

$$\ddot{x}_2 + \epsilon_2 \left(\frac{\dot{x}_2^3}{3} - \dot{x}_2\right) + \omega_2^2 x_2 = C(x_1 - x_2) + R(\dot{x}_1 - \dot{x}_2) \qquad (9)$$

$$C, \ \epsilon_1, \ \epsilon_2, \ R > 0.$$

using the Liapunov function

$$V(x_1, \dot{x}_1, x_2, \dot{x}_2) = 1/2\ (\dot{x}_1^2 + \dot{x}_2^2) + \frac{C}{2}(x_1 - x_2)^2 + \frac{\omega_1^2}{2} x_1^2 + \frac{\omega_2^2}{2} x_2^2 .$$

one finds that

$$\dot{V}(x_1, \dot{x}_1, x_2, \dot{x}_2) = -\epsilon_1 \left(\dot{x}_1^2 - \frac{\dot{x}_1^4}{3}\right) - \epsilon_2 \left(\dot{x}_2^2 - \frac{\dot{x}_2^4}{3}\right) + R(\dot{x}_1 - \dot{x}_2)^2 .$$

On converting $\dot{V} = 0$ to polar coordinates in the $(\dot{x}_1, \dot{x}_2)$ plane it is found that if an R is taken such that

$$0 \leq R < \frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}$$

then an inner amplitude bound exists. The general inner amplitude bound is the hypersurface V = a constant = $1/2\ r^2$ where r is the radius of the circle centered at the origin in the $(\dot{x}_1, \dot{x}_2)$ plane that can be inscribed in the curve $\dot{V} = 0$, that is

$$r^2 = \min \left[3\left(\frac{\epsilon_1 \cos^2\theta + \epsilon_2 \sin^2\theta - R(\cos\theta - \sin\theta)^2}{\epsilon_1 \cos^4\theta + \epsilon_2 \sin^4\theta}\right)\right]$$

$$0 \leq \theta \leq 2\pi .$$

As an example, solutions were computed numerically for system (9) with

$$\epsilon_1=1,\ \epsilon_2=3,\ R=0.5.$$

The solutions do not rapidly converge to a periodic solution. If R is increased to 0.8, the previous established condition on R is violated, and an inner amplitude bound may not be computed by this method.

The previous discussion has shown that even though the existence of periodic solutions cannot be proven theoretically one can prove that given systems can never decay to their singular point at the origin.

4. Conservatively Coupled van der Pol Oscillator and a Stable System

The behavior of the coupled system when a damped simple harmonic oscillator and a van der Pol oscillator are conservatively coupled has been investigated. The system

$$\begin{aligned} \ddot{x}_1 + \dot{x}_1 + x_1 &= (x_2-x_1) \\ \ddot{x}_2 + \epsilon(\dot{x}_2^3/3 - \dot{x}_2) + x_2 &= (x_1-x_2) \end{aligned} \qquad (10)$$

may be investigated using the positive definite function

$$V(x_1,\dot{x}_1,x_2,\dot{x}_2) = 1/2\ \{\dot{x}_1^2+\dot{x}_2^2+(x_1-x_2)^2+x_1^2+x_2^2\}.$$

Here $\dot{V} = -\dot{x}_1^2 - \epsilon(\frac{\dot{x}_2^4}{3} - \dot{x}_2^2)$

and $\dot{V}=0$ can be seen to go through the origin of the $(\dot{x}_1,\dot{x}_2)$ plane as $\dot{x}_2$ goes through zero. Thus no circle can be drawn in the $(\dot{x}_1,\dot{x}_2)$ plane around the origin and inside the $\dot{V}=0$ curve. This means that no inner amplitude bound may be found from $\dot{V}$ under consideration but for $\epsilon=10$, numerical solutions show that the two equations lock into a periodic solution. The above absence of results may be observed for systems of the form

$$\begin{aligned} \ddot{x}_1 + \dot{x}_1 g_1(\dot{x}_1) + \frac{\partial U^*(x_1, x_2)}{\partial x_1} &= 0 \\ \ddot{x}_2 + \epsilon\left(\frac{\dot{x}_2^3}{3} - \dot{x}_2\right) + \frac{\partial U^*(x_1, x_2)}{\partial x_2} &= 0 \end{aligned} \qquad (11)$$

where $U^*(x_1, x_2)$ is a positive definite radially unbounded function and $g_1(\dot{x}_1) > 0$ for all $\dot{x}_1$, $g_1(-\dot{x}_1) = g_1(\dot{x}_1)$ and $g_1(0)$ is finite.

5. Discussion

Stability constraints have been derived for two general forms of coupled systems of second order nonlinear differential equations. Examples of these systems have been investigated for oscillatory behavior, and general conditions have been derived to assure that their solutions never decay even though the systems have dissapative coupling. Further work is being undertaken on oscillatory behavior of such systems so as to find a result similar to (**) for higher order systems. The hypersurface computed to find an inner amplitude bound is the first step. Numerical results are not presented here for lack of space.

6. Bibliography

[1] L. Cesari, Asymptotic Behavior and Stability Problems in Ordinary Differential Equations, Academic Press, N.Y., 1963.

[2] J. P. LaSalle, "Some Extensions of Liapunov's Second Method," IRE Trans. on Circuit Theory, vol. CT-7, pp. 520-527, Dec. 1960.

[3] N. Wax, "On Some Periodic Solutions of the Lienard Equation," IEEE Trans. on Circuit Theory, vol. CT-13, pp. 419-423, Dec. 1966.

[4] J. K. Aggarwal and C. G. Richie, "On Coupled van der Pol Oscillators," IEEE Trans. on Circuit Theory, vol. CT-13, pp. 465-466, Dec. 1966.

The research reported herein was sponsored by JSEP Research Grant AF-AFOSR-766-65, NSF Grant GK 261 and NASA Grant NAS8-18120.