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Application of Modern Control and Nonlinear Estimation Techniques

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INTRODUCTION

This report presents the work done and the results obtained during the year December 1966 - November 1967. Certain parts of the material contained in this report have been presented in the three quarterly reports submitted previously. Repetition of this material was deemed desirable, however, in the interest of making this report a presentation of the totality of the work done on the research contract up to the present time with no necessity for referring to the previous quarterly reports.

Each chapter is intended to be a complete presentation of its own material, with no cross-referencing among the individual chapters. For convenience, the references associated with each chapter are included at the end of the particular chapter. In the interest of clarity and continuity of presentation, some duplication will be noted in various chapters.

Chapter 1 contains the work related to optimal guidance of low-thrust, interplanetary space vehicles. Both the deterministic and stochastic versions of this problem are discussed.

The estimation and control aspects associated with the problem of soft landing on a planet with unknown atmospheres is discussed in Chapter 2. One and two degrees of freedom for the space vehicle are considered. An example is given in an appendix to illustrate methods by which the estimation equations may be simplified.

The problem of minimum energy control of electric propulsion systems is considered in Chapter 3. The controllers for various types of terminal constraints is discussed.

Chapter 4 contains a discussion of sensitivity considerations in the design of feedback systems.

CHAPTER 1

OPTIMAL GUIDANCE OF LOW-THRUST, INTERPLANETARY SPACE VEHICLES

1.1. Introduction and General Discussion.

In the past few years much interest has been developed in the use of ion-propulsion for space missions. The low-thrust ion engine will probably find its most important application in missions to the outer planets where the retarding effect of the sun's gravity will require a large space vehicle energy. Up to the present, all the energy (velocity) has been provided by the launch vehicle. For high energy missions, such as those to the outer planets, it seems desirable to use high impulse low-thrust engines to augment the energy supplied by the boost vehicle. These low-thrust devices would operate during the long flight times between launch and encounter supplying a higher specific impulse than that available from present chemical boosters.

For any space mission, a nominal or desired trajectory is determined. This trajectory is completely specified in terms of a set of injection conditions, and also a nominal thrust program. The nominal path selected usually represents a compromise between many conflicting factors such as launch energy required, arrival date, telemetering and tracking considerations, ion-engine fuel required, etc. In many cases, the nominal thrust program represents the "optimal control" in the sense that it minimizes a particular mathematical performance functional. However, if we use the same performance functional to synthesize a guidance system, and temporarily ignore the influence of noise inputs to the system, then rather large errors may result at encounter. The reason for this will be brought out in Section 3-D.

Because of launch energy dispersion and random effects in flight, the spacecraft will inevitably be perturbed away from its standard path. It is then the job of the guidance system to provide trajectory corrections which not only ensure that the vehicle approaches its destination in the intended fashion, but also provide that the vehicle remains as close as is practical to the nominal orbit in order to guarantee the compromises chosen.[2] It is also desirable that the guidance system be as efficient as possible in performing these tasks. The following are three guidance schemes that one may consider:

(a) Midcourse guidance - This method, currently being employed in space missions, uses high thrust impulsive forces, applied at one or two points, in order to nullify the injection errors. This technique would not be practical in low-thrust missions, however, since the random disturbances that could be expected are enormously greater than those encountered on ballistic trajectories. It is because of these large disturbances that a continuous guidance strategy would be necessary.

(b) Second variation technique[3],[4],[5] - This method essentially represents a partial solution to a general feedback optimal control problem. By application of this technique, one obtains all optimal trajectories, and their associated optimal thrust programs, in a region near the nominal path in the state space. Although it is very elegant, this method unfortunately suffers from some inherent drawbacks. One of these is that time varying feedback gains are usually required, and implementation problems become apparent. Another disadvantage, which is far more serious, is that intolerably large errors may result at encounter owing to the presence of disturbances. An analysis of how this phenomenon arises is given in Section 3-D.

(c) Return to the nominal trajectory[6],[7] - This approach presumes that small biases are allowed on the ion engine thrust vector magnitude and orientation. Such biases are utilized in control strategies, and permit returning the space vehicle to the neighborhood of the nominal trajectory using a minimum amount of fuel. As one has undoubtedly inferred from the above discussion, noise plays a decisive role in the low-thrust guidance problem. We saw how the above mentioned guidance methods, which are certainly accurate in the small noise case, turn out to be quite undesirable in the problem we are now confronted with. The method we now propose achieves the desired guidance accuracy even in the event that the expected disturbances become as large as the control which we have available.

One might use the following analogy to describe the strategy now being suggested. We conceive of a small car enroute across a desert on which high velocity omnidirectional winds are present. As every small car owner knows, a strong gust of wind can cause a non-negligible deviation from the intended course. However, instead of planning an entirely new route, we are much more tempted to return to the original course in order to prepare for future gusts. If we didn't use such a strategy, several properly oriented gusts could bring very unfortunate consequences.

In addition to its accuracy, this guidance technique also has other advantages:

(i) Low fuel consumption. In fact, the guidance system is designed to minimize the fuel required to make trajectory corrections. For instance, approximately .2% of the total available fuel would be necessary to correct the largest expected injection errors of a Mar's mission.

(ii) Mechanization simplicity. This is because only discrete values of the control variables are needed. Therefore a guidance command would represent switching from one of nine different control configurations to another. It is important to realize that the discrete nature of the control policy is not a constraint, but rather is the result of employing a minimum fuel controller.

(iii) Independent of nominal thrust program. This means that the ion engine could be preprogrammed to execute a particular thrust function, and the guidance system in no way modifies this program.

We now set out to concisely formulate the problem mathematically. We shall consider both the deterministic and stochastic problems, in that order.

1.2. Formulation and Solution of the Deterministic Problem.

A. Assumptions.

The system model, or plant, is based on the following assumptions:

(a) The space vehicle is in heliocentric flight. This assumption is based on the fact that the ion propulsion would be initiated three days after launch,[1] and hence the vehicle would be free of the earth's gravitational field.

(b) Spacecraft motion is constrained to one plane. This is considered a valid initial assumption and a desirable characteristic of space trajectories.

(c) The nominal thrust program consists of an angle history, $\alpha(t)$, and a thrust history, $u_n(t)$ (Fig. 1-1). Mass and power availability variations are accounted for in $u_n(t)$.

B. Available Control.

As one degree of freedom we will allow small, discrete thrust level changes. We need only consider discrete controls since the minimum fuel controller is a "bang-bang" controller. Ion propulsion systems will consist of an array of thrustors as depicted in Fig. 1-2. This configuration lends itself very nicely to discretely throttling the engine. Since modules will be held in reserve,[8] these could be used to provide a step increase in the thrust level. Similarly, a step decrease in thrust could be obtained by shutting down modules which are symmetric with respect to the spacecraft's center of gravity.

In addition to these thrust variations, we will also allow small discrete attitude variations about the yaw axis. More precisely, we will allow a rotation of the spacecraft itself, independent of the nominal attitude history obtained by rotating only the engine. This method leads to much simpler system implementation. We may only allow small variations in the yaw angle because the solar panels, from which the spacecraft receives its power, must be oriented toward the sun. In summary, the nine allowable states of the thrust vector are represented in Fig. 1-3.

At this point we may also include the observation that the nominal thrust program would very likely be designed as a step function in both the angle and thrust level variables - that is, both of these quantities would be held fixed over a finite number of time intervals. This is especially probable in the thrust level program as a result of the design of electric propulsion devices as an array of individual engines. Although we do not incorporate the discrete nature of the thrust program as a constraint, it will be useful, in some instances, to consider this possibility.

C. Performance Criteria.

Basically there are two performance indices which are meaningful in this problem. The first is the minimum time criterion which indicates the desirability of returning the space vehicle to nominal trajectory as quickly as possible, and thereby maintaining flight along the standard path for as much of the voyage as possible. The second performance index which is extremely meaningful here is that of minimum fuel, or returning the space vehicle to the standard orbit with as little expenditure of propellant as possible. It will presently be shown that these two criteria are, for all practical purposes, equivalent in this problem. For that reason they will be used interchangeably.

D. First Solution.

The coordinate frames we will be considering appear in Fig. 1-1. The coordinate frame (x_1, x_3) has its origin at time t at the point in space where a vehicle on the nominal trajectory would at time t, assuming flight begins at $t = 0$. Note that the angle $\beta(t)$ is determined by the nominal trajectory desired and is thus a function of time only. The differential equations of vehicle motion in the (x_1', x_3') frame are as follows:

$$\dot{x}_1' = x_2' \triangleq F_1$$

$$\dot{x}_2' = \ddot{x}_1' = \frac{-GM_s(x_1' + D)}{((x_1' + D)^2 + (x_3')^2)^{3/2}} - \frac{u(t)\ (x_3' \cos(\gamma + \alpha) + (x_1' + D) \sin(\gamma + \alpha)}{((x_1' + D)^2 + (x_3')^2)^{1/2}} \triangleq F_2 \tag{1-1}$$

$$\dot{x}_3' = x_4' \triangleq F_3$$

$$\dot{x}_4' = \ddot{x}_3' = \frac{-GM_s\ x_3'}{((x_1' + D)^2 + (x_3')^2)^{3/2}} + \frac{u(t)\ ((x_1' + D) \cos(\alpha + \gamma) - x_3' \sin(\alpha + \gamma))}{((x_1' + D)^2 + (x_3')^2)^{1/2}} \triangleq F_4$$

where G is a constant of gravitation, M_s is the mass of the sun, and u, γ, α, and D are defined in Figs. 1-1 and 1-3.

In vector notation, Eq.(1-1) becomes

$$\dot{X}' = F(u, \gamma, X')$$

where $X' = \mathrm{col}(x_1', x_2', x_3', x_4')$ and $F = \mathrm{col}(F_1, F_2, F_3, F_4)$. If at time τ we have deviations $X(\tau) = \mathrm{col}[x_1(\tau), x_2(\tau), x_3(\tau), x_4(\tau)]$ from the nominal trajectory, then the problem is to find the controls,

$$u(t),\ \gamma(t) \qquad 0 \le \tau \le t \le T$$

subject to control constraint shown in Fig. 1-3 such that

$$X(T) = 0$$

and either of the performance indices

$$\text{(a)} \quad \int_0^T dt \qquad\qquad \text{(b)} \quad \int_0^T u(t)\, dt$$

is minimized.

Referring to Fig. 1-4, consider the following coordinate transformation:

$$\begin{aligned} x_1'' &= x_1' \cos(\beta + \alpha) + x_3' \sin(\beta + \alpha) \\ x_3'' &= - x_1' \sin(\beta + \alpha) + x_3' \cos(\beta + \alpha) \end{aligned} \tag{1-2}$$

Differentiating these equations twice yields

$$\dot{x}_1'' = \dot{x}_1' \cos(\beta + \alpha) + \dot{x}_3' \sin(\beta + \alpha) - x_1'(\dot{\beta} + \dot{\alpha}) \sin(\beta + \alpha) + x_3'(\dot{\beta} + \dot{\alpha}) \cos(\beta + \alpha) \triangleq x_2''$$

$$\begin{aligned} \ddot{x}_1'' = \dot{x}_2'' = \ddot{x}_1' \cos(\beta + \alpha) + \ddot{x}_3' \sin(\beta + \alpha) - 2\dot{x}_1'(\dot{\beta} + \dot{\alpha}) \sin(\beta + \alpha) \\ - x_1'(\dot{\beta} + \dot{\alpha})^2 \cos(\beta + \alpha) - x_1'(\ddot{\beta} + \ddot{\alpha}) \sin(\beta + \alpha) + 2\dot{x}_3'(\dot{\beta} + \dot{\alpha}) \cos(\beta + \alpha) \\ - x_3'(\dot{\beta} + \dot{\alpha})^2 \sin(\beta + \alpha) + x_3'(\ddot{\beta} + \ddot{\alpha}) \cos(\beta + \alpha) \end{aligned}$$

$$\dot{x}_3'' = -\dot{x}_1 \sin(\beta + \alpha) + \dot{x}_3' \cos(\beta + \alpha) - x_1'(\dot{\beta} + \dot{\alpha}) \cos(\beta + \alpha)$$
$$- x_3'(\dot{\beta} + \dot{\alpha}) \sin(\beta + \alpha) \triangleq x_4''$$

$$\ddot{x}_3'' = \dot{x}_4'' = -\ddot{x}_1' \sin(\beta + \alpha) + \ddot{x}_3' \cos(\beta + \alpha) - 2\dot{x}_1'(\dot{\beta} + \dot{\alpha}) \cos(\beta + \alpha)$$
$$+ x_1'(\dot{\beta} + \dot{\alpha})^2 \sin(\beta + \alpha) - x_1'(\ddot{\beta} + \ddot{\alpha}) \cos(\beta + \alpha) - 2\dot{x}_3'(\dot{\beta} + \dot{\alpha}) \sin(\beta + \alpha)$$
$$- x_3'(\dot{\beta} + \dot{\alpha}) \cos(\beta + \alpha) - x_3'(\ddot{\beta} + \ddot{\alpha}) \sin(\beta + \alpha) \qquad (1\text{-}3)$$

Letting the subscript n denote the nominal values of the variables $\dot{X}''$, u, γ, and X', Eq.(1-3) can be written

$$\dot{X}_n'' + \delta\dot{X}'' = R(t)\, F(u_n + \delta_u, \gamma_n + \delta\gamma, X_n' + X) + S(t)\, (X_n' + X) \qquad (1\text{-}4)$$

where $\delta X''$, δ_u, and $\delta\gamma$ are deviations from nominal values, and the matrices $R(t)$ and $S(t)$ are defined by

$$R(t) \triangleq \begin{bmatrix} \cos(\beta + \alpha) & 0 & \sin(\beta + \alpha) & 0 \\ 0 & \cos(\beta + \alpha) & 0 & \sin(\beta + \alpha) \\ -\sin(\beta + \alpha) & 0 & \cos(\beta + \alpha) & 0 \\ 0 & -\sin(\beta + \alpha) & 0 & \cos(\beta + \alpha) \end{bmatrix} \qquad (1\text{-}5a)$$

$$S(t) \triangleq \begin{bmatrix} -(\dot{\beta}+\dot{\alpha}) \sin(\beta+\alpha) & 0 & (\dot{\beta}+\dot{\alpha}) \cos(\beta+\alpha) & 0 \\ \begin{matrix} -(\ddot{\beta}+\ddot{\alpha}) \sin(\beta+\alpha) \\ -(\dot{\beta}+\dot{\alpha})^2 \sin(\beta+\alpha) \end{matrix} & -2(\dot{\beta}+\dot{\alpha}) \sin(\beta+\alpha) & \begin{matrix} -(\ddot{\beta}+\ddot{\alpha}) \cos(\beta+\alpha) \\ -(\dot{\beta}+\dot{\alpha})^2 \sin(\beta+\alpha) \end{matrix} & 2(\dot{\beta}+\dot{\alpha}) \cos(\beta+\alpha) \\ -(\dot{\beta}+\dot{\alpha}) \cos(\beta+\alpha) & 0 & -(\dot{\beta}+\dot{\alpha}) \sin(\beta+\alpha) & 0 \\ \begin{matrix} -(\ddot{\beta}+\ddot{\alpha}) \cos(\beta+\alpha) \\ +(\dot{\beta}+\dot{\alpha})^2 \sin(\beta+\alpha) \end{matrix} & -2(\dot{\beta}+\dot{\alpha}) \cos(\beta+\alpha) & \begin{matrix} -(\ddot{\beta}+\ddot{\alpha}) \sin(\beta+\alpha) \\ -(\dot{\beta}+\dot{\alpha})^2 \cos(\beta+\alpha) \end{matrix} & -2(\dot{\beta}+\dot{\alpha}) \sin(\beta+\alpha) \end{bmatrix}$$

(1-5b)

In space trajectory problems such as this one, linearization of the dynamical equations about the nominal path often yields satisfactory approximations to the true physical situation. This statement is supported by the fact that we are striving to keep the vehicle in the neighborhood of the nominal trajectory, and also that the quantities δu and $\delta\gamma$ are "small" in relation to their nominal values. Hence, we carry out a Taylor series expansion of Eq.(1-4), which yields

$$\dot{X}_n'' + \delta\dot{X}'' = R(t)\ (F(u_n,\gamma_n,X_n') + F_{X'}(u_n,\gamma_n,X_n')\ X + F_u(u_n,\gamma_n,X_n')\delta_u$$
$$+ F_\gamma(u_n,\gamma_n,X_n')\ \delta\gamma + (\text{higher-order terms})) + S(t)\ (X_n' + X) \tag{1-6}$$

where $F_{X'}$ is the Jacobian matrix whose ij-th element, $\partial F_i/\partial x_j'$, is the partial derivative of the ith component of F with respect to the jth component of X'. Also, $F_u = \text{col}[\partial F_1/\partial u,\ \partial F_2/\partial u,\ \partial F_3/\partial u,\ \partial F_4/\partial u]$ and similarly for F_γ:

$$F_{X'} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ A & 0 & B & 0 \\ 0 & 0 & 0 & 1 \\ C & 0 & D & 0 \end{bmatrix} ; \qquad F_u = \begin{bmatrix} 0 \\ -\sin(\alpha+\beta) \\ 0 \\ \cos(\alpha+\beta) \end{bmatrix} ;$$

$$F_\gamma = \begin{bmatrix} 0 \\ -u_n\cos(\beta+\alpha) \\ 0 \\ -u_n\sin(\alpha+\beta) \end{bmatrix}$$

where

$$A = \frac{-GM_s + u_n(x'_{1n} + D)}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{3/2}} + \frac{3GM_s(x'_{1n} + D)^2}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{5/2}}$$

$$B = \frac{-u_n(x'_{1n} + D)^2}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{3/2}} + \frac{3GM_s(x'_{1n} + D)\, x'_{3n}}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{3/2}}$$

$$C = \frac{3GM_s x'_{3n}(x'_{1n} + D)}{((x'_{1n} + D)^2 + ((x'_{3n})^2)^{5/2}} + \frac{u_n(2(x'_{1n} + D)^2 + (x'_{3n})^2)}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{3/2}}$$

$$D = \frac{GM_s(2(x'_{3n})^2 - (x'_{1n} + D)^2}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{5/2}} - \frac{u_n x'_{3n}(x'_{1n} + D)}{((x'_{1n} + D)^2 + (x'_{3n})^2)^{3/2}}$$

Defining the controls

$$u_1 = - u_n \delta\gamma \qquad\qquad u_2 = \delta u \qquad\qquad (1\text{-}7)$$

and making suitable manipulations, Eq.(1-6) becomes

$$\delta\dot{X}''(t) = \begin{bmatrix} 0 & \cos(\beta+\alpha) & 0 & \sin(\beta+\alpha) \\ A\cos(\beta+\alpha)+C\sin(\beta+\alpha) & 0 & A\cos(\beta+\alpha)+D\sin(\beta+\alpha) & 0 \\ 0 & -\sin(\beta+\alpha) & 0 & \cos(\beta+\alpha) \\ -A\sin(\beta+\alpha)+C\cos(\beta+\alpha) & 0 & -B\sin(\beta+\alpha)+D\cos(\beta+\alpha) & 0 \end{bmatrix} X(t)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1(t)$$

$$+ \begin{bmatrix} -\dot{\mu} \sin \mu & 0 & \dot{\mu} \cos \mu & 0 \\ -\ddot{\mu} \sin \mu - \dot{\mu}^2 \cos \mu & -2\dot{\mu} \sin \mu & \ddot{\mu} \cos \mu - \dot{\mu}^2 \sin \mu & 2\dot{\mu} \cos \mu \\ -\dot{\mu} \cos \mu & 0 & -\dot{\mu} \sin \mu & 0 \\ -\ddot{\mu} \cos \mu + \dot{\mu}^2 \sin \mu & -2\dot{\mu} \cos \mu & -\ddot{\mu} \sin \mu - \dot{\mu}^2 \cos \mu & -2\dot{\mu} \sin \mu \end{bmatrix} X \qquad (1\text{-}8)$$

where we have set $\mu = \beta + \alpha$ and have neglected higher order terms.

We note that in general u_1 is a time varying function if u_n is. However, as has been mentioned above, $u_n(t)$ would probably be a step function as a practical consideration (if it is not, we approximate it with a step function), and hence u_1 is piecewise constant. To gain more insight into the problem, Eq.(1-8) will be simplified by neglecting small terms. The quantities A, B, C and D are proportional to changes in the sun's gravity and the angle β over a region in space near the nominal trajectory. These quantities are of the order of 10^{-12} in mks units, and will therefore be neglected. Similarly the quantities $\ddot{\beta}$ and $\dot{\beta}^2$ are of the order of 10^{-14}, and $\dot{\alpha}$ and $\ddot{\alpha}$ would be smaller than that or zero. It can be seen, for example, that in the Jupiter mission shown in Fig. 1-5 that α varies at a slower rate than β. Of course, when α is a step function, $\dot{\alpha}$ and $\ddot{\alpha}$ are both zero. Finally it will be assumed that the quantities $(\dot{\alpha} + \dot{\beta})x_2$ and $(\dot{\alpha} + \dot{\beta})x_4$ are negligible with respect to u_1 and u_2. Actually, typical values would be 10^{-6} for $(\dot{\beta} + \dot{\alpha})x_2$ and $(\dot{\beta} + \dot{\alpha})x_4$ and 10^{-4} for u_1 and u_2. We expect the latter assumption to yield the largest error.

We use the physical reasoning above and the fact that

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \delta X''(t) = \begin{bmatrix} -\dot{\mu}\sin\mu & \cos\mu & \dot{\mu}\cos\mu & \sin\mu \\ 0 & 0 & 0 & 0 \\ -\dot{\mu}\cos\mu & -\sin\mu & -\dot{\mu}\sin\mu & \cos\mu \\ 0 & 0 & 0 & 0 \end{bmatrix} X(t)$$

to obtain

$$\delta\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \delta X''(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_2(t) \qquad (1\text{-}9)$$

It is evident from examining Eq.(1-9) that the four-dimensional minimum-time problem has been reduced to two two-dimensional problems, since the $\delta x_1''$ and $\delta x_2''$ equations are decoupled from the $\delta x_3''$ and $\delta x_4''$ equations. The (a) minimum-time and (b) minimum-fuel problems will now be solved by applying the principles of optimal control theory. We will consider one two-dimensional problem for each performance index.

Let

$$\delta x_1'' = y_1 \qquad \delta x_2'' = y_2$$

then

$$\dot{y}_1 = y_2 \qquad \dot{y}_2 = u \qquad (1\text{-}10)$$

The Hamiltonians for the two problems are

(a) $\quad H = 1 + \lambda_1 y_2 + \lambda_2 u$

(b) $\quad H = u + \lambda_1 y_2 + \lambda_2 u$

The optimal controls minimize the Hamiltonian. Hence

$$\text{(a)} \qquad u^{*} = -\, k \,\text{sgn}(\lambda_2)$$
$$\text{(b)} \qquad u^{*} = -\, k \,\text{sgn}(1 + \lambda_2)$$

where k is the maximum attainable value of the control variable u. The Lagrange multiplier equations for both problems are

$$\dot{\lambda}_1 = 0 \qquad\qquad \dot{\lambda}_2 = -\,\lambda_1$$

whose solutions are

$$\lambda_1(t) = \lambda_1(0) \qquad\qquad \lambda_2(t) = -\,\lambda_1(0)\; t + \lambda_2(0) \tag{1-11}$$

Equations (1-11) implies that one switching of the control u is possible in each case. Solving Eq.(1-10) for constant u yields

$$y_1(t) = \frac{1}{2}\, ut^2 + y_2(0)\; t + y_1(0)$$

$$y_2(t) = u_1 t + y_2(0)$$

Eliminating t from these equations, we find that

$$2u_1(y_1 - y_1(0)) = (y_2 - y_2(0))^2 + 2y_2(0)\;(y_2 - y_2(0)) \tag{1-12}$$

Equation (1-12) shows that the vehicle will follow a parabolic trajectory in the (y_1, y_2) plane for constant u. Coupling this fact with the fact that only one switching is optimal, the "switching boundary" is obtained, as shown in Fig. 1-6. Hence, we conclude that the same control strategy is optimal for both performance indices. The expected trajectory for a set of initial deviations from the nominal trajectory is also shown in Fig. 1-6. Various numerical experiments were performed using this solution for specific space missions, and the results are reported in references (6) and (7).

E. Second Solution.

The motivation for the second solution is the desirability of obtaining a more accurate approximation to the minimum-time (minimum-fuel) solution by making larger use of digital computer capabilities.

The first step is to linearize Eq.(1-1) as follows:

$$\dot{X}'(t) = \dot{X}'_n + \dot{X}(t) = F(u_n+\delta u,\ \gamma_n+\delta\gamma,\ X'_n+X) = F(u_n, \gamma_n, X'_n) + F_u\dot{\delta}_u + F_\gamma\delta\gamma + F_{X'}X$$
$$+\ (\text{higher-order terms})$$

As before, we neglect higher-order terms, and the terms A, B, C, and D in $F_{X'}$. Also, we use the definitions of u_1 (where $u_n(t)$ is regarded as a step function) and u_2 to obtain

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ \cos(\beta+\alpha) \\ 0 \\ \sin(\beta+\alpha) \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ -\sin(\beta+\alpha) \\ 0 \\ \cos(\beta+\alpha) \end{bmatrix} u_2(t) \qquad \text{(1-13a)}$$

We now will regard $\alpha(t)$ as a step function and consider the following transformation:

$$TX = \hat{X} \quad ; \quad T = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha & 0 \\ 0 & \cos\alpha & 0 & \sin\alpha \\ -\sin\alpha & 0 & \cos\alpha & 0 \\ 0 & -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Then Eq.(1-13a) becomes

$$(\dot{T^{-1}\hat{X}}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} T^{-1}\hat{X} + \begin{bmatrix} 0 & 0 \\ \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ 0 & 0 \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

and therefore

$$\dot{\hat{X}} = T \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} T^{-1}\hat{X} + T \begin{bmatrix} 0 & 0 \\ \cos(\beta+\alpha) & -\sin(\beta+\alpha) \\ 0 & 0 \\ \sin(\beta+\alpha) & \cos(\beta+\alpha) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (1\text{-}13b)$$

where

$$T^{-1} = T^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha & 0 \\ 0 & \cos\alpha & 0 & -\sin\alpha \\ \sin\alpha & 0 & \cos\alpha & 0 \\ 0 & \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Eq.(1-13b) thus becomes

$$\dot{\hat{X}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \hat{X} + \begin{bmatrix} 0 \\ \cos\beta \\ 0 \\ \sin\beta \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ -\sin\beta \\ 0 \\ \cos\beta \end{bmatrix} u_2(t)$$

For notational convenience we redefine

$$\hat{X} = X$$

and obtain

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} X(t) + \begin{bmatrix} 0 \\ \cos\beta(t) \\ 0 \\ \sin\beta(t) \end{bmatrix} u_1(t) + \begin{bmatrix} 0 \\ -\sin\beta(t) \\ 0 \\ \cos\beta(t) \end{bmatrix} u_2(t) \qquad (1\text{-}13c)$$

The differential equations for the Lagrange multipliers are then

$$\dot{\lambda}_1 = 0 \qquad \dot{\lambda}_2 = -\lambda_1 \qquad \dot{\lambda}_3 = 0 \qquad \dot{\lambda}_4 = -\lambda_3$$

which have the solutions

$$\lambda_1(t) = \lambda_1(0) \qquad \lambda_2(t) = -\lambda_1(0)t + \lambda_2(0) \qquad \lambda_3(t) = \lambda_3(0) \qquad \lambda_4(t) = -\lambda_3(0)t + \lambda_4(0)$$

The optimal controls are therefore given by

$$\begin{aligned} u_1^*(t) &= -k \operatorname{sgn}((-\lambda_1(0)t + \lambda_2(0)) \cos\beta(t) + (-\lambda_3(0)t + \lambda_4(0)) \sin\beta(t)) \\ u_2^*(t) &= -k \operatorname{sgn}((-\lambda_1(0)t + \lambda_2(0))(-\sin\beta(t)) + (-\lambda_3(0)t + \lambda_4(0)) \cos\beta(t)) \end{aligned} \tag{1-14}$$

Some possible realizations of Eq.(1-14) would be as in Fig. 1-7. (Note that β is not expected to exceed 90 deg before nominal trajectory acquisition.) These realizations suggest that each control would have a maximum of two switchings.

Now, given the initial conditions on Eq.(1-12), we can write the explicit solution for $X(T)$, where T is the nominal trajectory acquisition time. That is,

$$X(T) = \Phi(T,0)X(0) + \int_0^T \Phi(T,t) \begin{bmatrix} 0 \\ \cos\beta(t) \\ 0 \\ \sin\beta(t) \end{bmatrix} u_1(t)dt + \int_0^T \Phi(T,t) \begin{bmatrix} 0 \\ -\sin\beta(t) \\ 0 \\ \cos\beta(t) \end{bmatrix} u_2(t)dt \tag{1-15}$$

where $\Phi(t_2, t_1)$ is the fundamental matrix that satisfies the matrix differential equation

$$\dot{\Phi}(t_2,t_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Phi(t_2,t_1) \tag{1-16}$$

with $\Phi(t_1,t_1) = I$. The solution of Eq.(1-16) is

$$\Phi(t_2,t_1) = \begin{bmatrix} 1 & (t_2-t_1) & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & (t_2-t_1) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1-17}$$

Since the absolute values of u_1 and u_2 are constant, only the sign of these quantities is needed inside the integrals of Eq.(1-15). If we designate $u_1(0)$ and $u_2(0)$ as the initial values of u_1 and u_2, t_1 and t_2 as the switching times of u_1, and t_3 and t_4 as the switching times of u_2, then Eq.(1-15) becomes

$$X(T) = \Phi(T,0)X(0) + u_1(0)\left[\int_0^{t_1} - \int_{t_1}^{t_2} + \int_{t_2}^{T}\right] + u_2(0)\left[\int_0^{t_3} - \int_{t_3}^{t_4} + \int_{t_4}^{T}\right] \tag{1-18}$$

The integrals of Eq.(1-18) can be explicitly evaluated if we assume that β varies at a constant rate. This is an excellent approximation for the trajectories of interest. Hence, if we assume that

$$\beta(t) = \omega t \qquad \omega = \text{constant} \doteq \dot{\beta}$$

and if we define

$$I_1 = u_1(0)\ \frac{2t_1}{\omega}\sin\omega t_1 + \frac{2}{\omega^2}\cos\omega t_1 - \frac{2t_2}{\omega}\sin\omega t_2 - \frac{2\cos\omega t_2}{\omega^2} + \frac{T\sin\omega T}{\omega}$$

$$+ \frac{1}{\omega^2}\cos\omega T - \frac{1}{\omega^2} - u_2(0)\ - \frac{2t_3\cos\omega t_3}{\omega} + \frac{2\sin\omega t_3}{\omega^2} + \frac{2t_4\cos\omega t_4}{\omega} - \frac{2\sin\omega t_4}{\omega^2}$$

$$- \frac{T\cos\omega T}{\omega} + \frac{\sin\omega T}{\omega^2}$$

$$I_2 = u_1(0)\ \frac{2}{\omega}\sin\omega t_1 - \frac{2}{\omega}\sin\omega t_2 + \frac{1}{\omega}\sin\omega T - u_2(0) - \frac{2}{\omega}\cos\omega t_3 + \frac{2}{\omega}\cos\omega t_4$$

$$- \frac{1}{\omega}\cos\omega T + \frac{1}{\omega}$$

$$I_3 = u_1(0)\ \frac{-2t_1\cos\omega t_1}{\omega} + \frac{2\sin\omega t_1}{\omega^2} + \frac{2t_2\cos\omega t_2}{\omega} - \frac{2\sin\omega t_2}{\omega^2} - \frac{T\cos\omega T}{\omega} + \frac{\sin\omega T}{\omega^2}$$

$$+ u_2(0)\ \frac{2t_3}{\omega}\sin\omega t_3 + \frac{2}{\omega^2}\sin\omega t_3 - \frac{2t_4}{\omega}\sin\omega t_4 - \frac{2\cos\omega t_4}{\omega^2} + \frac{T\sin\omega T}{\omega}$$

$$+ \frac{1}{\omega^2}\cos\omega T - \frac{1}{\omega^2}$$

$$I_4 = u_1(0)\ \frac{-2\cos\omega t_1}{\omega} + \frac{2}{\omega}\cos\omega t_2 - \frac{1}{\omega}\cos\omega T + \frac{1}{\omega} + u_2(0)\ \frac{2}{\omega}\sin\omega t_3 - \frac{2}{\omega}\sin\omega t_4$$

$$+ \frac{1}{\omega}\sin\omega T$$

then Eq.(1-18) becomes

$$
\begin{aligned}
x_1(T) &= x_1(0) + Tx_2(0) - I_1 + TI_2 \triangleq G_1(t_1, t_2, t_3, t_4, T) \\
x_2(T) &= x_2(0) + I_2 \qquad\qquad\qquad\quad \triangleq G_2(t_1, t_2, t_3, t_4, T) \\
x_3(T) &= x_3(0) + Tx_4(0) - I_3 + TI_4 \triangleq G_3(t_1, t_2, t_3, t_4, T) \\
x_4(T) &= x_4(0) + I_4 \qquad\qquad\qquad\quad \triangleq G_4(t_1, t_2, t_3, t_4, T)
\end{aligned}
\tag{1-19}
$$

Equations (1-19) are four equations in five unknowns. Since it is desired that $X(T) = 0$, the problem is now to find the minimum value of T for which Eqs.(1-19) can be satisfied. Fortunately enough, these equations can be solved by the Newton-Raphson technique, and such analysis indicates that the minimum value of T is achieved either when $T = t_2$ or $T = t_4$. Hence one control will have one switching, and the other will have two switchings. It is fairly easy to determine the correct $u_1(0)$ and $u_2(0)$, and thereby Eqs.(1-19) can be solved for the minimum value of T and for the switching times of the control variables.

The experimental results for the second solution, and how they relate to the same experiments using the first solution, are given in References (6) and (7).

F. The Open Loop Problem - An Algorithm for Determining Minimum-Fuel and Minimum-Time Trajectories.

At this point it is necessary to compute the exact open loop trajectories in order to determine the accuracy of the closed loop systems already derived. To do this, we first consider the differential equations for the deviations of the state vector from nominal values:

$$
\begin{aligned}
\delta\dot{X} &= \dot{X} - \dot{X}_n = F(X, u, \gamma) - F(X_n, u_n, \gamma_n) \\
&\triangleq G(t, X, u, \gamma)
\end{aligned}
\tag{1-20}
$$

The performance indices we are considering are the following:

$$\text{(a)} \int_0^T u\,dt \qquad \text{(b)} \int_0^T dt \qquad T \text{ free}, \quad \delta X(T) = 0$$

where we are given $\delta X(0) = \delta X_0$.

The Hamiltonians for the two problems are

$$\text{(a)}\quad H = u + \langle \lambda, G(t,X,u,\gamma) \rangle \qquad \text{(b)}\quad H = 1 + \langle \lambda, G(t,X,u,\gamma) \rangle \tag{1-21}$$

The optimal control minimizes the Hamiltonian at each instant of time. In particular, it minimizes

$$\text{(a)}\quad M(u,\gamma,x_1,x_3,\lambda_2,\lambda_4) = u\left[1 - \frac{\lambda_2(x_3 \cos(\gamma+\alpha) + (x_1+D)\sin(\gamma+\alpha))}{((x_1+D)^2 + x_3^2)^{1/2}} + \frac{\lambda_4((x_1+D)\cos(\gamma+\alpha) + x_3 \sin(\gamma+\alpha))}{((x_1+D)^2 + x_3^2)^{1/2}}\right]$$

(1-22)

$$\text{(b)}\quad M(u,\gamma,x_1,x_3,\lambda_2,\lambda_4) = 1 - \frac{u\lambda_2(x_3 \cos(\gamma+\alpha) + (x_1+D)\sin(\gamma+\alpha))}{((x_1+D)^2 + x_3^2)^{1/2}} + \frac{u\lambda_4((x_1+D)\cos(\gamma+\alpha) + x_3 \sin(\gamma+\alpha))}{((x_1+D)^2 + x_3^2)^{1/2}}$$

We define for both cases

$$\Phi_1(x_1,x_3,\lambda_2,\lambda_4)=u_i \in U = \{u_1,u_2,u_3\} \quad u_i \text{ minimizes } M$$
$$\Phi_2(x_1,x_3,\lambda_2,\lambda_4)=\gamma_i \in \Lambda = \{\gamma_1,\gamma_2,\gamma_3\} \quad \gamma_i \text{ minimizes } M \tag{1-23}$$

Note that Φ_1 and Φ_2 are discontinuous functions whose partial derivatives are zero with respect to all arguments (except at discontinuities). Also note that Φ_1 and Φ_2 are not explicitly known functions, but can be easily calculated on the computer since only nine combinations of u and γ need to be checked. Substituting Eq.(1-23) into Eq.(1-21) we obtain

$$\text{(a)} \qquad H^* = \Phi_1 + \langle \lambda, G(t,X,\Phi_1,\Phi_2)\rangle$$
$$\text{(b)} \qquad H^* = 1 + \langle \lambda, G(t,X,\Phi_1,\Phi_2)\rangle \tag{1-24}$$

The canonic equations are

$$\delta\dot{X} = H^*_\lambda$$
$$\dot{\lambda} = - H^*_X \tag{1-25}$$

and the transversality condition gives

$$H^*(T) = 0 \tag{1-26}$$

We note here that quasilinearization cannot be used for this problem, since Φ_1 and Φ_2 are discontinuous functions. It should also be mentioned that the approximation-in-policy-space algorithm was tried, but does not converge for this problem. The main reason for this is probably because of the discontinuities in the control. The motivation for the algorithm which follows is the Newton-Raphson technique for solving non-linear equations. The basic idea is to determine how the end conditions on δX and H^* vary as functions of T and the initial conditions on λ. One could normally approximate such behavior by first linearizing the non-linear equations, and then using linear differential equation techniques. This is the general attack in the quasilinearization method. As we have already pointed out, however, the method fails here owing to the discontinuity of Φ_1 and Φ_2.

We now define the following quantities:

$$\begin{bmatrix} \lambda(0) \\ T \end{bmatrix} \triangleq C_0$$

$$\begin{bmatrix} \delta X(T) \\ H^*(T) \end{bmatrix} \triangleq E(\lambda(0), T) = E(C_0) \tag{1-27}$$

Let $C_0^{(n)}$ be the nth estimate of C_0. Then in general $E(C_0^{(n)}) \neq 0$, or

$$E(C_0^{(n)}) = \epsilon^{(n)} \neq 0$$

We would like to find $\Delta C_0^{(n)}$ such that

$$E(C_0^{(n)} + \Delta C_0^{(n)}) = 0$$

Expanding this equation to first order about $C_0^{(n)}$ yields

$$E(C_0^{(n)}) + E_{C_0} \Delta C_0^{(n)} = 0$$

This implies

$$\Delta C_0^{(n)} = -E_{C_0}^{-1} \, \epsilon^{(n)} \qquad (1\text{-}28)$$

Equation (1-28) would be easy to solve for $\Delta C_0^{(n)}$ except for the fact that we do not have an explicit expression for E_{C_0}. This matrix is approximated in the algorithm by a perturbation technique. The algorithm is as follows:

(a) Guess $C_0^{(1)}$.

(b) Integrate Eqs.(1-25) to obtain $\epsilon^{(1)} = E(C_0^{(1)})$.

(c) Perterb $C_0^{(1)}$ by an amount $f\, C_0^{(1)}$ where the scalar $f \ll 1$.

(d) Compute E_{C_0} according to the following approximate formula:

$$E_{C_0} = \begin{bmatrix} \dfrac{E_1(C_0^{(1)} + f\, C_0^{(1)}) - E_1(C_0^{(1)})}{f\, C_{01}^{(1)}} & \cdots & \dfrac{E_1(C_0^{(1)} + f\, C_0^{(1)}) - E_1(C_0^{(1)})}{f\, C_{05}^{(1)}} \\ \vdots & & \vdots \\ \dfrac{E_5(C_0^{(1)} + f\, C_0^{(1)}) - E_5(C_0^{(1)})}{f\, C_{01}^{(1)}} & \cdots & \dfrac{E_5(C_0^{(1)} + f\, C_0^{(1)}) - E_5(C_0^{(1)})}{f\, C_{05}^{(1)}} \end{bmatrix}$$

(e) Calculate $E_{C_0}^{-1}$, then obtain $\Delta C_0^{(1)}$ from Eq.(1-28).

(f) Repeat this process until the solution converges.

The computer results for both the minimum-fuel and minimum-time problems are shown in Figs.(1-8)-(1-13). In addition, the trajectories obtained by using the closed loop controller derived in the second solution are included for comparison purposes. We first note that the differences between the trajectories for the non-linear minimum-fuel and minimum-time problems are small, as was predicted by the analysis at the end of Section D. As a second consideration, we observe that the linearized controller gives a very good approximation to the exact optimal solution. One aspect of the extremal trajectories that was lost by linearization, however, is the time interval when $\gamma = 0$. It is interesting to note, though, that this "coast period" has little effect on the performance index. We conclude that the rather negligible degradation in system performance which results from linearization is more than compensated for by the comparative simplicity of controller implementation.

1.3. Formulation and Solution of the Stochastic Problem.

A. Introduction.

A spacecraft which is propelled and guided by a low-thrust ion-engine will be subjected to random disturbances. Undoubtedly the sources of these disturbances are many, but they will contribute to produce the two stochastic processes of interest:

(a) the attitude (yaw) angle of the vehicle

(b) the low-thrust acceleration magnitude.

These processes will have a significant effect on the trajectory which the space vehicle follows, and for that reason the guidance problem becomes crucial. Herein is a study of that problem.

B. The Problem.

In previous sections the low thrust guidance problem has been formulated for the deterministic case. The underlying idea was to return the vehicle to the nominal trajectory in a minimum amount of time (or, equivalently, using minimum fuel). The reasons for doing this have already been pointed out and will not be reiterated here. The stochastic formulation will be different for two reasons. First, it is unrealistic in the stochastic problem to require the state deviations to be zero at the terminal time. In many cases, such a constraint would lead to meaningless results. Instead, a neighborhood of the origin will be defined as the set of desired final states. Second, the minimum time criterion is no longer applicable since, in general, the time required to reach the target set is a random variable. A reasonable approach to this problem is to minimize the average time required for this task.

It was shown in the first deterministic solution that the guidance problem could be analytically reduced to controlling two independent, double-integrator plants. This "separation" property will be exploited here since it closely approximates the dynamical characteristics of the space vehicle.

C. The Noise Model.

One of the most important questions is how to model the noises acting on the system. Invariably one is forced to make approximations and assumptions since a complete characterization of a random process is virtually impossible. However, the essential features of a stochastic process can and should be retained by an examination of available data. The essential features we have strived to include in our stochastic model are the following:

(a) the mean value (u)

(b) the range of variation, or standard deviation (σ)

(c) the rate at which the process varies, or the correlation time $(1/\beta)$.

If these quantities are not available a priori, it would be necessary to estimate them in flight. It is not unrealistic, however, to assume that they are available from preflight test data.

A stochastic model which retains all of these features as parameters is the Ornstein-Uhlenbeck (OU) process.[10] The control variable u_1 (or u_2) can therefore be represented as follows:

$$u_1(t) = u + x_3(t) \tag{1-29}$$

where x_3 is an OU process with zero mean, variance σ^2, and correlation time $1/\beta$. A typical sample function for this stochastic process is given in Fig. 1- 17. If a suitable interpretation is given to the solution of a stochastic differential equation, it can be shown that the process $x_3(t)$ satisfies the following Langevin equation

$$\dot{x}_3 = - \beta x_3 + \xi(t) \tag{1-30}$$

where $\xi(t)$ is zero mean, Gaussian, white noise with variance σ^2.

We can summarize the discussions above with the following problem formulation (note that only one of the two two-dimensional problems of Eq.(1-9), Section 2, need be considered):

PLANT:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + x_3 \\ \dot{x}_3 &= - \beta x_3 + \xi \end{aligned} \tag{1-31}$$

PERFORMANCE INDEX:

$$E \int_0^T dt \quad , \qquad X(T) \in N_0$$

where N_0 is the neighborhood of the origin referred to previously, and $X = \text{col}(x_1, x_2, x_3)$. It will initially be assumed that the state is known exactly. Then we will consider the more realistic case where the state is not precisely known. Before we begin the analysis of these problems, however, it would be of considerable interest to determine the performance of other guidance systems under the influence of the same stochastic disturbances.

D. Analysis of the Second Variation Guidance Technique in the Presence of Stochastic Disturbances.

One guidance technique that has attracted much attention recently is the method of neighboring optimal trajectories, otherwise known as the second variation technique.[(3),(4),(5)] As was pointed out in Section 1, this scheme essentially yields the feedback solution of an optimization problem in a small region of the state space around the nominal trajectory. Of course, it is not necessary to use the second variation technique to obtain this feedback solution. For example, if at each instant of time one could recompute the open loop nominal trajectory based on the current state of the system, the same effect would be achieved. When this deterministic controller is blindly used in a noisy system, however, one may be surprised to find that a rather undesirable system performance results. The reason for this will become clear in the following analysis.

As a starting point we assume that the plant equations and the performance index have been specified. In addition, we presume that the associated nominal trajectory has been calculated. According to the theory of the second variation, the approximate feedback solution is obtained by solving the following equivalent linear problem:

PLANT:

$$\delta\dot{X} = \left|\frac{\partial^2 H}{\partial\lambda\partial X}\right|^* \delta X + \left|\frac{\partial^2 H}{\partial\lambda\partial u}\right|^* \delta u \qquad (1\text{-}32)$$

PERFORMANCE INDEX:

$$\int_0^T \left|\frac{\partial^2 H}{\partial u^2}\right|^* \delta u^2 + \left|\frac{\partial^2 H}{\partial X^2}\right|^* \delta X^2 + \left|\frac{\partial^2 H}{\partial u \partial X}\right|^* \delta X \delta u \qquad (1\text{-}33)$$

along with suitably specified (linearized) terminal boundary conditions. Here, H represents the prehamiltonian of the original problem, and the * indicates that a quantity is to be evaluated along the nominal trajectory. Thus Eqs.(1-32) and (1-33) yield a linear regulator problem whose solution can be exactly determined. We now wish to investigate this problem within a fairly general framework so as to obtain the dominant characteristics of the class of problems of interest.

First we need to make the observation that the weighting matrices in Eq.(1-33) act in such a way that δu and δX are penalized with approximately equal value for a given percentage change in nominal values. Another way of stating the same thing is that within a field of neighboring optimal trajectories small changes in the state X are caused by proportionately small changes in the control u. We will therefore assume that the weighting matrices can be approximated by constants whose values tend to produce the effect described above. Of course, an exact analysis would be totally dependent on the individual problem. Here we are attempting only to obtain the gross effects in a wide class of problems, and justification depends on comparison with particular cases.**

Since Eq.(1-32) simply represents the linearized plant equations, we will use the equations obtained in Section 2 (Eq.(1-9)). In keeping with the discussion above, we will use the following performance index:

$$\int_0^T \delta X \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & C_1 \end{bmatrix} \delta X + \delta u \begin{bmatrix} C_2^{-2} & 0 \\ 0 & C_2^{-2} \end{bmatrix} \delta u \; dt \qquad (1\text{-}34)$$

where the constants C_1 and C_2 are chosen such that X^2_{nom} and u^2_{nom} are given equal weight. Now, from the theory of the linear regulator problem, the optimal control δu^* is given by

** See Appendix A.

$$\delta u^* = -\frac{1}{2} R^{-1} b^T P \, \delta X \tag{1-35}$$

where the matrix P satisfies the differential equation

$$\dot{P} + PA + A^T P - \frac{1}{2} PbR^{-1}b^T P + 2Q = 0 \tag{1-36}$$

Here

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} ; \quad b = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} ; \quad R = \begin{bmatrix} C_2^{-2} & 0 \\ 0 & C_2^{-2} \end{bmatrix} ; \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 \end{bmatrix}$$

and the boundary condition on Eq.(1-36) depends on the transversality condition of the particular optimization problem. Since for all space missions the value of the terminal time T is very large, the matrix P would assume its stationary, or asymptotic value through most of the flight. We therefore solve Eq.(1-36) as an algebraic equation by setting $\dot{P} = 0$. Doing this yields the following control law (because of symmetry, we only consider one two-dimensional problem):

$$\delta u_1^* = - C_2 \delta x_1 - \sqrt{2C_2 + C_1 C_2^2} \, \delta x_2$$

Therefore, the spacecraft state deviations will obey the following differential equations:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -C_2 & -\sqrt{2C_2 + C_1C_2^2} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \end{bmatrix} \tag{1-37}$$

We now must analyze the performance of this system in the presence of noise. Using the same noise model described in Section C, we are led to the following Langevin equation:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -C_2 & -\sqrt{2C_2 + C_1C_2^2} & 1 \\ 0 & 0 & -\beta \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sigma\,\xi(t) \end{bmatrix} \tag{1-38}$$

where the new state, x_3, is the Ornstein-Uhlenbeck process, and $\xi(t)$ is a zero mean, Gaussian, white noise with unit variance. Now, according to the Fokker-Planck theory, the state of the system described by Eq.(1-38) is completely represented at each instant of time by a Gaussian density function, and the covariance matrix, M, of this density satisfies the following differential equation[9]:

$$\dot{M} = DM + MD^T + GG^T \tag{1-39}$$

where

$$D = \begin{bmatrix} 0 & 1 & 0 \\ -C_2 & -\sqrt{2C_2 + C_1C_2^2} & 1 \\ 0 & 0 & -\beta \end{bmatrix} ; \quad G = \begin{bmatrix} 0 \\ 0 \\ \sigma \end{bmatrix}$$

and the boundary condition is the prespecified initial covariance matrix, $M(0)$. We again use the fact that the time necessary for the space flight is very large, and M will necessarily converge to its asymptotic value. Therefore we solve Eq.(1-39) with $\dot{M} = 0$ and find

$$M = \begin{bmatrix} \left(\dfrac{1}{C_2} + \dfrac{\sqrt{2C_2 + C_1C_2^2}}{\beta C_2}\right) M_{22} & 0 & \dfrac{\sqrt{2C_2 + C_1C_2^2}}{\beta} M_{22} \\ 0 & M_{22} & \sqrt{2C_2 + C_1C_2^2}\, M_{22} \\ \dfrac{\sqrt{2C_2 + C_1C_2^2}}{\beta} M_{22} & \sqrt{2C_2 + C_1C_2^2}\, M_{22} & \sigma^2/2\beta \end{bmatrix}$$

where

$$M_{22} = \frac{\sigma^2/2\beta}{\dfrac{C_2\sqrt{2C_2 + C_1C_2^2}}{\beta} + 2C_2 + C_1C_2^2 + \beta\sqrt{2C_2 + C_1C_2^2}}$$

The values of M_{11} and M_{22}, which represent the standard deviations of the spacecraft position deviations and velocity deviations respectively, are plotted in Figs. (1-14) and (1-15) as a function of the attitude control limit cycle time. For the purpose of evaluating these quantities, the following nominal system parameters have been assumed

THRUST = 1 oz.

SPACECRAFT WEIGHT = 2500 lbs.

ATTITUDE CONTROL DEADBAND = 1°

Examination of Figs. (1-14) and (1-15) reveals that truly enormous errors may result from using this guidance strategy. An intuitive picture of why this happens is fairly easy to see. That is, when the spacecraft deviates from the nominal trajectory, the new optimal thrust program assumes a form such that the course of the vehicle is corrected in a relatively gradual fashion. This is perfectly all right to do in a deterministic system where there is no noise present to produce any further deviations. However, since large disturbances do act on the system we can only expect that by correcting the space vehicle's course very slowly, we give the noise more time to draw the vehicle even farther away from the nominal trajectory. This effect builds upon itself until steady state deviations are attained, as shown in Figs. (1-14) and (1-15). We must emphasize here that the errors we are considering are those resulting from the control policy only, and do not include the covariance of the state estimation, which would represent a lower bound on the accuracy of the space mission.

The conclusion we can draw from this analysis is that as the magnitude of the disturbances approaches the magnitude of the control available to the system, then a deterministic guidance policy seems to be quite inaccurate. It is apparent that a more aggressive error correction policy should be used in order to achieve the accuracy desired. Intuitively we sense that in problems of this type, where the available control is very small, that we must use <u>all</u> the control available <u>all</u> of the time in order to combat the effect of the noise inputs to the system. We now return to considering an approach which does just that.

E. The Stochastic Minimum Time Problem with Known State Variables.

In order to solve the problem formulated in Section C, we consider the method of dynamic programming. The first step is to define

$$V(\tau, C) = \underset{\substack{u(t) \\ \tau \le t \le T}}{\text{Min}} \; E \left[\int_{\tau}^{T} dt \;\middle|\; X(\tau) = C \right] \tag{1-40}$$

where we have regarded u - the mean value of u_1 - as a control variable. We can rewrite the right hand side of Eq.(1-40) as follows:

$$V(\tau,C) = \underset{\substack{u(t)\\ \tau\le t\le \Delta+\tau}}{\text{Min}} \quad \Delta + E[(\tau+\Delta,\ X(\tau+\Delta))|X(\tau) = C] + O(\Delta^2)$$

$$= \Delta + \underset{u}{\text{Min}} \int_X p(X(\tau+\Delta)|X(\tau) = C)\ V(\tau+\Delta, X(\tau+\Delta))dX + O(\Delta^2) \qquad (1\text{-}41)$$

where $p(X(\tau+\Delta)|X(\tau) = C)$ is the probability density of the state vector at time $\tau + \Delta$ given $X(\tau) = C$ (the transition probability density function). This quantity is obtained by solving the associated Fokker-Planck equation. Equation (1-41) is the building block by which we can generate numerical solutions to the problem. The approximation in policy space algorithm is employed, and basically this method involves the following steps:

1. Guess $V_o(\tau,C)$
2. Compute $V_n(\tau,C) = \Delta + \underset{u}{\text{Min}}\ E[V_{n-1}(\tau+\Delta,\ X(\tau+\Delta))|X(\tau)]$
3. Iterate until the solution converges.

We are now ready to solve for the transition probability density function.

F. The Fokker-Planck Equation.

Doob[11] has shown that the solution of Eq.(1-31) will be a Markov process which can be defined by its transition probability density function

$$p(o,\ X_o;\ t,\ X) \qquad (1\text{-}42)$$

which is the probability density that $X(t) = X$ given $X(o) = X_o$. In addition, it can be shown that this probability density function satisfies the Fokker-Planck equation associated with Eq.(1-31):

$$\frac{\partial p}{\partial t} = b\frac{\partial^2 p}{\partial x_3^2} - x_2\frac{\partial p}{\partial x_1} - u\frac{\partial p}{\partial x_2} - x_3\frac{\partial p}{\partial x_2} + \beta p + \beta x_3\frac{\partial p}{\partial x_3} \qquad (1\text{-}43)$$

where $b = \frac{\sigma^2}{2}$. In order to solve Eq.(1-43) we regard u as a constant. This is consistent with the fact that only discrete values of control are allowed, and also that u is held constant over the time interval Δ in the dynamic programming solution. The boundary condition on Eq.(1-43) is

$$\lim_{t\to 0} p(0,X_o;\ t,X) = \delta(x_1-x_{10})\ \delta(x_2-x_{20})\ \delta(x_3-x_{30}) \tag{1-44}$$

We proceed now to solve Eq.(1-43) using the Fourier transform technique. First we transform Eq.(1-43) in x_1

$$\int \frac{\partial p}{\partial t} e^{-ik_1x_1}dx_1 - \int b \frac{\partial^2 p}{\partial x_3^2} e^{-ik_1x_1}dx_1 + \int x_2 \frac{\partial p}{\partial x_1} e^{-ik_1x_1}dx_1 + \int u \frac{\partial p}{\partial x_2} e^{-ik_1x_1}dx_1$$

$$+ \int x_3 \frac{\partial p}{\partial x_2} e^{-ik_1x_1}dx_1 - \beta \int pe^{-ik_1x_1}dx_1 - \beta \int x_3 \frac{\partial p}{\partial x_3} e^{-ik_1x_1}dx_1 = 0 \tag{1-45}$$

If we define

$$\int pe^{-ik_1x_1}dx_1 \triangleq p'(t,k_1,x_2,x_3)$$

then Eq.(1-45) becomes

$$\frac{\partial p'}{\partial t} - b \frac{\partial^2 p'}{\partial x_3^2} + ik_1x_2p' + u \frac{\partial p'}{\partial x_2} + x_3 \frac{\partial p'}{\partial x_2} - \beta p' - \beta x_3 \frac{\partial p'}{\partial x_3} = 0 \tag{1-46}$$

Transforming Eq.(1-46) in x_2 and defining

$$\int p'e^{-ik_2x_2}dx_2 \triangleq p''(t,k_1,k_2,x_3)$$

we obtain

$$\frac{\partial p''}{\partial t} - b\,\frac{\partial^2 p''}{\partial x_3{}^2} + ik_1\,\frac{\partial p''}{\partial k_2}\,(i) + u\;ik_2 p'' + x_3 ik_2 p'' - \beta p'' - \beta x_3\,\frac{\partial p''}{\partial x_3} = 0 \qquad (1\text{-}47)$$

Finally, defining

$$\int p'' e^{-ik_3 x_3} dx_3 \triangleq \hat{p}(t,k_1,k_2,k_3)$$

we have

$$\frac{\partial \hat{p}}{\partial t} - b(ik_3)^2\,\hat{p} - k_1\,\frac{\partial \hat{p}}{\partial k_2} + iuk_2\hat{p} + ik_2(i)\,\frac{\partial \hat{p}}{\partial k_3} - \beta\hat{p} + \beta\hat{p} + \beta k_3\,\frac{\partial \hat{p}}{\partial k_3} = 0$$

which becomes

$$\frac{\partial \hat{p}}{\partial t} - k_1\,\frac{\partial \hat{p}}{\partial k_2} + (\beta k_3 - k_2)\,\frac{\partial \hat{p}}{\partial k_3} + (bk_3{}^2 + iuk_2)\,\hat{p} = 0 \qquad (1\text{-}48)$$

Equation (1-48) is a linear first order partial differential equation which has the characteristic equations

$$\frac{dt}{ds} = 1 \quad ; \quad \frac{dk_1}{ds} = 0 \quad ; \quad \frac{dk_2}{ds} = -\,k_1$$

$$\frac{dk_3}{ds} = \beta k_3 - k_2 \quad ; \quad \frac{d\hat{p}}{ds} = -\,(bk_3{}^2 + iuk_2)\,\hat{p}$$

These equations have the solutions (subscript o indicates initial values)

$$t = s \quad ; \quad k_1 = k_{10} \quad ; \quad k_2 = -\,k_{10}s + k_{20} \quad ;$$

$$k_3 = e^{\beta s}k_{30} - \frac{k_{10}s}{\beta} - \frac{k_{10}}{\beta^2} + \frac{k_{10}e^{\beta s}}{\beta^2} + \frac{k_{20}}{\beta}\,(1-e^{\beta s}) \quad ; \qquad (1\text{-}49)$$

$$\hat{p} = \hat{p}(0) \exp - \left\{ \int_0^s b\, e^{\beta s'} k_{30} - \frac{k_{10}s'}{\beta} - \frac{k_{10}}{\beta^2} + \frac{k_{10}e^{\beta s'}}{\beta^2} + \frac{k_{20}}{\beta}(1-e^{\beta s'}) \right)^2 + iu(-k_{10}s' + k_{20})\, ds' \Bigg\} \qquad (1\text{-}50)$$

Performing the integration in Eq.(1-50) and collecting terms yields

$$\hat{p} = \hat{p}(o) \exp - \Bigg\{ b\, k_{10}^{\,2} \left(\frac{s^3}{3\beta^2} + \frac{s}{\beta^4} - \frac{2se^{\beta s}}{\beta^4} + \frac{s^2}{\beta^3} + \frac{e^{2\beta s}}{2\beta^5} - \frac{1}{2\beta^5} \right)$$

$$+ k_{20}^{\,2} \left(\frac{s}{\beta^2} + \frac{e^{2\beta s}}{2\beta^3} - \frac{2e^{\beta s}}{\beta^3} + \frac{3}{2\beta^3} \right) + k_{30}^{\,2} \left(\frac{e^{2\beta s}}{2\beta} - \frac{1}{2\beta} \right) + k_{10}k_{20} \left(- \frac{s^2}{\beta^2} + \frac{2se^{\beta s}}{\beta^3} + \frac{2e^{\beta s}}{\beta^4} \right.$$

$$\left. - \frac{1}{\beta^4} - \frac{2s}{\beta^3} - \frac{e^{2\beta s}}{\beta^4} \right) + k_{10}k_{30} \left(- \frac{2se^{\beta s}}{\beta^2} + \frac{e^{2\beta s}}{\beta^3} - \frac{1}{\beta^3} \right) + k_{20}k_{30} \left(\frac{2e^{\beta s}}{\beta^2} - \frac{1}{\beta^2} - \frac{e^{2\beta s}}{\beta^2} \right)$$

$$- \frac{ius^2}{2} k_{10} + iu\, k_{20}s \Bigg\} \qquad (1\text{-}51)$$

We now impose the boundary condition as follows:

$$p(0,X_o;s,x_1,x_2,x_3) = \left(\frac{1}{2\pi}\right)^3 \iiint \hat{p}(s,k_{10},k_{20},k_{30}) e^{ik_{10}x_1} e^{ik_{20}x_2} e^{ik_{30}x_3} dk_{10}dk_{20}dk_{30}$$

But

$$p(0,x_{10},x_{20},x_{30};0,x_{10},x_{20},x_{30}) = \delta(x_1-x_{10})\, \delta(x_2-x_{20})\, \delta(x_3-x_{30})$$

$$= \left(\frac{1}{2\pi}\right)^3 \iiint \hat{p}(0,k_{10},k_{20},k_{30}) e^{ik_{10}x_{10}} e^{ik_{20}x_{20}} e^{ik_{30}x_{30}} dk_{10}dk_{20}dk_{30}$$

Thus we must have

$$\int \delta(x_1-x_{10})\delta(x_2-x_{20})\delta(x_3-x_{30})e^{-ik_{10}x_1}e^{-ik_{20}x_2}e^{-ik_{30}x_3}dx_1dx_2dx_3 =$$

$$= \hat{p}(0,k_{10},k_{20},k_{30}) = e^{-ik_{10}x_{10}}e^{-ik_{20}x_{20}}e^{-ik_{30}x_{30}} \quad (1\text{-}52)$$

Using Eq.(1-52), and solving Eqs.(1-49) for s, k_{10}, k_{20}, and k_{30}, Eq.(1-51) becomes

$$\hat{p} = \exp - \Big[ik_1x_{10} + i(k_2+k_1t)\, x_{20} + i \Big(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta} + k_3e^{-\beta t}\Big) x_{30}$$

$$+ b\, k_1^{\,2} \Big(\frac{t^3}{3\beta^2} + \frac{t}{\beta^4} - \frac{2te^{\beta t}}{\beta^4} + \frac{t^2}{\beta^3} + \frac{e^{2\beta t}}{2\beta^5} - \frac{1}{2\beta^5}\Big) + (k_2+k_1t)^2 \Big(\frac{t}{\beta^2} + \frac{e^{2\beta t}}{2\beta^3} - \frac{2e^{\beta t}}{\beta^3} + \frac{3}{2\beta^3}\Big)$$

$$+ \Big(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta} + k_3e^{-\beta t}\Big)^2 \Big(\frac{e^{2\beta t}}{2\beta} - \frac{1}{2\beta}\Big) + k_1(k_2+k_1t)$$

$$\Big(- \frac{t^2}{\beta^2} + \frac{2te^{\beta t}}{\beta^3} + \frac{2e^{\beta t}}{\beta^4} - \frac{1}{\beta^4} - \frac{2t}{\beta^3} - \frac{e^{2\beta t}}{\beta^4}\Big) + k_1 \Big(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta}$$

$$+ k_3e^{-\beta t}\Big) \Big(\frac{-2te^{\beta t}}{\beta^2} + \frac{e^{2\beta t}}{\beta^3} - \frac{1}{\beta^3}\Big) + (k_2+k_1t) \Big(\frac{k_1e^{-\beta t}}{\beta^2} - \frac{k_1}{\beta^2} - \frac{e^{-\beta t}k_2}{\beta} + \frac{k_2}{\beta} + \frac{k_1t}{\beta}$$

$$+ k_3e^{-\beta t}\Big) \Big(\frac{2e^{\beta t}}{\beta^2} - \frac{1}{\beta^2} - \frac{e^{2\beta t}}{\beta^2}\Big) - \frac{iut^2}{2} k_1 + iu(k_2+k_1t)t \Big] \quad (1\text{-}53)$$

Simplifying Eq.(1-53) and collecting terms, we finally obtain

$$\hat{p} = \exp - \Big[k_1^{\,2} \Big(\frac{bt^3}{3\beta^2} - \frac{bt^2}{\beta^3} + \frac{bt}{\beta^4}(1-2e^{-\beta t}) + \frac{b}{\beta^5}\Big(\frac{1}{2} - \frac{e^{-2\beta t}}{2}\Big)\Big) + k_2^{\,2} \Big(\frac{bt}{\beta^2} + \frac{b}{\beta^3}\Big(- 3/2$$

$$+ 2e^{-\beta t} - \frac{e^{-2\beta t}}{2}\Big)\Big) + k_3^{\,2} \Big(\frac{b}{2\beta} - \frac{be^{-2\beta t}}{2\beta}\Big) + k_1k_2 \Big(\frac{bt^2}{\beta^2} + \frac{bt}{\beta^3}\big(- 2 + e2^{-\beta t}\big) + \frac{b}{\beta^4}\big(1 + e^{-2\beta t}$$

$$- 2e^{-\beta t}\big)\Big) + k_2k_3 \frac{b}{\beta^3}\big(1 + e^{-2\beta t} - 2e^{-\beta t}\big) + k_1k_3 \Big(\frac{bt}{\beta^2}\big(-2e^{-\beta t}\big) + \frac{b}{\beta^3}\big(1 - e^{-2\beta t}\big)\Big)$$

$$+ ik_1 \left(x_{10} + x_{20}t + x_{30} \left(\frac{e^{-\beta t}}{\beta^2} - \frac{1}{\beta^2} + \frac{t}{\beta} \right) + \frac{ut^2}{2} \right) + ik_2 \left(x_{20} + x_{30} \left(\frac{1}{\beta} - \frac{e^{-\beta t}}{\beta} \right) + ut \right)$$

$$+ ik_3 (e^{-\beta t} x_{30}) \tag{1-54}$$

This expression can be recognized as the characteristic function of a Gaussian density with mean values

$$\bar{x}_1 = x_{10} + x_{20}t + x_{30} \left(\frac{e^{-\beta t}}{\beta^2} - \frac{1}{\beta^2} + \frac{t}{\beta} \right) + \frac{ut^2}{2}$$

$$\bar{x}_2 = x_{20} + x_{30} \left(\frac{1}{\beta} - \frac{e^{-\beta t}}{\beta} \right) + ut$$

$$\bar{x}_3 = e^{-\beta t} x_{30}$$

and covariance matrix (K)

$$k_{11} = E[(x_1 - \bar{x}_1)^2] = 2 \left(\frac{bt^3}{3\beta^2} - \frac{bt^2}{\beta^3} + \frac{bt}{\beta^4} (1 - 2e^{-\beta t}) + \frac{b}{\beta^5} \left(\frac{1}{2} - \frac{e^{-2\beta t}}{2} \right) \right)$$

$$k_{22} = E[(x_2 - \bar{x}_2)^2] = 2 \left(\frac{bt}{\beta^2} + \frac{b}{\beta^3} \left(-3/2 + 2e^{-\beta t} - \frac{e^{-2\beta t}}{2} \right) \right)$$

$$k_{33} = E[(x_3 - \bar{x}_3)^2] = 2 \left(\frac{b}{2\beta} - \frac{be^{-2\beta t}}{2\beta} \right)$$

$$k_{12} = E[(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)] = \frac{bt^2}{\beta^2} + \frac{bt}{\beta^3} (-2 + 2e^{-\beta t}) + \frac{b}{\beta^4} (1 + e^{-2\beta t} - 2e^{-\beta t})$$

$$k_{23} = E[(x_2-\bar{x}_2)(x_3-\bar{x}_3)] = \frac{b}{\beta^2}(1 + e^{-2\beta t} - 2e^{-\beta t})$$

$$k_{13} = E[(x_1-\bar{x}_1)(x_3-\bar{x}_3)] = \frac{bt}{\beta^2}(-2e^{-\beta t}) + \frac{b}{\beta^3}(1 - e^{-2\beta t})$$

Of course, x_3 could be carried along in the analysis as a possible refinement, but here we will only concern ourselves with the marginal density of x_1 and x_2, and x_3 will be integrated out of these equations. The result is $(KK = K^{-1};\ X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix})$

$$p(0,x_{10},x_{20},x_{30};t,x_1,x_2) = \frac{1}{2\pi|K|^{1/2}(kk_{33})^{1/2}} e^{-\frac{1}{2}(X-\bar{X})^T \begin{bmatrix} kk_{11} - \frac{kk_{13}^2}{kk_{33}} & kk_{12} - \frac{kk_{23}kk_{13}}{kk_{33}} \\ kk_{12} - \frac{kk_{23}kk_{13}}{kk_{33}} & kk_{22} - \frac{kk_{23}^2}{kk_{33}} \end{bmatrix} (X-\bar{X})} \tag{1-55}$$

Equation (1-55) is used in conjunction with Eq.(1-41) to carry out the numerical analysis of the stochastic minimum time problem.

G. Numerical Results with Known State.

As has already been indicated, the approximation in policy space algorithm was used in the analysis. The initial guess for the value function was given by the deterministic solution to the same problem. The target neighborhood of the origin was taken as a rectangular area with boundaries at $\pm$ 40 meters in the x_1 direction, and $\pm$ .05 meter/sec. in the x_2 direction. Grid sizes of 10 meters and .025 meter/sec. were used and gave the desired accuracy. Due to considerations of computing time, the area of interest was limited to ± 1 km. by $\pm .25$ m.sec. Of course, solutions in larger regions could be obtained if desired. Insight into the general solution characteristics can be acquired, however, from the results in the area that was considered.

The results appear in Fig. 1-16. A total of four runs were made and the resulting switching curves are shown for each case. Two runs were made using a correlation time of 20 minutes for the attitude control limit cycle. It should be noted that decreasing the correlation time has the same effect as decreasing the variance of the noise. It is interesting to note that as the noise gets large -- i.e., with its standard deviation equal to the value of the control magnitude -- then the switching curve is pushed back near to the x_1 axis. This is reasonable, though, since when the noise is the same order of magnitude as the control we wish to avoid "wandering" as demonstrated in Fig. 1-18.

H. The Stochastic Minimum Time Problem with Estimated State Variables.

1. State Estimation.

We now turn to the more realistic case of when the state vector is not known exactly, but must be estimated. For the purposes of analyzing the stochastic minimum time problem, we would be interested in having the state estimates in the rotated coordinate frame of Fig. (1-4). We therefore formulate the estimation problem using polar coordinates and note that for particular cases the additional $\alpha(t)$ degrees of rotation could subsequently be made conforming with the definitions given in Eqs.(1-2) and (1-3), Section 2. Referring to Fig. 1-19, the plant equations are given as follows:

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} - a \sin\alpha + x_5 \cos\alpha - x_6 \sin\alpha$$

$$\dot{v} = -\frac{uv}{r} + a\cos\alpha + x_6\cos\alpha + x_5\sin\alpha$$

$$\dot{r} = u \qquad (1\text{-}56)$$

$$\dot{\theta} = v/r$$

$$\dot{x}_5 = -\beta_1 x_5 + \xi_1(t)$$

$$\dot{x}_6 = -\beta_2 x_6 + \xi_2(t)$$

where x_5 and x_6 are the attitude and acceleration stochastic processes, respectively, which are used in accordance with the previous discussion. Furthermore, we presume that the following measurements are allowed on the system: (R = Earth's orbital radius; ω = Earth's orbital angular velocity):

(a) Range: $h_1 = \rho = [r^2 + R^2 - 2rR\cos(\theta - \omega t)]^{1/2} + \xi_3(t)$

(b) Range Rate: $h_2 = \dot{\rho} = \dfrac{ru - Ru\cos(\theta - \omega t) + rR(\frac{v}{r} - \omega)\sin(\theta - \omega t)}{\rho} + \xi_4(t)$

(c) Yaw Angle: $h_3 = x_5 + \xi_5(t)$

(d) Low-Thrust Acceleration: $h_4 = x_6 + \xi_6(t)$

where ξ_3, ξ_4, ξ_5, and ξ_6 are the error processes associated with the observation measurements. We now assume that the linearized plant and observations equations are sufficiently accurate for our purposes, and therefore the Kalman(12) filter equations yield the minimum variance estimate of the state of the system. The error covariance matrix, P, satisfies the following Ricatti differential equation:

$$\dot{P} = AP + PA^T - PH^TK^{-1}HP + GQG^T \qquad (1\text{-}57)$$

where

$$A = \begin{bmatrix} 0 & \frac{2v}{r} & \frac{-v^2}{r^2} + \frac{2\mu}{r^3} & 0 & \cos\alpha & -\sin\alpha \\ \frac{-v}{r} & \frac{-u}{r} & \frac{vu}{r} & 0 & \sin\alpha & \cos\alpha \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r} & -\frac{v}{r^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\beta_2 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & \frac{r - R\cos(\theta-\omega t)}{\rho} & \frac{rR\sin(\theta-\omega t)}{\rho} & 0 & 0 \\ \frac{r-R\cos(\theta-\omega t)}{\rho} & \frac{rR\sin(\theta-\omega t)}{\rho} & \frac{u - \sin(\theta-\omega t)\omega R - \dot{\rho}\frac{\partial\rho}{\partial r}}{\rho} & \frac{Ru\sin(\theta-\omega t) + rR\frac{v}{r} - \omega\cos(\theta-\omega t) - \dot{\rho}\frac{\partial\rho}{\partial\theta}}{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} \sigma_3^2 & 0 & 0 & 0 \\ 0 & \sigma_4^2 & 0 & 0 \\ 0 & 0 & \sigma_5^2 & 0 \\ 0 & 0 & 0 & \sigma_6^2 \end{bmatrix} ; \quad G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} ; \quad Q = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$$

where σ_i^2 is the covariance of the process $\xi_i(t)$. Hence, with the specification of the initial error covariance matrix, we may calculate $P(t)$ before launch using Eq.(1-57).

2. Optimization of Measurements.

Before continuing with the analysis of the stochastic minimum time controller, we turn to the very meaningful and interesting question of measurement optimization. We have specified in the previous section that accelerometer and angular measurements should be available. If they are not, then the state deviation uncertainties grow to enormous values by the time a mission would be complete. We therefore ask to which of the two measurements, accelerometer or angular, we should devote the most money. Putting this another way, suppose that we are given a fixed total cost for the sensors of the type mentioned. Then the question is how should we allocate the funds to derive a maximum return from the sensors? Of course, the definition of the return is somewhat arbitrary, and therefore we consider two possibilities.

$$\begin{aligned} &(a) \quad \int_0^T P_{11}^2 + P_{22}^2 + k^2(P_{33}^2 + P_{44}^2)\, dt \\ &(b) \quad P_{11}(T) + P_{22}(T) + k(P_{23}(T) + P_{44}(T)) \end{aligned} \qquad (1\text{-}58)$$

These performance indices give an indication of how accurately the state of the system is determined. We now assume that we pay a fixed price for each order of magnitude of accuracy of the angular and acceleration measurements. That is to say, if we are allocating a fixed amount of money, then

$$\log \sigma_5 + \log \sigma_6 = \text{constant} \qquad (1\text{-}59)$$

We note here that it would be a simple matter to consider other cost specifications, but to demonstrate the technique, Eq.(1-59) is assumed. It is quite reasonable to assume that σ_5 and σ_6 possess lower bounds which are necessitated by the level of sensor technology. The numerical results for this problem are shown in Fig. 1-20. The values of Eqs.(1-58) are represented using -12 for the constant in Eq.(1-59), and 10^{-10} as the lower bound on the standard deviation of both sensors. The optimization is made over approximately 50 days of the Mars mission considered in earlier experiments. It is obvious from examining Fig. 1-20 that accurate acceleration measurements are to be preferred over accurate angular measurements for the mission under consideration. The diagonal elements of the covariance matrix, as well as the nominal state vector, are shown in Fig. 1-21 - 1-24 where the optimum observations have been assumed. For comparative purposes, the same quantities are plotted in Figs. 1-25 - 1-28 using non-optimum, but admissible, values of the measurement accuracies.

We conclude that optimization of the state estimation yields definite gains for the system that was considered. This result lends motivation for a similar analysis of other space missions.

3. An Algorithm for Determining the Optimum Switching Strategy for the Stochastic Minimum Time Controller with State Estimation.

We now are ready to determine in what way our switching strategy would change relative to uncertainties in the state variables. We saw in the last section that the "state" of the system is actually given by a Gaussian density function with a time varying covariance matrix. In order to use the numerical approach described in Section E, we must determine in what way the transition probability density function needs to be modified in order to account for imprecisely known state variables. This will be done in the following section, where it will be assumed that $P(t)$ can be approximated by a step function with a fixed interval size Δ. It is now apparent that the transition probability density function will vary with time along the nominal trajectory. For this reason, we need to augment the algorithm described in Section E. The technique we employ here will be to first compute the switching strategy at nominal encounter time using the method of Section E. We then step backwards in time, at intervals of Δ, and determine the switching strategy at each interval by the following method:

(1) Compute $V(T,C)$ by the method in Section E.

(2) Compute $V(T-n\Delta,C) = \Delta + \min_u E_n[V(T-(n-1)\Delta, X(T-(n-1)\Delta) | X(T-n\Delta)]$, and store $u(T-n\Delta,C)$, for $n = 1, 2, \cdots N$ where N is the smallest number such that $T - N\Delta \le 0$. In this formula, E_n indicates that the expectation is to be calculated using the transition probability density function at time $T - n\Delta$.

We are now ready to solve for the modified transition probability density function.

4. The Transition Probability Density Function with State Variable Uncertainties.

The new transition probability density function can be obtained using the following integral:

$$P(X(t)|P(X(0))) = \int_{\text{Initial States } X(0)} P(\text{transition to } X(t)|X(0))P(X_0)dX_0 \qquad (1\text{-}60)$$

The first term in the integral is given by Eq.(1-55), and can be represented as

$$P(0,X_0;t,X) = \frac{1}{\pi|M|^{1/2}} e^{-(X-(AX_0+B))^T M^{-1}(X-(AX_0+B))}$$

where

$$A = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \quad ; \quad B = \begin{bmatrix} X_{30}\dfrac{e^{-\beta t}}{\beta^2} & -\dfrac{1}{\beta} & +\dfrac{t}{\beta}+\dfrac{ut^2}{2} \\ X_{30}\dfrac{1}{\beta} & -\dfrac{e^{-\beta t}}{\beta} & +ut \end{bmatrix}$$

The second term in the integrand is given by the Kalman estimate which can be represented as

$$P(X_o) = \frac{1}{\pi|P|^{1/2}} e^{-(X_o-\mu)^T P^{-1}(X_o-\mu)}$$

where μ is the estimated state, and $2P$ is the error covariance matrix which satisfies Eq.(1-57). We have noted that P is to be approximated by a step function. Therefore we let $P(T-n\Delta) = C_n$. Eq.(1-60) now becomes

$$P(X(t)|P(X(0))) = \int_{X(0)} \frac{1}{\pi|M|^{1/2}} e^{-(X-(AX_o+B))^T M^{-1}(X-(AX_o+B))}$$

$$\frac{1}{\pi|C_n|^{1/2}} e^{-(X_o-\mu)^T C_n^{-1}(X_o-\mu)} \, dX_o \qquad (1\text{-}61)$$

To carry out this integration we define

$$A^{-1}X - A^{-1}B \triangleq D$$

Then Eq.(1-61) becomes

$$P(X(t)|P(X(0))) = \frac{1}{\pi^2|M|^{1/2}|C_n|^{1/2}} \int e^{-(A(X_o-D))^T M^{-1}(A(X_o-D))} e^{-(X_o-\mu)^T C_n^{-1}(X_o-\mu)} \, dX_o$$

$$(1\text{-}62)$$

Now let

$$E = D-\mu = A^{-1}X - A^{-1}B-\mu \quad ; \quad \hat{X} = X_o-\mu$$

and Eq.(1-62) becomes

$$P(X(t)|P(X(0))) = \frac{1}{\pi^2|M|^{1/2}|C_n|^{1/2}}\int e^{-(\hat{X}-E)^TA^TM^{-1}A(\hat{X}-E) - \hat{X}\, C_n^{-1}\hat{X}}\, d\hat{X}$$

$$= \frac{1}{\pi^2|MC_n|^{1/2}}\int e^{\{-\hat{X}^T(C_n^{-1}+A^TM^{-1}A)\hat{X} + 2E^TA^TM^{-1}A\hat{X}-E^T(A^TM^{-1}A)E\}}d\hat{X} \tag{1-63}$$

Letting

$$(C_n^{-1} + A^TM^{-1}A)^{1/2}\hat{X} = X^*$$

equation (1-63) becomes

$$P(X(t)|P(X(0))) = \frac{e^{-E^T(A^TM^{-1}A)E}}{\pi^2|MC_n|^{1/2}|C_n^{-1}+A^TM^{-1}A|^{1/2}}\int e^{-X^{*T}X^*+E^T2A^TM^{-1}A(C_n^{-1}+A^TM^{-1}A)^{-1/2}X^*}\,dX^* \tag{1-64}$$

Completing the square and performing the integration yields

$$P(X(t)|P(X(0)) = \frac{1}{\pi|MC(C_n^{-1}+A^TM^{-1}A)|^{1/2}}\exp\{-E^T(A^TM^{-1}A)E+E^TA^TM^{-1}A(C_n^{-1}+A^TM^{-1}A)^{-1/2}$$

$$[E^TA^TM^{-1}A(C_n^{-1}+A^TM^{-1}A)^{-1/2}]^T\} = \frac{1}{\pi|MC_n(C_n^{-1}+A^TM^{-1}A)|^{1/2}}$$

$$\exp\{-E^T(A^TM^{-1}A+A^TM^{-1}A(C_n^{-1}+A^TM^{-1}A)^{-1}A^TM^{-1}A)E\}$$

We now substitute for E to obtain

$$P(X(t)|P(X(0)) = \frac{1}{\pi|MC_n(C_n^{-1}+A^TM^{-1}A)|^{1/2}} \exp\{-(X-B-A\mu)^T A^{-1T}(A^TM^{-1}A-A^TM^{-1}A(K^{-1}+A^TM^{-1}A)^{-1} A^TM^{-1}A)A^{-1}(X-(B+A\mu))\}$$

This simplifies to finally yield

$$P(X(t)|P(X(0))) = \frac{1}{\pi|MC(C_n^{-1}+A^TM^{-1}A)|^{1/2}} \exp\{-(X-(A\mu+B))^TM^{-1}(I-A(C_n^{-1}+A^TM^{-1}A)^{-1}A^TM^{-1}) (X-(A\mu+B))\} \qquad (1\text{-}65)$$

It is interesting to note the differences between Eq.(1-65) and Eq.(1-55). In Eq.(1-65) we note that the true state is now replaced by its estimate, μ, and also that the original covariance matrix inverse, M^{-1}, in Eq.(1-19) is degraded in Eq.(1-65) by a factor

$$I - A(C_n^{-1}+A^TM^{-1}A)^{-1}A^TM^{-1}$$

5. Numerical Results for the Case of Estimated State Variables.

Numerical experiments were performed in the unknown state case in exactly the same way as previously described except that this time Eq.(1-65) was used for the transition probability density function. Generally speaking, introducing uncertaintities in the state variables has the same general effect as increasing the magnitude of the disturbances on the space vehicle. Three numerical experiments were made using selected system parameters which are indicated in Fig. 1-29.

1.4. Summary and Future Work.

In this report we have emphasized the influence of noisy actuators on a low-thrust guidance system. It has been shown that the somewhat "classical" system determined via the second variation technique fails to yield adequate terminal accuracy when subjected to the noise inputs which one could normally expect. In fact, the noise level would most likely be comparable in magnitude to the control forces, and for that reason "small noise" assumptions are invalid.

The approach taken here has been to design a guidance system which not only demands the desired terminal accuracy, but also minimizes the fuel required to perform that task. The analysis of this system when it is subjected to stochastic inputs led to rather interesting results. It was shown that the "separation" property of estimation and control -- which one obtains with the usual case of linear dynamics, Gaussian noise, and least squares performance index -- is not obtained using a nonlinear controller of the type considered. In other words, one cannot merely substitute the estimated state variables for their actual values in a deterministic controller. In fact, for the important aspect of this problem where the noise becomes comparable in magnitude to the control, it was found that a switching strategy far different from the deterministic solution must be used. To use the deterministic strategy in such cases might bring a disasterous ending to the space mission.

In Appendix A, a particular example of the inadequacy of the "optimum", or neighboring-optimal guidance system is demonstrated. In future reports, it is planned to carry out a similar analysis of other missions of current interest to JPL.

APPENDIX A

ANALYSIS OF THE NEIGHBORING-OPTIMAL GUIDANCE SYSTEM ACCURACY FOR A CONSTANT THRUST, MINIMUM-TIME MARS RENDEZVOUS MISSION

In the following, the complete numerical analysis of the accuracy of a guidance system obtained by means of the second-variation optimization technique is described. The first step is to obtain the nominal, or open loop trajectory, and for that purpose a constant acceleration level of $.78 \times 10^{-3}$ m./sec^2 is assumed which corresponds to a 3 oz. thrust applied to a 2500 lb. space vehicle. Since the minimum time Mars rendezvous is a free terminal time problem, we use the analytical artifice of normalized time to convert the free terminal time problem into the more usual fixed time problem. This is done by defining

$$t = T\tau \qquad \tau \in [0,1] \tag{A-1}$$

where t is the true time and τ is the normalized time. Here T represents the unknown terminal time which is treated as a state variable by adjoining its dynamical equation

$$\dot{T} = 0$$

It is easily seen that

$$\frac{dx}{d\tau} = T\frac{dx}{dt}$$

and thus we can consider the following equivalent dynamical system (note: dotted variables represent derivatives with respect to τ)

$$\dot{u} = \frac{v^2}{r} - \frac{\mu}{r^2} - a \sin \alpha \, T$$

$$\dot{v} = \left(- \frac{uv}{r} + a \cos \alpha\right) T$$

$$\dot{r} = uT$$

$$\dot{\theta} = \frac{vT}{r}$$

$$\dot{T} = 0$$

If at this point we form the Hamiltonian of the optimization problem, and carry out its minimization with respect to α according to Pontryagin's maximum principle, we can obtain the canonic differential equations for the system state variables and Lagrange multipliers:

$$\dot{u} = \left(\frac{v^2}{r} - \frac{\mu}{r^2} - \frac{a\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right) T$$

$$\dot{v} = \left(- \frac{uv}{r} - \frac{a\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} \right) T$$

$$\dot{r} = uT$$

$$\dot{\theta} = \frac{vT}{r} \qquad \text{(A-2)}$$

$$\dot{T} = 0$$

$$\dot{\lambda}_1 = \frac{\lambda_2 vT}{r} - \lambda_3 T$$

$$\dot{\lambda}_2 = - \frac{2\lambda_1 Tv}{r} + \frac{\lambda_2 Tu}{r} - \frac{\lambda_4 T}{r}$$

$$\dot{\lambda}_3 = \frac{\lambda_1 v^2 T}{r^2} - \frac{2\mu\lambda_1 T}{r^3} - \frac{\lambda_2 vuT}{r^2} + \frac{\lambda_4 vT}{r^2}$$

$$\dot{\lambda}_4 = 0$$

$$\dot{\lambda}_5 = -1 - \lambda_1\left(\frac{v^2}{r} - \frac{\mu}{r^2} - \frac{a\lambda_1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\right) - \lambda_2\left(-\frac{uv}{r} - \frac{a\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}\right)$$

$$- \lambda_3 u - \frac{\lambda_4 v}{r}$$

The given boundary conditions, and the transversality conditions associated with the optimization problem yield the boundary values necessary for the solution of Eq.(A-2):

$$u(0) = 0$$

$$u(1) = 0$$

$$v(0) = V_{EARTH}$$

$$v(1) = V_{MARS} \qquad \text{(A-3)}$$

$$r(0) = r_{EARTH}$$

$$r(1) = r_{MARS}$$

$$\theta(0) = \theta_{EARTH}$$

$$\theta(1) = \theta_{MARS}(T)$$

$$\lambda_5(0) = 0$$

$$\lambda_5(1) = - \lambda_4(1)\ \dot{\theta}_{MARS}$$

Using the quasilinearization method, a solution of Eq.(A-2) was determined which satisfies the required boundary conditions.

At this point we are ready to consider the second variation, or neighboring optimal guidance system. Although more elegant derivations have been given, the method is quite equivalent to linearizing Eqs.(A-2), and using these to approximate the behavior of the system. Therefore suppose that at time τ we have known state variable deviations equal to col[$\delta u(\tau)$, $\delta v(\tau)$, $\delta r(\tau)$, $\delta\theta(\tau)$], and we wish to determine the manner in which these errors are nulled in the optimum system. From the theory of linear differential equations, we must have

$$\Phi(1,\tau)\begin{bmatrix}\delta u(\tau)\\ \delta v(\tau)\\ \delta r(\tau)\\ \delta\theta(\tau)\\ \delta T(\tau)\\ \delta\lambda_1(\tau)\\ \delta\lambda_2(\tau)\\ \delta\lambda_3(\tau)\\ \delta\lambda_4(\tau)\\ \delta\lambda_5(\tau)\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ \dot{\theta}_{MARS}\ \delta T(1)\\ \delta T(1)\\ \delta\lambda_1(1)\\ \delta\lambda_2(1)\\ \delta\lambda_3(1)\\ \delta\lambda_4(1)\\ -\delta\lambda_4(1)\ \dot{\theta}_{MARS}\end{bmatrix} \tag{A-4}$$

where Φ is the (10 x 10) fundamental matrix of the linearized equations, and satisfies

$$\dot{\Phi} = A(\tau)\ \Phi \qquad\qquad \Phi(0) = I$$

Here A is the Jacobian matrix of Eqs.(A-2) evaluated along the nominal state and Lagrange multiplier vectors. Since Eqs.(A-4) are 10 linear equations in 10 unknowns, we may determine the unknown initial values in the following form

$$\begin{bmatrix} \delta T(\tau) \\ \delta\lambda_1(\tau) \\ \delta\lambda_2(\tau) \\ \delta\lambda_3(\tau) \\ \delta\lambda_4(\tau) \end{bmatrix} = B(\tau) \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix}$$

where $B(\tau)$ is a 5×4 matrix (note that $\delta\lambda_5(\tau) = 0$). Hence we may write

$$\begin{bmatrix} \delta\dot{u} \\ \delta\dot{v} \\ \delta\dot{r} \\ \delta\dot{\theta} \end{bmatrix} = \left[\Phi^{(1)}(1,\tau) + \Phi^{(2)}(1,\tau)\, B(\tau) \right] \begin{bmatrix} \delta u \\ \delta v \\ \delta r \\ \delta\theta \end{bmatrix} \tag{A-5}$$

where $\Phi^{(1)}$ is defined to be the first four terms of the first four rows of the fundamental matrix, and $\Phi^{(2)}$ represents the fifth through the ninth terms of the first four rows of the fundamental matrix. Eq.(A-5) therefore represents the differential equation satisfied by the state variable deviations.

In order to complete the analysis, we must now adjoin to the system the noise terms representing the attitude and thrust vector variations. One obtains a result completely analogous to Eq.(1-56):

$$\dot{X} = CX + G \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{A-6}$$

where $X \triangleq \text{col}[\delta u,\ \delta v,\ \delta r,\ \delta\theta,\ x_5,\ x_6]$ and

$$C = \begin{bmatrix} & & \cos\alpha & -\sin\alpha \\ & & \sin\alpha & \cos\alpha \\ \Phi^1 + \Phi^2 B & & 0 & 0 \\ & & 0 & 0 \\ 0\ 0\ 0\ 0 & & -\beta_1 & 0 \\ 0\ 0\ 0\ 0 & & 0 & -\beta_2 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and ξ_1 and ξ_2 are independent, zero mean, Gaussian, white noises with variances $\sigma_1^{\,2}$ and $\sigma_2^{\,2}$. The state deviation at time τ is therefore given by

$$\int_0^{\tau} \hat{\Phi}(\tau,t)\, G \begin{bmatrix} \xi_1(t) \\ \xi_2(t) \end{bmatrix} dt \tag{A-7}$$

where $\hat{\Phi}$ is the fundamental matrix of Eq.(A-6). The covariance of the deviations is then given by

$$E\left[\left(\int_0^{\tau} \hat{\Phi}(\tau,t)\, G(t_1) \begin{bmatrix} \xi_1(t_1) \\ \xi_2(t_2) \end{bmatrix} dt_1\right)\left(\int_0^{\tau} \hat{\Phi}\,(\tau,t_2)\, G(t_2) \begin{bmatrix} \xi_1(t_2) \\ \xi_2(t_2) \end{bmatrix} dt_2\right)^T\right]$$

$$= \int_0^{\tau} \hat{\Phi}\,(\tau,t)\, K\, \hat{\Phi}^T(\tau,t)\, dt \tag{A-8}$$

$$\text{where} \quad K = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_1^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_2^2 \end{bmatrix}$$

Using Eq.(A-8), we can numerically evaluate the covariance of the state deviations. For the case when $\sigma_1 = \sigma_2 = a/600$, and $\beta_1 = \beta_2 = 1/100$ hrs., we find that the standard deviations of the state variables after 90% of the mission has been completed are about 100 million kilometers in position, and 30 kilometers/sec. in velocity. These errors are far greater than predicted in Section 3-D probably because thrust level control is not permitted in this system. It is the tendency of linear-regular controllers to make strong corrections near the terminal mission time -- this amounts to approximately the final 10% of the mission in this example. However, in order to fully correct the enormous deviations cited above, the control demanded would definitely not be available, and the errors could not be significantly reduced. A run was also made using $\beta_1 = \beta_2 = 0$, and the resulting state deviations were comparable to those cited above. Finally, the case when x_5 and x_6 are white noises (with the same variances as given above) was investigated, and the resulting errors were approximately one half as great as with the correlated noise.

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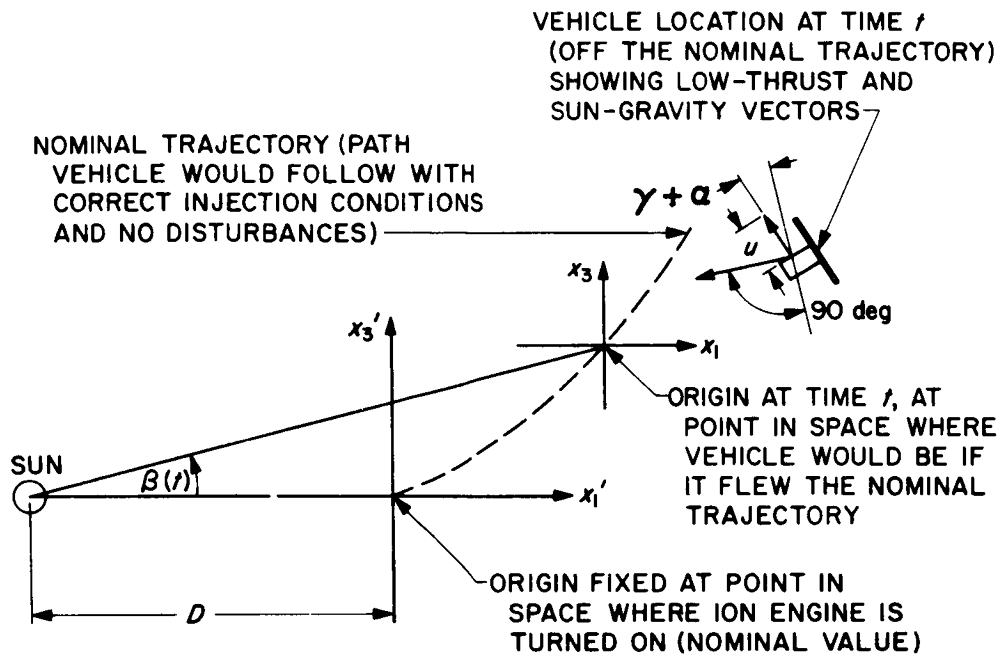


Fig.1-1. Definition of the coordinate frames (x_1', x_3') and (x_1, x_3).

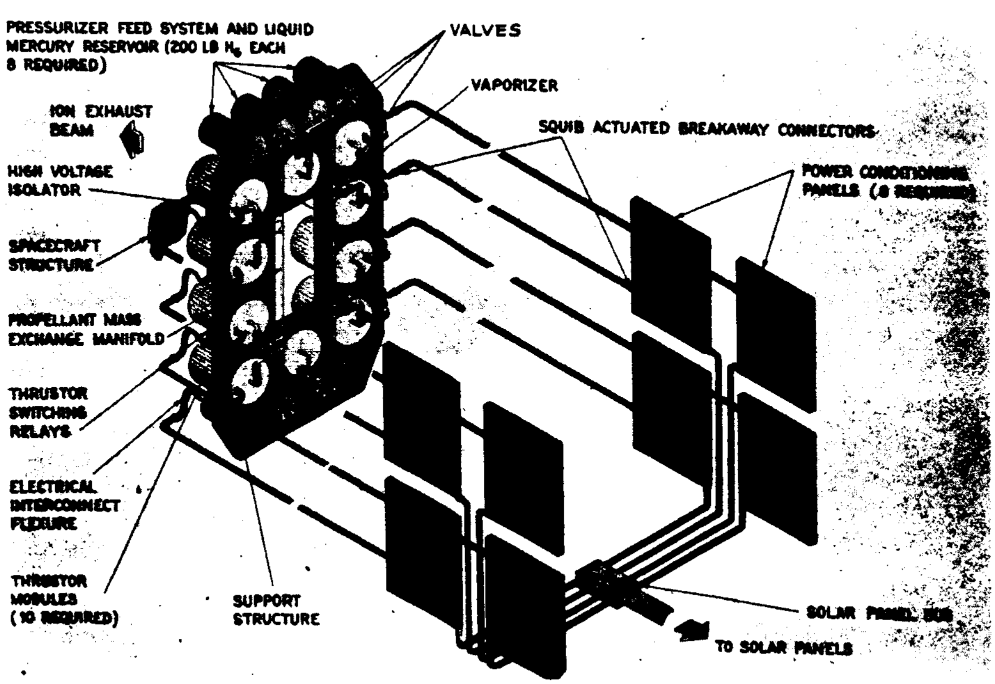


Fig.1-2. A low-thrust ion-engine.

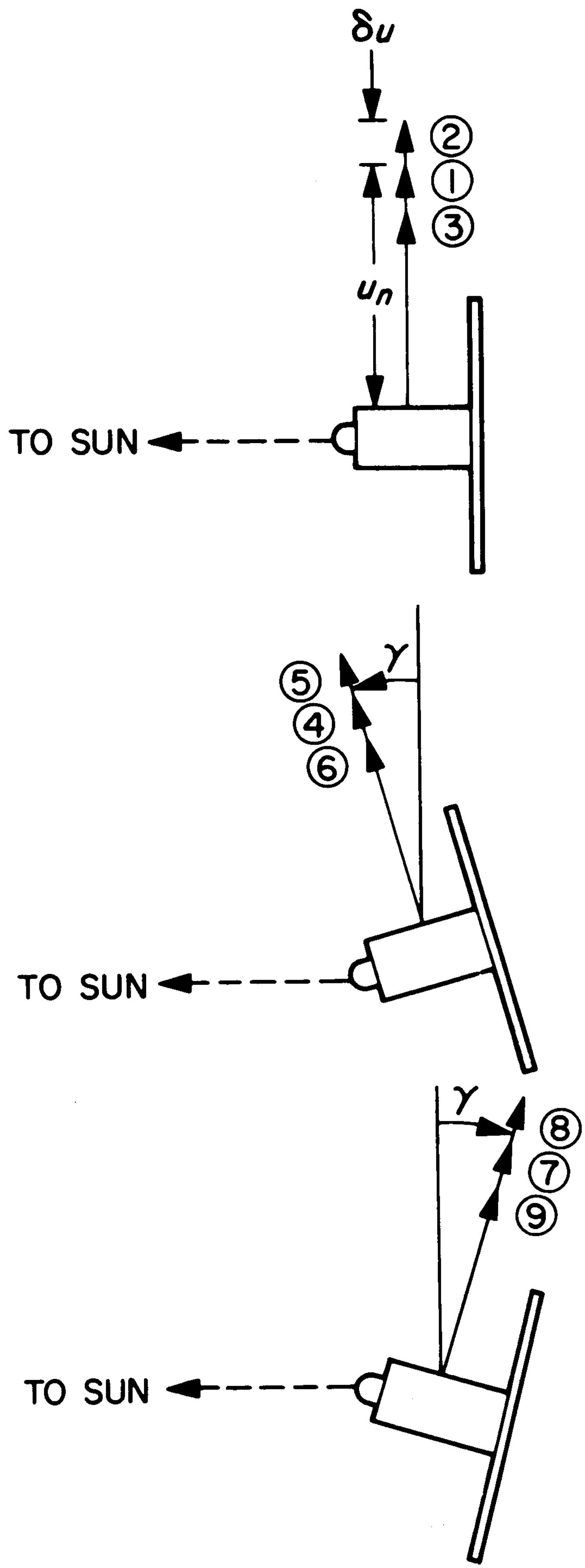


Fig.1-3. The nine allowable states of the ion-engine thrust vector. α = 0.

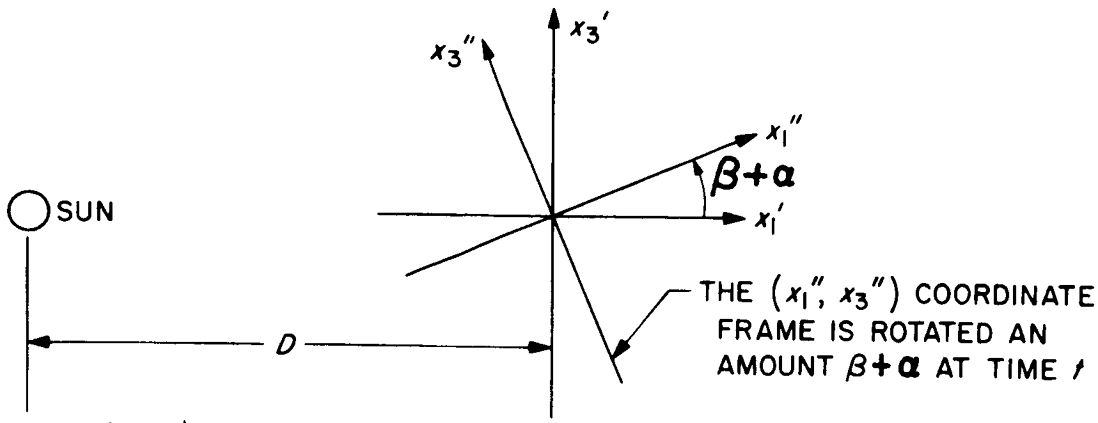


Fig.1-4. Definition of the (x_1'',x_3'') coordinate frame.

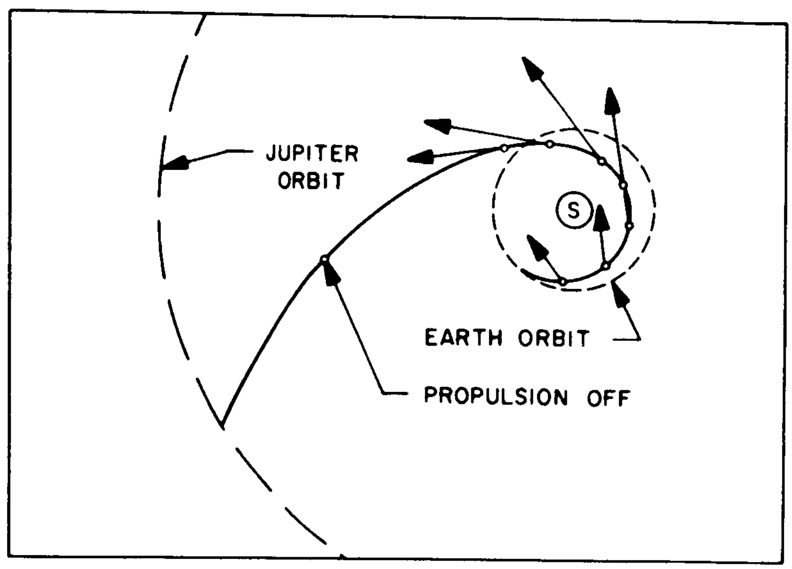


Fig.1-5. A low-thrust trajectory for a Jupiter mission.

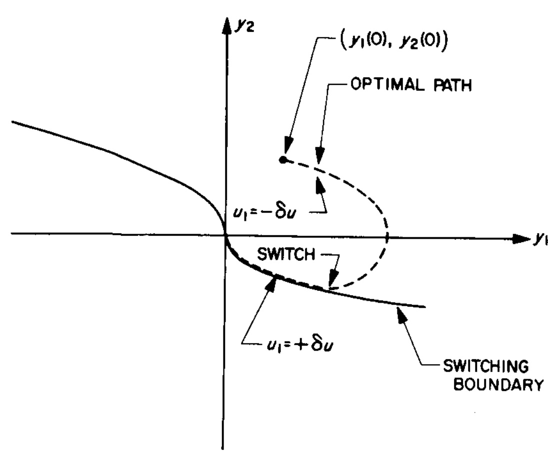


Fig.1-6. Definition of the "switching boundary" in the (y_1,y_2) plane.

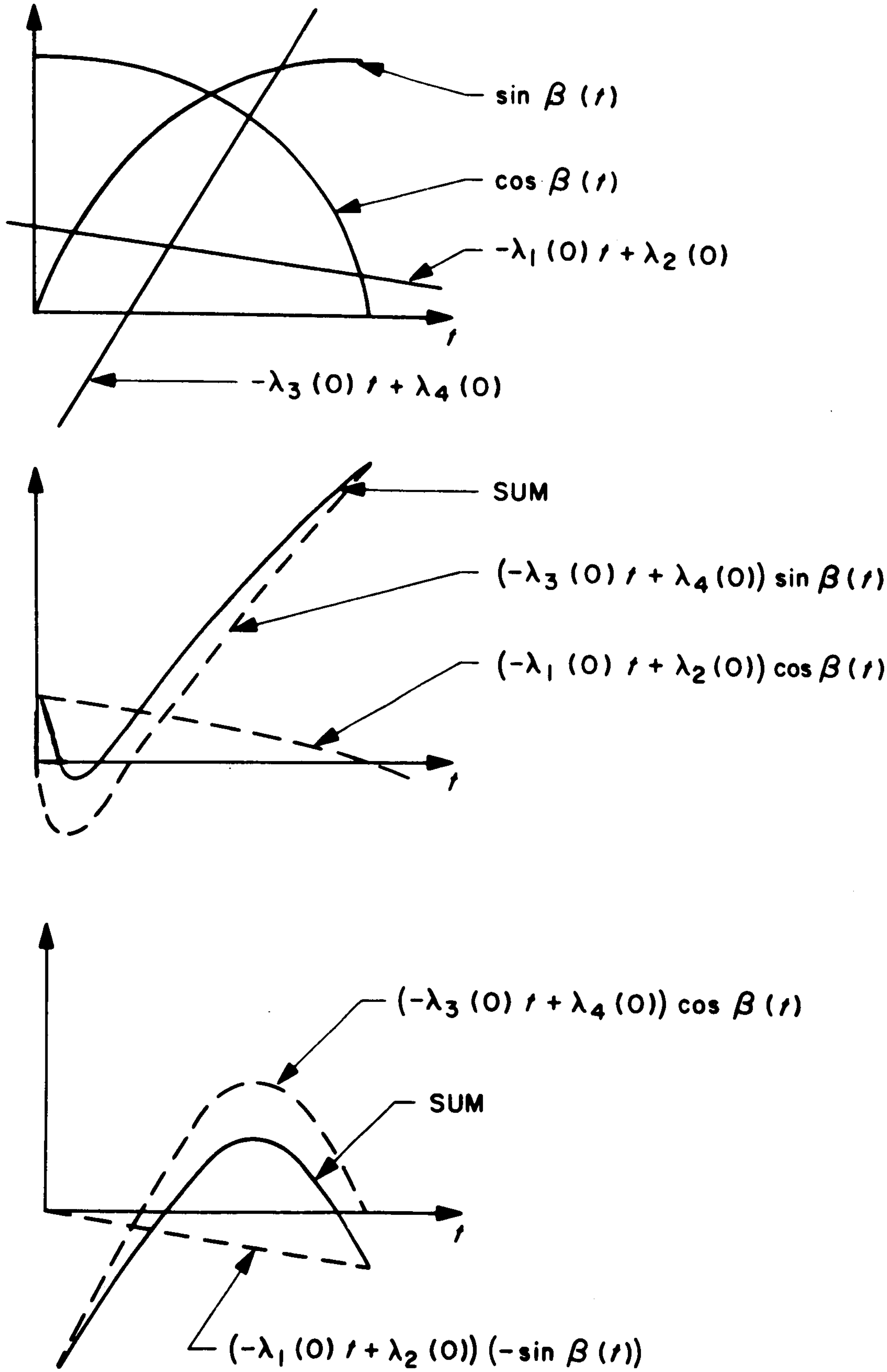


Fig.1-7. Control function switchings for the second solution.

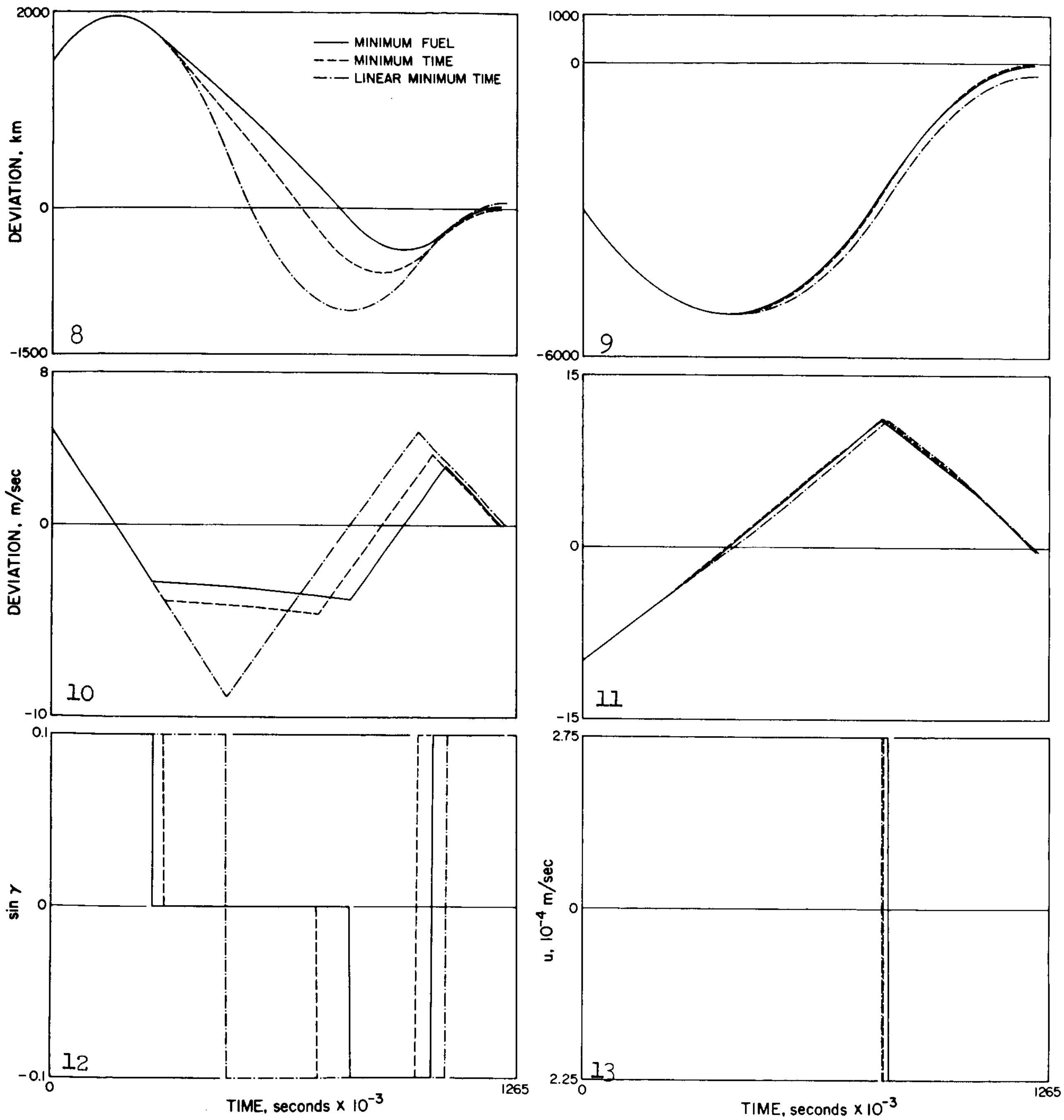


Fig.1-8 - Fig.1-13. Results obtained for the nonlinear minimum-time, nonlinear minimum-fuel, and linear control systems.(1-8) The x_1 position deviation;(1-9) the x_3 position deviation;(1-10) the x_1 velocity deviation;(1-11) the x_3 velocity deviation;(1-12) the control variable u_1; and (1-13) the control variable u_2.

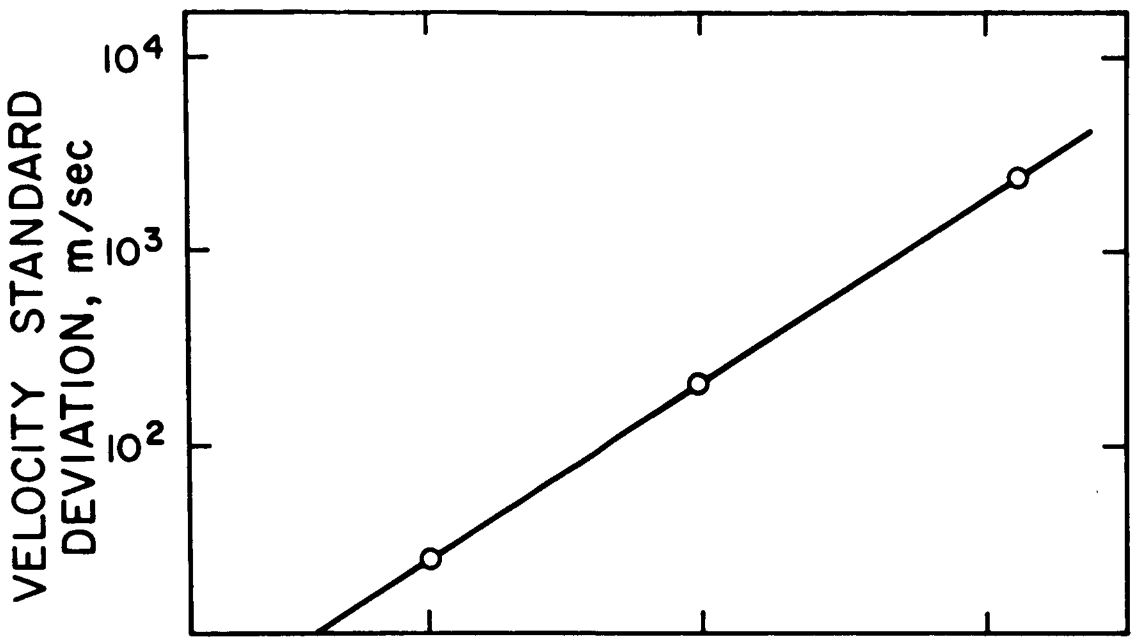


Fig.1-14. Steady state velocity deviations for the second-variation control system.

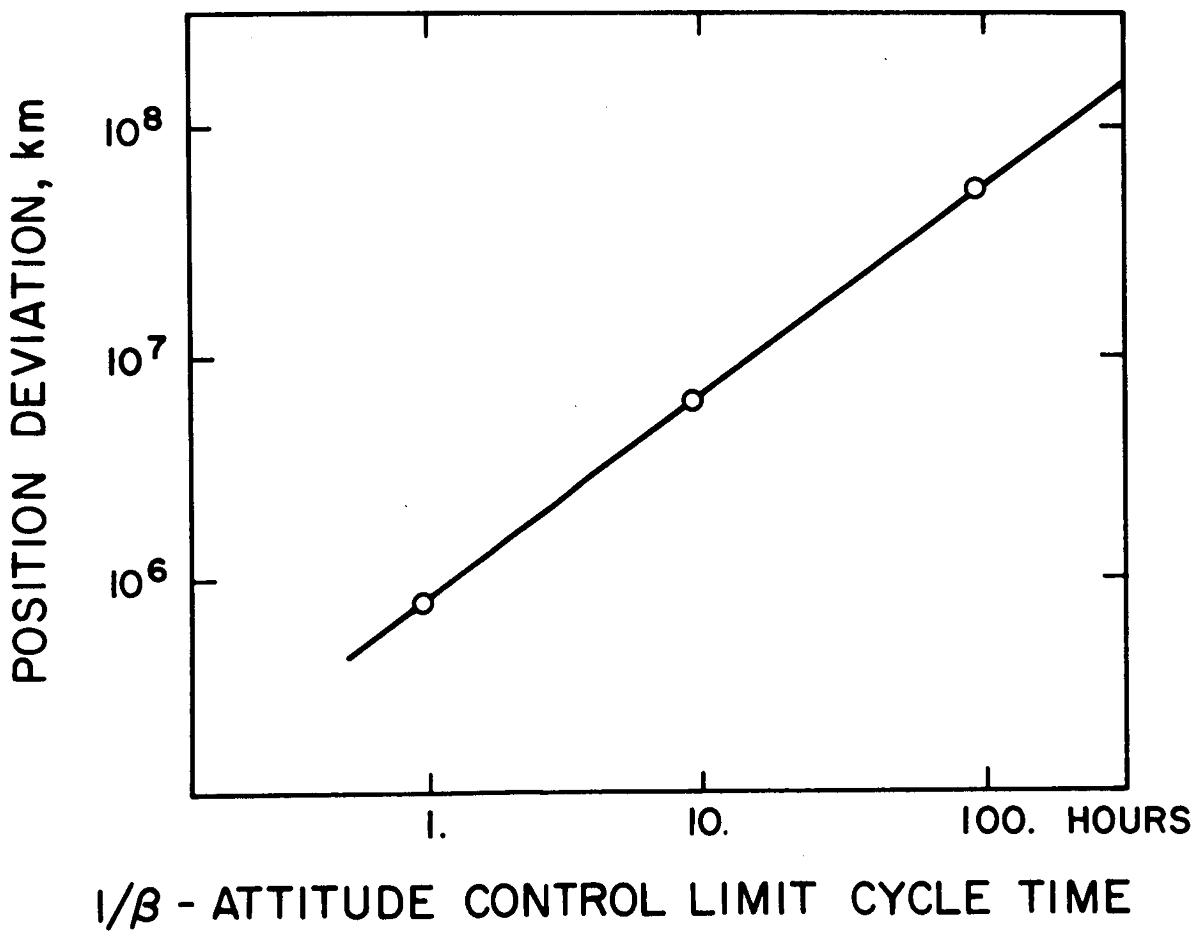


Fig.1-15. Steady state position deviations for the second-variation control system.

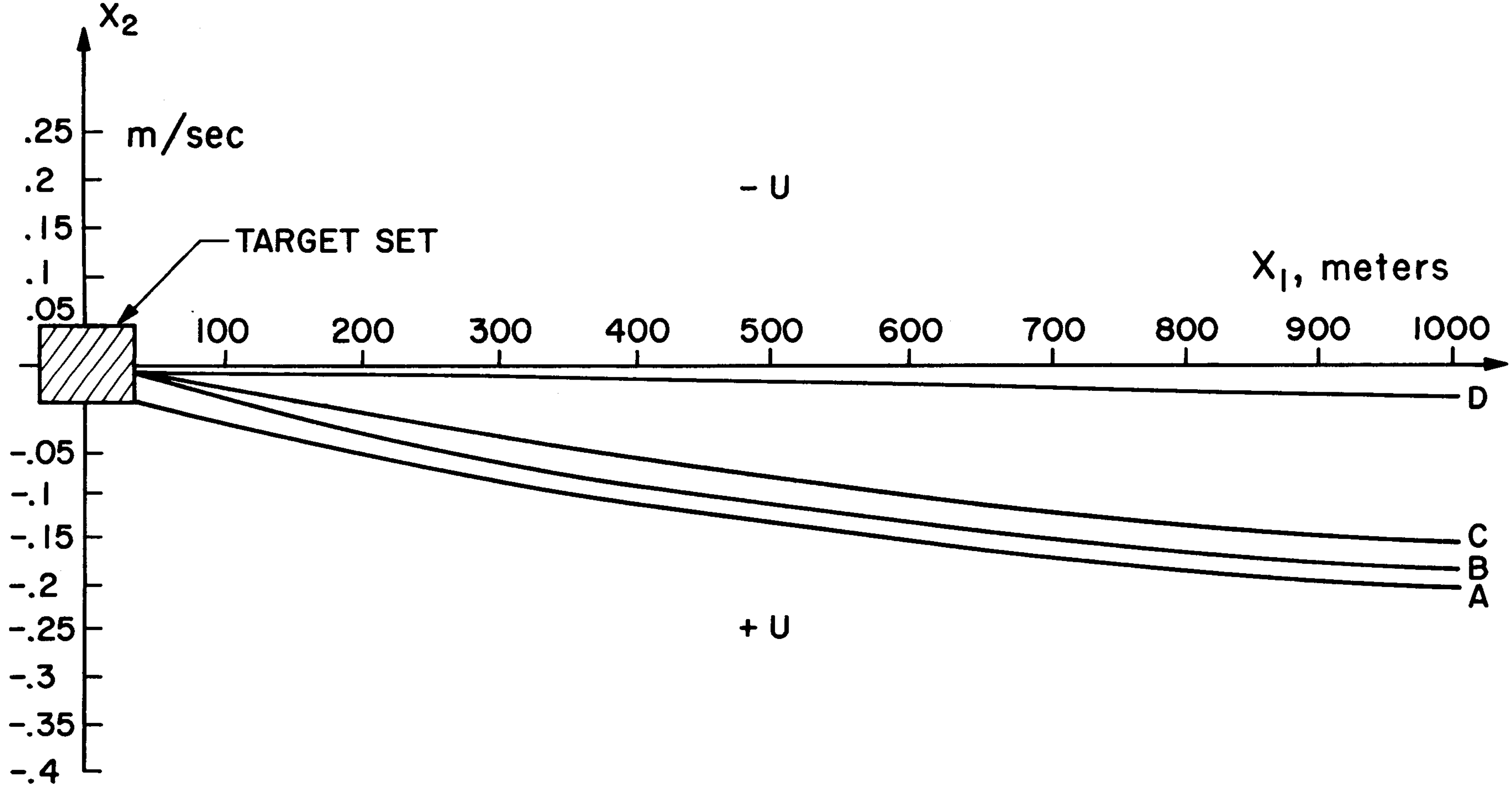


Fig.1-16. Switching curves for known state stochastic system: (a) Deterministic case, $\sigma = 0$; $u = .25 \times 10^{-4}$ m./sec^2; (b) $\beta = 1/1200$, $\sigma = .2u$, $u = .25 \times 10^{-4}$; (c) $\beta = 1/60$, $\sigma = u$, $u = .25 \times 10^{-4}$; (d) $\beta = 1/1200$, $\sigma = u$, $u = .25 \times 10^{-4}$.

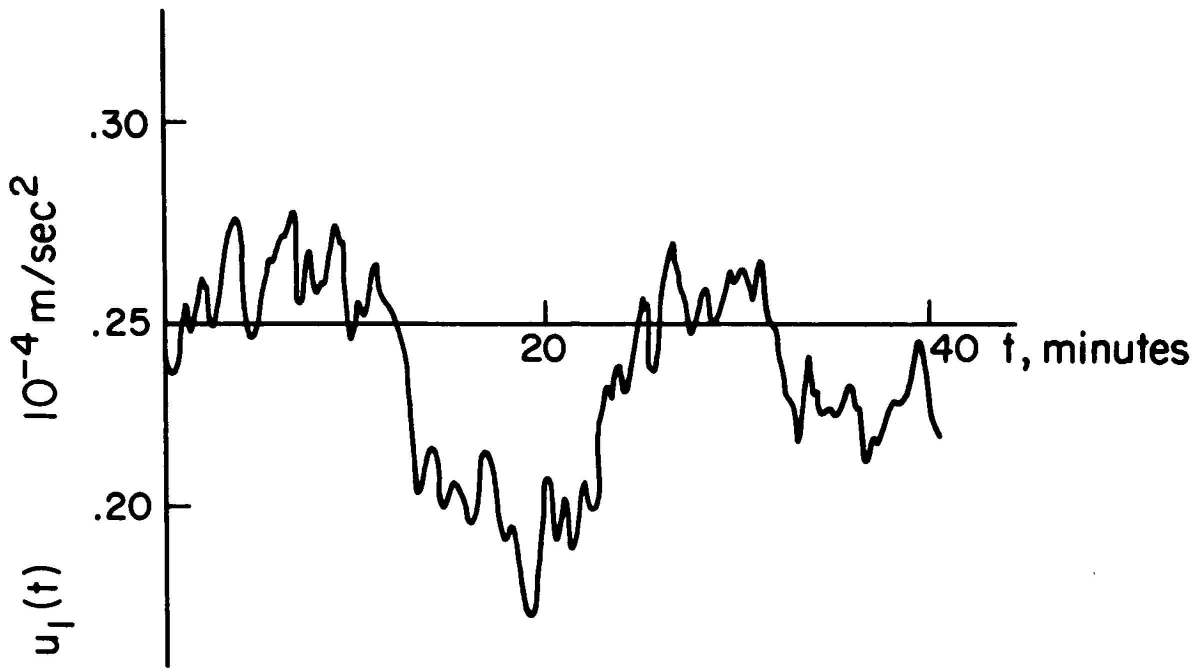


Fig.1-17. A typical sample function for the Ornstein-Uhlenbeck stochastic process ($1/\beta$ = 20 minutes, $u = .25 \times 10^{-4}$ m./sec.^{2}, $\sigma = .2u = .05 \times 10^{-4}$ m./sec.^{2})

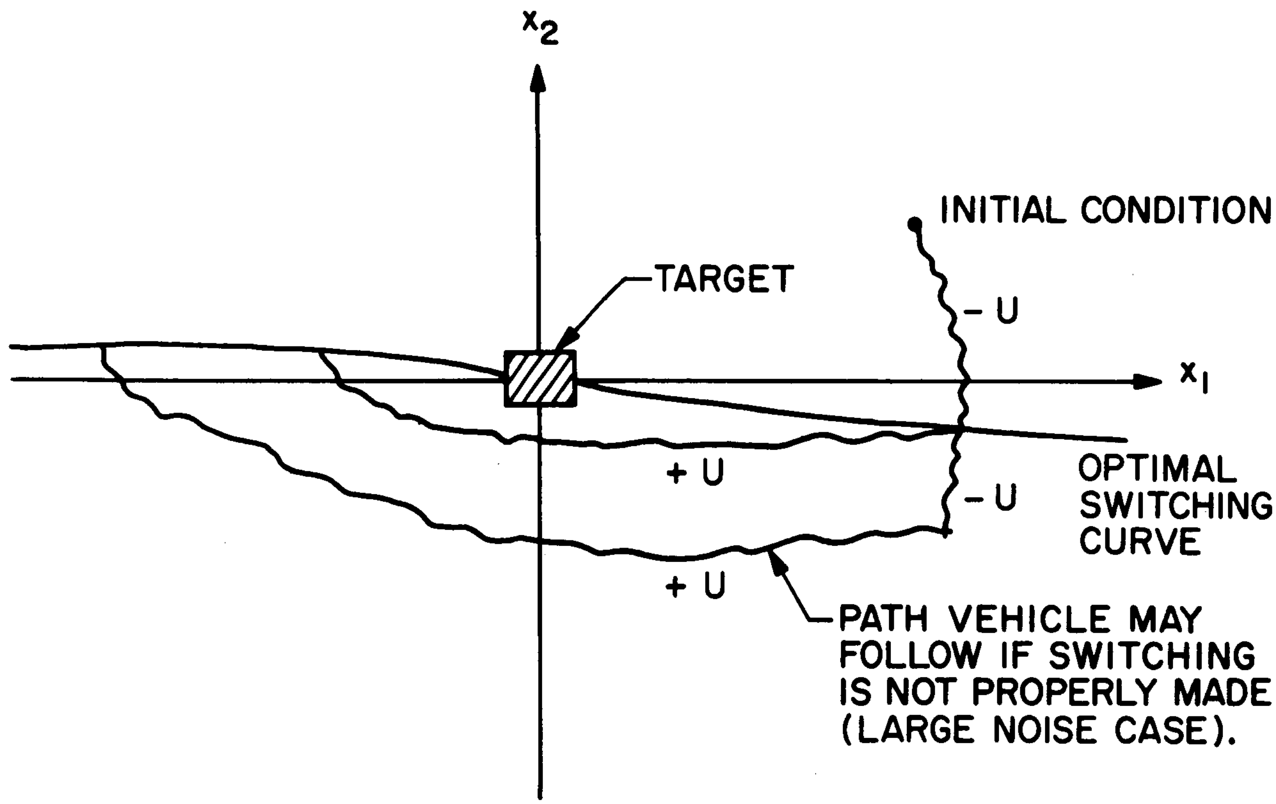


Fig.1-18. Demonstration of "wandering" that may occur in presence of large noises.

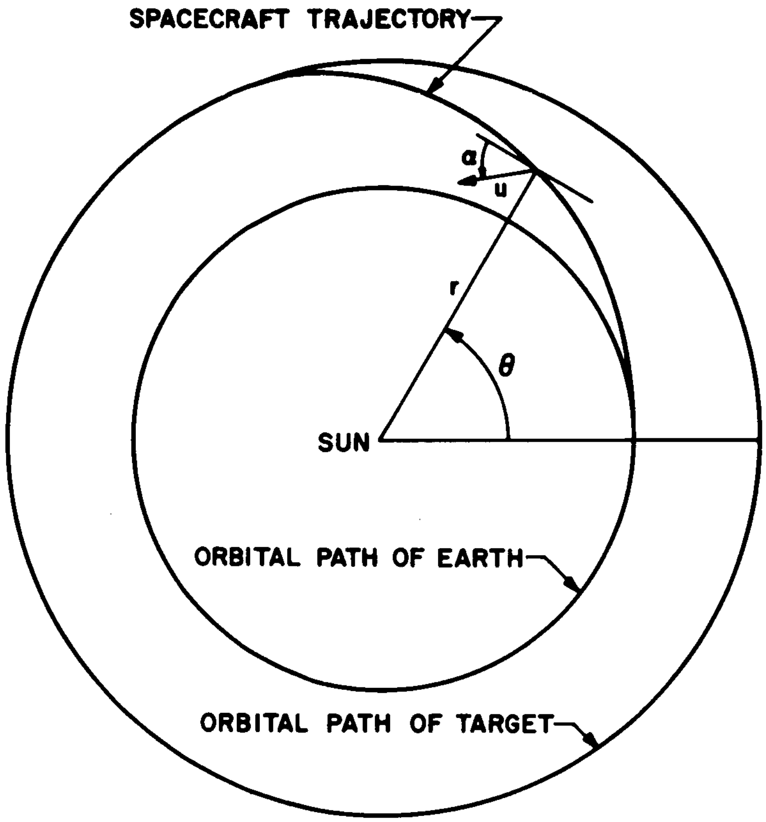


Fig.1-19. Definition of the polar coordinate frame.

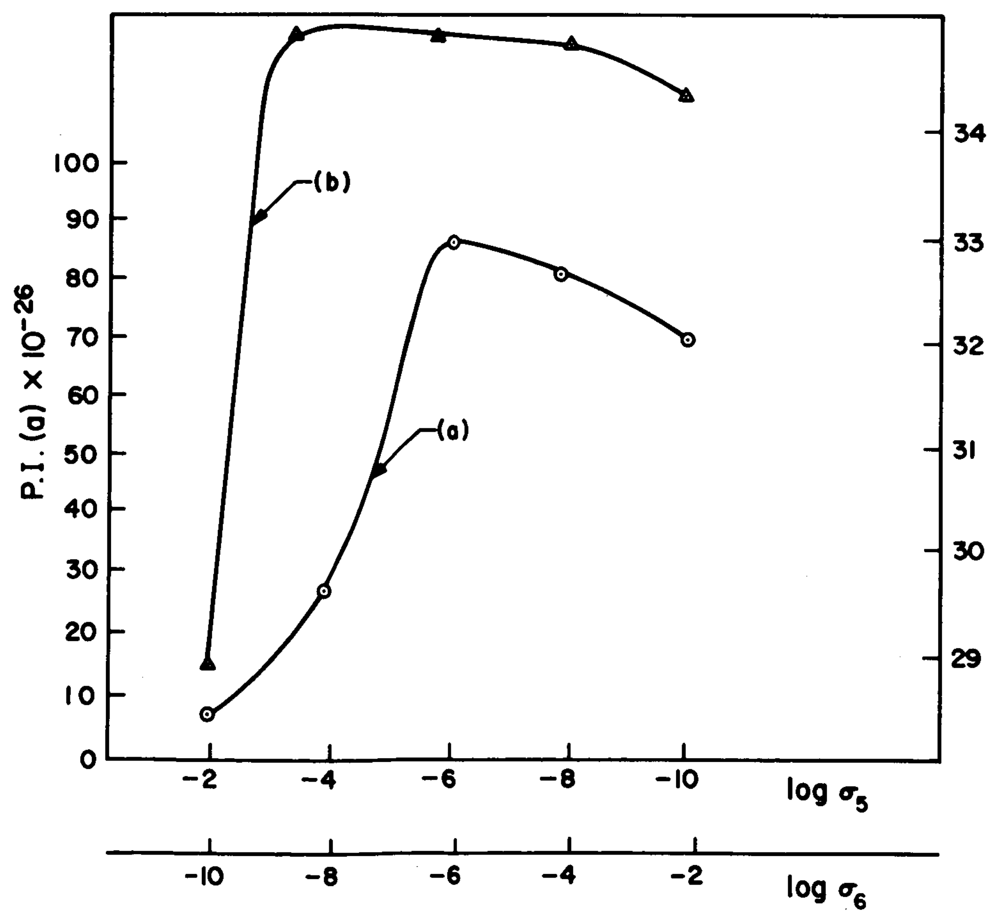


Fig.1-20. Optimization of yaw angle (σ_5) and low-thrust acceleration (σ_6) accuracies for performance indices (a) $\int_0^T p_{11}^2 + p_{22}^2 + k^2(p_{33}^2 + p_{44}^2)\,dt$ and (b) $p_{11}(T) + p_{22}(T) + k(p_{33}(T) + p_{44}(T))$.

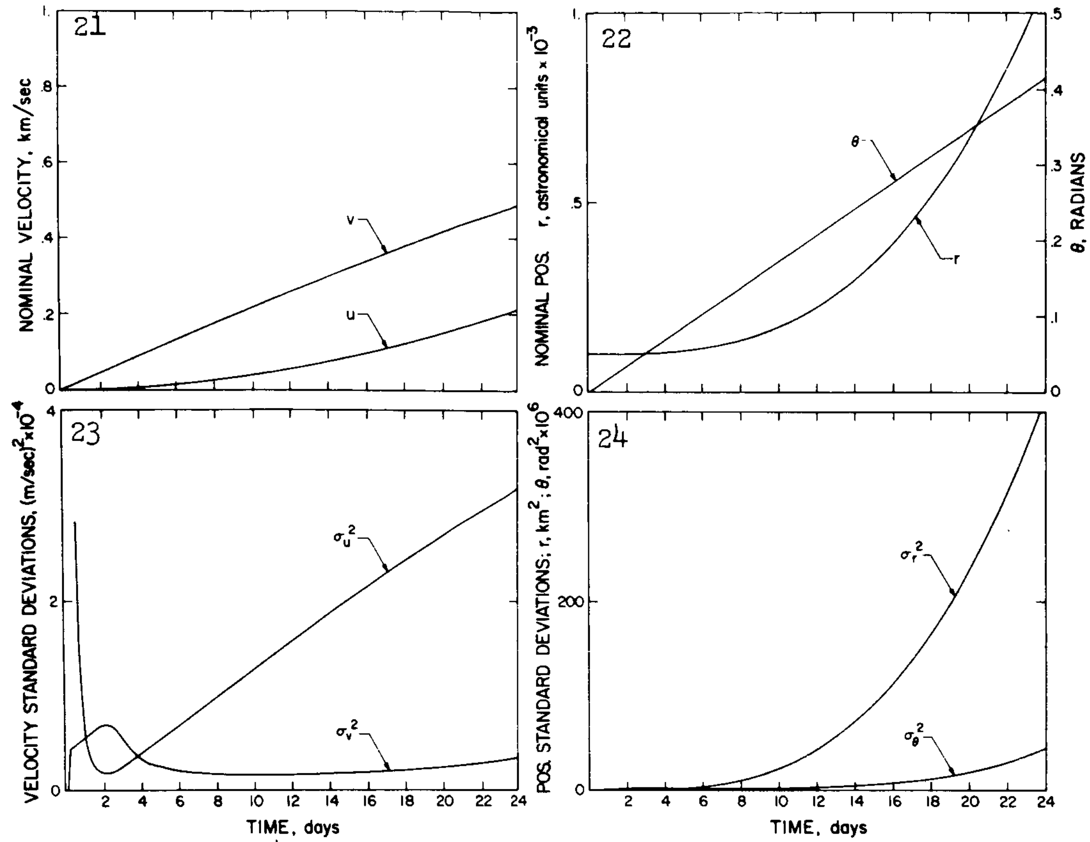


Fig.1-21 - Fig.1-24. State estimation using optimum measurements: (1-21) nominal velocities; (1-22) nominal positions; (1-23) standard deviations of velocities; (1-24) standard deviations of positions.

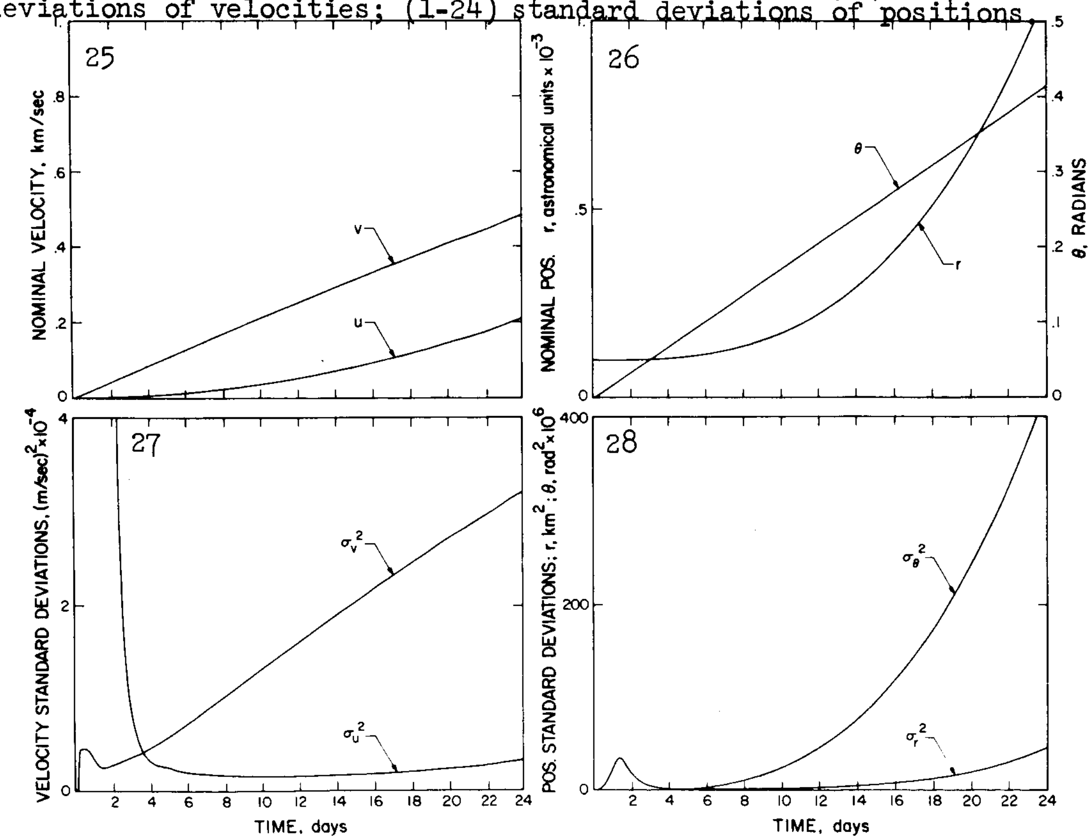


Fig.1-25 - Fig.1-28. State estimation using non-optimum measurements: (1-25) nominal velocities; (1-26) nominal positions; (1-27) standard deviations of velocities; (1-28) standard deviations of positions.

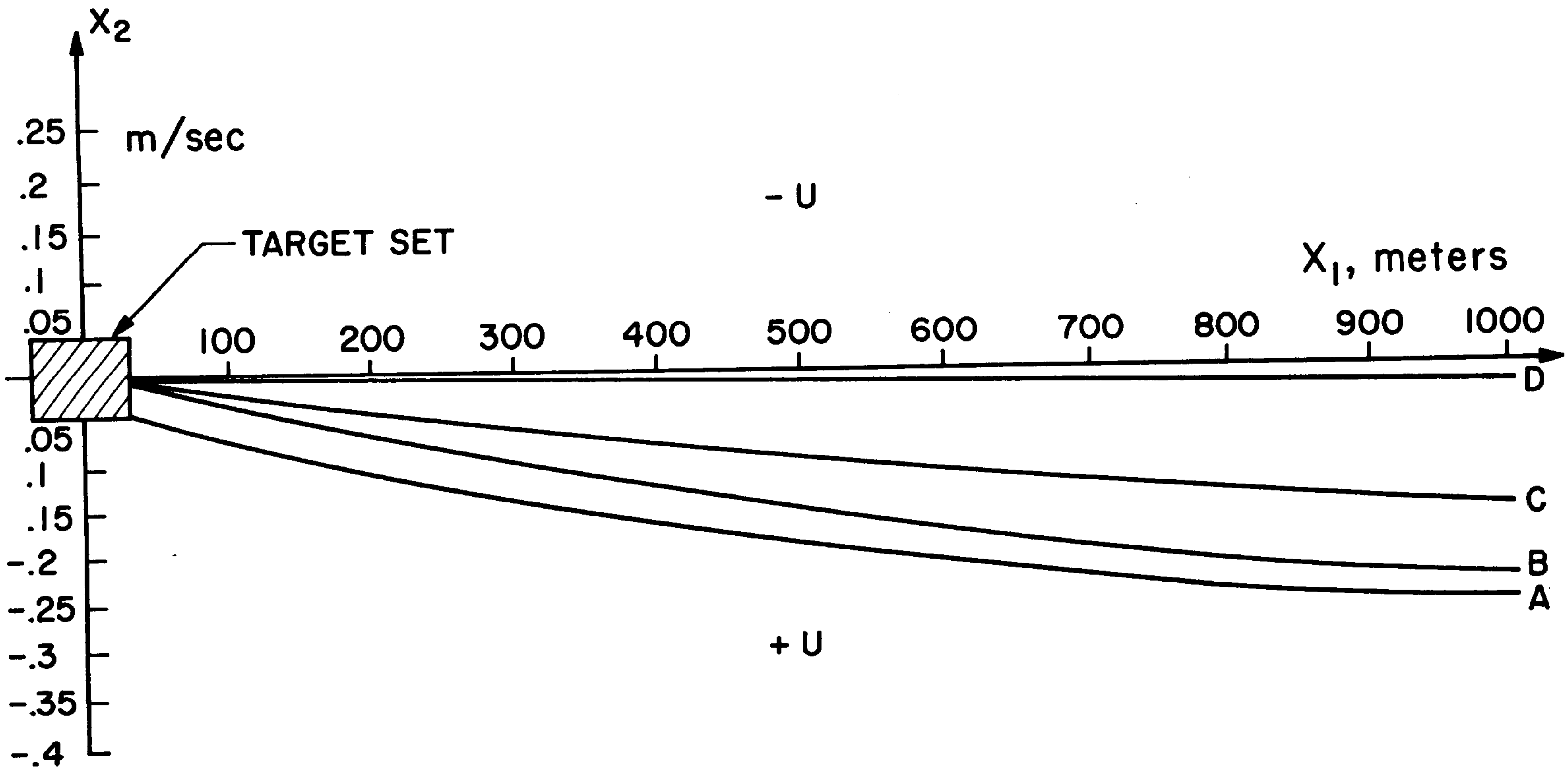


Fig.1-26. Switching curves for unknown states:

(a) deterministic case, $u = .25 \times 10^{-4}$ m./sec.^{2};

(b) $\beta = 1/1200$, $\sigma = .2u$, $u = .25 \times 10^{-4}$, $C = \begin{bmatrix} 100 & 0 \\ 0 & .0025 \end{bmatrix}$;

(c) $\beta = 1/1200$; $\sigma = .2u$, $u = .25 \times 10^{-4}$, $C = \begin{bmatrix} 100 & 0 \\ 0 & .01 \end{bmatrix}$;

(d) $\beta = 1/1200$, $\sigma = u$, $u = .25 \times 10^{-4}$, $C = \begin{bmatrix} 100 & 0 \\ 0 & .0025 \end{bmatrix}$.

OPTIMALLY CONTROLLED SOFT LANDING IN IMPERFECTLY KNOWN PLANETARY ATMOSPHERE

- A Preliminary Study for Adaptive Soft Landing -

2.1. Introduction.

By optimally controlled soft landing we mean a controlled landing with minimum fuel consumption, and, prescribed terminal conditions on the vehicle's trajectory which must be satisfied.

The space vehicle is supposed to be a known dynamic system in that sense that the functional form of the dynamic equations governing the vehicle's behaviour are known. The gas-dynamic forces, however, which act on the vehicle descending through a planetary atmosphere are known with very limited accuracy due to our very imperfect and, for the moment, hardly improvable knowledge on planetary atmospheric data. (The basic data are: atmospheric mass density, atmospheric pressure and the velocity of sound; all these data are functions of the height above the planet's surface.) For that very reason we have to assume that at least some of the basic parameters of the dynamic equations governing the vehicle's behaviour during the phase of atmospheric flight are very imperfectly known. We have to assume, furthermore, that there would be unknown (external) disturbances acting on the descending vehicle, and, that the given measurements on the state of the vehicle would be corrupted with significant noise. Needless to emphasize that the very limitations of our knowledge on the relevant planetary atmospheric data have a considerable effect on trajectory and performance calculations for the atmospheric entry and landing phase of a planetary mission.

In our study we separate the motion of the center of gravity from the motion of the vehicle about its center of gravity and will consider only the motion of the vehicle's center of gravity. This separation is justified in the case of undertaking trajectory and performance analysis only. (In the case of undertaking stability and attitude control analysis one mainly considers the motion about the vehicle's center of gravity.) We will consider, furthermore, that the vehicle's trajectory lies in one plane, defined by the radius vector from the planet's center to the landing point and by the initial (entry) azimuth direction.

In the present study the given problems are considered from the point of view of Modern Control Theory, and, in two, seemingly distinct parts. After outlining the relevant dynamic equations of atmospheric entry (Section 2.2) we deal (1) with Optimal Thrust Programs by using Pontryagin's Maximum Principle and assuming known parameters in the state equations (Sections 2.3-4), and, (2) with Non-linear Sequential State and Parameter Estimation based on Dynamic Programming (Sections 2.5-10). New results are presented in form of Asymptotic Non-linear Filter (Section 2.11). In the Summary Section of this Chapter the connection between Optimal Thrust Programs and Sequential Estimation is briefly explained and the future work is indicated.

The necessary data we have applied in the presented numerical calculations are only order-of-magnitude data for some typical planetary mission. (But some of the applied data are quite close to a Mars Mission.) In this study we have mainly intended to present and elaborate modern viewpoints and techniques which are relevant to handle the complex problem of optimally controlled soft landing in imperfectly known planetary atmosphere.

2.2. Dynamics of Atmospheric Entry.

During atmospheric entry the motion of the space vehicle is governed by its own inertia, by the gravitational force and by gas-dynamic forces. The gravitational force acts toward the planet's center. The gas-dynamic drag force acts antiparallel to the vehicle's motion. The gas-dynamic lift force (and centrifugal force) act normal to the vehicle's motion.

In deriving the equations of motion of the space vehicle we only consider (a) the motion of the center of gravity of the vehicle, and (b) two-degrees-of-freedom (planar) motions of the center of gravity of the vehicle. As a convenient reference system we will use a trajectory-fixed coordinate system with its origin located at the center of gravity of the space vehicle, and, with unit vectors $\underline{e}_D$ and $\underline{e}_L$ parallel and perpendicular to the vehicle's motion. The path angle will be defined as the angle between the velocity vector and the local (instantaneous) horizontal. (See Fig. 2.2.1.) We assume, furthermore, a non-rotating, spherical planet with quiet (non-moving) atmosphere. The forces acting on the space vehicle are shown in Fig. 2.2.2. We assume that the rocket thrust is designed so that it always acts in the tangential direction.

According to Newton's second law the momentum balances in the tangential ($\underline{e}_D$) and normal ($\underline{e}_L$) direction of the motion give the following differential equations:

$$m \frac{dv}{dt} = m \frac{\Gamma}{r^2} \sin \alpha - D - T \tag{2.2.1}$$

$$mv \frac{d\alpha}{dt} = m \left(\frac{\Gamma}{r^2} - \frac{v^2}{r} \right) \cos \alpha - L \tag{2.2.2}$$

where m = mass of the vehicle;

v = velocity of the vehicle;

$\frac{\Gamma}{r^2} = g$, acceleration of gravity ($\Gamma = GM$, with M: mass of the planet and G: gravitational constant);

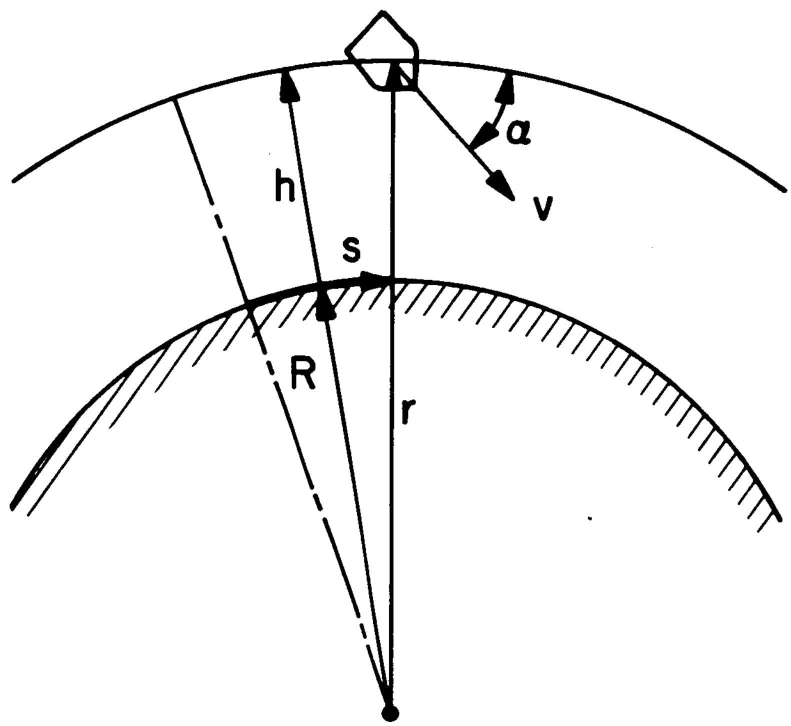


Fig. 2.2.1

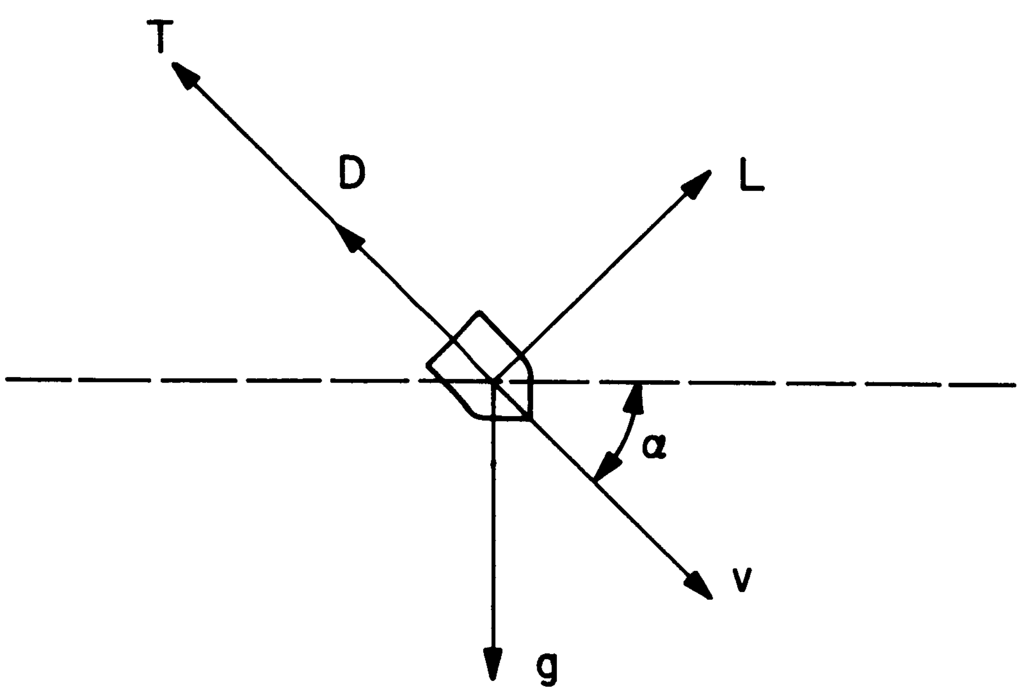


Fig. 2.2.2

D, L = gas-dynamic drag and lift forces, respectively;

T = rocket thrust force, designed to be acting always in the tangential direction of the vehicle's motion;

r, α = defined on Fig. 2.2.1.

The term $(v^2/r) \cos \alpha$ in Eq.(2.2.2) is a fictitious centrifugal force which compensates for the curvature of the spherical planet.

The altitude "h" of the vehicle above the planet's surface and the ground range "s" of the vehicle, measured from the entry reference vertical, are given by

$$\frac{dh}{dt} = - v \sin \alpha \tag{2.2.3}$$

$$\frac{ds}{dt} = \frac{R}{r} v \cos \alpha \tag{2.2.4}$$

where R is defined on Fig. 2.2.1.

The gas-dynamic forces, D and L, are dependent on the dynamic pressure $(\rho v^2)/2$, where ρ = atmospheric density and v = the vehicle's velocity. We have

$$D = C_D A \frac{1}{2} \rho v^2 \tag{2.2.5}$$

$$L = C_L A \frac{1}{2} \rho v^2 \tag{2.2.6}$$

where C_D, C_L = dimensionless gas-dynamic coefficients in the tangential and normal direction of the motion, characterizing the geometry of a given body, and, are functions of the Mach-number as well as of the angle of attack. (The Mach number is a function of the local speed of sound.)

A = relevant reference area of the moving body.

For the atmosphere's density variation with the altitude "h" we will assume the exponential distribution:

$$\rho = \rho_0 \exp(-\, bh) \tag{2.2.7}$$

which is derived for an isothermal atmosphere in hydrostatic equilibrium. In Eq.(2.2.7):

ρ_0 = density of the atmosphere on the planets surface.

b = inverse scale factor. (Atmospheric constant, is dependent on the planet's gravitational attraction and on the temperature and composition of the planetary atmosphere.)

The rocket thrust force, T, in Eq.(2.2.1) is given by

$$T = \frac{dm}{dt}\,(v_e - v); \quad 0 \le T \le T_{max} \tag{2.2.8}$$

where v_e = velocity of the exhaust gas relative to the vehicle

T_{max} = maximum value of T.

In our calculations we will make the following assumptions:

(a) $\frac{\Gamma}{r^2} = g = \text{constant}$

(b) $R \gg h$, hence $r \simeq R = \text{constant}$

(c) $|v_e| \gg |v|$, hence $T \simeq \frac{dm}{dt}\, v_e$

(d) $\frac{L}{D} = 0$, which implies a pure gravity-turn ballistic descent, since $T \| v$.

By introducing the following notations

$$K' \triangleq \frac{1}{2} C_D A \rho_o \; ; \quad \beta = T \text{ (control force)} \tag{2.2.9}$$

and substituting Eqs.(2.2.5-9) into Eqs.(2.2.1-4) and remembering the assumptions specified above, we obtain the following equations:

$$\dot{h} = - v \sin \alpha \tag{2.2.10}$$

$$\dot{s} = v \cos \alpha \tag{2.2.11}$$

$$\dot{v} = g \sin \alpha - \frac{1}{m} K' \exp(- bh) \, v^2 - \frac{1}{m} \beta \tag{2.2.12}$$

$$\dot{\alpha} = \left(\frac{g}{v} - \frac{v}{R} \right) \cos \alpha \tag{2.2.13}$$

which are four, coupled, ordinary non-linear differential equations describing the planar, descending motion of a pure gravity-turn ballistic vehicle in a planetary atmosphere.

In the case of a solely vertical descent $(\alpha \triangleq 90^\circ, \;\; \dot{\alpha} \triangleq 0)$ Eqs.(2.2.10-13) are simplified to

$$\dot{h} = - v \tag{2.2.14}$$

$$\dot{v} = g - \frac{1}{m} K' \exp(- bh) \, v^2 - \frac{1}{m} \beta \tag{2.2.15}$$

2.3. Optimal Thrust Program for Vertical Atmospheric Descent and Soft Landing.

In this Section the term "optimality" will be used in the sense of Modern Control Theory. Hence Optimal Thrust Program is called an admissible thrust program (= admissible control) which transfers the space vehicle from a given initial state to the prescribed terminal state and minimizes the "cost function". As a natural "cost function" we consider the consumed fuel:

$$\mathfrak{J} = -\int_0^{\tau} \dot{m}\, dt \tag{2.3.1}$$

where τ = terminal time (free)

$\dot{m}$ = mass flow rate (decreasing)

The corresponding motion of the vehicle is called an Optimal Trajectory. The Thrust Program (or, equivalently, the Control) is called admissible if it satisfies the imposed constraints in the period of control.

In view of Eqs.(2.2.8-9) and of Assumption (c) there, the Control β (the rocket thrust) is given by

$$\beta = \dot{m}v_e \ , \qquad 0 \le \beta \le \beta_{max} \tag{2.3.2}$$

(The control force β could also be written in terms of the specific impulse I_{sp} of the rocket engine, since $v_e = I_{sp}g$, and therefore $\beta = I_{sp}g\dot{m}$.)

In this Section we will consider one-level, on-off, non-stop thrust motors constituting the control force β. (By "non-stop" we mean motors which can stop just by burning out if once they are on.) Hence we have the following imposed constraint on the control:

$$\beta = \begin{cases} 0 \\ \beta_m & \text{in a finite time period } \tau. \end{cases} \tag{2.3.3}$$

As prescribed terminal conditions we consider

$$h(\tau) = 0$$
$$v(\tau) = 0 \qquad (2.3.4)$$
$$m(\tau) = m_f$$

where h = height above the planet's surface

v = velocity

m_f = final mass which has to be delivered to the planet's surface.

Taking Eqs.(2.3.1-4) into consideration one can see that the Optimal (= Minimal) Thrust Program is equivalent to the minimal time problem. (In fact the "cost function" $\mathfrak{J}$ is a monotone increasing function of the terminal time τ.) Thus Optimal Thrust Program can be assured by quoting existence theorems for time-optimal controls.[Ref.1]

By introducing

$$x_1 \triangleq h \quad \text{(vertical position above the planet's surface)}$$

$$x_2 \triangleq v \quad \text{(velocity)}$$

and augmenting the state space by defining

$$x_3 \triangleq m \quad \text{(mass of the vehicle)}$$

then, from Eqs.(2.2.14-15) and (2.3.2), we obtain the following system equations:

$$\dot{x}_1 = - x_2 \qquad (2.3.5)$$

$$\dot{x}_2 = g - \frac{1}{x_3} K' \exp(-bx_1)\, x_2^2 - \frac{1}{x_3} \beta \tag{2.3.6}$$

$$\dot{x}_3 = -\frac{\beta}{v_e} \tag{2.3.7}$$

with prescribed terminal conditions

$$x_1(\tau) = x_2(\tau) = 0 \quad , \quad x_3(\tau) = m_f \tag{2.3.8}$$

and the "cost function" (= consumed fuel) is given by

$$\mathfrak{F} = -\int_0^\tau \dot{x}_3 \, dt \tag{2.3.9}$$

In order to obtain the form of the Optimal Thrust Program we apply the Pontryagin Maximum Principle.[1] The Hamiltonian "H" for the minimal time problem becomes:

$$H = -\lambda_1 x_2 + \lambda_2 [g - \frac{1}{x_3} K' \exp(-bx_1)\, x_2^2 - \frac{1}{x_3} \beta] - \lambda_3 \frac{\beta}{v_e} \tag{2.3.10}$$

where the auxiliary variables $\lambda_1, \lambda_2, \lambda_3$ are nontrivial solutions of the system of adjoint equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -\lambda_2 \frac{K'}{x_3} b x_2^2 \exp(-bx_1) \tag{2.3.11}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = \lambda_1 + 2\lambda_2 x_2 \frac{K'}{x_3} \exp(-bx_1) \tag{2.3.12}$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial x_3} = -\frac{\lambda_2}{x_3^2} K' \exp(-bx_1) x_2^2 + \beta \tag{2.3.13}$$

Since $\frac{\partial H}{\partial \beta} = 0$ gives $\frac{\lambda_2}{x_3} + \frac{\lambda_3}{v_e} = 0$, we see that the Hamiltonian is maximized by taking

$$\beta = \begin{cases} \beta_{max}, & \text{whenever } \frac{\lambda_2}{x_3} + \frac{\lambda_3}{v_e} > 0 \\ 0, & \text{whenever } \frac{\lambda_2}{x_3} + \frac{\lambda_3}{v_e} < 0 \end{cases} \tag{2.3.14}$$

In the case of

$$\frac{\lambda_2}{x_3} + \frac{\lambda_3}{v_e} = 0 \tag{2.3.15}$$

β is indeterminate. Equation (2.3.15) expresses the singularity condition. It can be shown, however, (also by using physical reasoning) that there is no singular control for that problem we consider here since relation (2.3.15) cannot hold on any finite closed interval in $[0, \tau]$.

Relation (2.3.14) expresses the fact that the Optimal Thrust Program is of the Bang-Bang Type. This means that the Optimal Thrust Program will consist of either full thrust from the time considered (= initiation of the terminal phase of the soft landing mission) until touchdown, or a period of zero thrust (= free fall) followed by full thrust until touchdown.

In order to synthesize the Optimal Thrust Program we have to determine an appropriate switching function. The development of the switching function consists in determining a relation $F(x_1, x_2, x_3) = 0$. If the given maximum thrust is applied continuously from the moment when this relation is first satisfied, a soft landing can be achieved.

A theoretically possible way for obtaining the switching function would be to integrate the equations of motion, Eqs.(2.3.5-7) under the assumption $\beta = \beta_m$ in the time interval $[0, t']$, and determining the relation which must exist between the initial values of the state variables $[x_1(0) = x_0,\ x_2(0) = v_0,\ x_3(0) = m_0]$ in order to achieve a soft landing in a time t'. Then, by eliminating the free time parameter t', we could obtain an expression $F(x_0, v_0, m_0) = 0$ in closed Form for the switching function.

Unfortunately, it is impossible to carry out the integration of Eqs.(2.3.5-7) analytically. (Would it be possible, we were still faced with the problem of eliminating t'.)*

The only way of developing the switching function is to integrate Eqs.(2.3.5-7) numerically in backward time. Or equivalently: integrate Eqs.(2.3.5-7) as they are from $t = 0$ to $t = \tau$ (just change the sign of $\dot{x}_3$) and use the prescribed terminal conditions as initial conditions. Then we obtain the switching function $F(x_0, v_0, m_0) = 0$ as tabulated numbers for time "t" as parameter. ($0 \leq t \leq \tau$ = chosen maximum time.)

In the present stage of investigation we were mainly interested in the overall gas-dynamic braking effect of a planetary atmosphere as far as the development of the switching functions is concerned. We made two types of calculations:

---

* The equations of motion can be integrated analytically in the case of no gas dynamic drag, or, in the case of homogeneous atmosphere with gas dynamic drag linearly dependent on velocity. But eliminating t' would still require solution of transcendental algebraic equations in these cases, too.

(1) With gas-dynamic drag: (a) $K' = 3.10^{-3}$ (kg/m)

$b = 1.5\ 10^{-4}$ (m^{-1})

(b) $K' = 6.10^{-3}$ (kg/m)

$b = 1.5\ 10^{-4}$ (m^{-1})

(2) Without gas-dynamic drag: (See Eqs.(2.3.5-7), but without the velocity-dependent term in the second equation.)

For the other constants we used the following values:

$m_f = 120$ (kg)

$v_e = 1800$ (m/sec)

$g = 3$ (m/sec^2)

$|\dot{m}| = 1$ (kg/sec), constant, since we assume

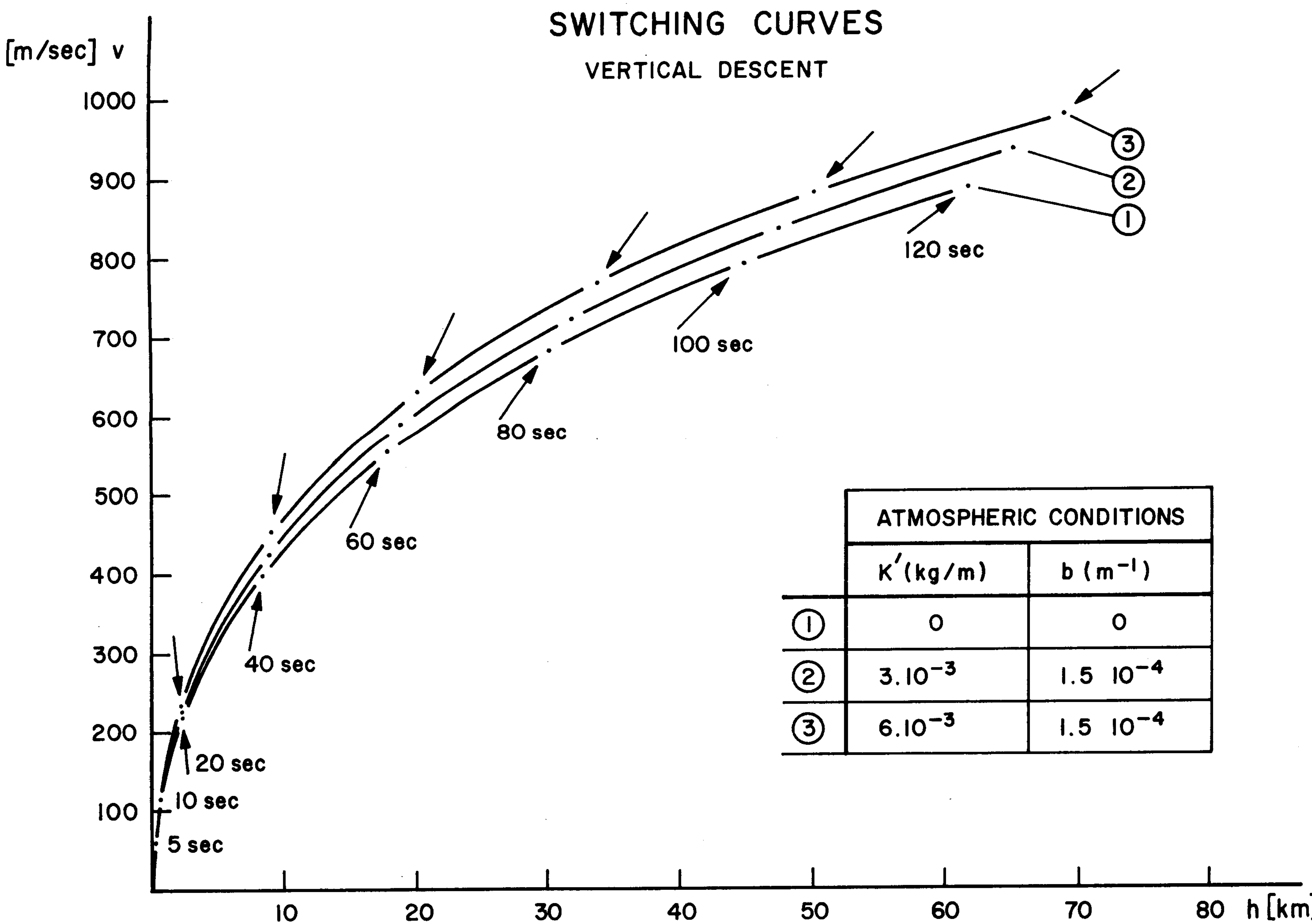
$$m(t) = m_0 - at \tag{2.3.16}$$

The integrations were carried out on the IBM 7094 computer using Runge-Kutta-Gill method together with the Adams-Moulton predictor-corrector formulas in the variable mode version (= automatic control of error) of the CIT subroutine called DEQ. The equations were integrated in a time interval $0 \leq t \leq 120$ (sec.) We used $\Delta t = 0.1$ sec for the integration step and $\bar{E} = 1.10^{-6}$ for the maximum allowable truncation error. Some representative results are tabulated below and are depicted in Fig. 2.3.1.

NB: the tabulated and depicted switching function is, at the same time, the actual optimal trajectory with time or mass as parameter; starting with a given set of values of h_0, v_0, m_0 (or t_0) and reversing the time we obtain the actual optimal trajectory for the chosen initial values and ending at $x_1 = x_2 = 0$ and $x_3 = m_f$.

SWITCHING FUNCTIONS (= OPTIMAL TRAJECTORIES) FOR VERTICAL SOFT-LANDING

		Given atmospheric conditions					
		$K' = 3.10^{-3}$ (kg/m) $b = 1.5\ 10^{-4}$ (m^{-1})		$K' = 6.10^{-3}$ (kg/m) $b = 1.5\ 10^{-4}$ (m^{-1})		Without atmospheric conditions	
t (sec)	$x_3 = m_0$ (kg)	$x_1 = h_0$ (m)	$x_2 = v_0$ (m/sec)	$x_1 = h_0$ (m)	$x_2 = v_0$ (m/sec)	$x_1 = h_0$ (m)	$x_2 = v_0$ (m/sec)
0	120	0	0	0	0	0	0
5	125	147.0	58.6	148.0	58.8	147.0	58.5
10	130	583.0	115.1	585.0	116.1	580.0	114.1
20	140	2281.0	223.7	2317.0	230.4	2246.0	217.5
40	160	8797.0	423.0	9180.0	451.5	8452.0	397.8
60	180	18960.0	587.2	20069.0	628.9	17971.0	549.8
80	200	32077.0	720.3	34057.0	764.8	30297.0	679.5
100	220	47633.0	832.4	50508.0	877.2	45030.0	791.0
120	240	65270.0	929.0	69042.0	973.9	61840.0	887.7



	ATMOSPHERIC CONDITIONS	
	K′(kg/m)	b (m^{-1})
(1)	0	0
(2)	3.10^{-3}	$1.5\ 10^{-4}$
(3)	6.10^{-3}	$1.5\ 10^{-4}$

Fig. 2.3.1

2.4. Optimal Thrust Program for Ballistic Descent.

Regarding Optimal Thrust Program for ballistic descent we follow the same considerations and assumptions on "cost function" and control constraint as we did for vertical descent in the previous Section. (See Eqs.(2.3.1-3))

As prescribed terminal conditions we consider now:

$$\begin{aligned} h(\tau) &= 3000 \text{ (m)} \\ s(\tau) &= 0 \\ \alpha(\tau) &= 84^{\circ} \\ v(\tau) &= 20 \text{ (m/sec)} \\ m(\tau) &= m_f = 120 \text{ (kg)} \end{aligned} \tag{2.4.1}$$

These terminal conditions mean that the rocket thrust stops at 3000 (m) above the planet's surface. (The ground range "s" is measured from that point.) In that moment the vehicle's path angle has to be 84° below the local horizontal, and, the vehicle's velocity and mass must be 20 (m/sec) and 120 (kg), respectively. From that moment on the vehicle continues a parachute mode descent to the planet's surface.

Taking Eqs.(2.3.1-3, 2.4.1) into consideration one can see that the Optimal (= Minimal) Thrust Program for ballistic descent is equivalent to the minimal time problem, as it was for vertical descent.

By introducing

$x_1 \triangleq h$ (altitude above the planet's surface)

$x_2 \triangleq s$ (ground range)

$x_3 \triangleq \alpha$ (path angle, measured below the local horizontal)

$x_4 \triangleq v$ (velocity)

and augmenting the state space by defining

$x_5 \triangleq m$ (mass of the vehicle)

then from Eqs.(2.2.10-13) and (2.3.2) we obtain the following system equations:

$$\dot{x}_1 = - x_4 \sin x_3 \tag{2.4.2}$$

$$\dot{x}_2 = x_4 \cos x_3 \tag{2.4.3}$$

$$\dot{x}_3 = \left(\frac{g}{x_4} - \frac{x_4}{R} \right) \cos x_3 \tag{2.4.4}$$

$$\dot{x}_4 = g \sin x_3 - \frac{1}{x_5} K' \exp(- bx_1) x_4^2 - \frac{1}{x_5} \beta \tag{2.4.5}$$

$$\dot{x}_5 = - \frac{\beta}{v_e} \tag{2.4.6}$$

with prescribed terminal conditions:

$x_1(\tau) = 3000$ (m); $x_2(\tau) = 0$

$x_3(\tau) = 84^\circ$; $x_4(\tau) = 20$ (m/sec), $x_5(\tau) = 120$ (kg)

and, the "cost function" is given by

$$\mathfrak{J} = -\int_0^T \dot{x}_5 \, dt$$

In order to obtain the form of the Optimal Thrust Program we again apply the Pontryagin Maximum Principle [Ref. 1]. The Hamiltonian "H" for the minimal time problem becomes:

$$H = -\lambda_1 x_4 \sin x_3 + \lambda_2 x_4 \cos x_3 + \lambda_3 \left(\frac{g}{x_4} - \frac{x_4}{R} \right) \cos x_3 + \lambda_4 \left[g \sin x_3 - \frac{1}{x_5} K' \cdot \exp(-bx_1)\, x_4^2 - \frac{1}{x_5} \beta \right] - \lambda_5 \frac{\beta}{v_e} \qquad (2.4.7)$$

where the auxiliary variables $\lambda_1, \cdots, \lambda_5$ are nontrivial solutions of the system of adjoint equations:

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -\lambda_4 \frac{K'}{x_5} b \exp(-bx_1)\, x_4^2 \qquad (2.4.8)$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = 0 \qquad (2.4.9)$$

$$\dot{\lambda}_3 = -\frac{\partial H}{\partial x_3} = \lambda_1 x_4 \cos x_3 + \lambda_2 x_4 \sin x_3 + \lambda_3 \left(\frac{g}{x_4} - \frac{x_4}{R} \right) \sin x_3 - \lambda_4 g \cos x_3 \qquad (2.4.10)$$

$$\dot{\lambda}_4 = -\frac{\partial H}{\partial x_4} = \lambda_1 \sin x_3 - \lambda_2 \cos x_3 + \lambda_3 \cos x_3 \left(\frac{1}{R} + \frac{g}{x_4^2} \right) + \lambda_4 \frac{2}{x_5} K' \exp(-bx_1)\, x_4 \qquad (2.4.11)$$

$$\dot{\lambda}_5 = -\frac{\partial H}{\partial x_5} = -\frac{\lambda_4}{x_5^2} K' \exp(-bx_1)\, x_4^2 + \beta \qquad (2.4.12)$$

Since $\frac{\partial H}{\partial \beta} = 0$ gives $\frac{\lambda_4}{x_5} + \frac{\lambda_5}{v_e} = 0$, we see that the Hamiltonian is maximized by taking

$$\beta = \begin{cases} \beta_{max}, & \text{whenever } \dfrac{\lambda_4}{x_5} + \dfrac{\lambda_5}{v_e} > 0 \\ 0, & \text{whenever } \dfrac{\lambda_4}{x_5} + \dfrac{\lambda_5}{v_e} < 0 \end{cases} \tag{2.4.13}$$

In the case of

$$\frac{\lambda_4}{x_5} + \frac{\lambda_5}{v_e} = 0 \tag{2.4.14}$$

β is indeterminate. Equation (2.4.14) expresses the singularity condition. But for this problem there is no singular control since relation (2.4.14) cannot hold on any finite closed intewall in $[0, \tau]$. (This fact is clear for physical reasons, too.)

Relation (2.4.13) expresses the fact that the Optimal Thrust Program for the considered problem is of the Bang-Bang type.

In order to synthesize the Optimal Thrust Program we have to determine an appropriate switching function. The evaluation of the switching function consists in determining a relation $F(x_1, x_2, x_3, x_4, x_5) = 0$. If the given maximum thrust is applied continuously from the moment when this relation is first satisfied, the prescribed terminal conditions can be achieved.

The only way of evaluating the switching function in this case is to integrate Eqs.(2.4.2-6) numerically in backward time. Or equivalently: integrate Eqs.(2.4.2-6) as they are from $t = 0$ to $t = \tau$ (just change the sign of $\dot{x}_5$) and use the prescribed terminal conditions as initial conditions (just change the sign of $x_4(\tau)$ in using it as an initial condition.) Then we obtain the switching function $F(h_0, s_0, \alpha_0, v_0, m_0) = 0$ as tabulated numbers for time "t" as parameter. ($0 \le t \le \tau$ = chosen maximum time.)

In order to investigate the gas-dynamic slowing-down effect of a planetary atmosphere as far as the evaluation of the switching function is concerned we made the following calculations:

(1) With gas-dynamic drag: (a) $K' = 3.10^{-3}$ (kg/m)

$b = 1.5\ 10^{-4}$ (m^{-1})

(b) $K' = 6.10^{-3}$ (kg/m)

$b = 1.5\ 10^{-4}$ (m^{-1})

(2) Without atmosphere. (See Eqs.(2.4.2-6), but using Eq.(2.4.5) without the velocity dependent term.)

We have applied the following values for the other constants:

$m_f = 120$ (kg)

$v_e = 1800$ (m/sec)

$g = 3$ (m/sec^2)

$R = 3.10^6$ (m)

$|\dot{m}| = 1$ (kg/sec), constant, since we assume

$$m(t) = m_0 - at \tag{2.4.15}$$

The integrations were carried out on the IBM 7094 computer using the variable mode version of the CIT subroutine called DEQ. The equations were integrated in $0 \le t \le 120$ (sec) by $\Delta t = 0.1$ step. $\bar{E} = 1.10^{-6}$ was used for the maximum allowable truncation error. Some representative results are tabulated below and are depicted on Fig. 2.4.1.

NB: The tabulated and depicted switching function are, at the same time, the actual optimal surfaces; starting with a given set of values of h_0, s_0, α_0, v_0, m_0 (or t_0) and reversing the time we obtain the actual optimal surface for the chosen initial values and ending at $h(\tau)$, $s(\tau)$, $\alpha(\tau)$, $v(\tau)$, $m(\tau)$.

SWITCHING FUNCTIONS (= OPTIMAL TRAJECTORIES) FOR BALLISTIC SOFT-LANDING.
(Given Atmospheric Conditions)

		$K' = 3.10^{-3}$ (kg/m), $b = 1.5\ 10^{-4}$ (m^{-1})				$K' = 6.10^{-3}$ (kg/m), $b = 1.5\ 10^{-4}$ (m^{-1})			
t (sec)	$x_5 = m_0$ (kg)	$x_1 = h_0$ (m)	$x_2 = s_0$ (m)	$x_3 = \alpha_0$ (deg)	$x_4 = v_0$ (m/sec)	$x_1 = h_0$ (m)	$x_2 = s_0$ (m)	$x_3 = \alpha_0$ (deg)	$x_4 = v_0$ (m/sec)
0	120	3000.0	0	84°	20.0	3000.0	0	84°	20.0
5	125	3245.0	-33.0	81°30'	78.8	3246.0	-33.0	81°30'	79.0
10	130	3775.0	-119.0	80°10'	135.4	3778.0	-120.0	80°10'	136.5
20	140	5641.0	-475.0	78°30'	243.3	5672.0	-480.0	78°30'	248.6
40	160	12333.0	-1993.0	76°20'	437.7	12602.0	-2042.0	76°25'	456.6
60	180	22418.0	-4618.0	74°40'	599.2	23153.0	-4755.0	74°50'	626.0
80	200	35261.0	-8310.0	73°20'	733.3	36552.0	-8542.0	73°40'	762.0
100	220	50384.0	-13021.0	72°10'	847.9	52254.0	-13333.0	72°30'	876.8
120	240	67438.0	-18709.0	71°	947.7	69895.0	-19082.0	71°30'	976.4

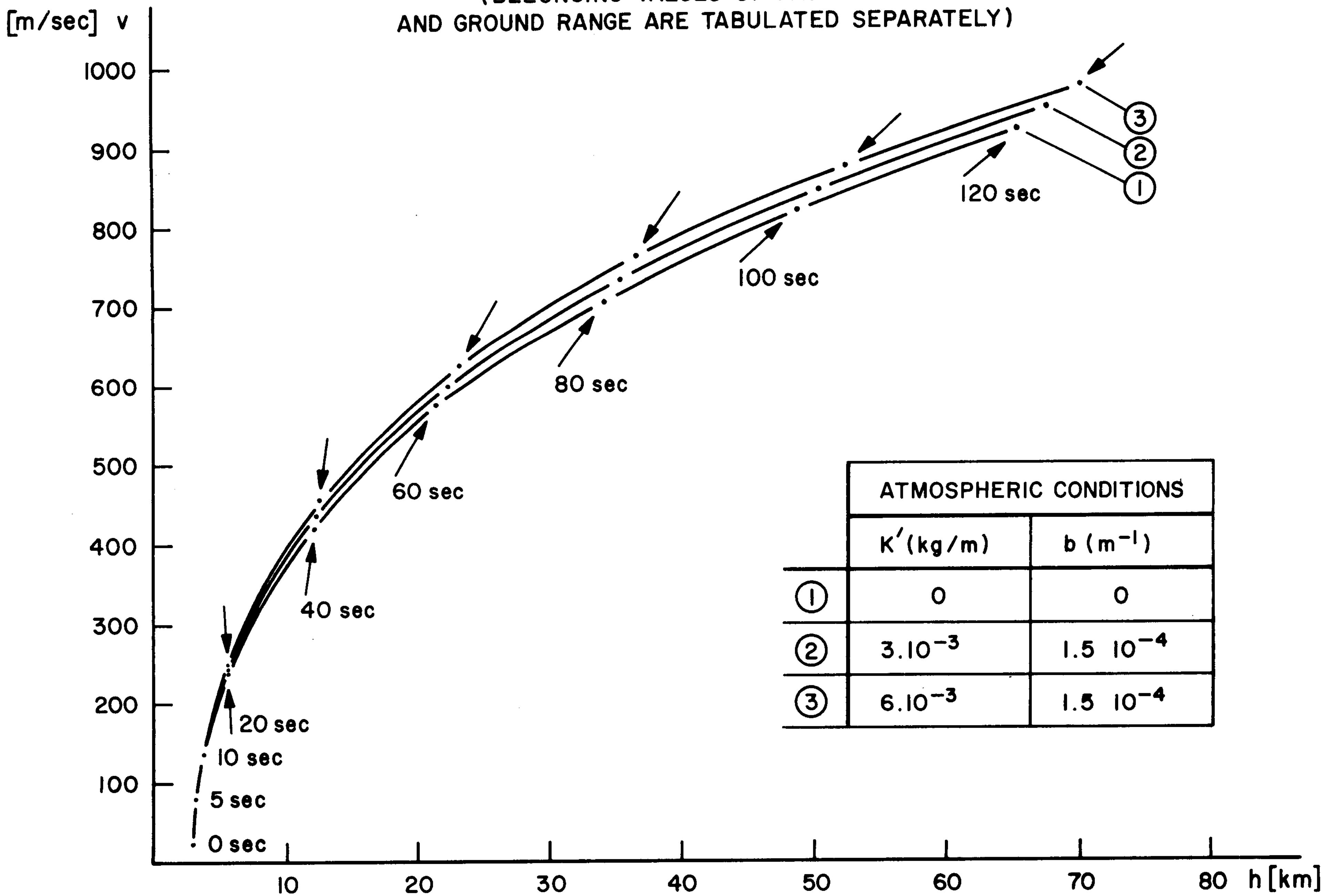
SWITCHING FUNCTIONS (= OPTIMAL TRAJECTORIES) FOR BALLISTIC SOFT-LANDING
(Without Atmospheric Conditions)

t (sec)	$x_5 = m_0$ (kg)	$x_1 = h_0$ (m)	$x_2 = s_0$ (m)	$x_3 = \alpha_0$ (deg)	$x_4 = v_0$ (m/sec)
0	120	3000.0	0	84°	20.0
5	125	3245.0	-33.0	$81^\circ 30'$	78.6
10	130	3772.0	-119.0	$80^\circ 10'$	134.4
20	140	5611.0	-470.0	$78^\circ 30'$	238.3
40	160	12081.0	-1947.0	$76^\circ 10'$	420.2
60	180	21735.0	-4491.0	$74^\circ 30'$	574.1
80	200	34055.0	-8093.0	73°	706.2
100	220	48629.0	-12728.0	$71^\circ 45'$	820.6
120	240	65126.0	-18392.0	$70^\circ 35'$	920.9

SWITCHING CURVES

BALLISTIC DESCENT

(BELONGING VALUES OF PATH ANGLE AND GROUND RANGE ARE TABULATED SEPARATELY)



	ATMOSPHERIC CONDITIONS	
	K' (kg/m)	b (m^{-1})
(1)	0	0
(2)	3.10^{-3}	1.5 10^{-4}
(3)	6.10^{-3}	1.5 10^{-4}

Fig. 2.4.1

The purpose of a fuel optimization study (such as those which are presented in this and in the previous sections) is to develop a system which accomplishes the terminal phase of the mission with an efficient utilization of fuel. Computation of the minimum fuel required for a given mission may be used as a guide in specifying fuel requirements.

It is obvious, however, that arriving the prescribed terminal manifold along the optimal trajectory (= switching curve) presupposes a precisely precalculated switching curve, and, a precise knowledge of the current state of the space vehicle. (The Pontryagin Maximum Principle determines an open-loop control.) Disregarding the limitations of the measuring instruments, the required precision is very much affected by the applied atmospheric parameter values. The switching curves and surfaces in Figs. 2.3.1 and 2.4.1, calculated for different K' values, illustrate how sensitive these trajectories are to uncertainties in the value of the parameter K' in the dynamic equations. The applied values of K', $3.10^{-3} \leq K' \leq 6.10^{-3}$ (kg/m), may be regarded as the uncertainty we have in the value of ground level pressure on Mars, which, by present knowledge is $\approx 10 \leq P_0 \leq 20$ (mb).

2.5. General Feature of a Proposed Scheme for Sequential Estimation of State and Parameters in Noisy Non-linear Systems.

An optimally controlled soft-landing maneuver in an imperfectly known atmosphere presupposes an "optimal knowledge" of the current state of the system. Generally we have to assume that

(a) there are unknown dynamic disturbances acting on the system (this is due either to the approximate character of the differential equations describing the actual behaviour of the system, or, to randomly acting external forces);

(b) the dynamic parameters of the system are imperfectly known;

(c) in some cases not all state variables are available for measurement;

(d) the observable state variables are corrupted by measurement noise;

(e) we have no information on the statistics of the acting dynamic and measurement noises.

Taking into account all these realistic assumptions we have to ask three basic questions:

(a) how to obtain "optimal knowledge" (or "true estimate") on the current state of the system;

(b) how a given scheme for sequential estimation does converge to the true state of the system, or, using practical terms: how much time is necessary to obtaining "true estimate" on the current state of the system.

(c) what are the practical implications of the given sequential estimation scheme as far as its implementation is concerned, or, in other words: whether it is possible to make reasonable simplifications on a (presumably) complicated scheme.

Since soft-landing under atmospheric influence is described by (ordinary) non-linear differential equations we are faced with the problem of non-linear filtering (or sequential estimation). Earlier investigators in the theory of optimal filtering have mostly dealt with linear systems and have assumed some (or complete) knowledge on the statistics of the relevant disturbances. Therefore their theories are inappropriate to handle the questions we are posing in connection with "optimally controlled soft-landing in an imperfectly known atmosphere".

Considerable results in the theory of sequential state estimation in noisy non-linear systems have been obtained only recently.[2][3] In the present report we essentially follow the general framework of Ref. 3 which seems to be adequate for handling our problem specified above.

In Ref. 3 a least-squares criterion is used for estimation purposes and the sequential nature of the estimation problem is brought out by applying the theory of invariant imbedding on the Euler-Lagrange equations which were formally obtained by using Pontryagin's maximum principle. The derived sequential estimator equations (which are ordinary differential equations) are approximations to a non-linear partial differential equation resulting from the invariant imbedding. (As a matter of fact this non-linear partial differential equation can also be obtained by using the dynamic programming approach.)

If we are given a system by

$$\dot{x} = f(x,t) + v(x,t)\ u \tag{2.5.1}$$

$$y = h(x,t) + (\text{Observation Error}) \tag{2.5.2}$$

where:

x = n-vector

$f(x,t)$ = n-vector function

u = p-vector random dynamic input

$v(x,t)$ = n·p-vector function

$h(x,t)$ = m-vector function

$y(t)$ = m-vector output (= observation)

then, according to Ref. 3, an appropriate set of sequential estimator equations are:

$$\frac{d}{dT}\hat{x} = f(\hat{x},T) = 2P(T)\ H(\hat{x},T)\ Q\{y(T)-h(\hat{x},T)\} \tag{2.5.3}$$

$$\frac{d}{dT}P = f_{\hat{x}}(\hat{x},T)\ P(T) + P(T)\ \bar{f}_{\hat{x}}(\hat{x},T) + 2P(T)[H(\hat{x},T)\ Q\{y(T)-h(\hat{x},T)\}]_{\hat{x}}P(T) + R^{-1} \tag{2.5.4}$$

where $\hat{x}$: denotes the least-squares estimate of x;

$f_{\hat{x}} = \left(\frac{\partial f_i}{\partial \hat{x}_j}\right)$, the Jacobian matrix of f;

$\bar{f}_{\hat{x}}$: denotes the transpose of $f_{\hat{x}}$;

$H = \left(\frac{\partial h_i}{\partial \hat{x}_j}\right)$, the Jacobian matrix of h;

$P = (P_{ij})$, n·n symmetric matrix;

Q = quasi-norm factor (we take $Q = 1$);

R = quasi-norm factor (n·n matrix);

T = running observation time;

$[H(\hat{x},T)\ Q\{y(T)-h(\hat{x},T)\}]_{\hat{x}}$ = n·n matrix with i^{th} column:

$$\frac{\partial}{\partial \hat{x}_i}[H(\hat{x},T)\ Q\{y(T)-h(\hat{x},T)\}]$$

Since in our cases we always will assume for the observation vector that

$$y_i = x_i + \text{(state-independent Random Observation Error)} \qquad (2.5.6)$$

$$i = 1, \cdots, m \le n$$

Therefore, we always will have for Eq.(2.5.4):

$$\dot{P} = f_{\hat{x}}P + P\bar{f}_{\hat{x}} - 2PHQ\bar{H}P + R^{-1} \qquad (2.5.7)$$

where

$$H = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \qquad (2.5.8)$$

and

$$HQ\bar{H} = \begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & 0 & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ \cdot & \cdot & & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix} \qquad (2.5.9)$$

In the H and $HQ\bar{H}$ matrices the number "m" of the diagonal elements different from zero is equal to the dimensionality "m" of the observation vector (2.5.6).

Equations (2.5.3) and (2.5.7) are the basic filter-equations which we will apply in the subsequent Sections for sequentially estimating states (and parameters) in the atmospheric descent problems. As one can see Eqs.(2.5.3) and (2.5.7) are ordinary, coupled, non-linear differential equations. The observations $y(T)$ appear as forcing terms in Eq.(2.5.3). The matrix R^{-1} in Eq.(2.5.7) can also be regarded as a forcing term for the P-equation (2.5.7) which can be named as a matrix gain equation.

In solving (or implementing) Eqs.(2.5.3) and (2.5.7) one can start with assumed (= freely estimated) values for $\hat{x}$ at $T = 0$, but, one has to find the proper starting values for the P-equations. At the same time one also has to select proper values for the (R_{ij}) matrix. Due to the non-linear character of the filter-equations (2.5.3) and (2.5.7) this question has to be investigated for each problem. The main problem in solving the non-linear filter-equations is, therefore, how to determine the appropriate $P_{ij}(0)$ and R_{ij} values which will assure that the estimated $\hat{x}(T)$ values, based on measurements $y(T)$ and on the dynamic description $f(x,t)$ of the system, will properly converge to the true values of $x(T)$.

2.6. Sequential Estimation of State in Vertical Descent. (Assuming perfectly known parameters.)

Considering a free fall trajectory (= no thrusting), and, making the same assumptions on coordinate system, gas-dynamic forces, atmospheric density distribution as they were outlined in Sections 2.2 and 2.3, we obtain the following differential equation governing the behaviour of the space vehicle.

$$\ddot{x} = K \exp(-bx)\,\dot{x}^2 - g + u(t) \; ; \quad K \triangleq \frac{K'}{m} \qquad (2.6.1)$$

where $x, \dot{x}, \ddot{x}$ = position, velocity and acceleration respectively;

g = acceleration of gravity (considered as a constant on a limited part of the trajectory);

m = mass of the space vehicle (constant, since we don't apply thrust in this limited part of the trajectory);

u(t) = random dynamic disturbance;

K', K = parameters (constants), reflecting the gas dynamic characteristics of the atmospheric flight;

b = parameter (constant), reflecting the physical characteristics of the planetary atmosphere.

Let $x_1 \triangleq x$ (altitude, measured as a positive distance from the landing-ground upward)

$x_2 \triangleq \dot{x}$ (velocity, measured as a negative quantity downward to the landing-ground)

Then Eq.(2.6.1) can be rewritten as

$$\dot{x}_1 = x_2 \tag{2.6.2}$$

$$\dot{x}_2 = K \exp(- bx_1)\, x_2^2 - g + u(t) \tag{2.6.3}$$

which constitute the system equations of the problem considered in this section.

A. Given noisy position measurements only.

In this case we have a specified one-dimensional observation vector:

$$\{\text{position}\} = y_1(T) = x_1(T) + (\text{Observation Noise}) \tag{2.6.4}$$

This gives for the H and $HQ\overline{H}$ matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad HQ\overline{H} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{2.6.5}$$

The Jacobian $f_{\hat{x}}$ becomes (from Eq.(2.6.2-3)):

$$f_{\hat{x}} = \begin{pmatrix} 0 & 1 \\ -Kb\hat{x}_2^2 \exp(-b\hat{x}_1) & 2K\hat{x}_2 \exp(-b\hat{x}_1) \end{pmatrix} \tag{2.6.6}$$

and we have a $P = 2 \times 2$ symmetric matrix. Thus the sequential estimator Eqs.(2.5.3) and (2.5.7) for this problem become:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(y_1-\hat{x}_1) \tag{2.6.7}$$

$$\dot{\hat{x}}_2 = K\hat{x}_2^2 \exp(-b\hat{x}_1) - g + 2P_{12}(y_1-\hat{x}_1) \tag{2.6.8}$$

$$\dot{P}_{11} = -2P_{12}^2 + 2P_{12} + R'_{11} \tag{2.6.9}$$

$$\dot{P}_{12} = P_{22} - 2P_{11}P_{12} + (2P_{12}-P_{11}b\hat{x}_2)\, K\hat{x}_2\exp(-b\hat{x}_1) + R'_{12} \tag{2.6.10}$$

$$\dot{P}_{22} = -2P_{12}^2 + 2(2P_{22}-P_{12}b\hat{x}_2) \cdot K\hat{x}_2\exp(-b\hat{x}_1) + R'_{22} \tag{2.6.11}$$

where R'_{ij} are elements of the $(R_{ij})^{-1}$ matrix.

B. Given noisy position and velocity measurements.

Since in this case we have a specified two-dimensional observation vector

$$\begin{pmatrix} \text{position} \\ \text{velocity} \end{pmatrix} = \begin{pmatrix} y_1(T) \\ y_2(T) \end{pmatrix} = \begin{pmatrix} x_1(T) + (\text{Observation Noise}) \\ x_2(T) + (\text{Observation Noise}) \end{pmatrix} \tag{2.6.12}$$

therefore the H and $HQ\overline{H}$ matrices become

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad HQ\overline{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.6.13}$$

The Jacobian $f_{\hat{x}}$ is the same as before, see Eq.(2.6.6), and, we have a $P = 2 \times 2$ symmetric matrix. Thus the sequential estimator equations (2.5.3) and (2.5.7) for this problem become:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(y_1-\hat{x}_1) + 2P_{12}(y_2-\hat{x}_2) \tag{2.6.14}$$

$$\dot{\hat{x}}_2 = K\hat{x}_2^2 \exp(-b\hat{x}_1) - g + 2P_{12}(y_1-\hat{x}_1) + 2P_{22}(y_2-\hat{x}_2) \tag{2.6.15}$$

$$\dot{P}_{11} = 2P_{12} - 2P_{11}^2 - 2P_{12}^2 + R'_{11} \tag{2.6.16}$$

$$\dot{P}_{12} = P_{22} - 2P_{12}(P_{11} + P_{22}) + (2P_{12} - P_{11}b\hat{x}_2)\, K\hat{x}_2 \exp(-b\hat{x}_1) + R'_{12} \tag{2.6.17}$$

$$\dot{P}_{22} = -2(P_{12}^2 + P_{22}^2) + 2(2P_{22}-P_{12}b\hat{x}_2)\, K\hat{x}_2\exp(-b\hat{x}_1) + R'_{22} \tag{2.6.18}$$

where R'_{ij} are elements of the $(R_{ij})^{-1}$ matrix in Eq.(2.5.7).

2.7. Sequential Estimation of State and One Parameter (either the atmospheric density or the gasdynamic drag parameter) in Vertical Descent.

In this section we again consider a free fall trajectory. The basic dynamic equation we will start with is the same as Eq.(2.6.1). But we take one of the parameters in Eq.(2.6.1) -- either K or b -- as imperfectly known.

By defining the imperfectly known parameter(s) in the dynamic equation as new state variable(s), -- which amounts to augmenting the problem's state space -- the imperfectly known parameter(s) can be modelled as solution(s) to ordinary differential equation(s) with unknown initial condition(s). This technique will allow us to handle all the sequential estimation problems in the unified view of state space.

Taking "K" in Eq.(2.6.1) as imperfectly known, we define it as a third state variable:

$$x_3 \triangleq K \quad ; \quad \dot{K} = 0$$

Thus the system equations become:

$$\dot{x}_1 = x_2 \tag{2.7.1}$$

$$\dot{x}_2 = x_3 \exp(-bx_1)\, x_2^2 - g + u(t) \tag{2.7.2}$$

$$\dot{x}_3 = 0 \tag{2.7.3}$$

Taking "b" in Eq.(2.6.1) as imperfectly known, we define it as a third state variable:

$$x_3 \triangleq b \quad ; \quad \dot{b} = 0$$

Thus the system equations become:

$$\dot{x}_1 = x_2 \tag{2.7.4}$$

$$\dot{x}_2 = K \exp(- x_3 x_1)\, x_2^2 - g + u(t) \tag{2.7.5}$$

$$\dot{x}_3 = 0 \tag{2.7.6}$$

A. Given noisy position measurements only.

The observation vector is a specified one-dimensional one in this case

$$\{\text{position}\} = y_1(T) = x_1(T) + (\text{Observation Noise}) \tag{2.7.8}$$

which gives for the H and $HQ\overline{H}$ matrices:

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad HQ\overline{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7.9}$$

In the case of $x_3 \triangleq K$, the Jacobian $f_{\hat{x}}$ becomes (from Eqs.(2.7.1-3)):

$$f_{\hat{x}} = \begin{pmatrix} 0 & 1 & 0 \\ - b\hat{x}_2^2\hat{x}_3 \exp(- b\hat{x}_1) \;, & 2\hat{x}_2\hat{x}_3 \exp(- b\hat{x}_1) \;, & \hat{x}_2^2 \exp(- b\hat{x}_1) \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7.10}$$

and we have a $P = 3 \times 3$ symmetric matrix. For the sequential estimator equations (2.5.3) and (2.5.7) we then obtain:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(y_1 - \hat{x}_1) \tag{2.7.11}$$

$$\dot{\hat{x}}_2 = \hat{x}_3 \exp(- b\hat{x}_1)\, \hat{x}_2^2 - g + 2P_{12}(y_1 - \hat{x}_1) \tag{2.7.12}$$

$$\dot{\hat{x}}_3 = 2P_{13}(y_1 - \hat{x}_1) \tag{2.7.13}$$

$$\dot{P}_{11} = - 2P_{11}^2 + 2P_{12} + R'_{11} \tag{2.7.14}$$

$$\dot{P}_{12} = - 2P_{11}P_{12} + P_{22} - b\hat{x}_2^2\hat{x}_3 \exp(- b\hat{x}_1) P_{11} + 2\hat{x}_2\hat{x}_3 \exp(- b\hat{x}_1) P_{12} + \hat{x}_2^2 \exp(- b\hat{x}_1) P_{13} + R'_{12} \quad (2.7.15)$$

$$\dot{P}_{13} = - 2P_{11}P_{13} + P_{23} + R'_{13} \quad (2.7.16)$$

$$\dot{P}_{22} = - 2P_{12}^2 - 2b\hat{x}_2^2\hat{x}_3 \exp(- b\hat{x}_1) P_{12} + 4\hat{x}_2\hat{x}_3 \exp(- b\hat{x}_1) P_{22} + 2\hat{x}_2^2 \exp(- b\hat{x}_1) P_{23} + R'_{22} \quad (2.7.17)$$

$$\dot{P}_{23} = - 2P_{12}P_{13} - b\hat{x}_2^2\hat{x}_3 \exp(- b\hat{x}_1) P_{13} + 2\hat{x}_2\hat{x}_3 \exp(- b\hat{x}_1) P_{23} + \hat{x}_2^2 \exp(- b\hat{x}_1) P_{33} + R'_{23} \quad (2.7.18)$$

$$P_{33} = - 2P_{13}^2 + R'_{33} \quad (2.7.19)$$

In the case of $x_3 \triangleq b$, the Jacobian $f_{\hat{x}}$ becomes (from Eqs.(2.7.4-6)):

$$f_{\hat{x}} = \begin{pmatrix} 0 & 1 & 0 \\ - K\hat{x}_2^2\hat{x}_3 \exp(- x_3x_1) , & 2\hat{x}_2K \exp(- x_3x_1) , & - K\hat{x}_1\hat{x}_2^2 \exp(-x_1x_3) \\ 0 & 0 & 0 \end{pmatrix} \quad (2.7.20)$$

and since we have a $P = 3 \times 3$ symmetric matrix, we obtain for the sequential estimator equations (2.5.3) and (2.5.4):

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(y_1-\hat{x}_1) \quad (2.7.21)$$

$$\dot{\hat{x}}_2 = K \exp(- \hat{x}_3\hat{x}_1) \hat{x}_2^2 - g + 2P_{12}(y_1-\hat{x}_1) \quad (2.7.22)$$

$$\dot{\hat{x}}_3 = 2P_{13}(y_1-\hat{x}_1) \tag{2.7.23}$$

$$\dot{P}_{11} = - 2P_{11}^2 + 2P_{12} + R'_{11} \tag{2.7.24}$$

$$\begin{aligned}\dot{P}_{12} = &- 2P_{11}P_{12} + P_{22} - K\hat{x}_2^2\hat{x}_3 \exp(- \hat{x}_3\hat{x}_1)\ P_{11} + \\ &2K\hat{x}_2 \exp(- \hat{x}_3\hat{x}_1)\ P_{12} - K\hat{x}_1\hat{x}_2^2 \exp(- \hat{x}_3\hat{x}_1)\ P_{13} + R'_{12}\end{aligned} \tag{2.7.25}$$

$$\dot{P}_{13} = - 2P_{11}P_{13} + 2P_{23} + R'_{13} \tag{2.7.26}$$

$$\begin{aligned}\dot{P}_{22} = &- 2P_{12}^2 - 2K\hat{x}_2^2\hat{x}_3 \exp(- \hat{x}_3\hat{x}_1)\ P_{12} + 4K\hat{x}_2 \exp(- \hat{x}_3\hat{x}_1)\ P_{22} - \\ &2K\hat{x}_1\hat{x}_2^2 \exp(- \hat{x}_3\hat{x}_1)\ P_{23} + R'_{22}\end{aligned} \tag{2.7.27}$$

$$\begin{aligned}\dot{P}_{23} = &- 2P_{12}P_{13} - K\hat{x}_2^2\hat{x}_3 \exp(- \hat{x}_3\hat{x}_1)\ P_{13} + 2K\hat{x}_2 \exp(- \hat{x}_3\hat{x}_1)\ P_{23} - \\ &K\hat{x}_1\hat{x}_2^2 \exp(- \hat{x}_3\hat{x}_1)\ P_{33} + R'_{23}\end{aligned} \tag{2.7.28}$$

$$\dot{P}_{33} = - 2P_{13}^2 + R'_{33} \tag{2.7.29}$$

B. <u>Given noisy position and velocity measurements</u>.

The observation vector is a specified two-dimensional one in this case (see Eq.(2.6.12)), and that gives

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H Q \overline{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.7.30}$$

In the case of $x_3 \triangleq K$ the Jacobian $f_{\hat{x}}$ is the same as Eq.(2.7.10), and, since we have a $P = 3 \times 3$ symmetric matrix, we obtain for the sequential estimator equations:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(y_1-\hat{x}_1) + 2P_{12}(y_2-\hat{x}_2) \tag{2.7.31}$$

$$\dot{\hat{x}}_2 = \hat{x}_3\hat{x}_2^2 \exp(-b\hat{x}_1) - g + 2P_{12}(y_1-\hat{x}_1) + 2P_{22}(y_2-\hat{x}_2) \tag{2.7.32}$$

$$\dot{\hat{x}}_3 = 2P_{13}(y_1-\hat{x}_1) + 2P_{23}(y_2-\hat{x}_2) \tag{2.7.33}$$

$$\dot{P}_{11} = -2P_{11}^2 - 2P_{12}^2 + 2P_{12} + R'_{11} \tag{2.7.34}$$

$$\dot{P}_{12} = -2P_{11}P_{12} - 2P_{12}P_{22} + P_{22} - b\hat{x}_2^2\hat{x}_3 \exp(-b\hat{x}_1) P_{11} + 2\hat{x}_2\hat{x}_3 \exp(-b\hat{x}_1) P_{12} + \hat{x}_2^2 \exp(-b\hat{x}_1) P_{13} + R'_{12} \tag{2.7.35}$$

$$\dot{P}_{13} = -2P_{11}P_{13} - 2P_{12}P_{23} + P_{23} + R'_{13} \tag{2.7.36}$$

$$\dot{P}_{22} = -2P_{12}^2 - 2P_{22}^2 - 2b\hat{x}_2^2\hat{x}_3 \exp(-b\hat{x}_1) P_{12} + 4\hat{x}_2\hat{x}_3 \exp(-b\hat{x}_1) P_{22} + 2\hat{x}_2^2 \exp(-b\hat{x}_1) P_{23} + R'_{22} \tag{2.7.37}$$

$$\dot{P}_{23} = -2P_{12}P_{13} - 2P_{22}P_{23} - b\hat{x}_2^2\hat{x}_3 \exp(-b\hat{x}_1) P_{13} + 2\hat{x}_2\hat{x}_3 \exp(-b\hat{x}_1) P_{23} + \hat{x}_2^2 \exp(-b\hat{x}_1) P_{33} + R'_{23} \tag{2.7.38}$$

$$\dot{P}_{33} = -2P_{13}^2 - 2P_{23}^2 + R'_{33} \tag{2.7.39}$$

In the case of $x_3 \triangleq b$ the Jacobian $f_{\hat{x}}$ is the same as Eq.(2.7.20). Since $P = 3 \times 3$ symmetric matrix, the sequential estimator equations become:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(y_1-\hat{x}_1) + 2P_{12}(y_2-\hat{x}_2) \tag{2.7.40}$$

$$\dot{\hat{x}}_2 = K \exp(-\hat{x}_3\hat{x}_1)\, \hat{x}_2^2 - g + 2P_{12}(y_1-\hat{x}_1) + 2P_{22}(y_2-\hat{x}_2) \tag{2.7.41}$$

$$\dot{\hat{x}}_3 = 2P_{13}(y_1-\hat{x}_1) + 2P_{23}(y_2-\hat{x}_2) \tag{2.7.42}$$

$$\dot{P}_{11} = -\ 2P_{11}^2 - 2P_{12}^2 + 2P_{12} + R'_{11} \tag{2.7.43}$$

$$\dot{P}_{12} = -\ 2P_{11}P_{12} - 2P_{12}P_{22} + P_{22} - K\hat{x}_2^2\hat{x}_3 \exp(-\hat{x}_3\hat{x}_1)\, P_{11} + 2K\hat{x}_2 \exp(-\hat{x}_3\hat{x}_1)\, P_{12} - K\hat{x}_1\hat{x}_2^2 \exp(-\hat{x}_3\hat{x}_1)\, P_{13} + R'_{12} \tag{2.7.44}$$

$$\dot{P}_{13} = -\ 2P_{11}P_{13} - 2P_{12}P_{23} + P_{23} + R'_{13} \tag{2.7.45}$$

$$\dot{P}_{22} = -\ 2P_{12}^2 - 2P_{22}^2 - 2K\hat{x}_2^2\hat{x}_3 \exp(-\hat{x}_3\hat{x}_1)\, P_{12} + 4K\hat{x}_2 \exp(-\hat{x}_3\hat{x}_1)\, P_{22} - 2K\hat{x}_1\hat{x}_2^2 \exp(-\hat{x}_3\hat{x}_1)\, P_{23} + R'_{22} \tag{2.7.46}$$

$$\dot{P}_{23} = -\ 2P_{12}P_{13} - 2P_{22}P_{23} - K\hat{x}_2^2\hat{x}_3 \exp(-\hat{x}_3\hat{x}_1)\, P_{13} + 2K\hat{x}_2 \exp(-\hat{x}_3\hat{x}_1)\, P_{23} - K\hat{x}_1\hat{x}_2^2 \exp(-\hat{x}_3\hat{x}_1)\, P_{33} + R'_{23} \tag{2.7.47}$$

$$\dot{P}_{33} = -\ 2P_{13}^2 + 2P_{23}^2 + R'_{33} \tag{2.7.48}$$

As one can see the only difference in the $\dot{P}$ equations for the specified one- and two-dimensional observation vectors comes from the term $2PHQ\overline{H}P$ of Eq.(2.5.7). This means

$$(2PHQ\overline{H}P)_{\substack{\text{in the case of}\\ \text{two dimensional}\\ \text{obs. vector}}} = (2PHQ\overline{H}P)_{\substack{\text{in the case of}\\ \text{one dimensional}\\ \text{obs. vector}}} + \begin{pmatrix} P_{12}^2 & P_{12}P_{22} & P_{12}P_{23} \\ \cdot/\cdot & P_{22}^2 & P_{22}P_{23} \\ \cdot/\cdot & \cdot/\cdot & P_{23}^2 \end{pmatrix}$$

The difference in the $\dot{\hat{x}}$ equations in the two cases is

$$\frac{d}{dT}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{pmatrix}_{\substack{\text{in the case of}\\ \text{two dimensional}\\ \text{obs. vector}}} = \frac{d}{dT}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\\ \hat{x}_3\end{pmatrix}_{\substack{\text{in the case of}\\ \text{one dimensional}\\ \text{obs. vector}}} + \begin{pmatrix}P_{12}\\ P_{22}\\ P_{23}\end{pmatrix}(y_2-\hat{x}_2)$$

2.8. Sequential Estimation of State and Two Parameters in Vertical Descent.

The free fall trajectory is considered in this Section, too. This means that the basic dynamic equation we will work with is the same as Eq.(2.6.1). But we will regard both "K" and "b" parameters in Eq.(2.6.1) as imperfectly known parameters.

Using the same arguments as they were outlined in the first part of Section 2.7, we augment the state space of the problem by defining

$$x_3 \triangleq b \quad , \quad \dot{b} = 0$$

$$x_4 \triangleq K \quad , \quad \dot{K} = 0$$

Thus the system equations become:

$$\dot{x}_1 = x_2 \tag{2.8.1}$$

$$\dot{x}_2 = x_4 \exp(-\, x_3x_1)\, x_2^2 - g + u(t) \tag{2.8.2}$$

$$\dot{x}_3 = 0 \tag{2.8.3}$$

$$\dot{x}_4 = 0 \tag{2.8.4}$$

A. Given Noisy Position Measurements Only.

Since the observation vector is one-dimensional (see Eq.(2.7.8)) we have for the H and $HQ\overline{H}$ matrices:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad HQ\overline{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For the Jacobian $f_{\hat{x}}$ we obtain from Eqs.(2.8.1-4):

$$f_{\hat{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\hat{x}_2^2\hat{x}_3\hat{x}_4 \exp(-\hat{x}_1\hat{x}_3) , & 2\hat{x}_2\hat{x}_4\exp(-\hat{x}_1\hat{x}_3) , & -\hat{x}_1\hat{x}_2^2\hat{x}_4\exp(-\hat{x}_1\hat{x}_3) , & \hat{x}_2^2\exp(\hat{x}_1\hat{x}_3) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.8.5}$$

Since $P = 4 \times 4$ symmetric matrix, the sequential estimator equations become:

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2\, P_{11}(y_1-\hat{x}_1) \tag{2.8.6}$$

$$\dot{\hat{x}}_2 = \hat{x}_4\hat{x}_2^2 e^{-\hat{x}_1\hat{x}_3} - g + 2P_{12}(y_1-\hat{x}_1) \tag{2.8.7}$$

$$\dot{\hat{x}}_3 = 2P_{13}(y_1-\hat{x}_1) \tag{2.8.8}$$

$$\dot{\hat{x}}_4 = 2P_{14}(y_1-\hat{x}_1) \tag{2.8.9}$$

$$\dot{P}_{11} = -\, 2P_{11}^2 + 2P_{12} + R'_{11} \tag{2.8.10}$$

$$\dot{P}_{12} = - 2P_{11}P_{12} + P_{22} + (- P_{11}\hat{x}_2^2\hat{x}_3\hat{x}_4 + 2P_{12}\hat{x}_2\hat{x}_4 - P_{13}\hat{x}_1\hat{x}_2^2\hat{x}_4 + P_{14}\hat{x}_2^2) \exp(-\hat{x}_1\hat{x}_3) + R'_{12} \quad (2.8.11)$$

$$\dot{P}_{13} = - 2P_{11}P_{13} + P_{23} + R'_{13} \quad (2.8.12)$$

$$\dot{P}_{14} = - 2P_{11}P_{14} + P_{24} + R'_{14} \quad (2.8.13)$$

$$\dot{P}_{22} = - 2P_{12}^2 + 2(- P_{12}\hat{x}_2^2\hat{x}_3\hat{x}_4 + 2P_{22}\hat{x}_2\hat{x}_4 - P_{23}\hat{x}_1\hat{x}_2^2\hat{x}_4 + P_{24}\hat{x}_2^2) \exp(- \hat{x}_1\hat{x}_3) + R'_{22} \quad (2.8.14)$$

$$\dot{P}_{23} = - 2P_{12}P_{13} + (- P_{13}\hat{x}_2^2\hat{x}_3\hat{x}_4 + 2P_{23}\hat{x}_2\hat{x}_4 - P_{33}\hat{x}_1\hat{x}_2^2\hat{x}_4 + P_{34}\hat{x}_2^2) \exp(- \hat{x}_1\hat{x}_3) + R'_{23} \quad (2.8.15)$$

$$\dot{P}_{24} = - 2P_{12}P_{14} + (- P_{14}\hat{x}_2^2\hat{x}_3\hat{x}_4 + 2P_{24}\hat{x}_2\hat{x}_4 - P_{34}\hat{x}_1\hat{x}_2^2\hat{x}_4 + P_{44}\hat{x}_2^2) \exp(- \hat{x}_1\hat{x}_3) + R'_{24} \quad (2.8.16)$$

$$\dot{P}_{33} = - 2P_{13}^2 + R'_{33} \quad (2.8.17)$$

$$\dot{P}_{34} = - 2P_{13}P_{14} + R'_{34} \quad (2.8.18)$$

$$\dot{P}_{44} = - 2P_{14}^2 + R'_{44} \quad (2.8.19)$$

B. Given noisy position and velocity measurements.

The observation vector in this case is a specified two-dimensional one (see Eq.(2.6.12)), and that gives:

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad HQ\overline{H} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Jacobian $f_{\hat{x}}$ is the same as Eq.(2.8.5), and, $P = 4 \times 4$ symmetric matrix.

The sequential estimator equations in this case would differ from those (Eqs.(2.8.6-19)) derived for the case of the specified one-dimensional observation vector by the following terms:

$$\dot{\hat{x}}_1 = \text{Eq.}(2.8.6) + 2P_{12}(y_2-\hat{x}_2) \tag{2.8.20}$$

$$\dot{\hat{x}}_2 = \text{Eq.}(2.8.7) + 2P_{22}(y_2-\hat{x}_2) \tag{2.8.21}$$

$$\dot{\hat{x}}_3 = \text{Eq.}(2.8.8) + 2P_{23}(y_2-\hat{x}_2) \tag{2.8.22}$$

$$\dot{\hat{x}}_4 = \text{Eq.}(2.8.9) + 2P_{24}(y_2-\hat{x}_2) \tag{2.8.23}$$

$$\dot{P}_{11} = \text{Eq.}(2.8.10) - 2P_{12}^2 \tag{2.8.24}$$

$$\dot{P}_{12} = \text{Eq.}(2.8.11) - 2P_{12}P_{22} \tag{2.8.25}$$

$$\dot{P}_{13} = \text{Eq.}(2.8.12) - 2P_{12}P_{23} \tag{2.8.26}$$

$$\dot{P}_{14} = \text{Eq.}(2.8.13) - 2P_{12}P_{24} \tag{2.8.27}$$

$$\dot{P}_{22} = \text{Eq.}(2.8.14) - 2P_{22}^2 \tag{2.8.28}$$

$$\dot{P}_{23} = \text{Eq.}(2.8.15) - 2P_{22}P_{23} \tag{2.8.29}$$

$$\dot{P}_{24} = \text{Eq.}(2.8.16) - 2P_{22}P_{24} \tag{2.8.30}$$

$$\dot{P}_{33} = \text{Eq.}(2.8.17) - 2P_{23}^2 \tag{2.8.31}$$

$$\dot{P}_{34} = \text{Eq.}(2.8.18) - 2P_{23}P_{24} \tag{2.8.32}$$

$$\dot{P}_{44} = \text{Eq.}(2.8.19) - 2P_{24}^2 \tag{2.8.33}$$

2.9. Sequential Estimation of State in Ballistic Descent (assuming perfectly known parameters).

Considering the free ballistic trajectory (= no thrusting) and omitting the differential equation describing the ground range "s" of the space vehicle (Eq.(2.2.11)), and, defining

$x_1 \triangleq h$ (altitude above surface)

$x_2 \triangleq \alpha$ (path angle)

$x_3 \triangleq v$ (velocity)

and using the same assumptions as we did in Section 2.2, we have the following differential equations (see also Eqs.(2.2.10), (2.2.12), (2.2.13)) for estimating the vehicle's state:

$$\dot{x}_1 = - x_2 \sin x_3 \tag{2.9.1}$$

$$\dot{x}_2 = \left(\frac{g}{x_2} - \frac{x_2}{R}\right) \cos x_3 \tag{2.9.2}$$

$$\dot{x}_3 = g \sin x_3 - K \exp(-bx_1)\, x_2^{\,2} + u(t) \qquad (2.9.3)$$

where $K \triangleq \frac{K'}{m}$; m = mass of the vehicle and K' is defined by Eq.(2.2.9)

$u(t)$ = random dynamic noise

g, b, R = as defined in Section 2.2. (Acceleration of gravity, inverse scale factor for atmospheric density, radius of the planet, respectively.)

A. Given Noise Position Measurements Only.

By "position" we mean in this case: altitude from the surface and path angle. Hence the observation vector becomes:

$$\begin{pmatrix} \text{altitude} \\ \\ \text{path angle} \end{pmatrix} = \begin{pmatrix} y_1(T) \\ \\ y_2(T) \end{pmatrix} = \begin{pmatrix} x_1(T) + \text{Observation Noise} \\ \\ x_2(T) + \text{Observation Noise} \end{pmatrix}$$

This gives for the H and $HQ\overline{H}$ matrices

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad HQ\overline{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (2.9.5)$$

The Jacobian $f_{\hat{x}}$ becomes (from Eqs.(2.9.1-3)):

$$f_{\hat{x}} = \begin{pmatrix} 0 & -\sin x_3 & -x_2 \cos x_3 \\ 0 & -\left(\frac{g}{x_2^{\,2}} + \frac{1}{R}\right)\cos x_3 & -\left(\frac{g}{x_2} - \frac{x_2}{R}\right)\sin x_3 \\ bKx_2^{\,2}\exp(-bx_1) & -2Kx_2\exp(-bx_1) & g \cos x_3 \end{pmatrix} \qquad (2.9.6)$$

For this problem we have a $P = 3 \times 3$ symmetric matrix. Thus the sequential estimator Eqs.(2.5.3), (2.5.7) in this case become:

$$\dot{\hat{x}}_1 = - \hat{x}_2 \sin \hat{x}_3 + 2\, P_{11}(y_1 - \hat{x}_1) + 2P_{12}(y_2 - \hat{x}_2) \tag{2.9.7}$$

$$\dot{\hat{x}}_2 = \frac{g}{\hat{x}_2} - \frac{\hat{x}_2}{R} \cos \hat{x}_3 + 2P_{12}(y_1 - \hat{x}_1) + 2P_{22}(y_2 - \hat{x}_2) \tag{2.9.8}$$

$$\dot{\hat{x}}_3 = g \sin \hat{x}_3 - K \exp(- b\hat{x}_1)\, \hat{x}_2^{\,2} + 2P_{13}(y_1 - \hat{x}_1) + 2P_{23}(y_2 - \hat{x}_2) \tag{2.9.9}$$

$$\dot{P}_{11} = - 2(P_{11}^2 + P_{12}^2) - 2(P_{12} \sin \hat{x}_3 + P_{13} \cos \hat{x}_3) + R_{11}^1 \tag{2.9.10}$$

$$\dot{P}_{12} = - 2(P_{11}P_{12} + P_{12}P_{22}) - P_{22} \sin \hat{x}_3 - P_{23}\hat{x}_2 \cos \hat{x}_3 - P_{12}\left(\frac{g}{\hat{x}_2^{\,2}} + \frac{1}{R}\right) \cdot \cos \hat{x}_3$$
$$- P_{13}\left(\frac{g}{\hat{x}_2} - \frac{\hat{x}_2}{R}\right) \sin \hat{x}_3 + R_{12}' \tag{2.9.11}$$

$$\dot{P}_{13} = - 2(P_{11}P_{13} + P_{12}P_{23}) - P_{23} \sin \hat{x}_3 - P_{33}\hat{x}_2 \cos \hat{x}_3 + P_{11} bK\hat{x}_2^2 \exp(- b\hat{x}_1)$$
$$- 2P_{12}K\hat{x}_2 \exp(- b\hat{x}_1) + P_{13} g \cos \hat{x}_3 + R_{13}' \tag{2.9.12}$$

$$\dot{P}_{22} = - 2(P_{12}^2 + P_{22}^2) - 2P_{22}\left(\frac{g}{\hat{x}_2^2} + \frac{1}{R}\right) \cos \hat{x}_3 - 2P_{23}\left(\frac{g}{\hat{x}_2} - \frac{\hat{x}_2}{R}\right) \sin \hat{x}_3 + R_{22}' \tag{2.9.13}$$

$$\dot{P}_{23} = - 2(P_{12}P_{13} + P_{22}P_{23}) - P_{23}\left(\frac{g}{\hat{x}_2^2} + \frac{1}{R}\right) \cos \hat{x}_3 - P_{23}\left(\frac{g}{\hat{x}_2} - \frac{\hat{x}_2}{R}\right) \sin \hat{x}_3$$
$$+ P_{12}\, bK\hat{x}_2^2 \exp(- b\hat{x}_1) - 2P_{22}K\hat{x}_2 \exp(- b\hat{x}_1) + P_{23} g \cos \hat{x}_3 + R_{23}' \tag{2.9.14}$$

$$\dot{P}_{33} = - 2(P_{13}^2 + P_{23}^2) + 2P_{13} bK\hat{x}_2^2 \exp(- b\hat{x}_1) - 4P_{23}K\hat{x}_2 \exp(- b\hat{x}_1) + 2P_{33} \cos \hat{x}_3 + R_{33}' \tag{2.9.15}$$

where R_{ij}' are elements of the R^{-1} matrix.

B. Given noisy position and velocity measurements.

In this case we have the following observation vector:

$$\begin{pmatrix} \text{altitude} \\ \text{path angle} \\ \text{velocity} \end{pmatrix} = \begin{pmatrix} y_1(T) \\ y_2(T) \\ y_3(T) \end{pmatrix} = \begin{pmatrix} x_1(T) + \text{Observation Noise} \\ x_2(T) + \text{Observation Noise} \\ x_3(T) + \text{Observation Noise} \end{pmatrix} \tag{2.9.16}$$

which gives for the H and $HQ\bar{H}$ matrices

$$H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad HQ\bar{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.9.17}$$

The Jacobian $f_{\hat{x}}$ is the same as in Eq.(2.9.6), and, $P = 3 \times 3$ symmetric matrix.

The sequential estimator equations in this case will be different from those Eqs.(2.9.7-15)) derived for the case of the specified two-dimensional observation vector by the following terms:

$$\dot{\hat{x}}_1 = \text{Eq.}(2.9.7) + 2P_{13}(y_3 - \hat{x}_3) \tag{2.9.18}$$

$$\dot{\hat{x}}_2 = \text{Eq.}(2.9.8) + 2P_{23}(y_3 - \hat{x}_3) \tag{2.9.19}$$

$$\hat{x}_3 = \text{Eq.}(2.9.9) + 2P_{33}(y_3 - \hat{x}_3) \tag{2.9.20}$$

$$\dot{P}_{11} = \text{Eq.}(2.9.10) - 2P_{13}^2 \tag{2.9.21}$$

$$\dot{P}_{12} = \text{Eq.}(2.9.11) - 2P_{13}P_{23} \tag{2.9.22}$$

$$\dot{P}_{13} = \text{Eq.}(2.9.12) - 2P_{13}P_{33} \qquad (2.9.23)$$

$$\dot{P}_{22} = \text{Eq.}(2.9.13) - 2P_{23}^2 \qquad (2.9.24)$$

$$\dot{P}_{23} = \text{Eq.}(2.9.14) - 2P_{23}P_{33} \qquad (2.9.25)$$

$$\dot{P}_{33} = \text{Eq.}(2.9.15) - 2P_{33} \qquad (2.9.26)$$

2.10. Numerical Results in Sequential Estimation of State and Parameters in Atmospheric Descent.

In order to be able to investigate the feasibility and convergence properties of the proposed non-linear filter equations discussed and derived in the previous Sections (Sections 2.5-9) we have made some numerical experiments (called "digital simulating") on the computer.

A. Digital simulating of the non-linear filter.

The dynamic noise (DN) and observation noise (ON) in the process of digital simulating have been modelled according to the following expressions:

For DN: $u_1 = c_1\xi_1(t)$ (2.10.1)

For ON: $u_2 = c_2\xi_2(t)$ (position measurement) (2.10.2)

$u_3 = c_3\xi_3(t)$ (velocity measurement) (2.10.3)

where $\xi_1(t), \xi_2(t), \xi_3(t)$ are, for each "t", statistically independent random variables, uniformly distributed between [+1, -1], and,

c_1, c_2, c_3 are constants, adjusted to the relative magnitude of the dynamic and observation noises, respectively.

This type of noise-modelling is in complete accordance with the general assumptions which were made in deriving the non-linear filter equations, and, reflects the fact that we have no information on the statistics of the acting dynamic and observation noises whatever.

Digital simulating of the non-linear filter (or sequential estimator) equations contains the following phases:

(a) We generate the system trajectories (for the dynamically perturbed systems, using Eq.(2.10.1) for the dynamic noise) by solving the relevant system equations for given initial conditions.

(b) We generate the noisy observations $y_i(t)$, which means: we corrupt the output data from Phase (a) with observation noise given by Eqs.(2.10.2-3).

(c) We use the generated $y_i(t)$ as input to the relevant sequential estimator equations which then are solved for assumed initial values for $\hat{x}_i$ and P_{ij}.

The procedure of digital simulating is schematically depicted on Fig. 2.10.1.

In the case of vertical descent the true system trajectories were generated for the following parameter values and initial conditions:

$$K \triangleq \frac{K'}{m} = 0.5\ 10^{-5}\ [m^{-1}]$$

$$b = 1.0\ 10^{-5}\ [m^{-1}]$$

$$g = 5.0\ [m/sec^2]$$

$$x_2(0) = 7.0\ 10^2\ [m/sec]$$

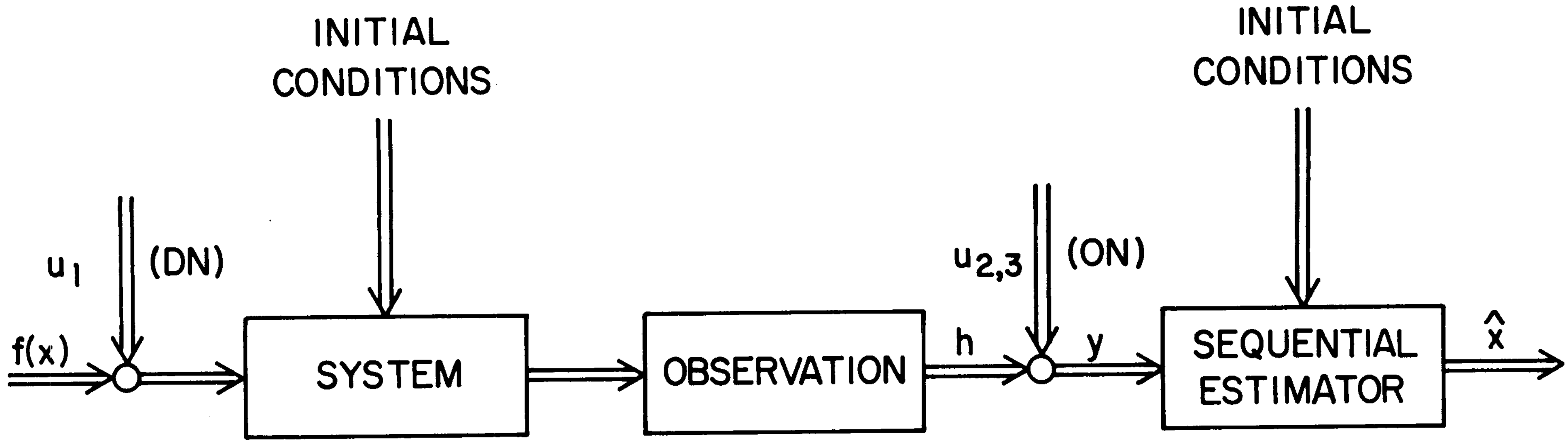

INITIAL
CONDITIONS
INITIAL
CONDITIONS
u1
(DN)
f(x)
SYSTEM
OBSERVATION
h
u2,3
(ON)
y
SEQUENTIAL
ESTIMATOR
x̂

Fig. 2.10.1

The selected values for "K" and "b" represent relatively thin atmospheric conditions. (These parameter values give about 0.2g deceleration due to the drag at the selected initial values on x_1 and x_2.) In the calculations we have assumed, furthermore, that "K" is constant on a limited part of the trajectory.

In generating noise we have assumed 4-8% dynamic noise, and, 0.5-1.0% observation noise in position measurements, and, 1.0-2.0% observation noise in velocity measurements. These %-s roughly give for the adjustable constants c_i in Eqs.(2.10.1-3):

$$c_1 = 0.12 - 0.24 \; [\mathrm{m/sec}^2]$$

$$c_2 = 200.0 - 400.0 \; [\mathrm{m}]$$

$$c_3 = 6.0 - 12.0 \; [\mathrm{m/sec}]$$

As one can see the maximum value of the dynamic noise (c_1) corresponds roughly to 10-20% of the deceleration due to the assumed value of the drag force.

In solving the sequential estimator equations we used ± 5-7% wrong estimates for the initial values of the state variables, and, ± 50% wrong estimates for the "initial values" of the parameters. These values roughly correspond to

$$0.93 \; 10^5 \le \hat{x}_1(0) \le 1.07 \; 10^5 \; [\mathrm{m}]$$

$$6.5 \; 10^2 \le \hat{x}_2(0) \le 7.5 \; 10^2 \; [\mathrm{m/sec}]$$

$$0.25 \; 10^{-5} \le \hat{K}(0) \le 0.75 \; 10^{-5} \; [\mathrm{m}^{-1}]$$

$$0.5\ 10^{-5} \le \hat{b}(0) \le 1.5\ 10^{-5}\ [m^{-1}]$$

(The assumed wrong initial estimates for "K" and "b" in the dynamic equations affect only the deceleration due to the drag. In view of Eq.(2.7.2) this means that we have $\dot{\hat{x}}_2(0) = 0.2g - g \pm 0.1g \pm 0.04g = -0.8g \pm 0.14g \rightarrow 20\%$ uncertainty in the acceleration at $t = 0$. And, at the same time, we also have the 5-7% uncertainties in the initial estimates of the velocity and the position.)

In solving the gain equations (the "P" equations derived from Eq.(2.5.7)):

(a) we have put the off-diagonal elements of the R^{-1} matrix equal to zero and just tried to select proper diagonal elements for R^{-1};

(b) we have always used zero as initial value for the off-diagonal P-equations and tried to select proper initial values for the diagonal P-equations only.

The "proper" R'_{ii} and $P_{ii}(0)$ values were selected by trial-and-error technique.

In selecting "proper" R'_{ii} and $P_{ii}(0)$ values we have used the following fairly obvious simultaneous criteria: a good sequential estimation must

(a) <u>converge fast</u> to the true trajectory;

(b) <u>have a stable (smooth) behaviour</u> along the true trajectory;

(c) <u>be insensitive</u> for a given class of "wrong" initial estimates;

(d) such that (a), (b), (c) be met <u>for all estimated variables</u>.

The digital simulating was carried out on the IBM 7094 computer using two CIT subroutines: NRAND for generating random numbers, and, DEQ (fixed mode version) for integrating the differential equations. For integration step we always used $\Delta T = 0.01$ sec. The total integration ("estimation") time was 20 sec in each case.

B. Discussion of results.

Regarding the "digital experiments" we have run we can make the following general remarks.

(a) The gain equations (the "P" equations) settle down on some asymptotic $P_{ij}(\infty)$ values in each case. This usually occurred after 10-15 sec estimation time.

(b) The dominating factors in achieving "good estimates" in the above-specified sense are the selected (constant) values of the R'_{ii} terms. (In this connection it is interesting to note that by choosing $R^{-1} \equiv 0$ some estimated trajectories did not converge to the true trajectories at all, but they run parallel to them in a distance determined by the wrong initial estimates.)

(c) Both the order of magnitude of the R'_{ii} terms and the ratios R'_{ii}/R'_{jj} are important factors in achieving "good estimates".

(d) Since the P_{ii} equations represent (second order) approximations to the optimal value of the "cost functional", one has to select the initial values for the P_{ii} equations in the neighborhood of the optimal solution. (Or equivalently: in the region of convergence.) But variations in the $P_{ii}(0)$ values in the region of convergence do not markably affect the "goodness" of the sequential estimation.

Some representative results of digital simulating we so far have obtained are shown in Figs. 2.10.2-11. Among those Figures we especially call attention to Figs. 2.10.8-8.a which display trajectory-characteristics markedly different from those depicted in Figs. 2.10.2-7. Marked differences are manifested not only in the transient part of the estimated trajectories but also in the behaviour of the asymptotic part (in the smoothness) of the trajectories. These marked differences have their source in reversing the order of magnitude of the R'_{ii} terms. Observing that fact we used it as a guide in selecting "proper" R'_{ii} values in our calculations.

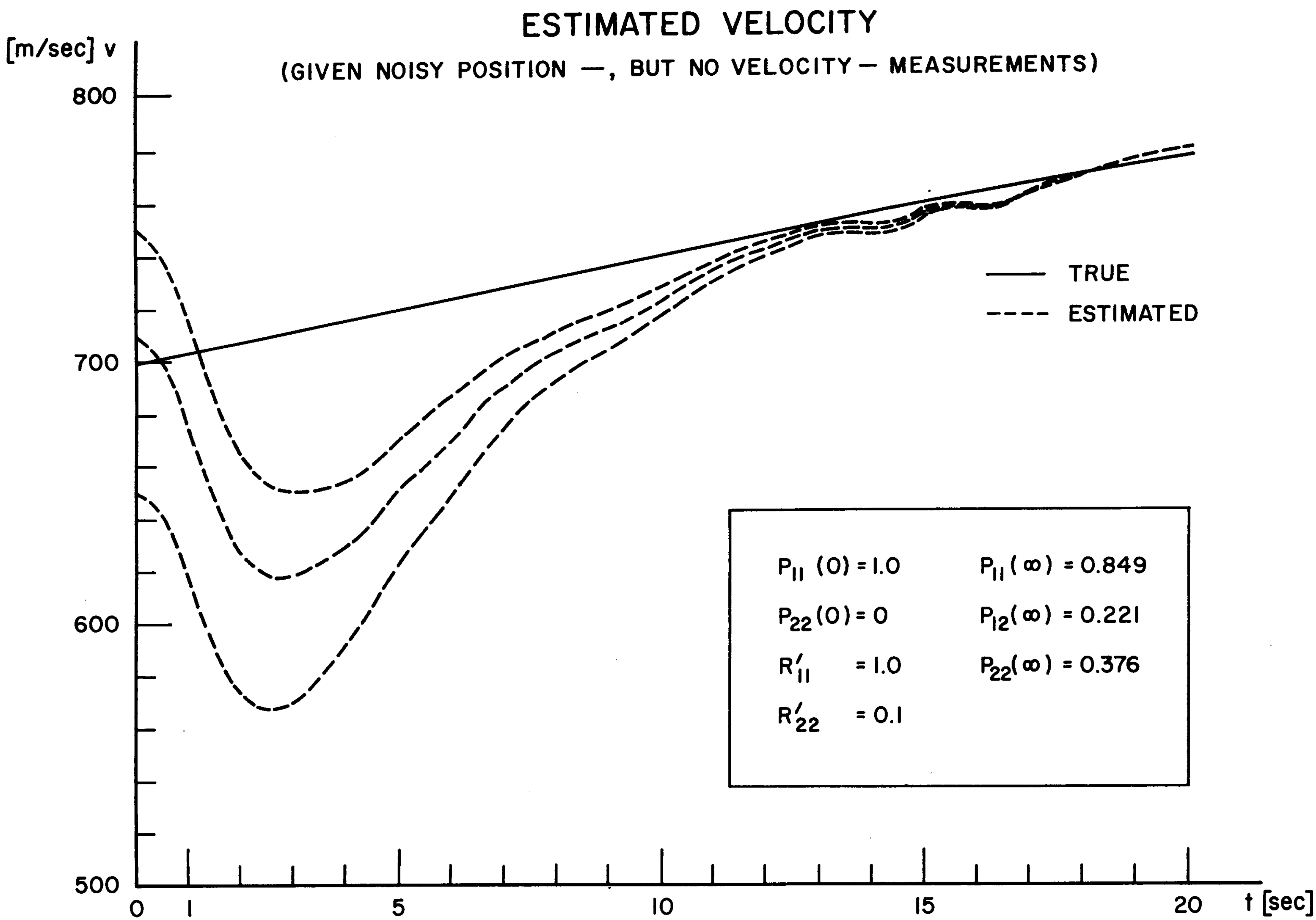

ESTIMATED VELOCITY
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m/sec] v
800
700
600
500
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
$P_{11}(0) = 1.0$
$P_{22}(0) = 0$
$R'_{11} = 1.0$
$R'_{22} = 0.1$
$P_{11}(\infty) = 0.849$
$P_{12}(\infty) = 0.221$
$P_{22}(\infty) = 0.376$

Fig. 2.10.2

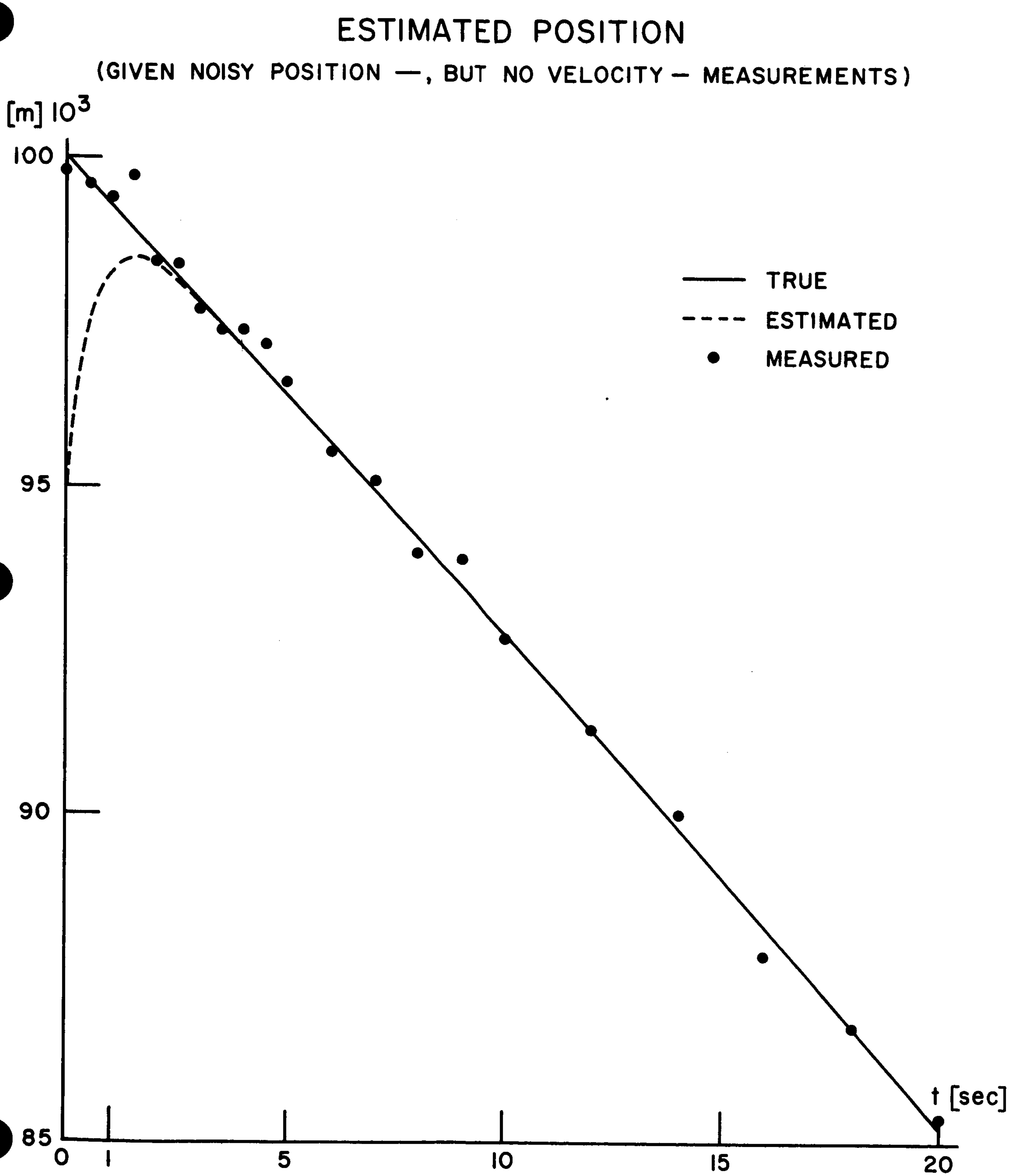

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10^3
100
95
90
85
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED

Fig. 2.10.2.a

ESTIMATED VELOCITY

(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)

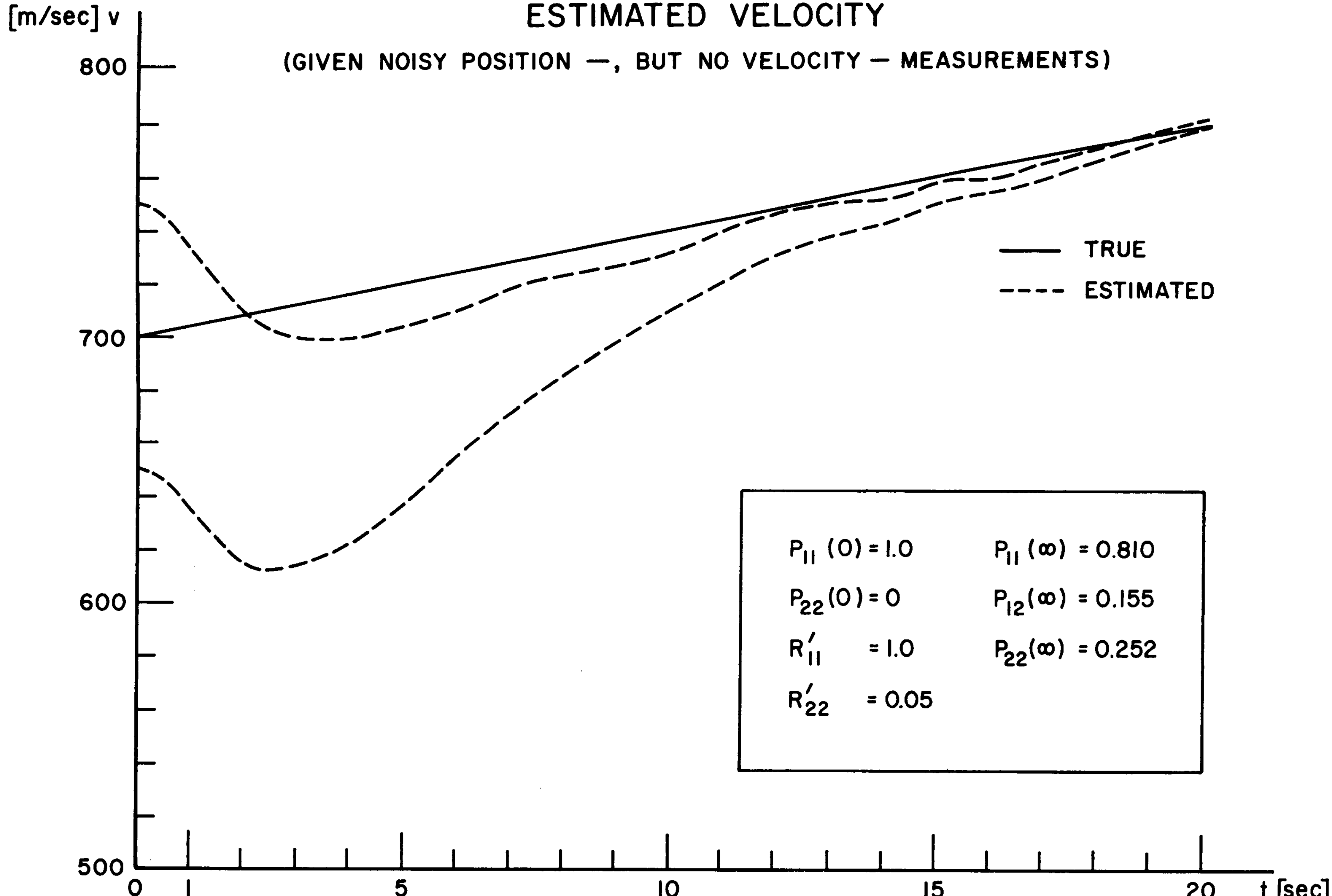


Fig. 2.10.3

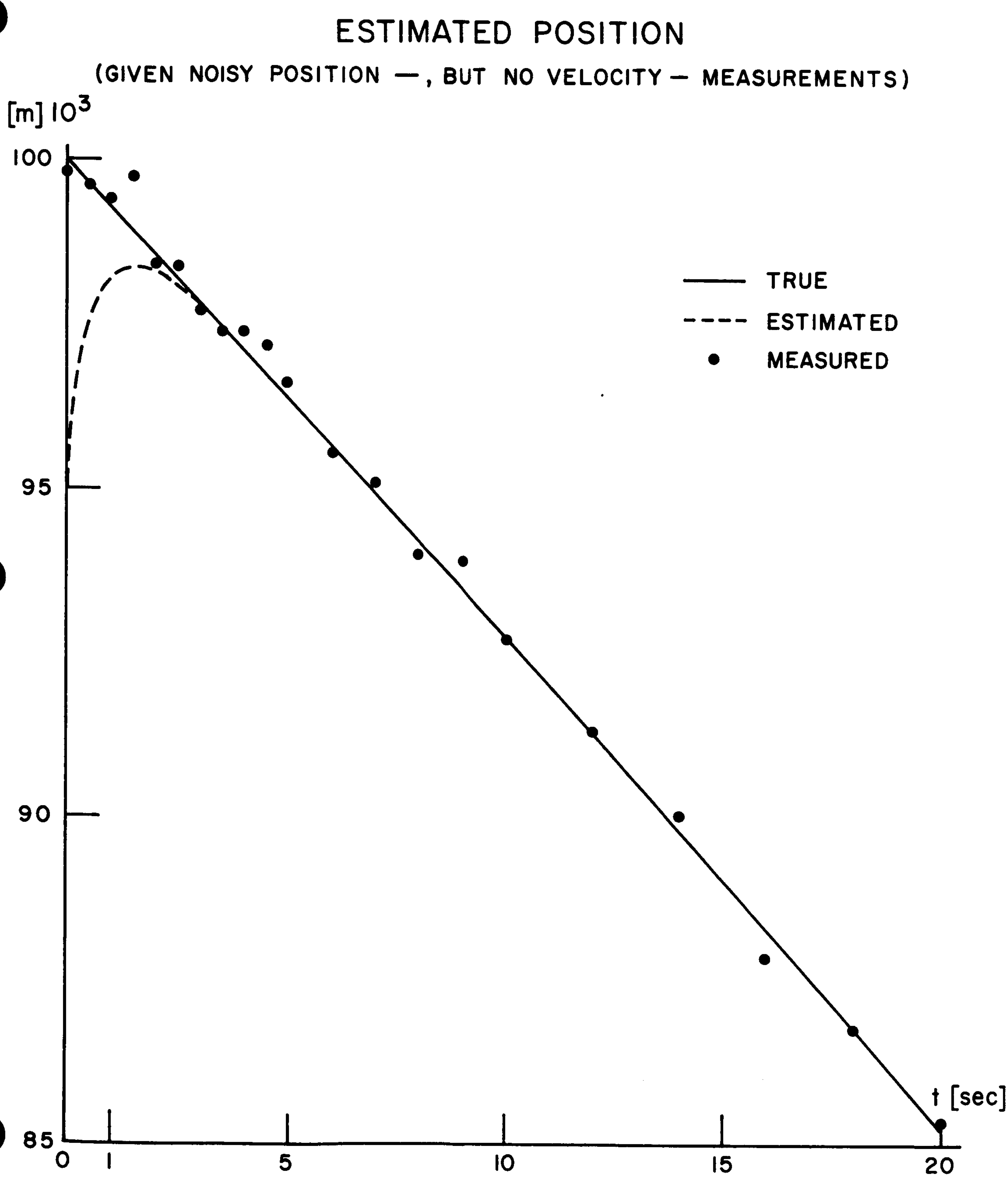
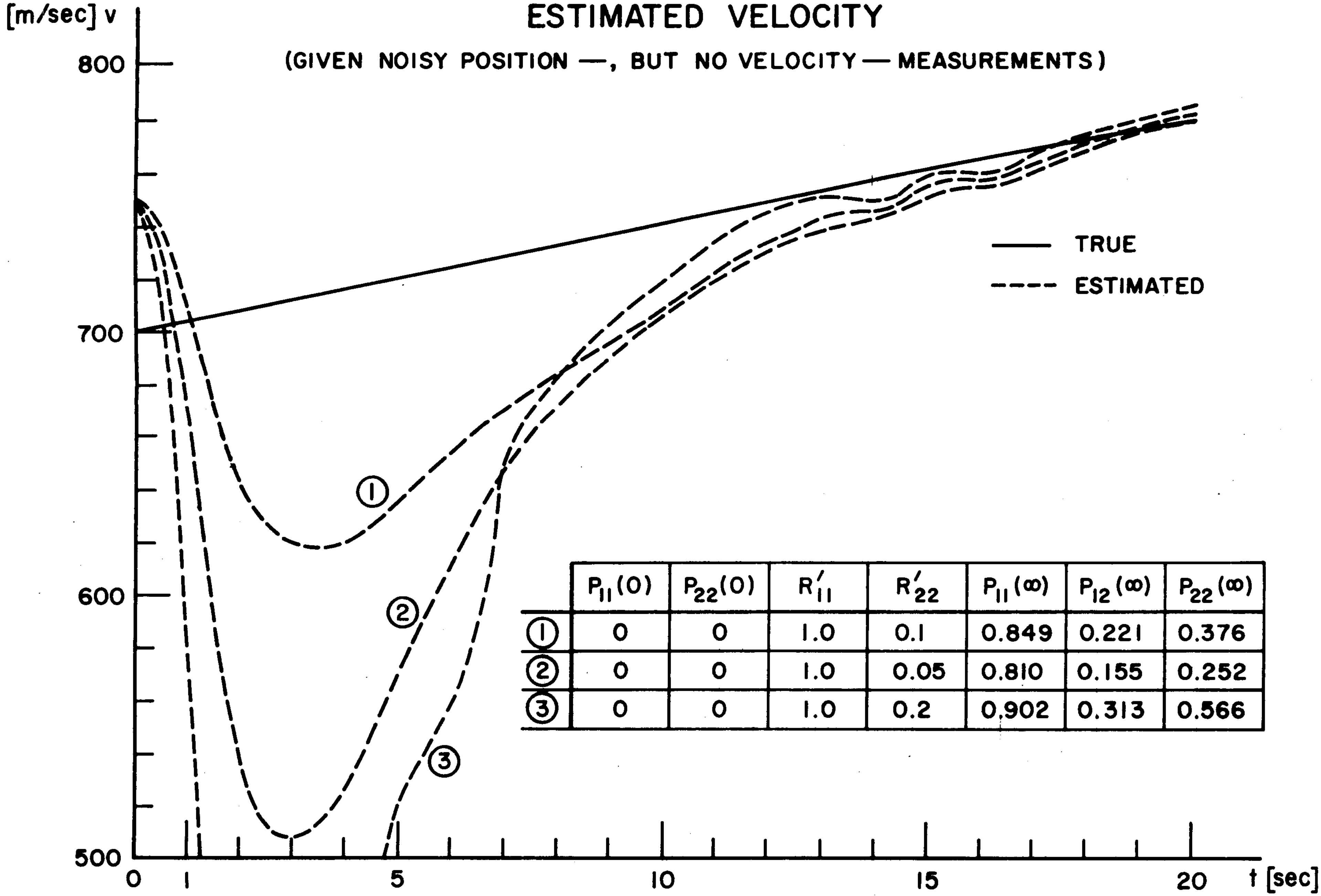

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10^3
100
95
90
85
TRUE
ESTIMATED
MEASURED
t [sec]
0
1
5
10
15
20

Fig. 2.10.3.a

ESTIMATED VELOCITY

(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)



	$P_{11}(0)$	$P_{22}(0)$	R'_{11}	R'_{22}	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	0	0	1.0	0.1	0.849	0.221	0.376
②	0	0	1.0	0.05	0.810	0.155	0.252
③	0	0	1.0	0.2	0.902	0.313	0.566

Fig. 2.10.4

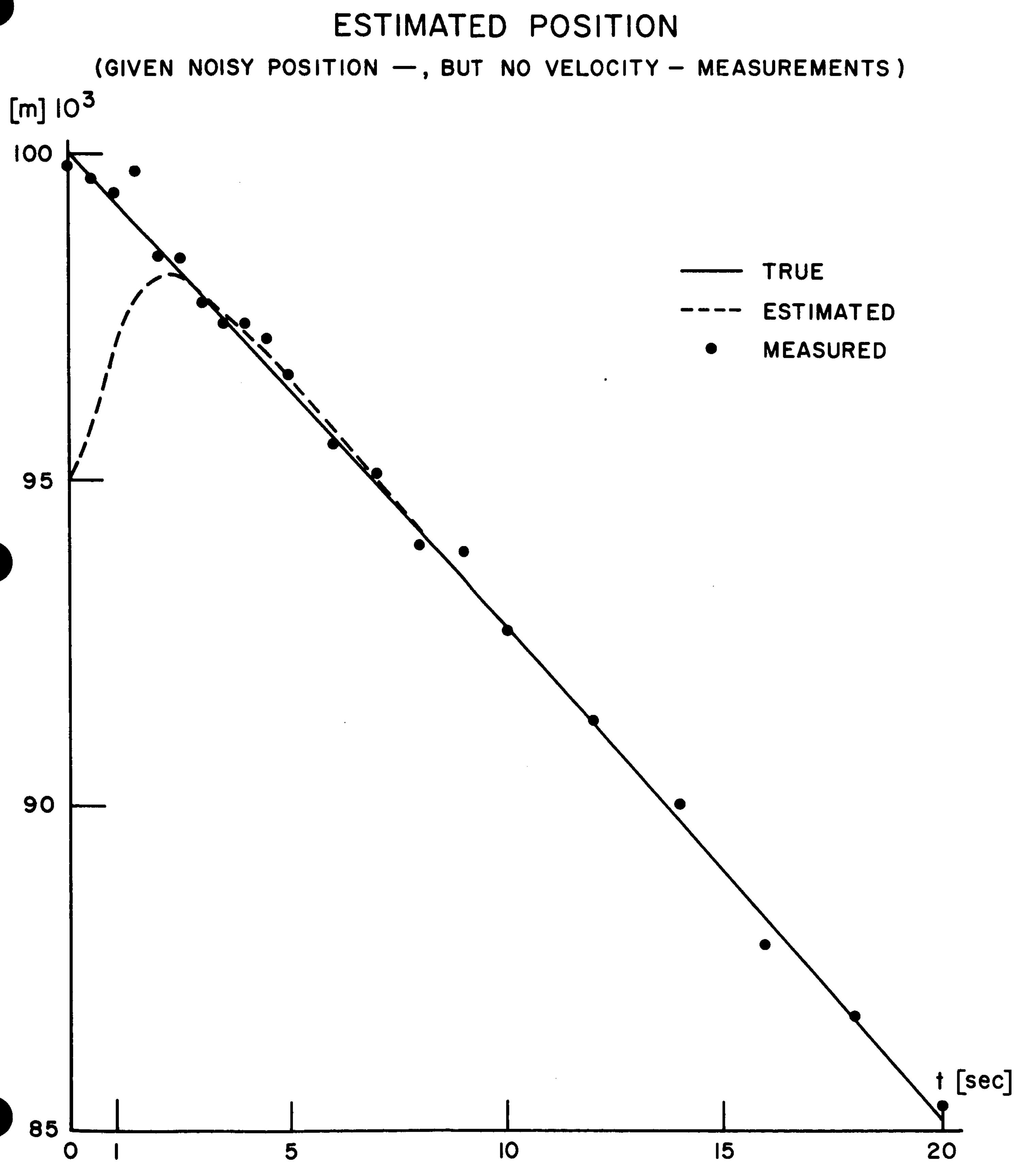
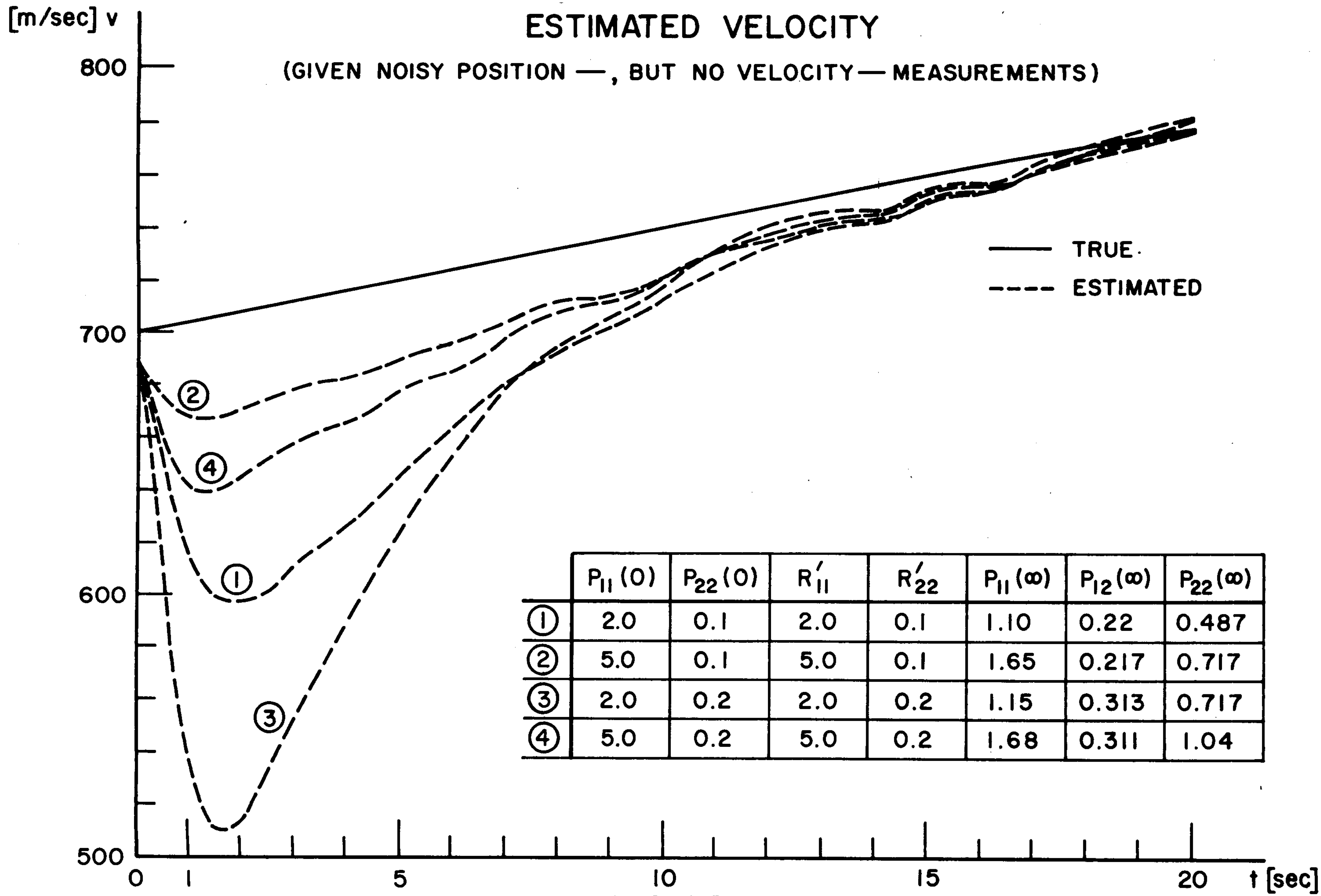

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY – MEASUREMENTS)
[m] 10³
100
95
90
85
TRUE
ESTIMATED
MEASURED
t [sec]
0
1
5
10
15
20

Fig. 2.10.4.a



	$P_{11}(0)$	$P_{22}(0)$	R'_{11}	R'_{22}	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	2.0	0.1	2.0	0.1	1.10	0.22	0.487
②	5.0	0.1	5.0	0.1	1.65	0.217	0.717
③	2.0	0.2	2.0	0.2	1.15	0.313	0.717
④	5.0	0.2	5.0	0.2	1.68	0.311	1.04

Fig. 2.10.5

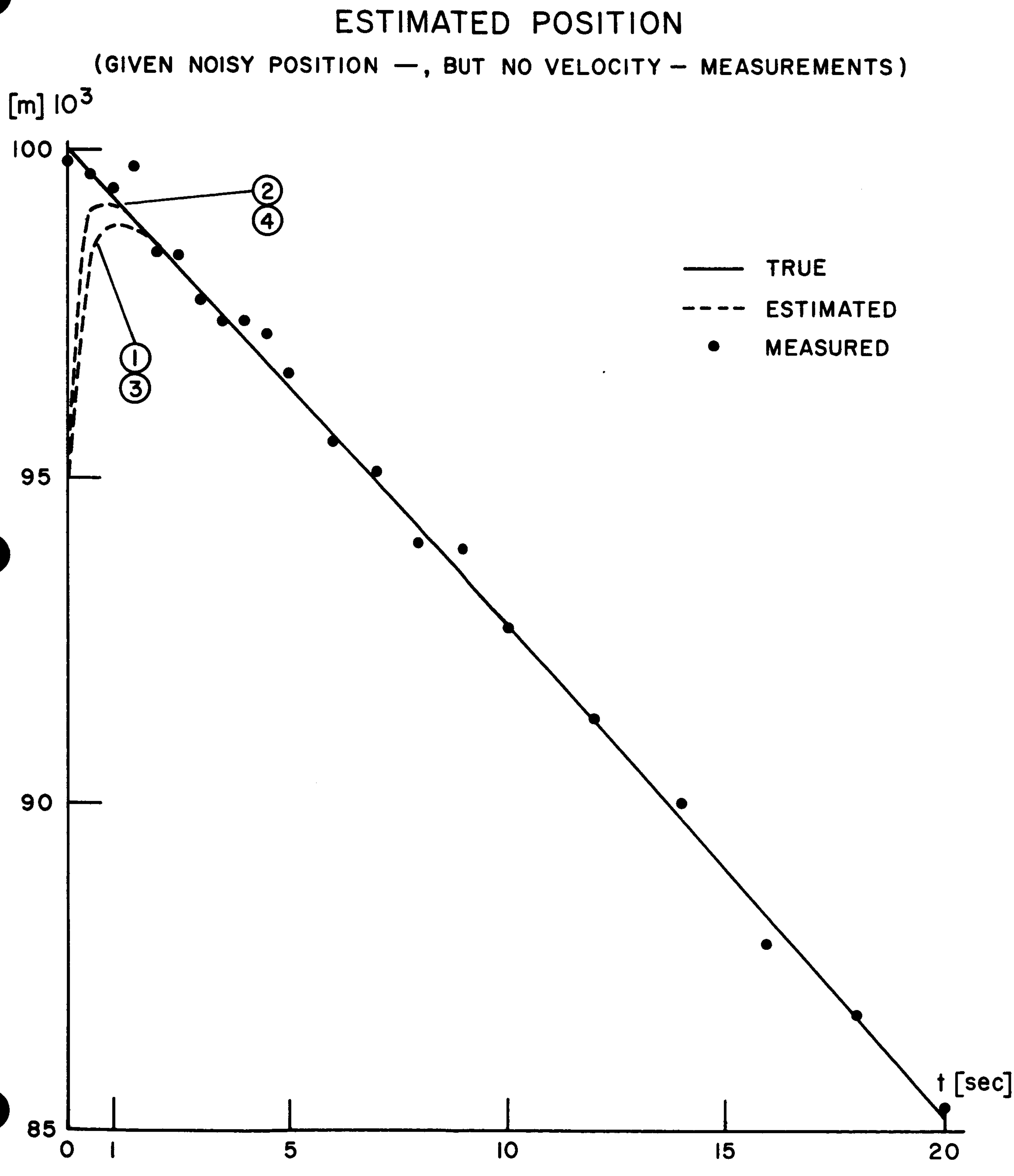

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10^3
100
95
90
85
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED
1
2
3
4

Fig. 2.10.5.a

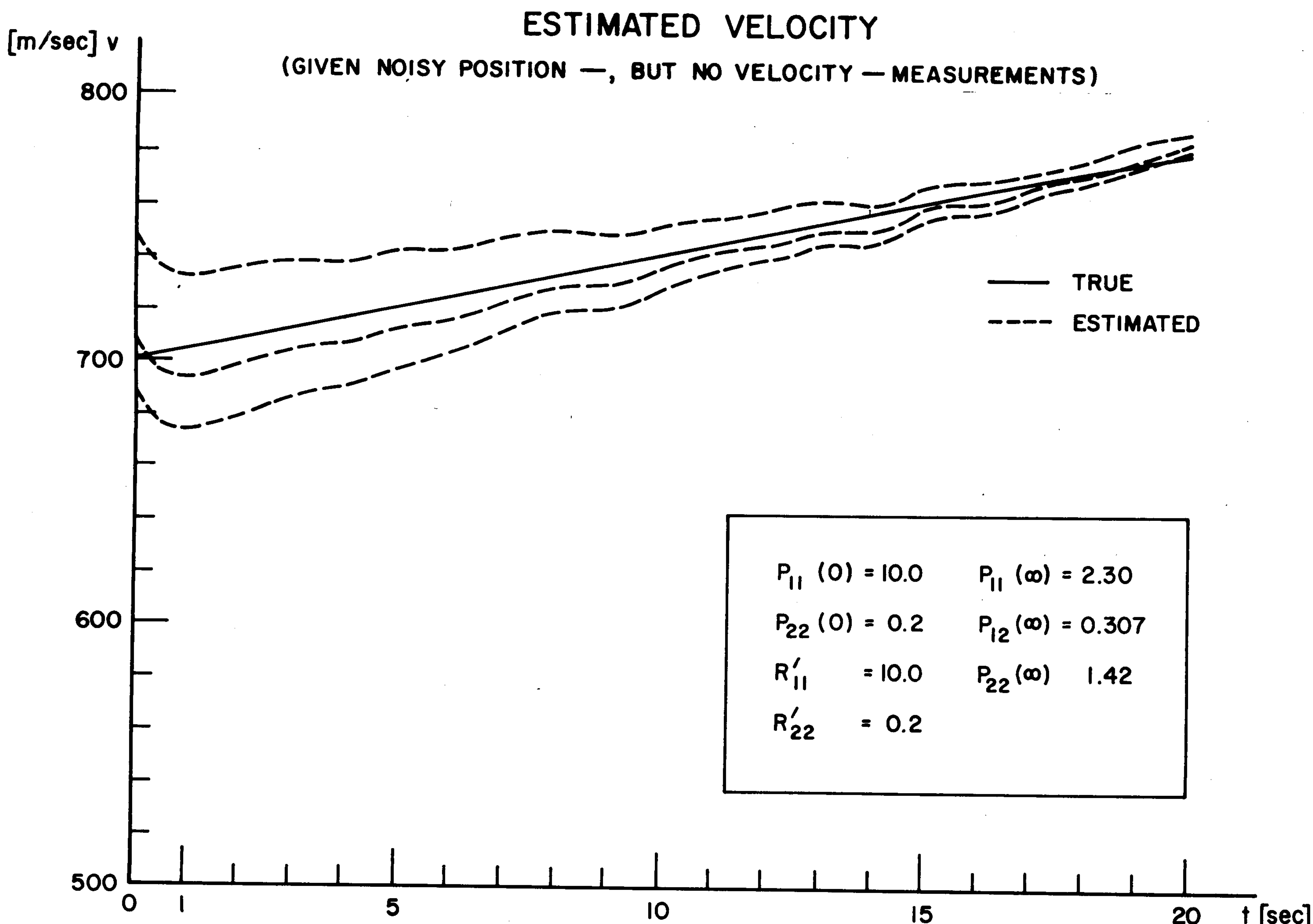

ESTIMATED VELOCITY
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m/sec] v
800
700
600
500
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
P11 (0) = 10.0
P22 (0) = 0.2
R'11 = 10.0
R'22 = 0.2
P11 (∞) = 2.30
P12 (∞) = 0.307
P22 (∞) 1.42

Fig. 2.10.6

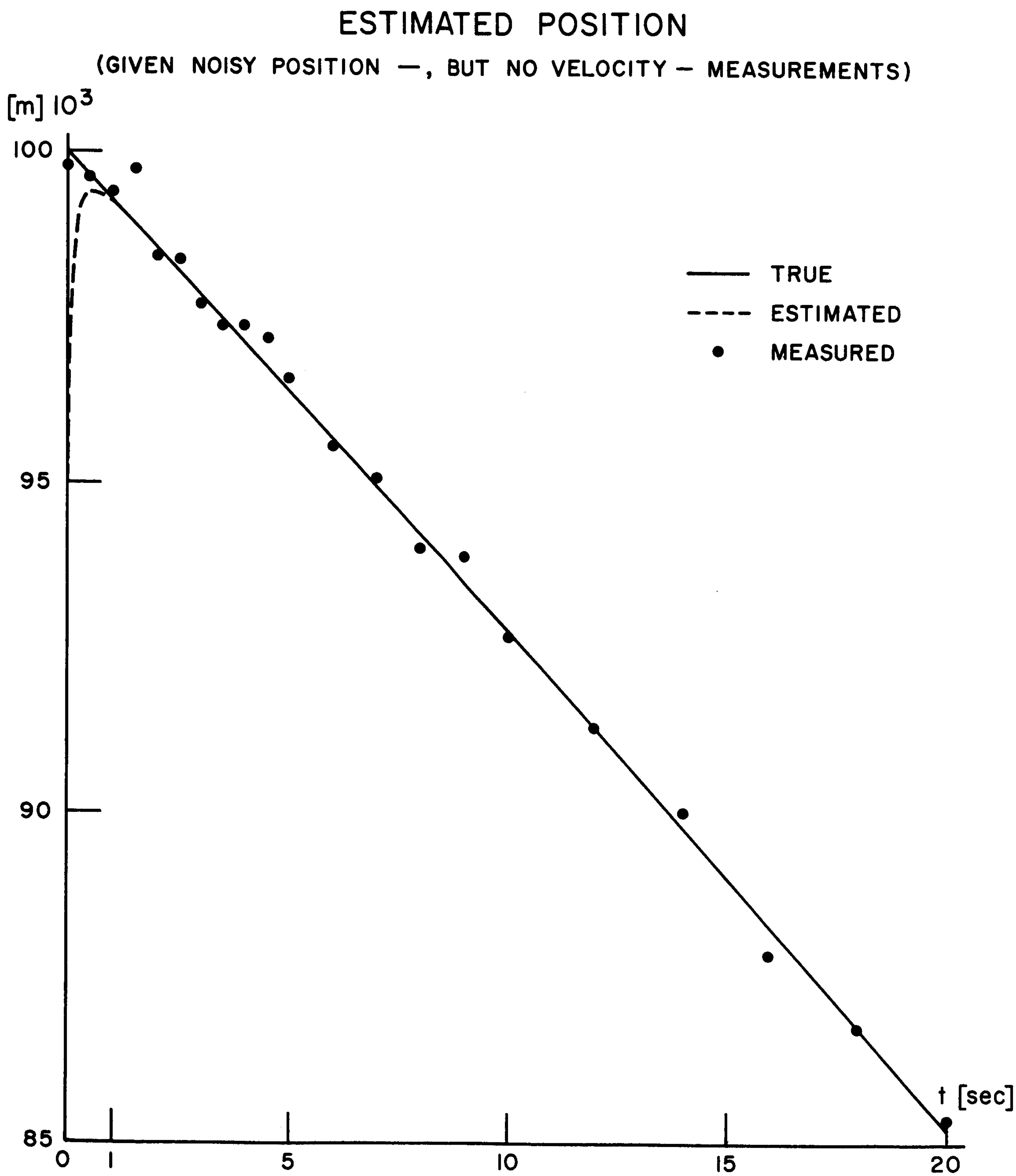
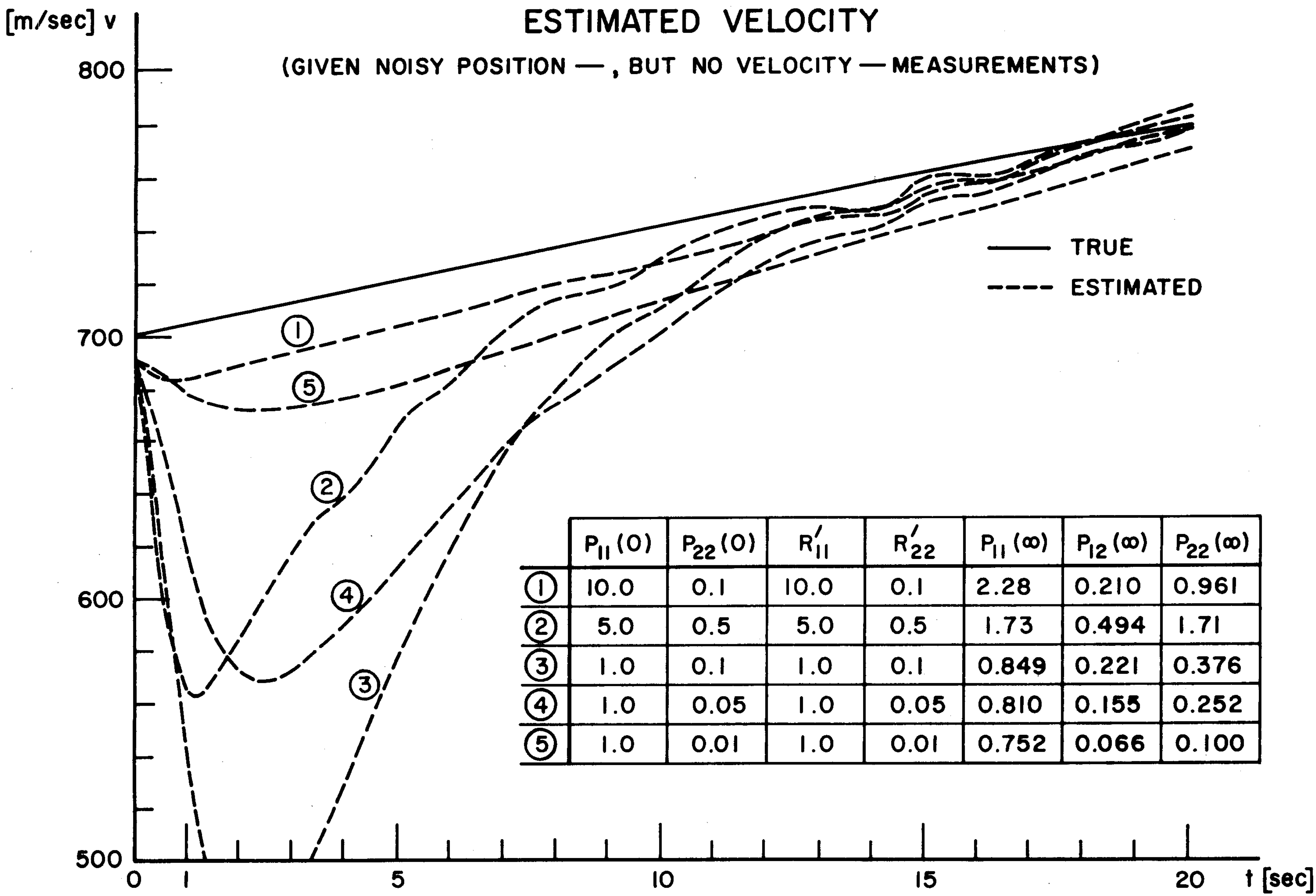

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10³
100
95
90
85
TRUE
ESTIMATED
MEASURED
t [sec]
0
1
5
10
15
20

Fig. 2.10.6.a



	$P_{11}(0)$	$P_{22}(0)$	R'_{11}	R'_{22}	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	10.0	0.1	10.0	0.1	2.28	0.210	0.961
②	5.0	0.5	5.0	0.5	1.73	0.494	1.71
③	1.0	0.1	1.0	0.1	0.849	0.221	0.376
④	1.0	0.05	1.0	0.05	0.810	0.155	0.252
⑤	1.0	0.01	1.0	0.01	0.752	0.066	0.100

Fig. 2.10.7

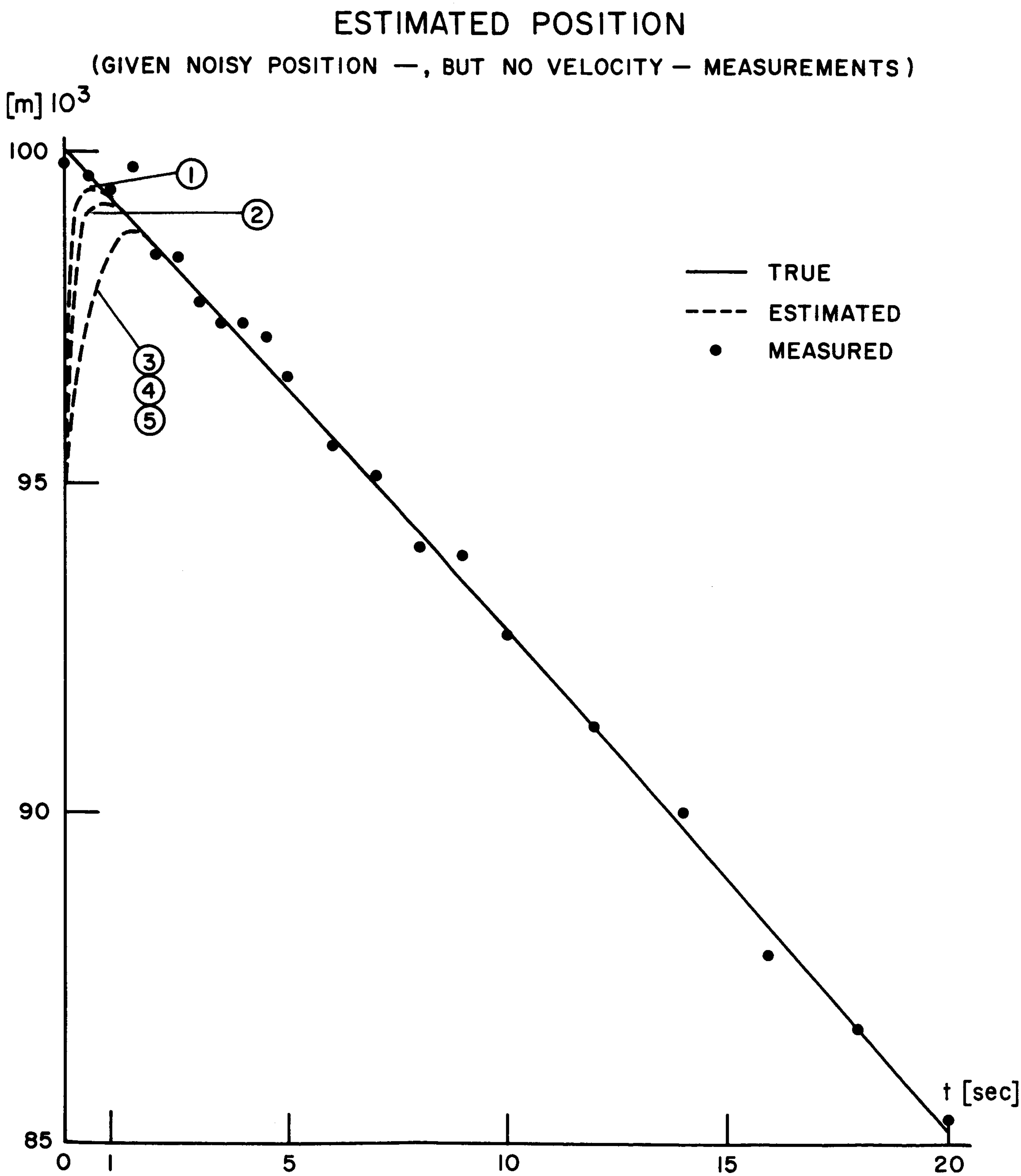
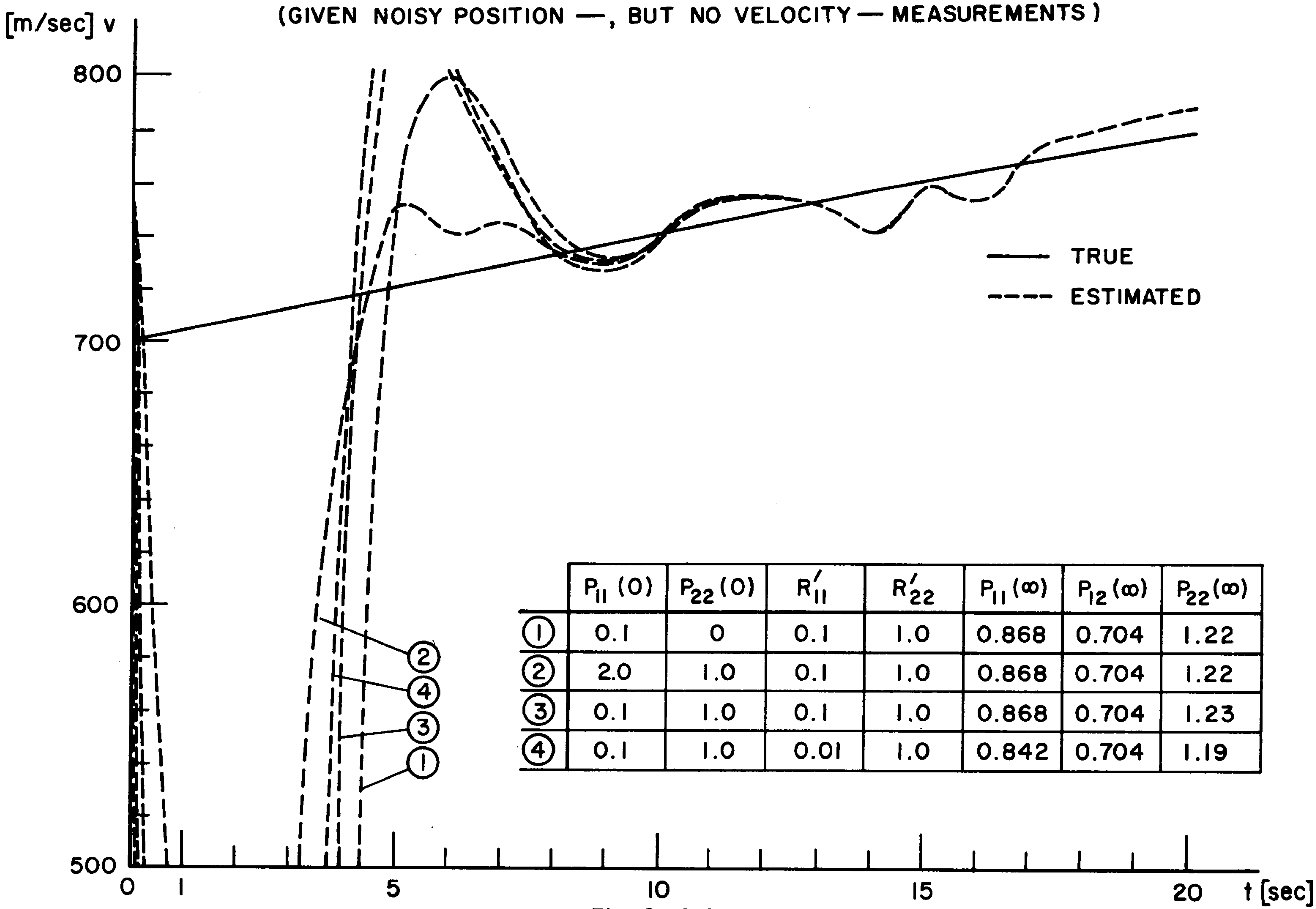

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10³
100
95
90
85
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED
1
2
3
4
5

Fig. 2.10.7.a

ESTIMATED VELOCITY

(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)



	$P_{11}(0)$	$P_{22}(0)$	R'_{11}	R'_{22}	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	0.1	0	0.1	1.0	0.868	0.704	1.22
②	2.0	1.0	0.1	1.0	0.868	0.704	1.22
③	0.1	1.0	0.1	1.0	0.868	0.704	1.23
④	0.1	1.0	0.01	1.0	0.842	0.704	1.19

Fig. 2.10.8

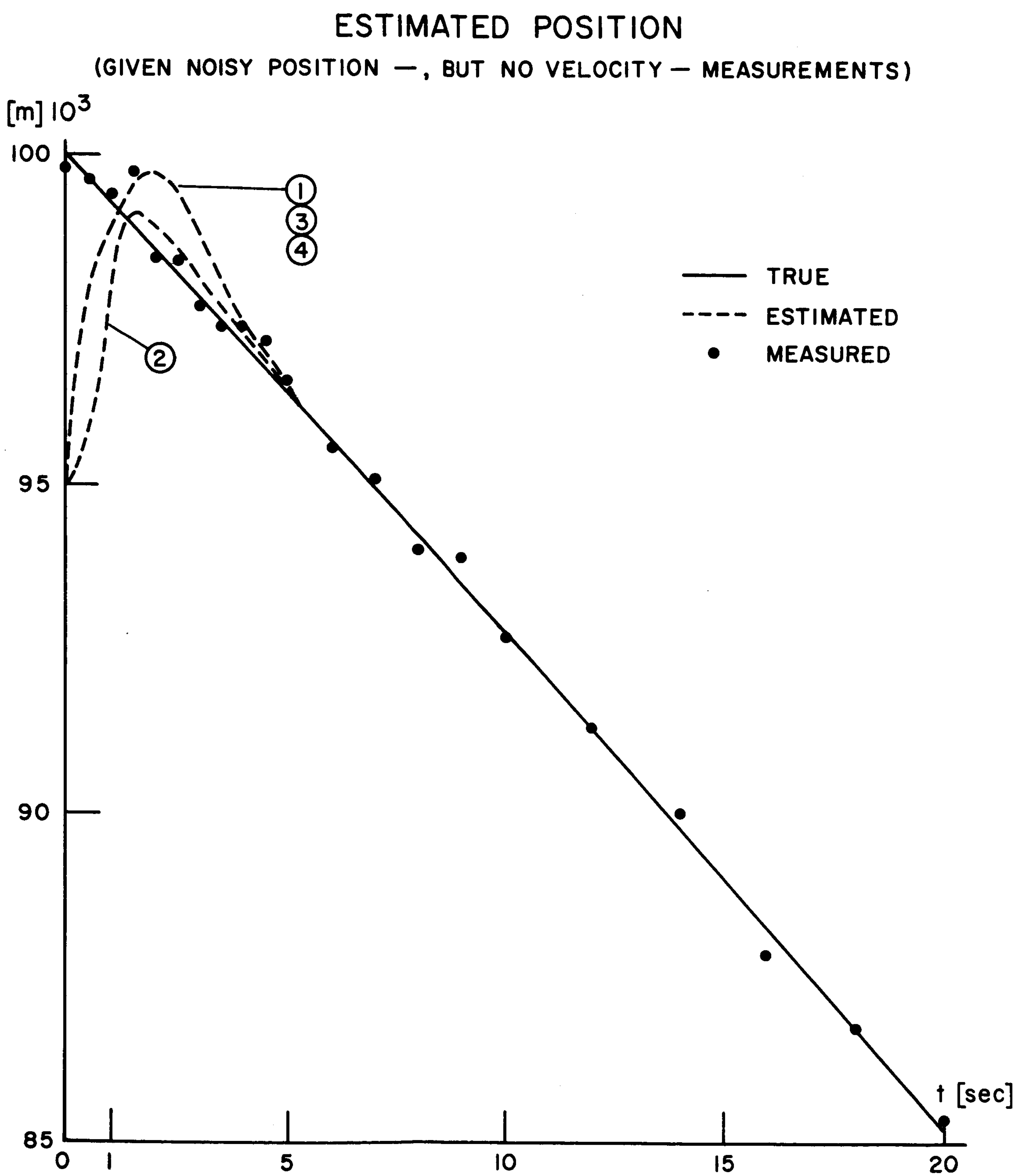

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10³
100
95
90
85
1
2
3
4
TRUE
ESTIMATED
MEASURED
t [sec]
0
1
5
10
15
20

Fig. 2.10.8.a

The best results we have obtained in estimating position and velocity having only (noisy) position measurements are displayed in Figs. 2.10.2-2.a. As we can see in these Figures after 3-4 sec estimation time (corresponding to the transient part of the estimated trajectory) we have a "smooth estimate" on position with an error amplitude corresponding to 1/10-th of the measurement noise (which is equivalent to 20-40 m in our numerical example), and, after 12-13 sec transient estimation time we have a "smooth estimate" on velocity with error amplitude $\approx$ 4-5 m/sec. The results shown in the other Figures (Figs. 2.10.3-7) display slower convergence to the true trajectory, or, they are associated with higher error amplitudes and frequencies, or, they are more sensitive to the chosen class of wrong initial estimates than those results which are depicted in Figs. 2.10.2-2.a.

The best results we have obtained in estimating position and velocity having both position and velocity measurements are shown in Figs. 2.10.9-9.a. As one can see in these Figures after 1.5-2.0 sec transient estimation time we have a "smooth estimate" on position with an error amplitude corresponding to 1/12-1/15-th of the measurement noise (which is equivalent to 10-30 m in our example), and, after 4-5 sec transient estimation time we have a "smooth estimate" on velocity with an error amplitude corresponding to 1/8-th of the measurement noise (which is equivalent to 1-2 m/sec in our example). The sequential estimation results, by using those R'_{ii} values which were applied for the "best trajectories" depicted in Figs. 2.10.9-9.2, are very much insensitive to the (wrong) initial estimates on the state variables as it is clearly demonstrated by Figs. 2.10.10-10.a. On the other hand, Figs. 2.10.11-11.a clearly show the importance of the order of magnitude of the R'_{ii} terms and their ratio in obtaining good sequential estimates on the state variables.

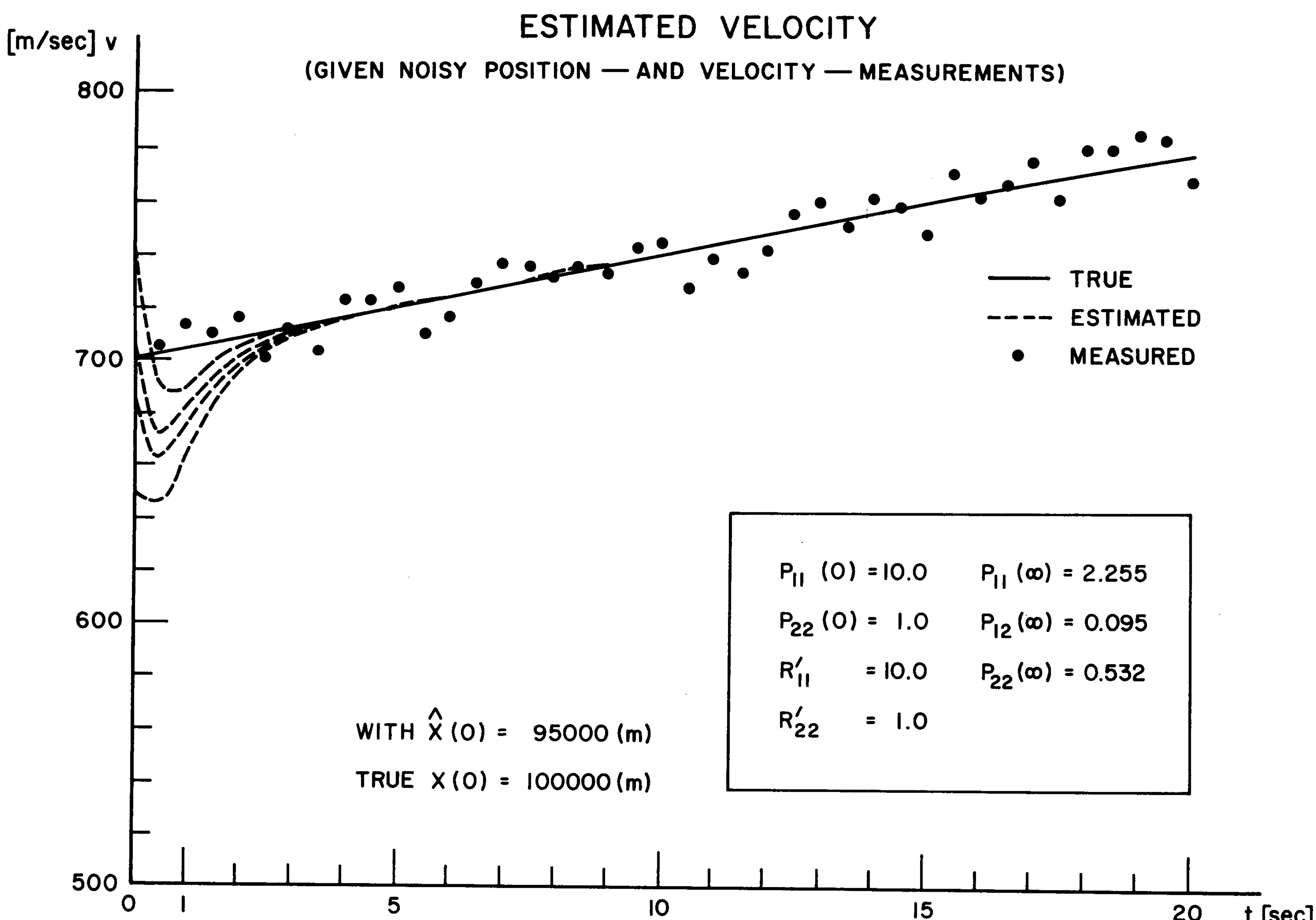

ESTIMATED VELOCITY
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m/sec] v
800
700
600
500
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED
WITH $\hat{X}(0)$ = 95000 (m)
TRUE X(0) = 100000 (m)
$P_{11}(0)$ = 10.0
$P_{22}(0)$ = 1.0
R'_{11} = 10.0
R'_{22} = 1.0
$P_{11}(\infty)$ = 2.255
$P_{12}(\infty)$ = 0.095
$P_{22}(\infty)$ = 0.532

Fig. 2.10.9

ESTIMATED POSITION

(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)

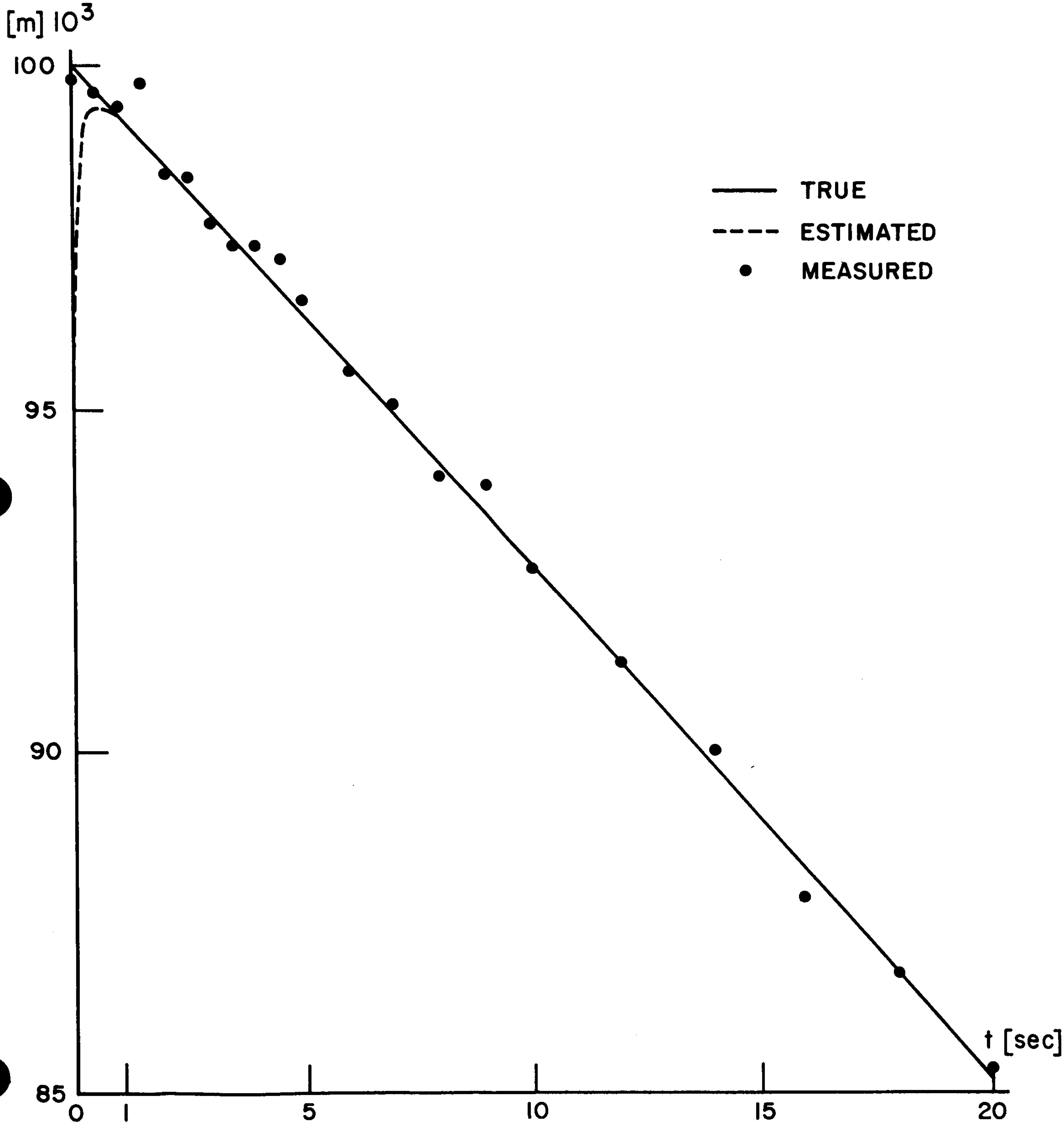
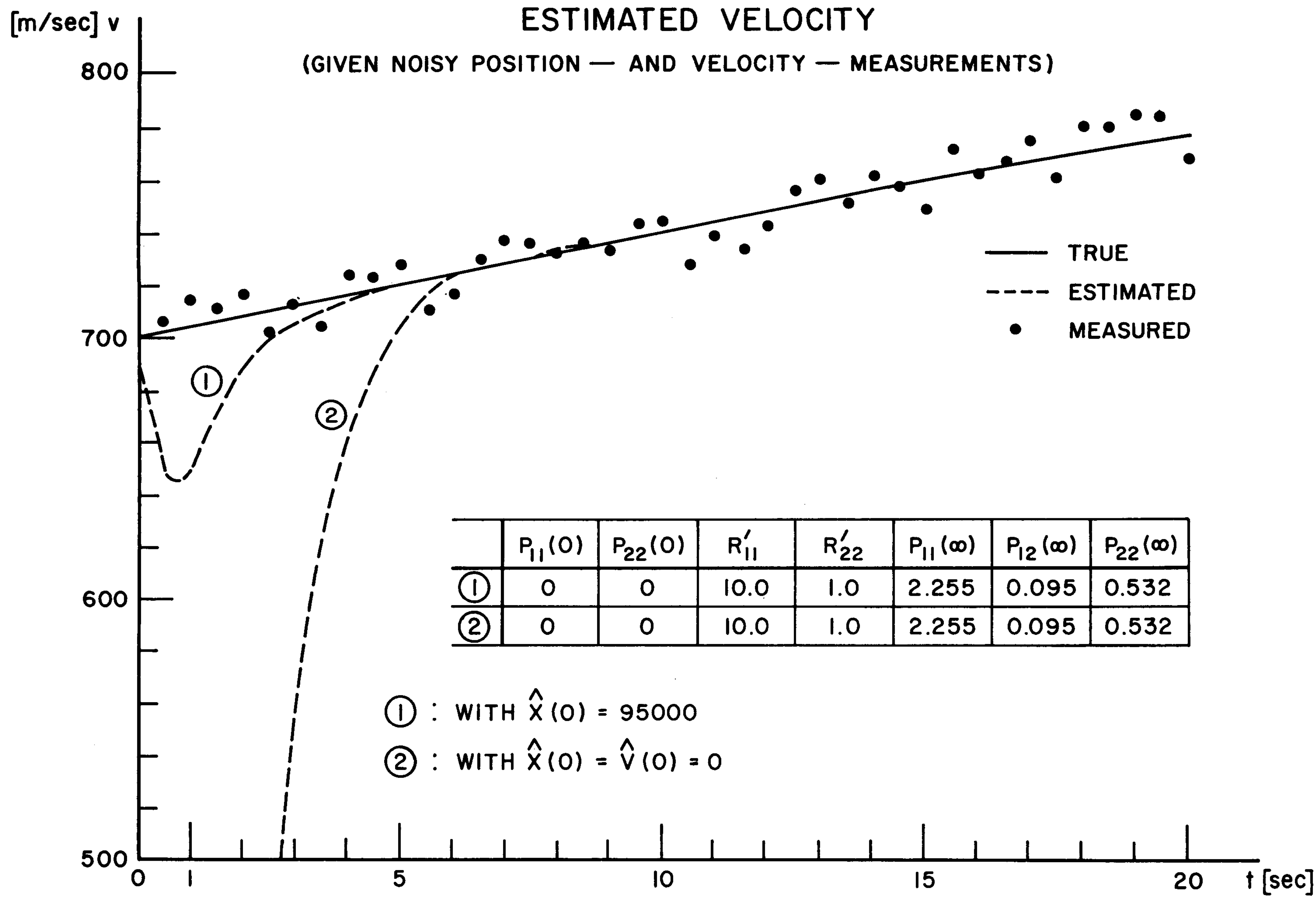


Fig. 2.10.9.a



	$P_{11}(0)$	$P_{22}(0)$	R'_{11}	R'_{22}	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	0	0	10.0	1.0	2.255	0.095	0.532
②	0	0	10.0	1.0	2.255	0.095	0.532

① : WITH $\hat{X}(0) = 95000$

② : WITH $\hat{X}(0) = \hat{V}(0) = 0$

Fig. 2.10.10

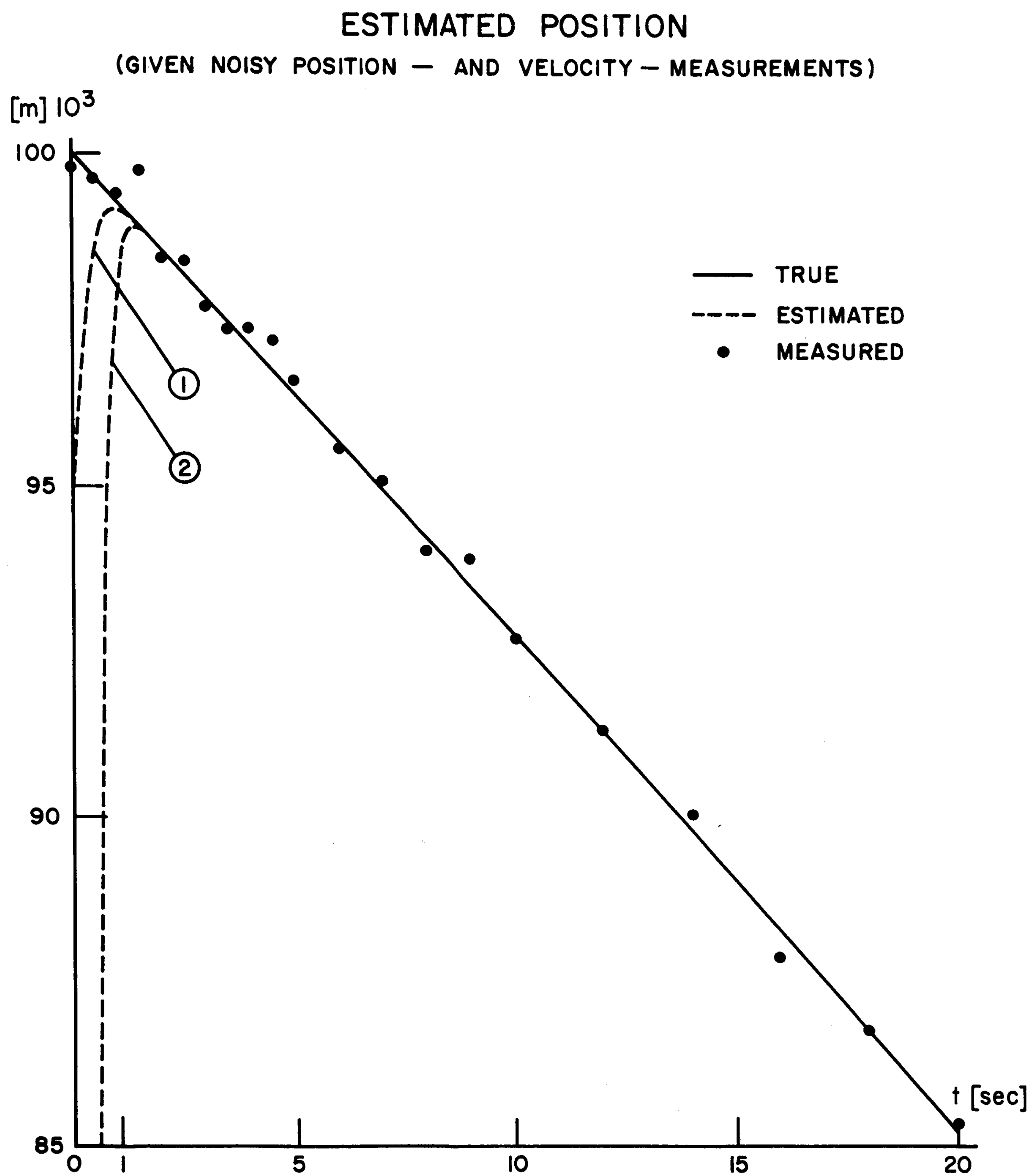
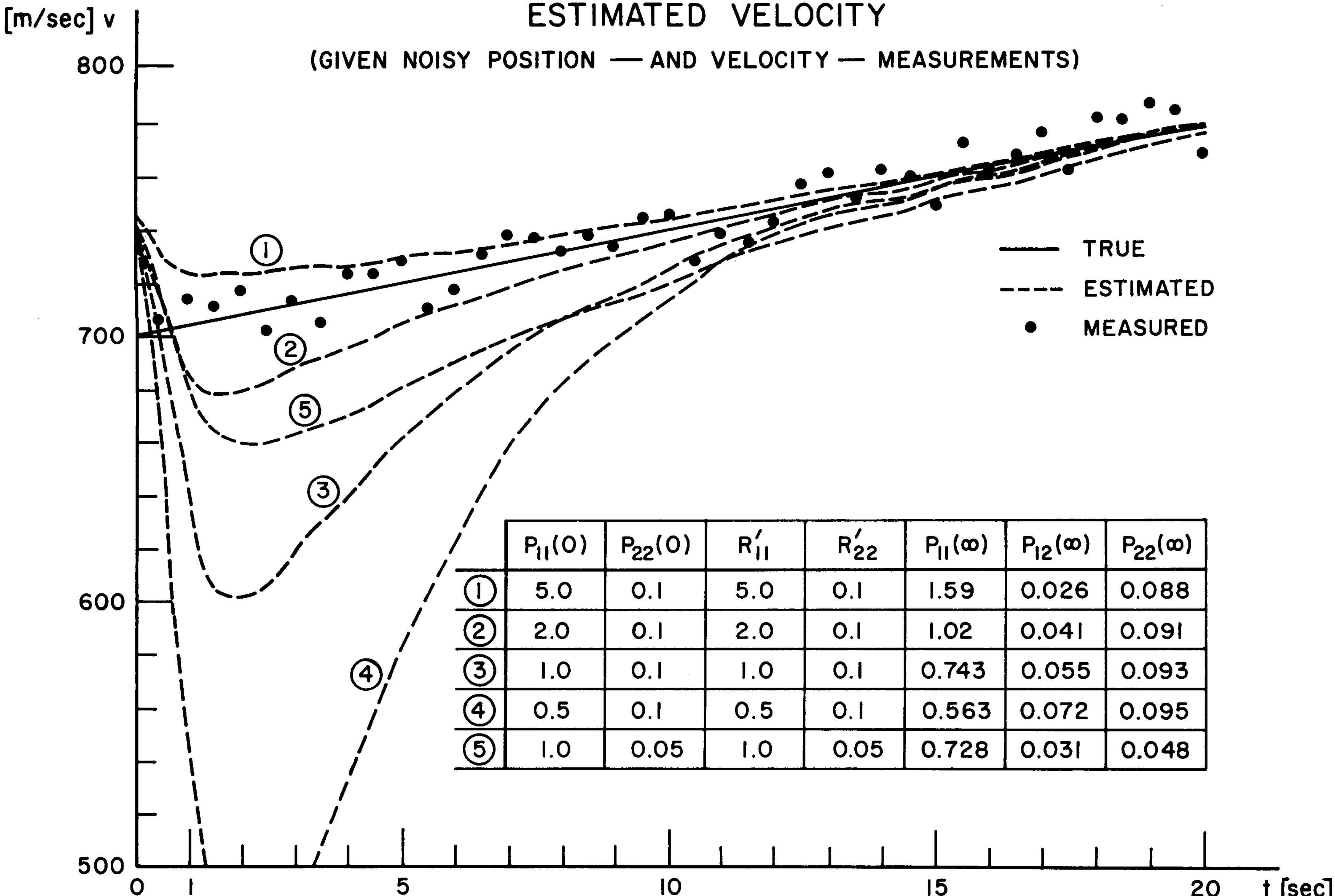

ESTIMATED POSITION
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m] 10³
100
95
90
85
TRUE
ESTIMATED
MEASURED
1
2
t [sec]
0
1
5
10
15
20

Fig. 2.10.10.a

ESTIMATED VELOCITY

(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)



	$P_{11}(0)$	$P_{22}(0)$	R'_{11}	R'_{22}	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	5.0	0.1	5.0	0.1	1.59	0.026	0.088
②	2.0	0.1	2.0	0.1	1.02	0.041	0.091
③	1.0	0.1	1.0	0.1	0.743	0.055	0.093
④	0.5	0.1	0.5	0.1	0.563	0.072	0.095
⑤	1.0	0.05	1.0	0.05	0.728	0.031	0.048

Fig. 2.10.11

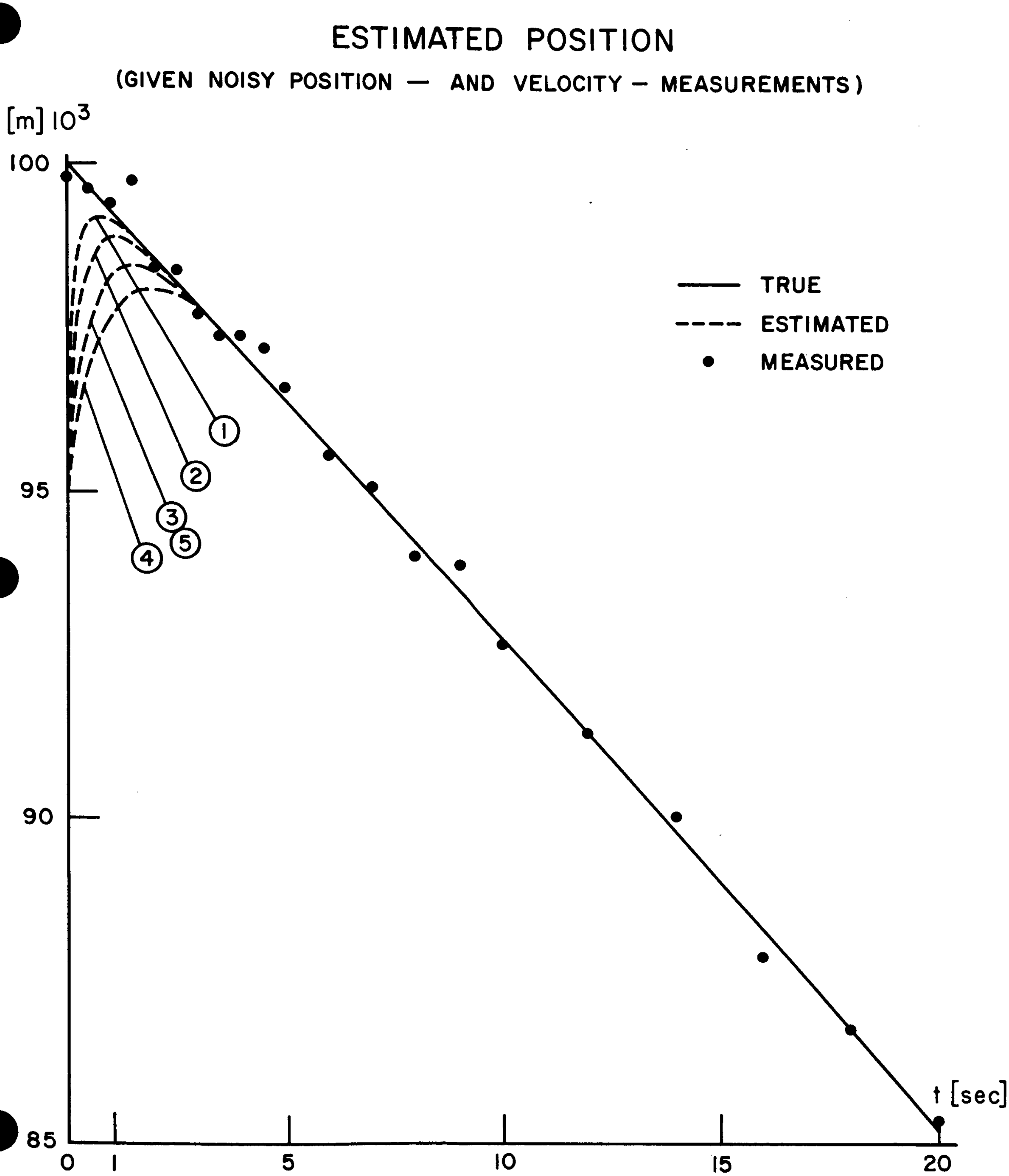

ESTIMATED POSITION
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m] 10³
100
95
90
85
TRUE
ESTIMATED
MEASURED
1
2
3
5
4
t [sec]
0
1
5
10
15
20

Fig. 2.10.11.a

2.11. New, Asymptotic Non-linear Filter Equations.

It turns out, as we have pointed out in the previous Section, that the Gain Equations of the proposed Non-linear Filter settle down on some asymptotic $P_{ij}(\infty)$ values in each case of our estimation problem. This very fact suggests the idea of simplifying the sequential Estimator Equations which were derived in Sections 2.6-9 by simply omitting the Gain Equations (the $\dot{P}_{ij}$ equations) from the filter equations and using only the State Estimator Equations (the $\dot{\hat{x}}$ equations) with the precomputed, proper asymptotic values of P_{ij}. The new, Asymptotic Non-linear Filter Equation, which we propose here, has the following general form:

$$\frac{d}{dT}\hat{x} = f(\hat{x},T) + 2P(\infty)\, H(\hat{x},T)\, Q\{y(T)-h(\hat{x},T)\} \qquad 2.11.1$$

where the symbols have the same meaning as it was explained in Section 2.5, and, $P(\infty)$: precomputed, proper asymptotic values of the gain matrix P.

We run some digital experiments by using Eq.(2.11.1) and obtained surprisingly good results. (In the subsequent computations we have used the same numerical values for constants, for initial values, for generating noise, etc. as we did in Section 2.10.)

In the case of estimating position and velocity having only position measurements we have, according to Eq.(2.11.1), the following simplified non-linear filter equations (obtained by omitting Eqs.(2.6.9-11) and using only Eqs. (2.6.7-8) with proper $P_{ij}(\infty)$ values):

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(\infty)\ (y_1-\hat{x}_1) \qquad (2.11.2)$$

$$\dot{\hat{x}}_2 = K\hat{x}_2^2 \exp(-\ b\hat{x}_1) - g + 2P_{12}(\infty)\ (y_1-\hat{x}_1) \qquad (2.11.3)$$

where we have applied the following asymptotic gain values:

$$P_{11}(\infty) = 0.849$$

$$P_{12}(\infty) = 0.221 \qquad (2.11.4)$$

These gain values were obtained by the computations which provided the "best results" depicted in Figs. 2.10.2-2.a. (The symbols in Eqs.(2.11.2-3) have the same meaning as in Section 2.6.)

The results obtained by the proposed, new, Asymptotic Non-linear Filter Equations (2.11.2-4) are displayed in Figs. 2.11.1-1.a. Comparing these Figures with Figs. 2.10.2-2.a we observe that the simplifications used in the Non-linear Filter Equations only affect the transient part of the estimated trajectories, keeping the "good properties" of the filter unchanged. (Relative insensitivity to wrong initial estimates on the state variables, accurate reproduction of all state variables, stable, smooth behaviour around the true trajectories.)

In the case of estimating positon and velocity having both position and velocity measurements, we can, according to Eq.(2.11.1), use the following simplified non-linear filter equations (obtained by omitting Eqs.(2.6.16-18)) from the Sequential Estimator Equations and using only Eqs.(2.6.14-15) with proper $P_{ij}(\infty)$ values):

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(\infty)(y_1-\hat{x}_1) + 2P_{12}(\infty)(y_2-\hat{x}_2) \qquad (2.11.5)$$

$$\dot{\hat{x}}_2 = K\hat{x}_2^2\exp(-\,b\hat{x}_1) - g + 2P_{12}(\infty)(y_1-\hat{x}_1) + 2P_{22}(\infty)(y_2-\hat{x}_2) \qquad (2.11.6)$$

where we have applied the following asymptotic gain values:

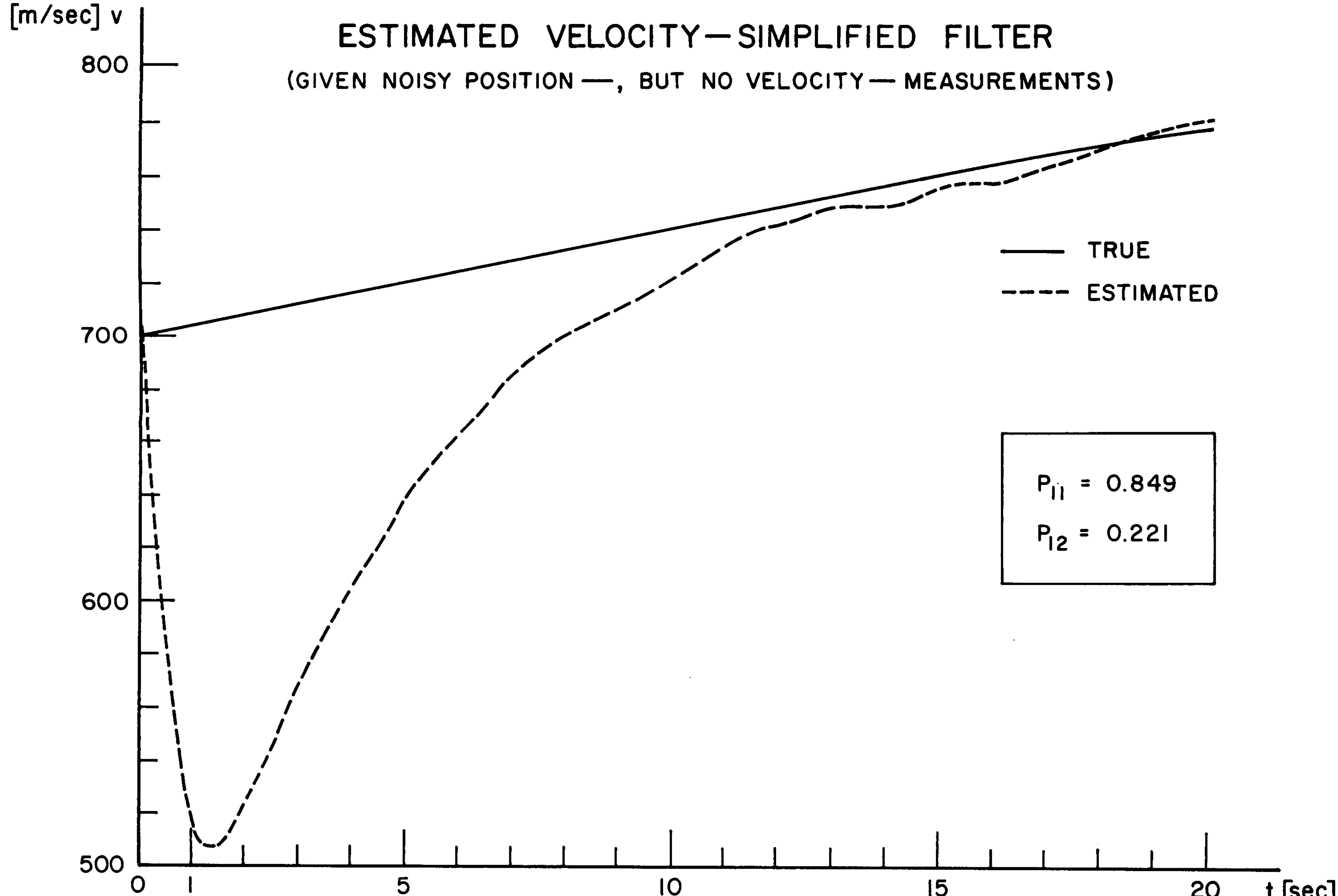

ESTIMATED VELOCITY—SIMPLIFIED FILTER
(GIVEN NOISY POSITION—, BUT NO VELOCITY—MEASUREMENTS)
[m/sec] v
800
700
600
500
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
P11 = 0.849
P12 = 0.221

Fig. 2.11.1

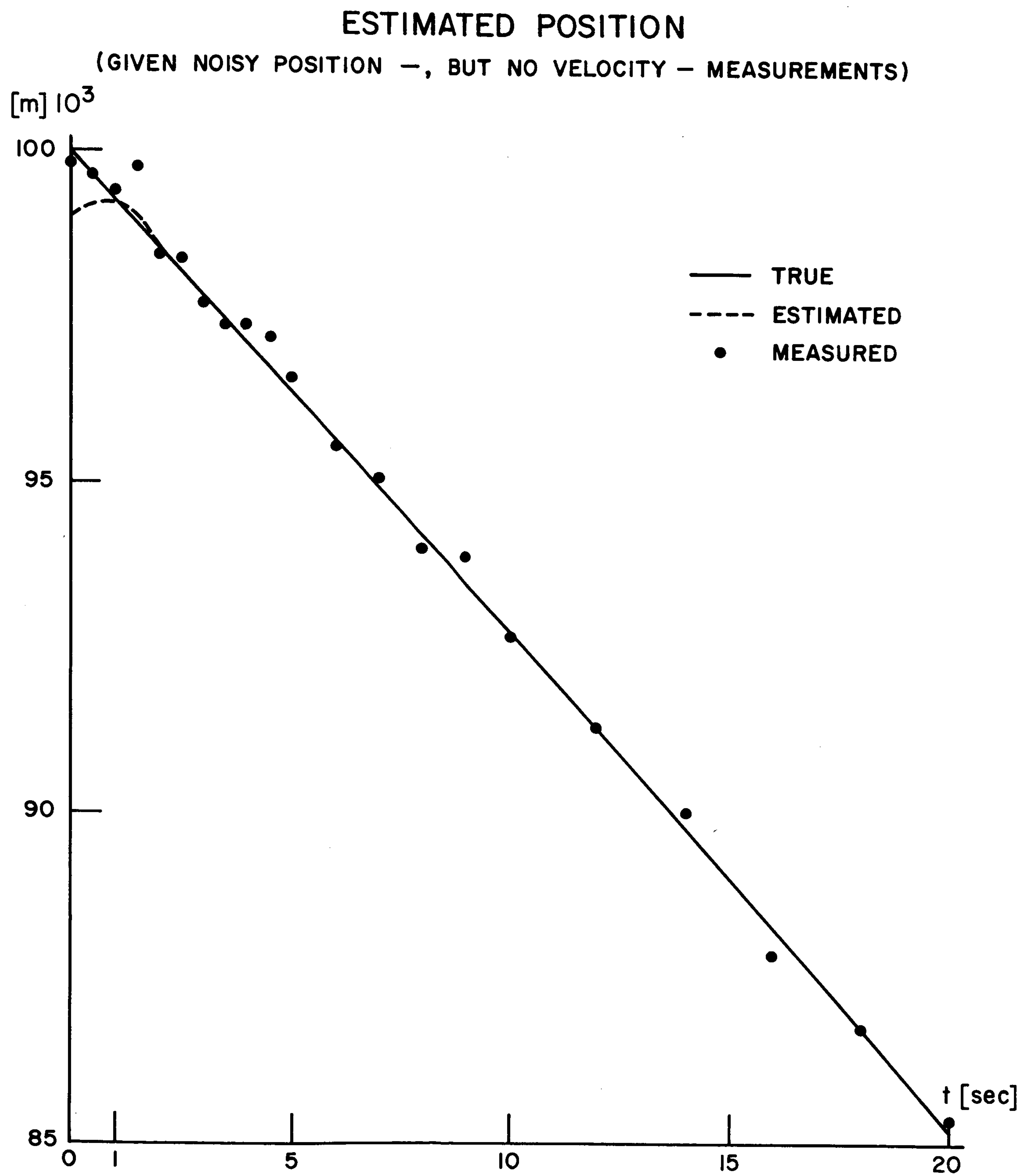

ESTIMATED POSITION
(GIVEN NOISY POSITION —, BUT NO VELOCITY — MEASUREMENTS)
[m] 10^3
100
95
90
85
TRUE
ESTIMATED
MEASURED
t [sec]
0
1
5
10
15
20

Fig. 2.11.1.a

$$P_{11}(\infty) = 2.255$$

$$P_{12}(\infty) = 0.095 \qquad (2.11.7)$$

$$P_{22}(\infty) = 0.532$$

These gain values were obtained by the computations which provided the "best results" depicted in Figs. 2.10.9-9.a. (The symbols in Eqs.(2.11.5-6) have the same meaning as in Section 2.6.)

The results obtained by the proposed, new Asymptotic non-linear Filter Equations (2.11.5-7) are shown in Figs. 2.11.2-2.a. Comparing these figures with Figs. 2.10.9-9.a we see again that the simplifications introduced into the Non-linear Filter Equations, resulting the new, Asymptotic Non-linear Filter Equations, affect only the transient part of the estimated trajectories, keeping the "good properties" of the filter unaltered.

In order to investigate how variations in the $P_{ij}(\infty)$ values influence the estimated trajectories belonging to the new, Asymptotic Non-linear Filter Equations we made several "digital experiments" applying different $P_{ij}(\infty)$ values which were previously obtained for different $P_{ij}(0)$ and R'_{ii} values. Figures 2.11.3-3.a display some of the estimated trajectories obtained by using different $P_{ij}(\infty)$ values. (The values $P_{11}(\infty) = 1.032$, $P_{12}(\infty) = 0.069$, $P_{22}(\infty) = 0.167$ used in obtaining Trajectory No. 1 in Figs. 2.11.3-3.a come from computations with $P_{11}(0) = 2.0$, $P_{22}(0) = 0.2$, $R'_{11} = 2.0$, $R'_{22} = 0.2$). Comparing Trajectories No. 1 in Figs. 2.11.3-3.a with the estimated trajectories in Figs. 2.11.2-2.a we see that altering the $P_{ij}(\infty)$ values affects significantly only the transient part of the estimated trajectories; other properties of the estimated trajectories are insignificantly changed. Note in this connection that the order of magnitude of the $P_{ij}(\infty)$ values used in obtaining Trajectories No. 1 in Figs. 2.11.3-3.a are the same as the order of magnitude of the corresponding $P_{ij}(\infty)$ values which were used in obtaining the estimated trajectories

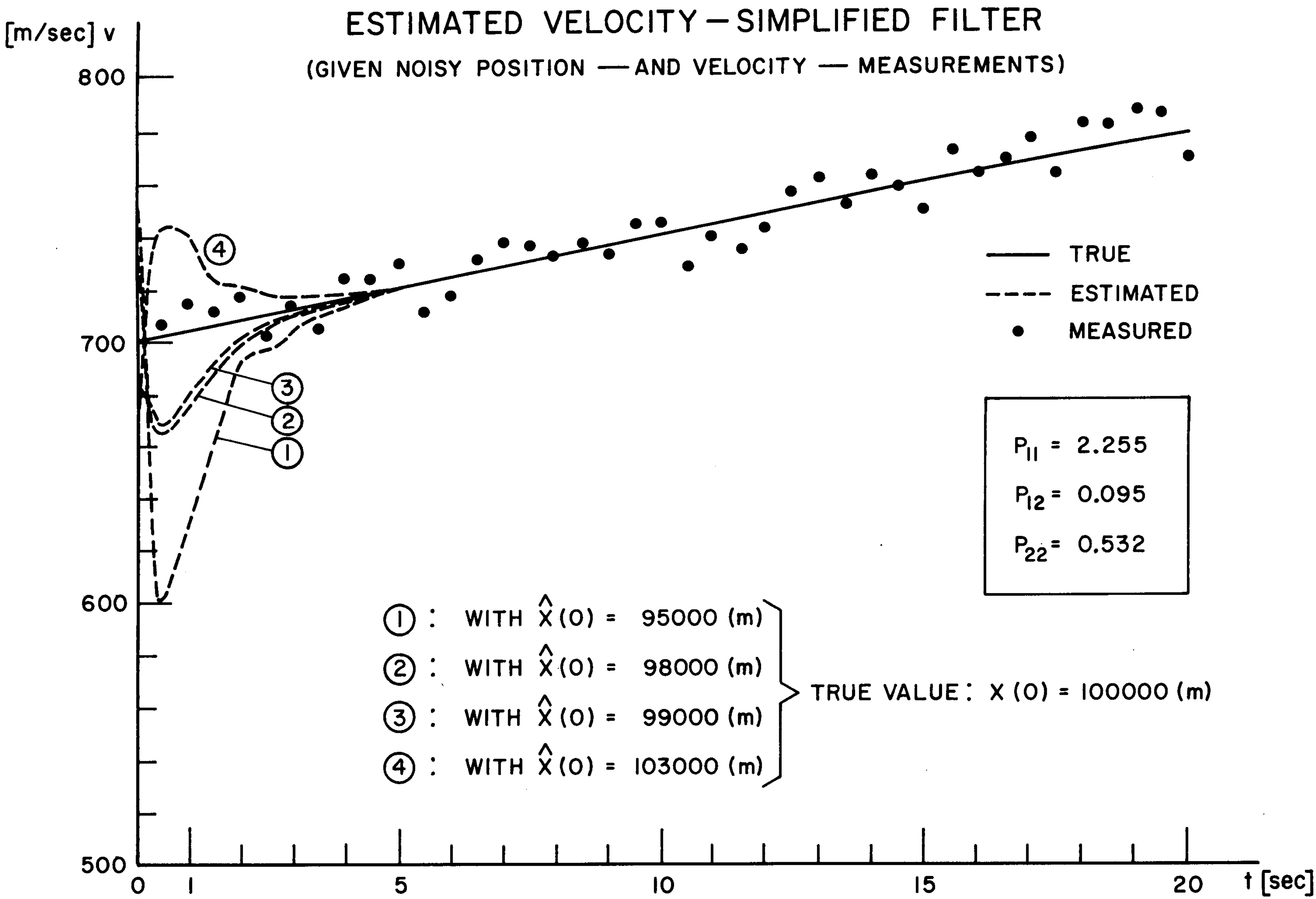

ESTIMATED VELOCITY — SIMPLIFIED FILTER
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m/sec] v
800
700
600
500
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED
P11 = 2.255
P12 = 0.095
P22 = 0.532
1 : WITH X̂(0) = 95000 (m)
2 : WITH X̂(0) = 98000 (m)
3 : WITH X̂(0) = 99000 (m)
4 : WITH X̂(0) = 103000 (m)
TRUE VALUE : X (0) = 100000 (m)

Fig. 2.11.2

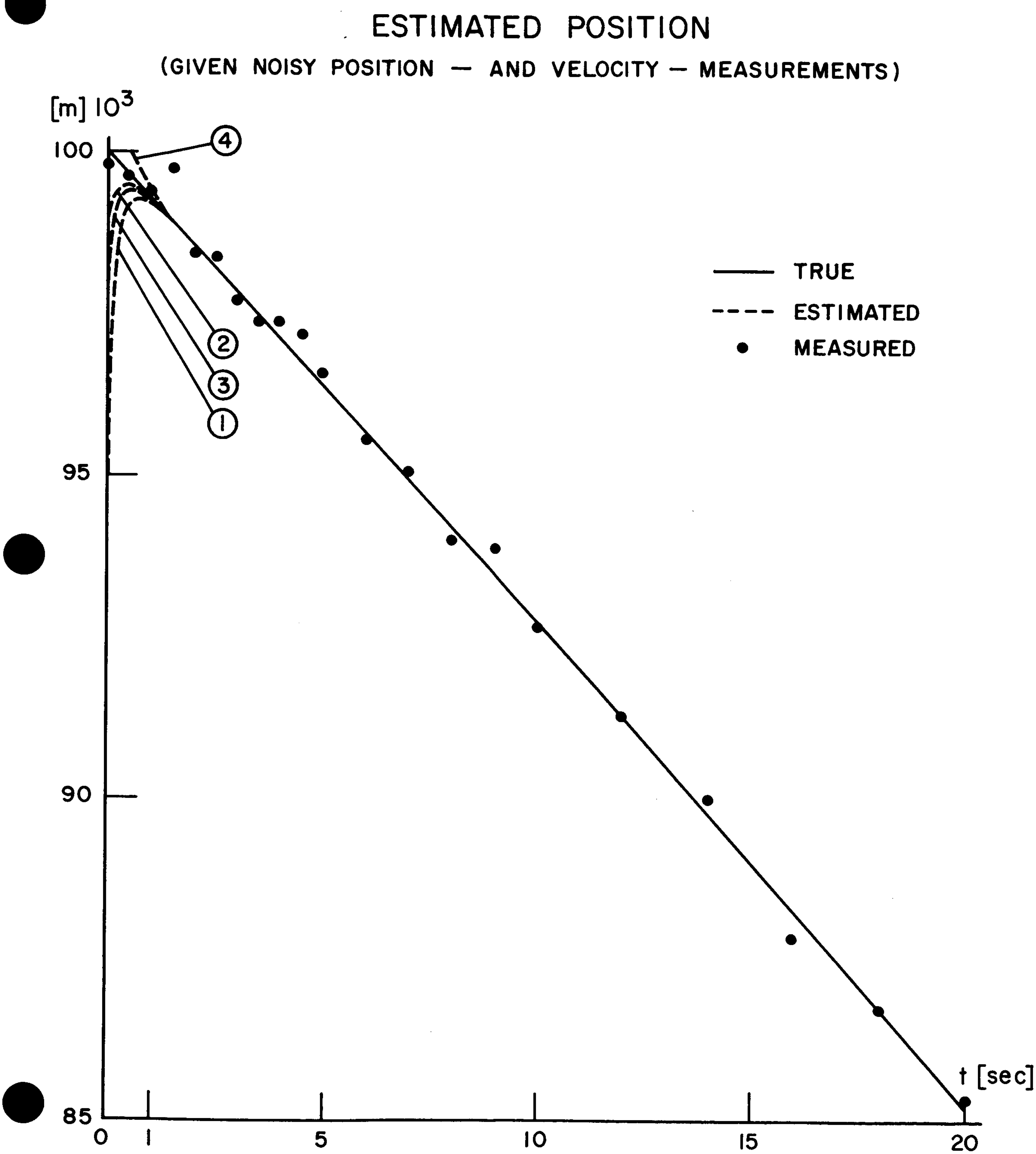

ESTIMATED POSITION
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m] 10^3
100
95
90
85
0
1
5
10
15
20
t [sec]
1
2
3
4
TRUE
ESTIMATED
MEASURED

Fig. 2.11.2.a

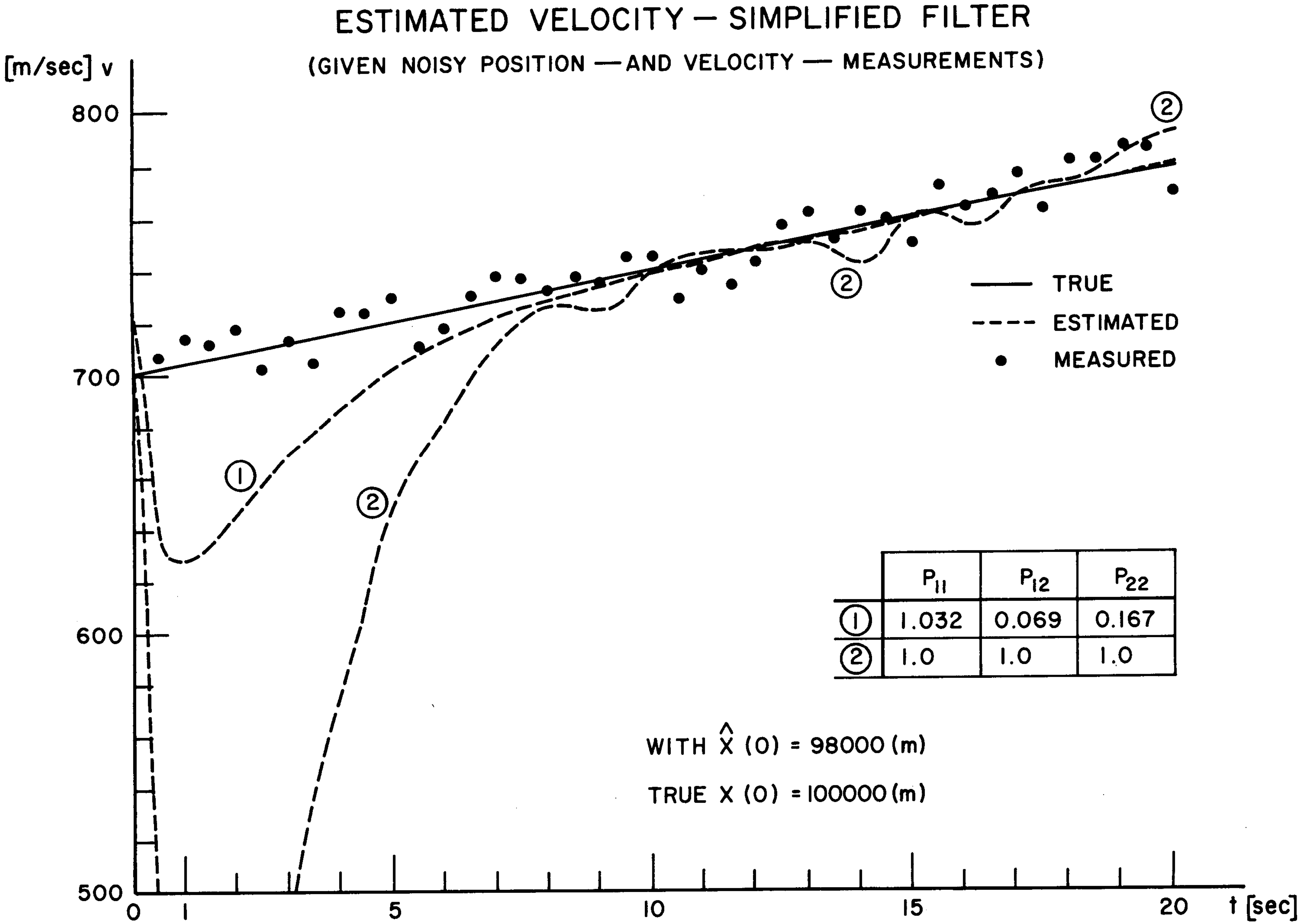

ESTIMATED VELOCITY — SIMPLIFIED FILTER
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m/sec] v
800
700
600
500
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED
①
②
P11 P12 P22
① 1.032 0.069 0.167
② 1.0 1.0 1.0
WITH X̂ (0) = 98000 (m)
TRUE X (0) = 100000 (m)

Fig. 2.11.3

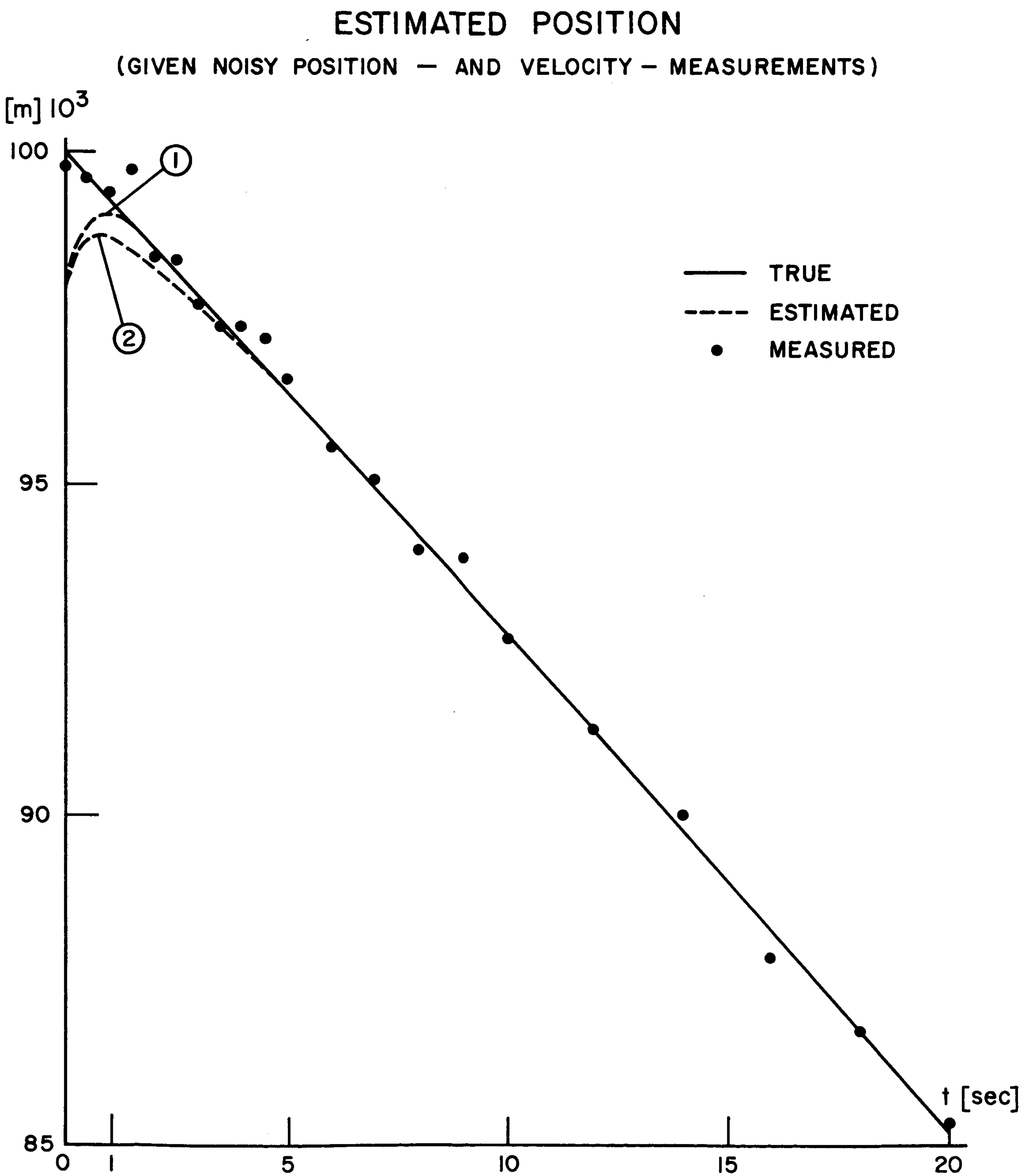

ESTIMATED POSITION
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m] 10³
100
95
90
85
0
1
5
10
15
20
t [sec]
1
2
TRUE
ESTIMATED
MEASURED

Fig. 2.11.3.a

depicted in Figs. 2.11.2-2.a. In Figs. 2.11.3-3.a we also show the estimated trajectories (labelled with No. 2) obtained by using $P_{11}(\infty) = P_{12}(\infty) = P_{22}(\infty) = 1.0$ as "guessed" values instead of applying some precomputed $P_{ij}(\infty)$ values. It is interesting to observe in what extents the properties of the estimated trajectories are changed in this case when the order of magnitude of $P_{ij}(\infty)$ corresponds only partially to the order of magnitude of the precomputed $P_{ij}(\infty)$ values.

We have also ran some "digital experiments" for estimating position, velocity and one parameter (K) having position and velocity measurements, by using the proposed, new, Asymptotic Non-linear Filter Equations. In this case, according to Eq.(2.11.1), we have the following simplified non-linear filter equations (obtained by omitting Eqs.(2.7.34-39) from the Sequential Estimator Equations and using only Eqs.(2.7.31-33) with proper $P_{ij}(\infty)$ values):

$$\dot{\hat{x}}_1 = \hat{x}_2 + 2P_{11}(\infty)(y_1-\hat{x}_1) + 2P_{12}(\infty)(y_2-\hat{x}_2) \tag{2.11.8}$$

$$\dot{\hat{x}}_2 = \hat{x}_3\hat{x}_2^2\exp(-\,b\hat{x}_1) - g + 2P_{12}(\infty)(y_1-\hat{x}_1) + 2P_{22}(\infty)(y_2-\hat{x}_2) \tag{2.11.9}$$

$$\dot{\hat{x}}_3 = 2P_{13}(\infty)(y_1-\hat{x}_1) + 2P_{23}(\infty)(y_2-\hat{x}_2) \tag{2.11.10}$$

The symbols in these equations have the same meaning as in Section 2.7.

Instead of applying precomputed values for $P_{ij}(\infty)$ in Eqs.(2.11.8-10) we now tried to use "proper $P_{ij}(\infty)$" determined by order of magnitude analysis (the justification of which was demonstrated previously in the present Section). We have found

$$P_{11}(\infty) = 10$$

$$P_{12}(\infty) = 10^{-2}$$

$$P_{13}(\infty) = 10^{-6} \qquad (2.11.11)$$

$$P_{22}(\infty) = 1$$

$$P_{23}(\infty) = 10^{-5}$$

as "proper $P_{ij}(\infty)$ values".

The results obtained by Eqs.(2.11.8-11) which we termed as "tentative Simplified Filter" because of the way the $P_{ij}(\infty)$ values were determined, are displayed in Fig. 2.11.4. After 2-3 sec transient estimation time, as one can see in that Figure, we have a "smooth estimate" on position with an error amplitude corresponding to $\approx$ 1/10-th of the measurement noise (which is equivalent to 40 m in our example), and, after 5-6 sec transient estimation time we have a "smooth estimate" on velocity with an error amplitude corresponding to 1/5-1/6-th of the measurement noise (which is equivalent to 2-3 m/sec in our example.) The "parameter trajectory" converges asymptotically to the true (constant) value, and, after 18-20 sec transient estimation time the estimated value of the parameter (K) differs from the true value only with 8-10%.

In order to demonstrate the feasibility of the proposed, new, Asymptotic Non-linear Filter more profoundly we have performed two interesting digital experiments": (1) by altering the dynamic state of the system (going from acceleration over to deceleration); (2) by having systematic error in the value of the gas dynamic parameter K'.

(1) Suppose we were wrong by a factor 10 in the value of K' when we precalculated the asymptotic value $P(\infty)$ of the gain matrix. In the present example this means that the true trajectories are given by applying

TENTATIVE FILTER (SIMPLIFIED)

(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)

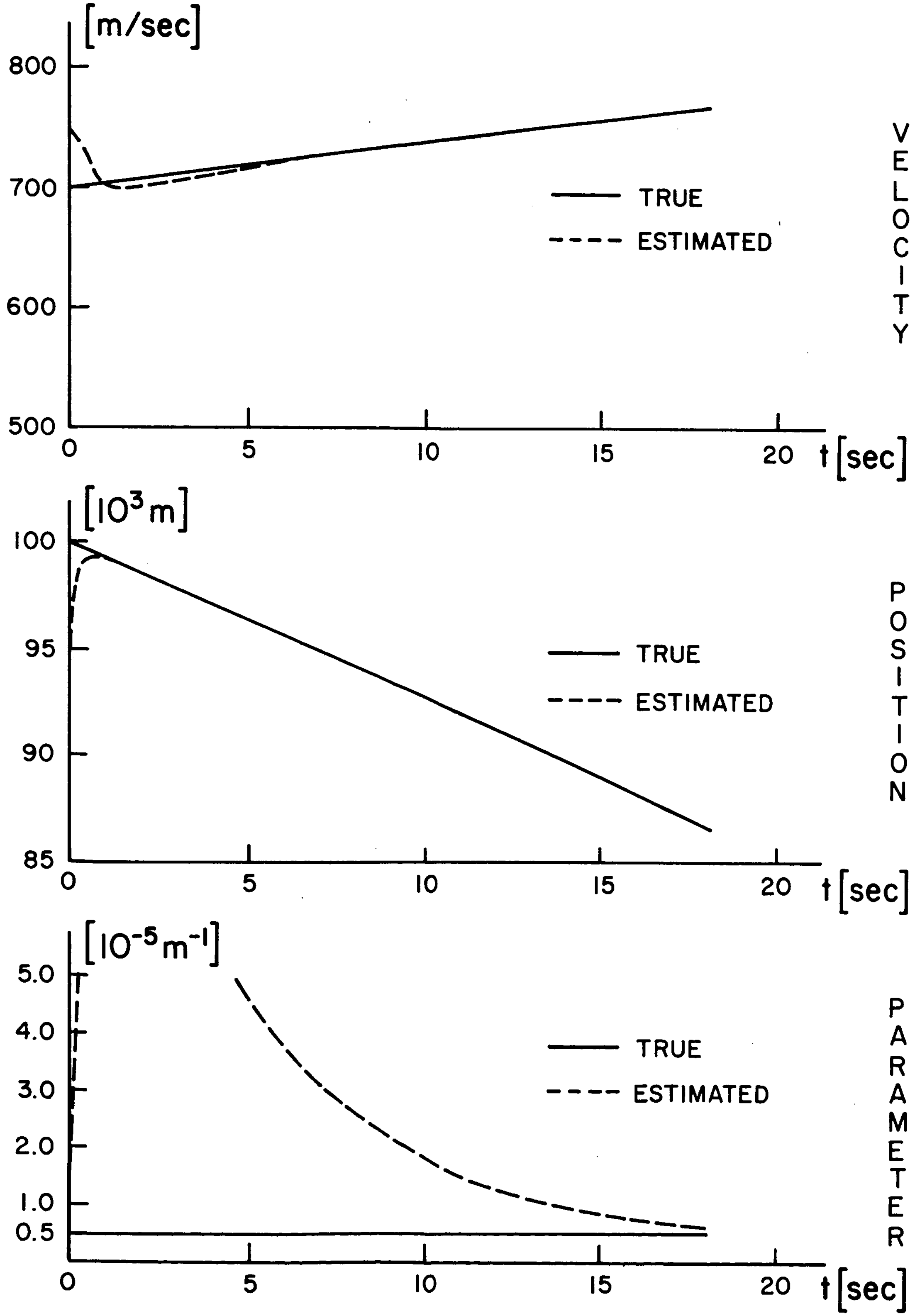


Fig. 2.11.4

$K' = 5.10^{-5}$ (kg/m) and not by using $K' = 5.10^{-6}$ (kg/m) as we did when we precalculated $P(\infty)$. But, in estimating the true state of the system by means of Eqs.(2.11.5-6), we still use that $P(\infty)$ value which was obtained by applying $K' = 5.10^{-6}$ (kg/m) in the dynamic equations of the system.

The results we obtained in this "digital simulating" are displayed in Figs. 2.11.5-5.a. As one can see in these figures the dynamic state of the system is essentially different from the previous state when we applied $K' = 5.10^{-6}$ (kg/m). Previously the accelerating force was greater than the decelerating force resulting <u>increasing velocity</u>, (see Fig. 2.11.2.). But now the decelerating force is greater than the accelerating force resulting <u>decreasing velocity</u>. Despite this difference in the dynamic state of the system the Asymptotic Filter, with $P(\infty)$ values obtained for the state of <u>increasing</u> velocity, does reproduce and estimate the system trajectories also in the state of <u>decreasing</u> velocity in the desired fashion. The "good properties" of the filter are not changed, and, the error amplitude of the estimated trajectories is the same as it was previously.

(2) Suppose we believe that the parameter K' has the value $K' = 5.10^{-5}$ (kg/m). By applying this value for K' in the <u>original</u> Sequential Estimator Equations (2.6.14-18) we obtained

$$P_{11}(\infty) = 2.255$$

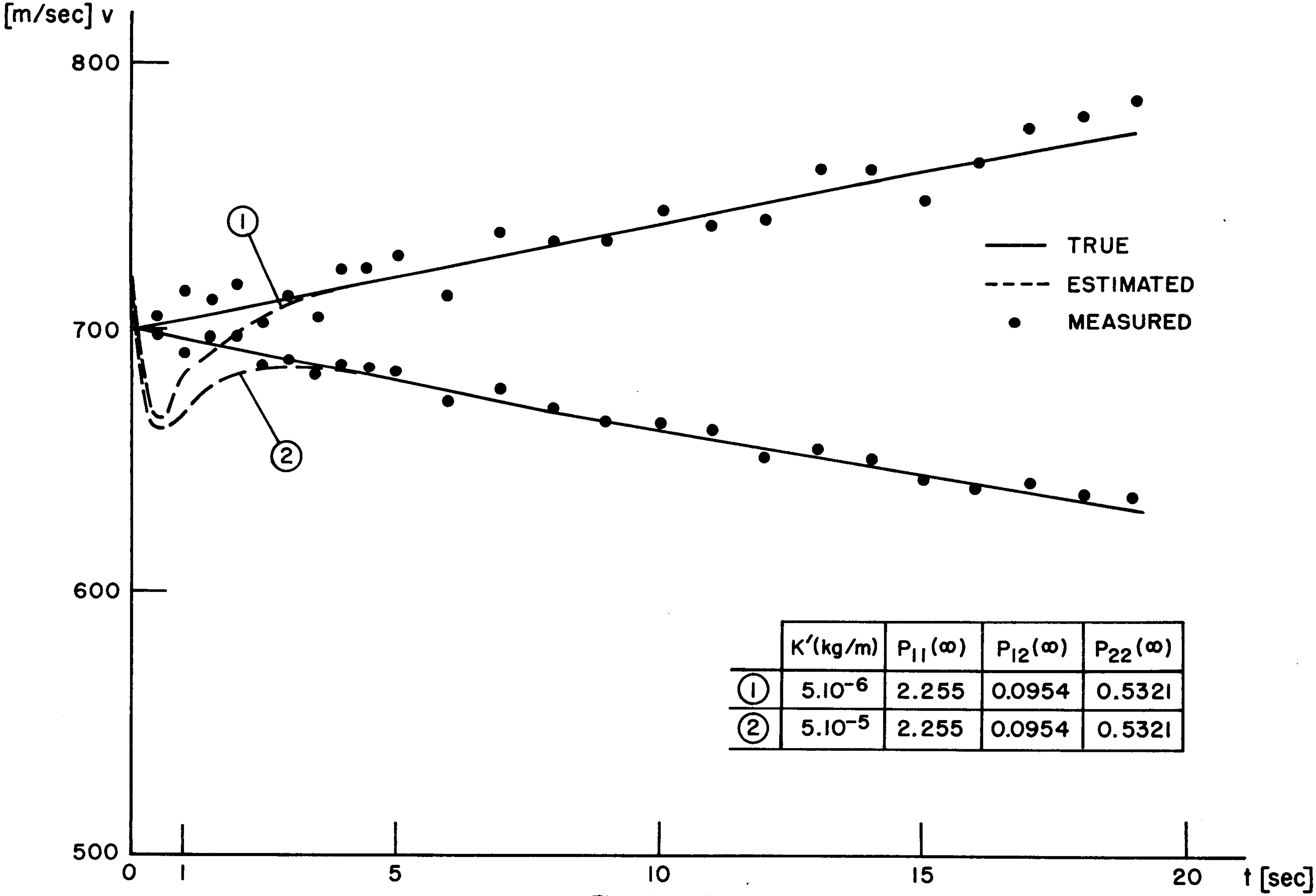
$$P_{12}(\infty) = 0.0938 \qquad (2.11.12)$$

$$P_{22}(\infty) = 0.5441$$

But let us assume that the true trajectories and the measurements based upon them, feeding them into the <u>Asymptotic</u> Filter Equations where we apply values of

ESTIMATED VELOCITY — SIMPLIFIED FILTER

(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)



	K'(kg/m)	$P_{11}(\infty)$	$P_{12}(\infty)$	$P_{22}(\infty)$
①	5.10^{-6}	2.255	0.0954	0.5321
②	5.10^{-5}	2.255	0.0954	0.5321

Fig. 2.11.5

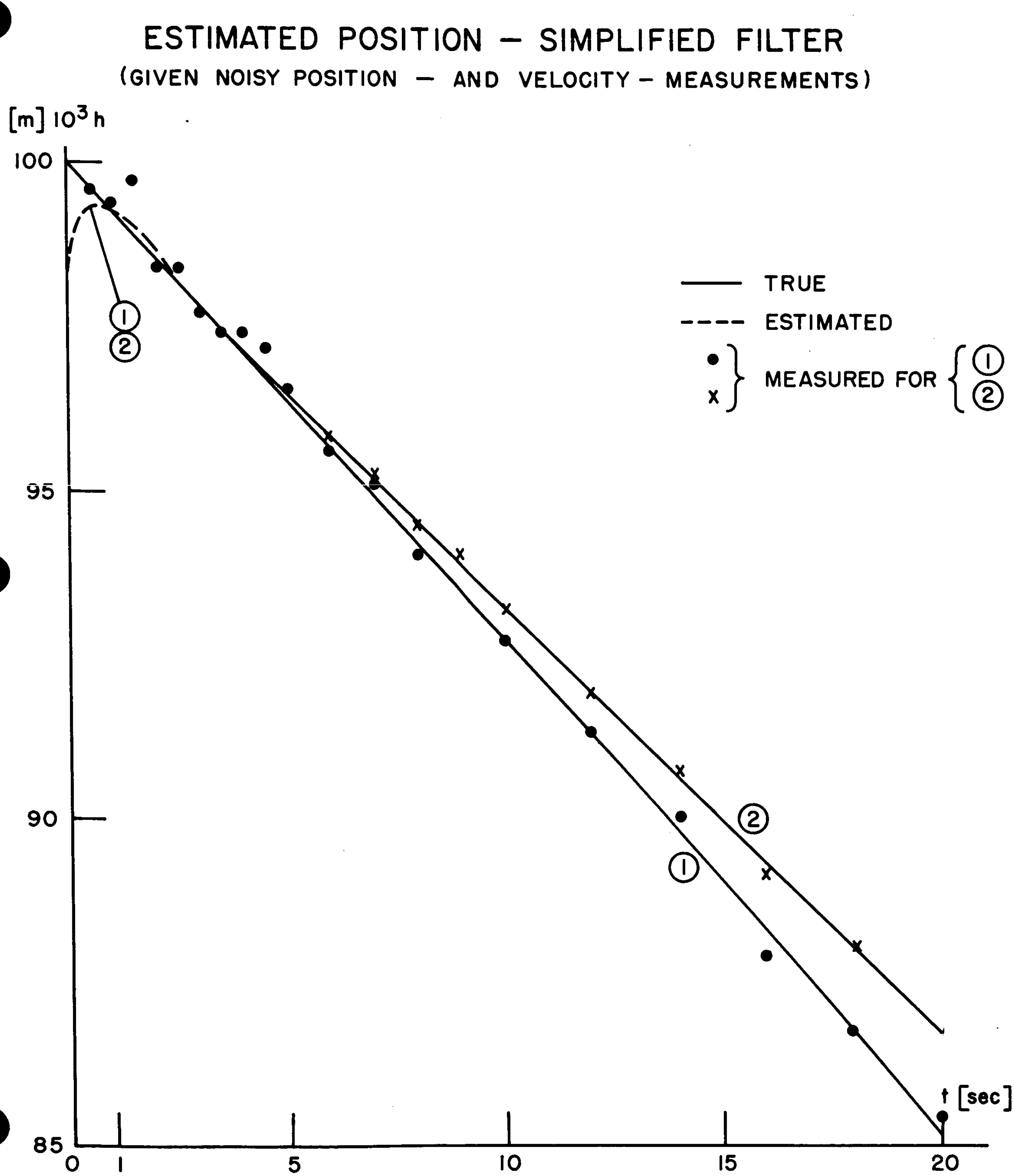

ESTIMATED POSITION – SIMPLIFIED FILTER
(GIVEN NOISY POSITION – AND VELOCITY – MEASUREMENTS)
[m] 10³ h
100
95
90
85
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED FOR
①
②

Fig. 2.11.5.a

$P_{ii}(\infty)$ shown in Eq.(2.11.12), are given by using $K' = 6.10^{-5}$ (kg/m). This means we have 20% systematic difference between the value of K' applied in the Asymptotic Filter Equations and the value of K' applied in generating true trajectories and measurements.

The results we obtained in this "digital simulating" are depicted in Figs. 2.11.6-6.a. Trajectories labelled with No. 1 in these figures belong to the original (unabbreviated) Sequential Estimator Equations, and, those labelled with No. 2 belong to the simplified, Asymptotic Filter Equations. As one can see in these Figures the basic "good properties" of the Asymptotic Filter are unaltered. The Asymptotic Filter does reproduce and estimate the true trajectories in the desired fashion, despite the 20% systematic error in the applied value of parameter K'.

The results we so far have obtained using the new, Asymptotic Non-linear Filter Equations are very promising, indeed. Thinking in terms of implementation of the Non-linear Filter Equations, it is hard to overemphasize the practical implications of the proposed, new, Asymptotic Non-linear Filter Equations.

2.12. Summary and Future Work.

Defining fuel consumption as a natural performance index (and not considering atmospheric heating effects as constraining factors), assuming, furthermore, known atmospheric data and a specified thrust engine we have shown how to obtain Optimal Thrust Programs for soft landing on an atmospheric planet. Optimal Thrust Programs are obtained by applying the Pontryagin Maximum Principle and are presented in form of Switching Functions. Calculation of Optimal Thrust Programs can be used as a guide in specifying fuel requirements for a given mission. Since the Pontryagin Maximum Principle provides an open-loop control the dependence of the Switching Functions on given planetary atmospheric data was emphasized and demonstrated.

ESTIMATED VELOCITY

(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS, AND 20% FAILURE IN THE VALUE OF PARAMETER K′)

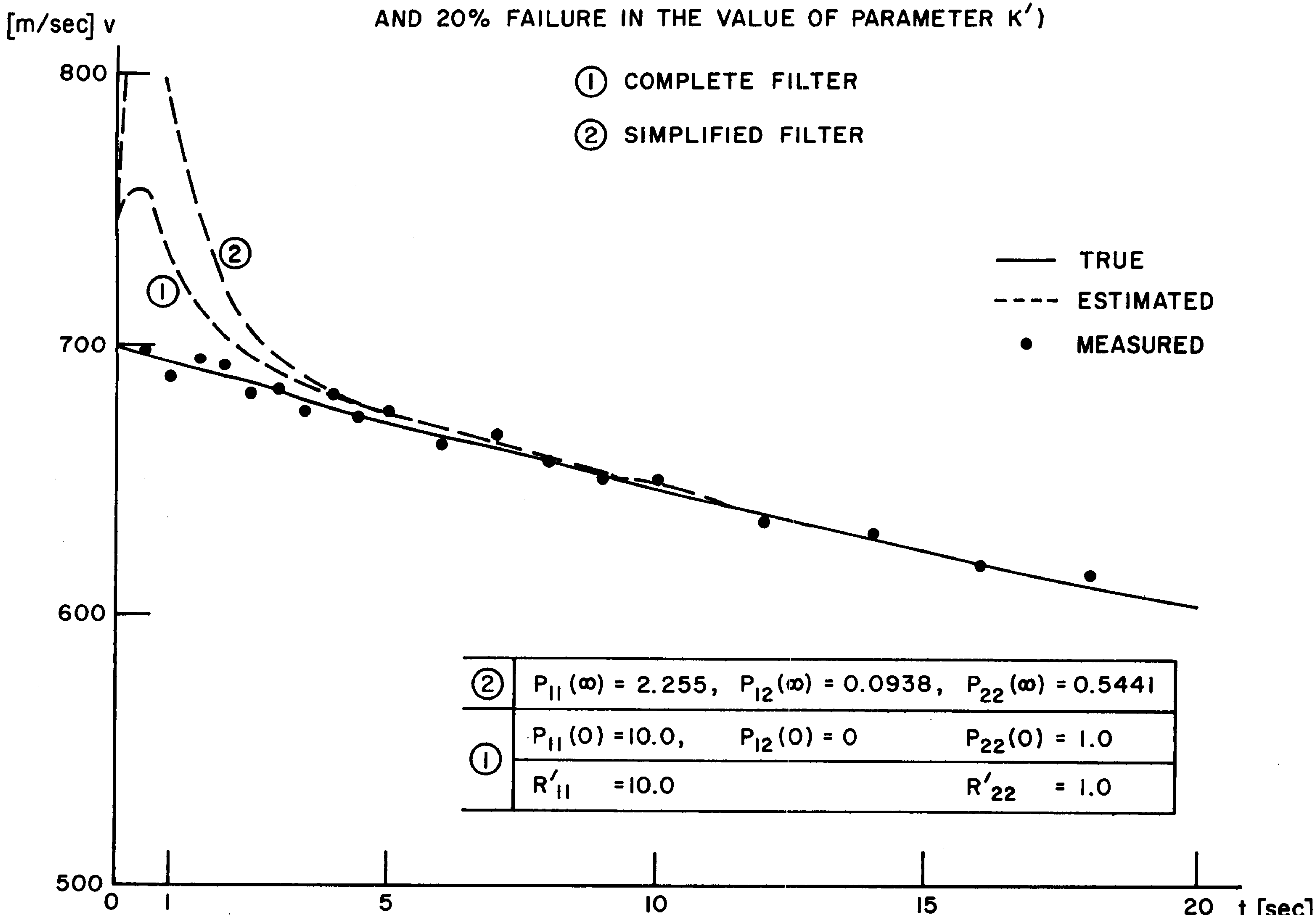


Fig. 2.11.6

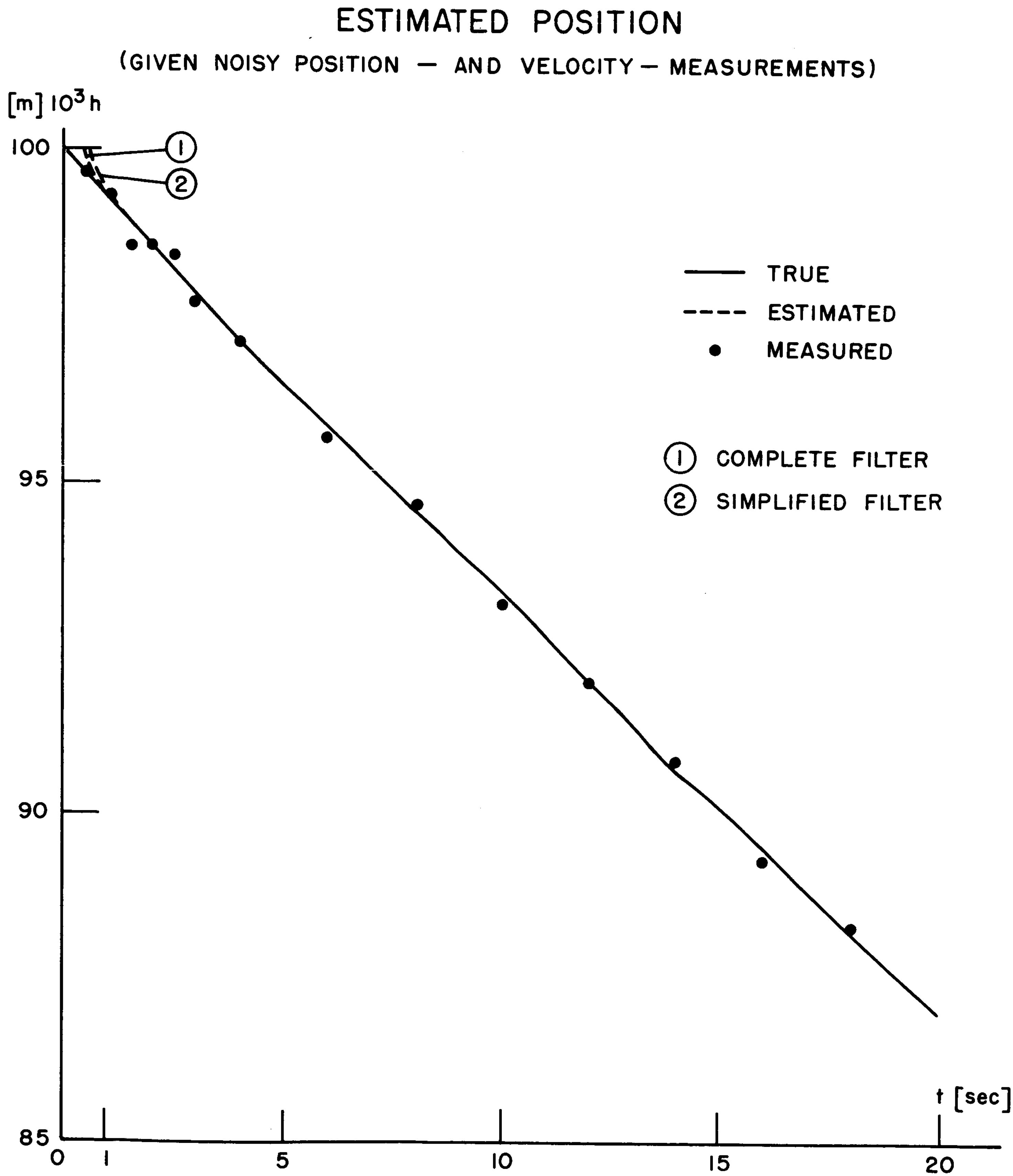

ESTIMATED POSITION
(GIVEN NOISY POSITION — AND VELOCITY — MEASUREMENTS)
[m] 10³ h
100
95
90
85
0
1
5
10
15
20
t [sec]
TRUE
ESTIMATED
MEASURED
1 COMPLETE FILTER
2 SIMPLIFIED FILTER

Fig. 2.11.6.a

The space vehicle encounts changing dynamic environments during the course of atmospheric entry and landing maneuver. This is essentially due to the change of atmospheric density from a dynamically insignificant value at high (orbital) altitudes to a dynamically dominating factor at low altitudes. Because of our very limited and hardly improvable knowledge on planetary atmospheric conditions any control program for any landing mission to atmospheric planets must exhibit adaptive features. By "adaptive features" we mean such elements in the control program which are designed

(1) to estimate the current state of the space vehicle in the presence of (external) dynamic and measurement noise and by starting with assumed (presumably wrong) atmospheric parameter values, and,

(2) to improve (or confirm) the assumed atmospheric parameter values during a limited phase (or limited phases) of the atmospheric flight

in order to be able to make the proper decisions in the control program to arriving the prescribed terminal state in an "optimal" way.

The main part of the present study was devoted to investigate the feasibility of a proposed Non-linear Filter, formulated as a system of coupled, ordinary non-linear differential equations with unknown initial conditions, for sequentially estimating state and parameters during a limited part of atmospheric flight. Several interesting properties of the Non-linear Filter, as applied to the present problem, are pointed out.

The main, new results of this investigation consist of demonstrating the possibility of simplifying the Non-linear Filter to a considerable extent as far as the involved mathematical operations are concerned. The introduced simplifications can be properly formulated in terms of an "Asymptotic Non-linear Filter" which exhibits the same merits as the original, complete filter does in the present problem. By "asymptotic" we mean the possibility of precomputing the settled-down values of the gain matrix, and, then using these values as constants in the State Estimator Equations from the initiation of sequential estimation in real time. In practical terms this means that, given an "n"-dimensional state vector, the Asymptotic Filter necessitates the solution (or implementation) of "n" coupled, ordinary, non-linear differential equations, while the original, complete filter necessitates the solution (or implementation of $N = n + \sum_{i=1}^{n} i$ coupled, ordinary, non-linear differential equations.

Besides additional computations for feasibility studies on the Non-linear Filter applied to the soft-lander problem, the future work will be concentrated on (1) existence problems related to the Non-linear Filter; (2) stability of the Asymptotic Non-linear Filter; (3) control based on estimated state, and, (4) reconsiderations of some fundamental aspects of the non-linear filtering problem.

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2. Bellman et.al.: Invariant Imbedding and Non-linear Filtering. (RAND, Research Memo. No. 4374-PR, December 1964)

3. Detchmendy and Sridhar: Sequential Estimation of State and Parameters in Noisy Nonlinear Dynamical Systems. (Journal of Basic Engineering, Vol. 88, Series D, No. 2, pp. 362-9.)

APPENDIX A

APPROXIMATING THE VECTOR, SEQUENTIAL, NONLINEAR, LEAST SQUARES FILTER FOR EASE OF IMPLEMENTATION

Introduction

Nonlinear Filter

Consider the system,

$$\dot{x} = g(t,x) + k(t,x)u \tag{A.1}$$

$$y(t) = h(t,x) + (\text{unknowables}) \tag{A.2}$$

where,

x:	$n \times 1$	state vector
$g(t,x)$:	$n \times 1$	vector function
$k(t,x)$:	$n \times p$	matrix function
u:	$p \times 1$	unknown input
$h(t,x)$:	$m \times 1$	vector function
y:	$m \times 1$	observation

No statistical assumptions are made concerning the observation error or the unknown input. Defining the vector residual errors,

$$\epsilon_1(t) = y(t) - h(t,\bar{x}) \tag{A.3}$$

$$\epsilon_2(t) = \dot{\bar{x}} - g(t,\bar{x}) \tag{A.4}$$

where $\bar{x}(t)$, $0 \le t \le T$ denotes a nominal trajectory, and criterion function,

$$\int_0^T [\|\epsilon_1(t)\|_Q^2 + \|\epsilon_2(t)\|_W^2]\, dt \tag{A.5}$$

then the estimation problem is to minimize (A.5) with respect to $\bar{x}(t)$, $0 \le t \le T$ given the observations $y(t)$, $0 \le t \le T$. If $\hat{x}(t)$ is the minimizing function then $\hat{x}(T)$ is the best least-squares estimate of $x(T)$.

The above is equivalent to minimizing, with respect to $\bar{x}(t)$ and $\bar{u}(t)$ $0 \le t \le T$, the expression,

$$\int_0^T \left[\|y-h(t,\bar{x})\|_Q^2 + \|\bar{u}\|^2_{k^T W k} \right] \tag{A.6}$$

subject to the differential constraint,

$$\dot{\bar{x}} = g(t,\bar{x}) + k(t,\bar{x})\,\bar{u} \tag{A.7}$$

with, $\bar{x}(0)$ and $\bar{x}(T)$ free, and T fixed.

By application of optimal control theory the above problem is seen to be equivalent to the two-point boundary value problem,

$$\dot{x}^* = \frac{\partial H^*}{\partial \lambda}(t,x^*,\lambda) \tag{A.8}$$

$$\dot{\lambda} = -\frac{\partial H^*}{\partial x^*}(t,x^*,\lambda) \quad ; \quad \lambda(0) = 0 \quad , \quad \lambda(T) = 0 \tag{A.9}$$

where,

$$H^*(t,x^*,\lambda) = \|y(t)-h(t,x^*)\|_Q^2 + \langle \lambda, g(t,x^*) \rangle - \frac{1}{4} \langle \lambda, kV^{-1}k^T\lambda \rangle \tag{A.10}$$

$$V(t,x^*) = k^T(t,x^*)\, W(t,x^*)\, k(t,x^*) \tag{A.11}$$

This problem is now converted to a sequential estimator problem by regarding T as a variable and replacing $\lambda(T) = 0$ by $\lambda(T) = c$ where $-\infty < c < \infty$, with T and c independent variables. If we define the missing terminal condition on x by $r(c,T)$, then $r(c,T)$ satisfies the partial differential equation resulting from the invariant imbedding procedure,

$$\frac{\partial r}{\partial T} - \frac{\partial r}{\partial c}\frac{\partial H^*}{\partial r}(T,r,c) = \frac{\partial H^*}{\partial c}(T,r,c) \tag{A.12}$$

A solution to this equation is sought of the form,

$$r(c,T) = \hat{x}(T) + P(T)\ c \tag{A.13}$$

where $P(T)$ is an $n \times n$ matrix and c is an n-vector. Substituting this equation in (A.12), expanding about $r(0,T)$, and retaining terms to first order in c yields the equations of the nonlinear filter,

$$\frac{d\hat{x}}{dT} = g(T,\hat{x}) + P(T)\ H(T,\hat{x})\ Q[y(T)-h(T,\hat{x})] \tag{A.14}$$

$$\frac{dP}{dT} = \left[\frac{\partial g}{\partial \hat{x}}(T,\hat{x})\right]P + P\left[\frac{\partial g}{\partial \hat{x}}(T,\hat{x})\right]^T + P\{HQ[y(T)-h(T,\hat{x})]\}_{\hat{x}}\ P$$
$$+\ k(T,\hat{x})\ V^{-1}(T,\hat{x})\ k^T(T,\hat{x}) \tag{A.15}$$

where,

$$H(T,\hat{x}) = \left[\frac{\partial h}{\partial \hat{x}}(T,\hat{x})\right]^T = \left[\frac{\partial h_i}{\partial \hat{x}_j}\right]^T$$

$\{HQ[y(T)-h(T,\hat{x})]\}_{\hat{x}}$ is an $n \times n$ matrix with i^{th} column

$$\frac{\partial}{\partial \hat{x}_i}\{HQ[y(T)-h(T,\hat{x})]\}$$

Discussion

In general for a system of dimension n, a mechanization of the nonlinear filter requires the solution of $n + n^2$ first order, nonlinear, ordinary differential equations. If an adaptive control design is sought then control is based on estimated state, i.e. the estimated state is viewed as the true state and the control problem is solved deterministically. In this way the estimation and control problems are uncoupled but the requirement remains that the estimation problem be solved on-line in real-time. In this situation either hardware and/or the economic consequences of computer requirements could make the realization of the full nonlinear filter (equations (A.14) and (A.15)) impractical. Hence economic or computer capability limitations force us to seek approximations to equations (A.14) and (A.15) in practical applications.

In the sequel we will present some results obtained for a specific system which illustrate a successful approach to the approximation problem.

Experimental Results

Introduction.

Let the plant and observations be described by,

$$\ddot{x} + 3\dot{x} + 2x + ax^3 = 5 \sin t + \eta_1(t) \qquad \text{(A.16)}$$

$$y(t) = x(t) + \eta_2(t) \qquad \text{(A.17)}$$

We wish to estimate $x(t)$, $\dot{x}(t)$, and the constant parameter a, sequentially. Adjoining the parameter to the original state equations yields,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - ax_1^3 - 3x_2 + 5 \sin t + \eta_1(t) \qquad \text{(A.18)}$$

$$\dot{a} = 0$$

Note that a statistical approach would require a knowledge of the mean and covariance of the noise terms, $\eta_1(t)$ and $\eta_2(t)$. Although this requirement is removed in the present approach, relative weightings for residual errors must still be established and these are obtained intuitively. Hence in Eq.(A.5) we will use $Q(t) = 1$ and $W(t) = I$. The noise terms will be generated as follows: $\eta_1(t)$ uniformly distributed in $[-0.5, 0.5]$ and

$$\eta_2(t) = 0.1\, k_1(t) + 0.1|x_1(t)|\, k_2(t) \qquad \text{(A.19)}$$

with k_1 and k_2 uniformly distributed in $[-0.5, 0.5]$. Note that $\eta_2(t)$ as generated depends on the state. In theory, a statistical approach dictates that $\eta_2(t)$ be independent of the state. In practice this is ignored since it doesn't seem to make an appreciable difference.

The nonlinear filter for this problem consists of the three-dimensional augmented state equations and the nine P equations, six of which are independent since the P matrix is symmetric. Nine initial conditions must be established before the (A.14) and (A.15) system can be integrated. If (A.6) is defined as the "return function" $J(c,T)$, and "dynamic programming" is used to derive the Hamilton-Jacobi equation for $J(c,T)$, it can be shown that the following relation exists,

$$P(T) = 2[J_{cc}(T,\hat{x})]^{-1} \qquad \text{(A.20)}$$

Near the minimum of the surface being searched we would expect J_{cc} to be a maximum and the diagonal terms of P to be a minimum. This fact allows a means to evaluate the "tracking" performance of the filter. For the computer simulations under discussion the estimated state is compared with the true state which was generated beforehand to evaluate tracking performance. In actual practice of course we don't know the true state, but we can evaluate tracking by observing the time evolution of the diagonal elements of the P matrix.

Equations (A.18) will be solved with initial conditions $x_1(0) = x_2(0) = 1$, $a(0) = 0.5$ to generate the true trajectories. For this system (A.14) and (A.15) (the carets on the states have been omitted for convenience) become,

$$\begin{aligned}
\dot{x}_1 &= x_2 + p_{11}(y-x_1) \\
\dot{x}_2 &= - 2x_1 - x_3x_1^3 - 3x_2 + 5 \sin t + p_{12}(y-x_1) \\
\dot{x}_3 &= p_{13}(y-x_1) \\
\dot{p}_{11} &= - p_{11}^2 + 2p_{12} + 1 \\
\dot{p}_{12} &= -(2 + 3x_3x_1^2)\, p_{11} - 3p_{12} - x_1^3p_{13} + p_{22} - p_{11}p_{12} \\
\dot{p}_{13} &= - p_{11}p_{13} + p_{23} \\
\dot{p}_{22} &= - 6p_{22} - (4 + 6x_3x_1^2)\, p_{12} - 2x_1^3p_{23} - p_{12}^2 + 1 \\
\dot{p}_{23} &= - 3p_{23} - (2 + 3x_3x_1^2)\, p_{13} - x_1^3p_{33} - p_{12}p_{13} \\
\dot{p}_{33} &= - p_{13}^2 + 1
\end{aligned} \qquad \text{(A.21)}$$

where the full system has been simplified by the relations $p_{12} = p_{21}$,

$p_{13} = p_{31}$, $p_{23} = p_{32}$, $x_3 \triangleq a$, and,

$$y(t) = x_1(t) + 0.1\, k_1(t) + 0.1|x_1(t)|\, k_2(t) \tag{A.22}$$

where the $x_1(t)$ appearing in (A.22) is that obtained by integrating system (A.18) with initial conditions,

$$\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 0 \\ x_3(0) &= 0.5 \end{aligned} \tag{A.23}$$

In the sequel the system (A.21) will be referred to as the "full" filter.

Initial Conditions.

The first thing to be determined is how the tracking performance of the full filter depends upon the assumed initial conditions $\hat{x}(0)$ and $P(0)$. From Eq.(A.20) and the subsequent discussion we realize that if,

$$\hat{x}(0) = m_0 \tag{A.24}$$

then,

$$P(0) = 2[J_{cc}(0,m_0)]^{-1} = P_0 \tag{A.25}$$

However, lacking any a priori information on $x(0)$ it is reasonable to choose $\hat{x}(0) = 0$. Similarly, lacking an analytic expression for $J(c,T)$, $P(0)$ must be chosen intuitively. As a first guess we chose,

$$P(0) = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \tag{A.26}$$

For the computations an integration interval size of .01 and total integration time of 10 were used.

The results for the full filter with the above choice of initial conditions are presented in Figures 1.1 through 1.4. It can be seen that $x(t)$ is tracking in approximately 4.5 seconds, $\dot{x}(t)$ in about 3 seconds, and the parameter a still has a small offset at the end of 10 seconds. The diagonal terms of the P matrix are shown in Figure 1.4.

A close inspection of Eqs.(A.21) and the "steady-state" oscillations of the diagonal terms of the P matrix indicates that tracking speed may be improved by making $p_{33} > p_{22} > p_{11}$ initially. In this case we chose $\hat{x}(0) = 0$ as before and,

$$P(0) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 10 & 1 \\ 1 & 1 & 20 \end{bmatrix} \tag{A.27}$$

The results are shown in Figures 2.1 through 2.4. Now $x(t)$ tracks in approximately 3 seconds, $\dot{x}(t)$ in 2.5 seconds, and the parameter a in 4 seconds. The operation of the full filter has been markedly enhanced by a more judicious choice of $P(0)$.

An examination of the behavior of the diagonal elements of the P matrix (Fig. 2.4) suggests another variation. These terms settle down to some steady nonsinusoidal oscillation about some average value after the filter is tracking. Another case was run with the diagonal elements of $P(0)$ set to these average values in order to determine if performance could be thereby improved. Accordingly we set

$$P(0) = \begin{bmatrix} 1.3 & 1 & 1 \\ 1 & 3.2 & 1 \\ 1 & 1 & 4.5 \end{bmatrix} \tag{A.28}$$

The results are shown in Figures 3.1 through 3.4. It can be seen that the tracking performance of the filter has not been noticeably affected by using initial conditions (A.28) in place of (A.27).

Filter Approximations.

Examination of the off-diagonal elements of $P(t)$ in the previous case reveals that these terms all oscillate about a zero average value. This suggests a possible approximate filter with the diagonal elements fixed at their initial values, viz. (A.28) and all of the filter dynamics supplies by the off-diagonal elements of $P(t)$. Setting $\dot{p}_{11} = \dot{p}_{22} = \dot{p}_{33} = 0$ results in the approximate filter,

$$\begin{aligned}
\dot{x}_1 &= x_2 + p_{11}(y-x_1) \\
\dot{x}_2 &= -2x_1 - x_3x_1^3 - 3x_2 + 5\sin t + p_{12}(y-x_1) \\
\dot{x}_3 &= p_{13}(y-x_1) \\
\dot{p}_{12} &= -(2 + 3x_3x_1^2)\, p_{11} - 3p_{12} - x_1^3 p_{13} + p_{22} - p_{11}p_{12} \\
\dot{p}_{13} &= -p_{11}p_{13} + p_{23} \\
\dot{p}_{23} &= -3p_{23} - (2 + 3x_3x_1^2)\, p_{13} - x_1^3 p_{33} - p_{12}p_{13}
\end{aligned} \tag{A.29}$$

with,

$$p_{11}(t) = p_{11}(0)$$

$$p_{22}(t) = p_{22}(0)$$

$$p_{33}(t) = p_{33}(0)$$

to be solved with the initial conditions,

$$x(0) = 0 \tag{A.30}$$

$$P(0) = \begin{bmatrix} 1.3 & 1 & 1 \\ 1 & 3.2 & 1 \\ 1 & 1 & 4.5 \end{bmatrix} \tag{A.31}$$

Figures 4.1 through 4.4 display the results for this case. A comparison with Figures 3.1 through 3.4 shows that the approximate filter, (A.29) works as well in all respects as the full filter, (A.21). An examination of the off-diagonal elements (not presented in this report) reveals different behavior between the two cases, as would be expected.

Another approximation is suggested by the fact that $p_{12}(t)$ oscillates about an average value of -0.5 over a smaller range than either $p_{13}(t)$ or $p_{23}(t)$. Accordingly, we set $\dot{p}_{12} = 0$ and set $p_{12}(t) = p_{12}(0)$ for the next approximation,

$$\begin{aligned}
\dot{x}_1 &= x_2 + p_{11}(y-x_1) \\
\dot{x}_2 &= -2x_1 - x_3x_1^3 - 3x_2 + 5 \sin t + p_{12}(y-x_1) \\
\dot{x}_3 &= p_{13}(y-x_1) \\
\dot{p}_{13} &= -p_{11}p_{13} + p_{23} \\
\dot{p}_{23} &= -3p_{23} - (2 + 3x_3x_1^2)\, p_{13} - x_1^3p_{33} - p_{12}p_{13}
\end{aligned} \tag{A.32}$$

with,

$$p_{11}(t) = p_{11}(0)$$

$$p_{12}(t) = p_{12}(0)$$

$$p_{22}(t) = p_{22}(0)$$

$$p_{33}(t) = p_{33}(0)$$

to be solved with the initial conditions,

$$x(0) = 0 \tag{A.33}$$

$$P(0) = \begin{bmatrix} 1.3 & -0.5 & 1 \\ -0.5 & 3.2 & 1 \\ 1 & 1 & 4.5 \end{bmatrix} \tag{A.34}$$

The results for this case are presented in Figures 5.1 through 5.3. A close comparison with Figures 4.1 through 4.3 reveals that this more approximate filter works even better (eg. $x(t)$ and a are tracking sooner) than the previous approximation!

The next approximation made consists of two steps. First, from the previous computation a value of .077 for $p_{13}(0)$ is suggested. Second, an attempt will be made to eliminate one of the remaining differential equations by replacing it with an algebraic equation. Accordingly the next approximate filter becomes,

$$\begin{aligned}
\dot{x}_1 &= x_2 + p_{11}(y-x_1) \\
\dot{x}_2 &= -\,2x_1 - x_3x_1^{\,3} - 3x_2 + 5\sin t + p_{12}(y-x_1) \\
\dot{x}_3 &= p_{13}(y-x_1) \\
\dot{p}_{23} &= -\,3p_{23} - (2 + 3x_3x_1^{\,2})\,p_{13} - x_1^{\,3}p_{33} - p_{12}p_{13} \\
p_{13} &= p_{23}/p_{11}
\end{aligned} \tag{A.35}$$

to be solved with the initial conditions,

$$x(0) = 0 \tag{A.36}$$

$$P(0) = \begin{bmatrix} 1.3 & -.05 & .077 \\ -.05 & 3.2 & 1 \\ .077 & 1 & 4.5 \end{bmatrix} \tag{A.37}$$

The results are presented in Figures 6.1 through 6.3. For comparison purposes the full filter (A.21) was used with initial conditions (A.36) and (A.37). These results are presented in Figures 7.1 through 7.3. It can be seen that the approximate filter (A.35) tracks even better than the full filter in this case!

The final approximation results from eliminating the remaining P matrix differential equation. In this case both p_{13} and p_{23} are computed from algebraic equations. The resulting equations will be called the "algebraic" filter, viz.,

$$\begin{aligned} \dot{x}_1 &= x_2 + p_{11}(y-x_1) \\ \dot{x}_2 &= -2x_1 - x_3 x_1^3 - 3x_2 + 5 \sin t + p_{12}(y-x_1) \\ \dot{x}_3 &= p_{13}(y-x_1) \\ p_{13} &= \frac{-(x_1^3 p_{33})}{(2 + 3x_3 x_1^2 + p_{12} + 3p_{11})} \\ p_{23} &= p_{11} p_{13} \end{aligned} \tag{A.38}$$

A solution of the algebraic filter equations with initial conditions,

$$x(0) = 0 \tag{A.37}$$

$$P(0) = \begin{bmatrix} 1.3 & -.05 & 0 \\ -.05 & 3.2 & 0 \\ 0 & 0 & 4.5 \end{bmatrix} \tag{A.40}$$

is presented in Figures 8.1 through 8.3. These should be compared with the operation of the full filter (A.21), with the same initial conditions, presented in Figures 9.1 through 9.3. It can be seen that the performance of the algebraic filter is superior to that of the full filter in this case!

In an attempt to ascertain whether the operation of the algebraic filter is input dependent the driving function to the system was changed. That is, Eq.(A.16) was replaced by,

$$\ddot{x} + 3\dot{x} + 2x + ax^3 = f(t) + \eta_1(t) \tag{A.41}$$

where,

$$f(t) = \begin{cases} 5(1 - e^{-t/5}) & , \quad t \le 10 \\ 5(1 - e^{-2})e^{-t/10} & , \quad t > 10 \end{cases} \tag{A.42}$$

The operation of the algebraic filter with 5 sin t in (A.38) replaced by (A.42) and initial conditions (A.39) and (A.40) is presented in Figures 10.1 through 10.3. For comparison purposes the full filter was used under the same conditions and those results are presented in Figures 11.1 through 11.3. It can be seen that the performance of the full filter and approximate filter is essentially identical!

Conclusions.

Although we have considered a specific example it is felt that the approach presented is one which will work for any problem for which the solution has been demonstrated to converge. Experiments with P(0) can be done to accelerate convergence, and then approximations to the filter equations can be undertaken in the systematic manner which we have illustrated.

It is evident from an examination of (A.21) and (A.38) that the latter form of the filter places much less stringent demands on real-time computation facilities and, as we have seen above, results in no deterioration of performance. In any specific estimation and control application with hardware, space, and economic constraints, our ability to derive feasible approximations to the filter equations such as (A.38) may be the factor which determines the overall quality of the solution to the control problem.

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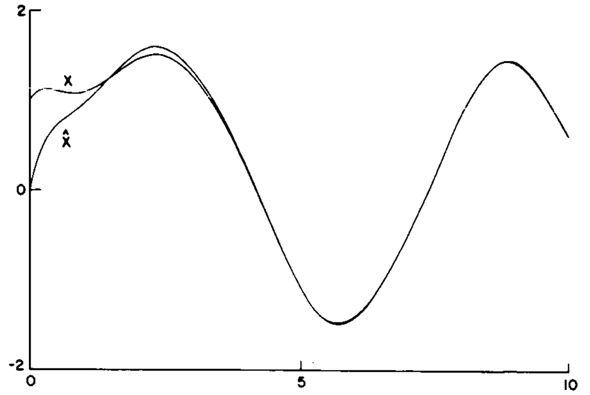


Fig. 1.1

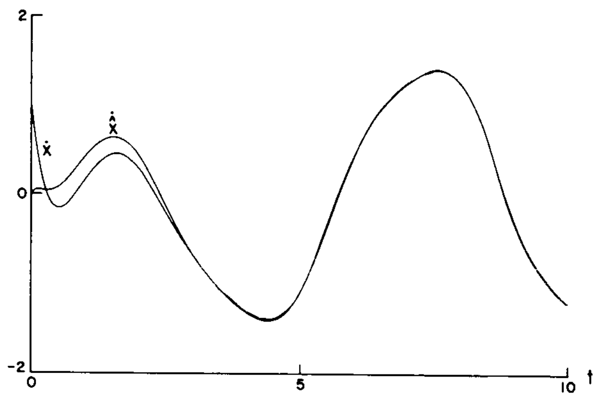


Fig. 1.2

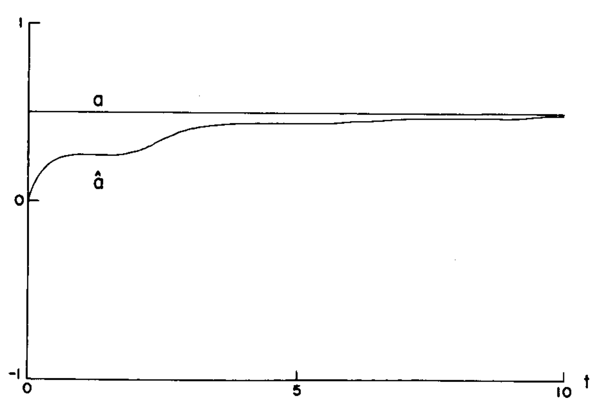
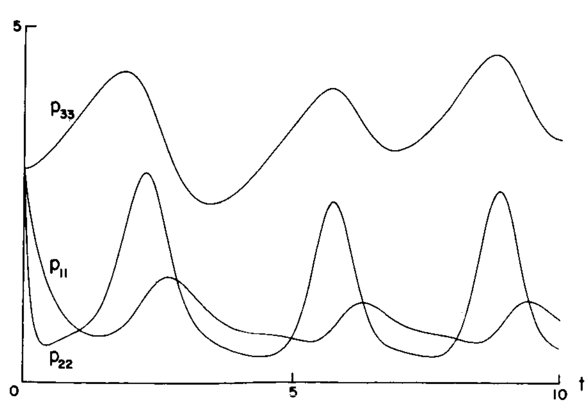


Fig. 1.3



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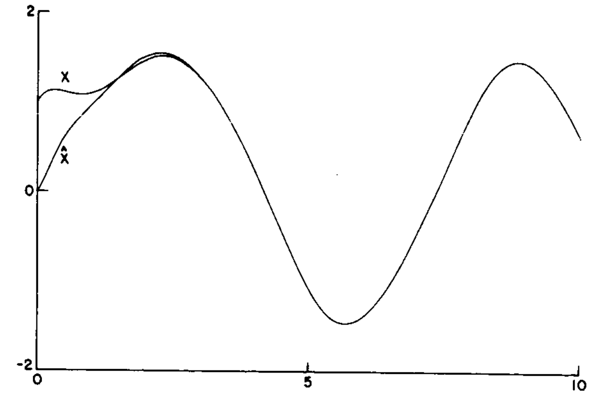

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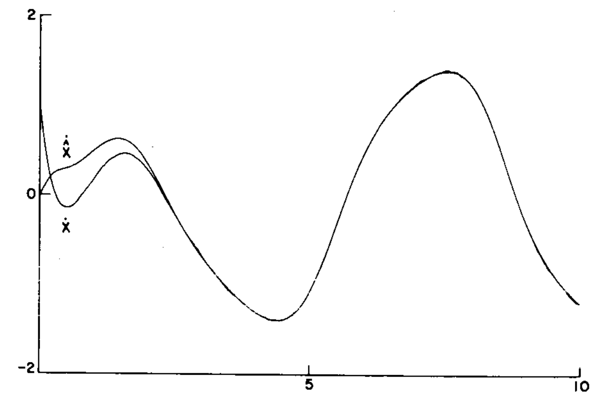

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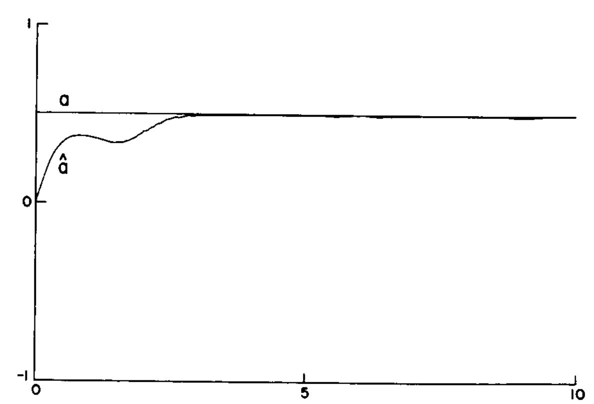

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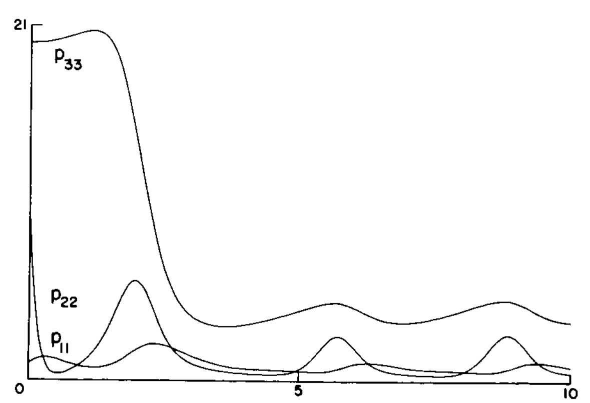
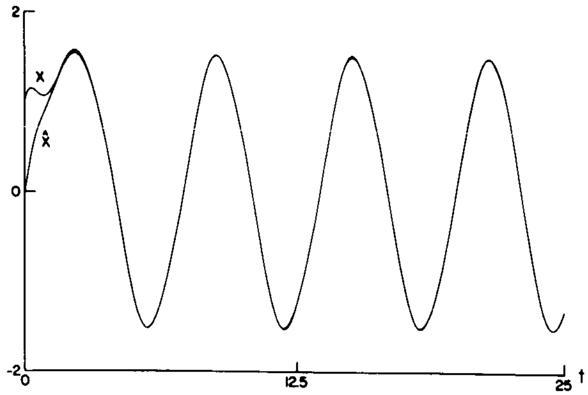


Fig. 2.4



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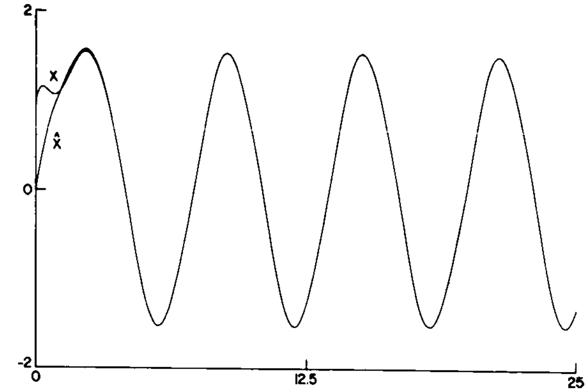

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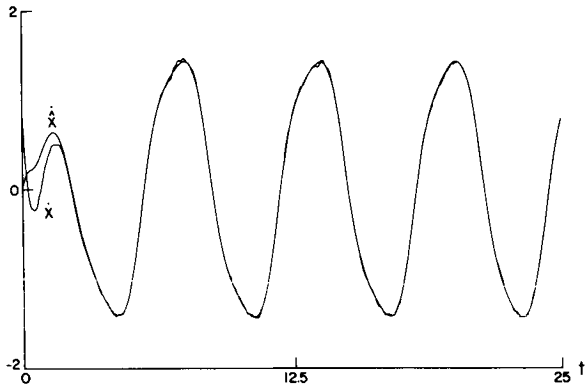

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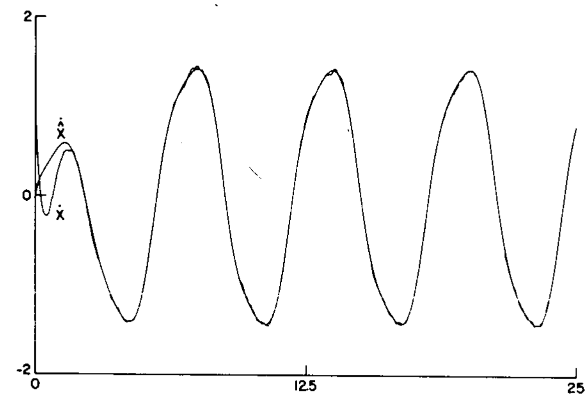

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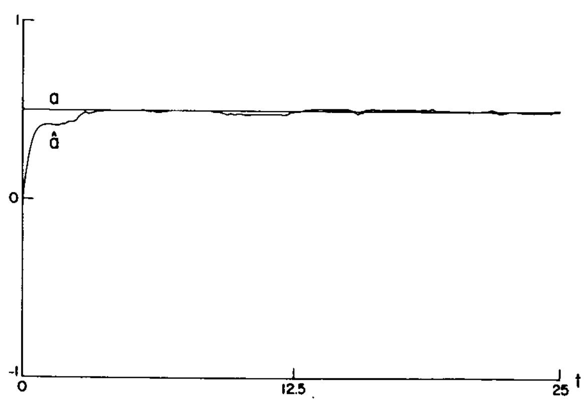

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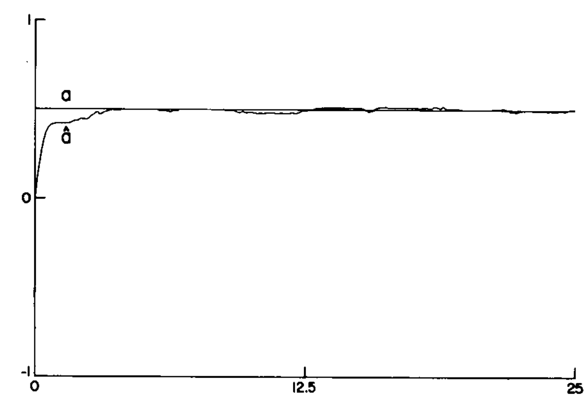

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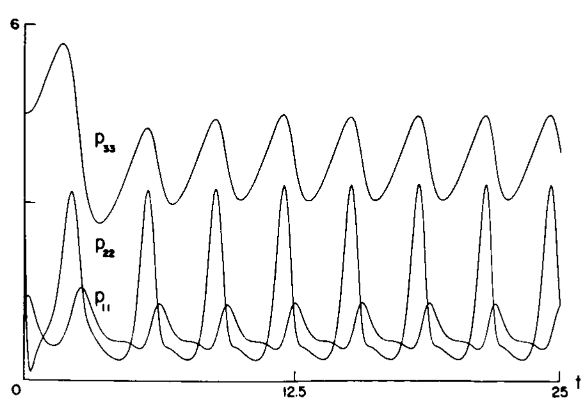

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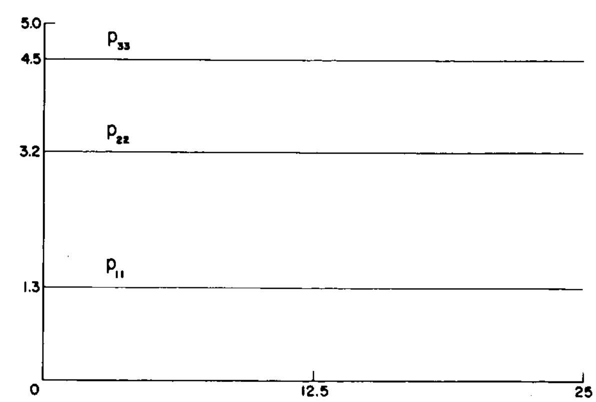

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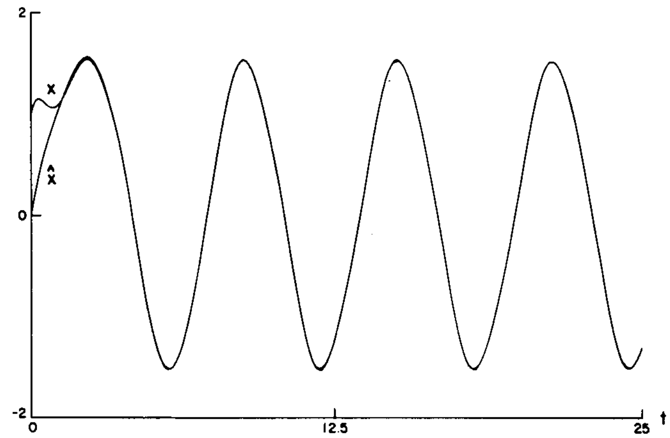

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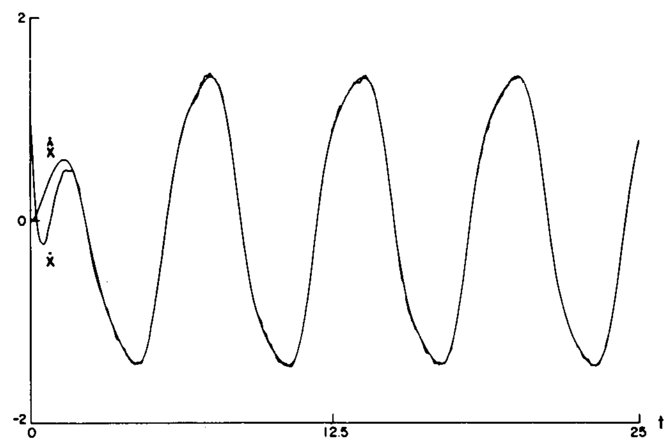

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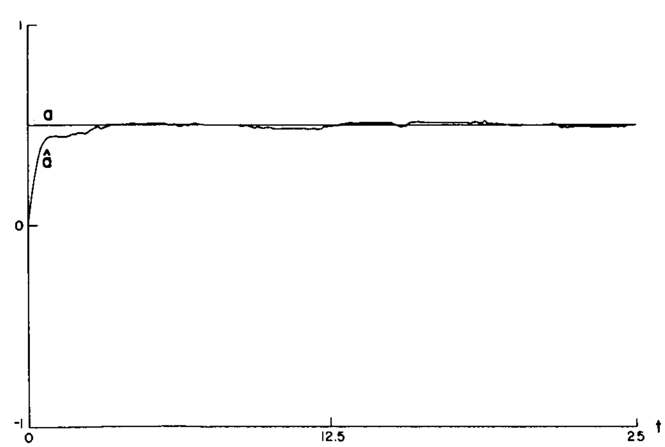

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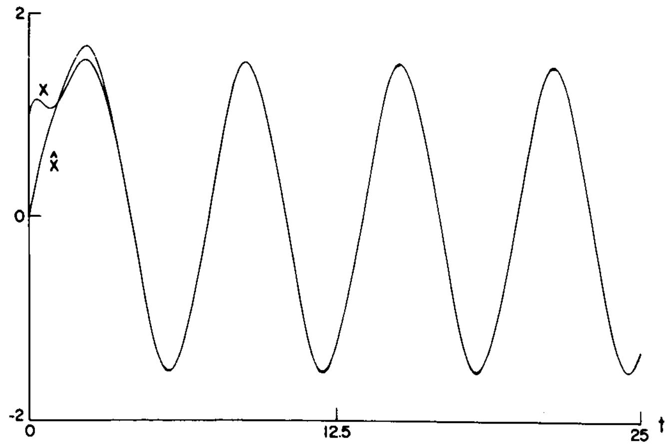


Fig. 6.1

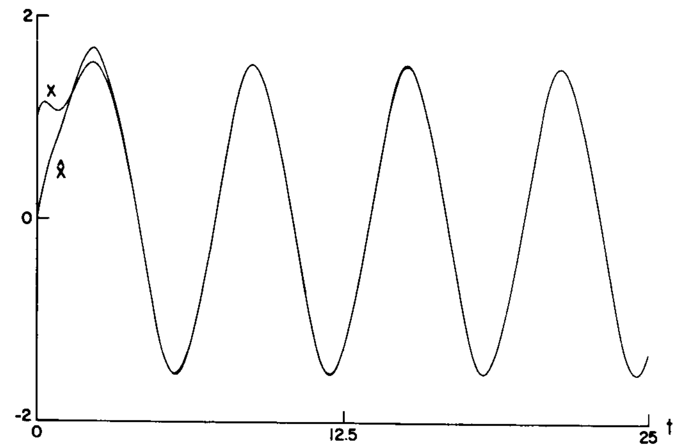
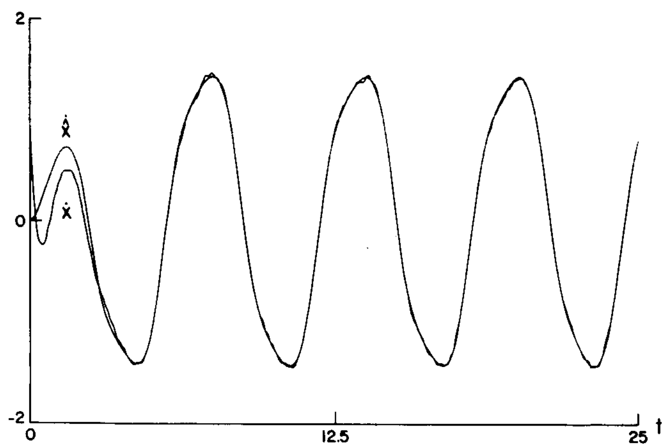


Fig. 7.1



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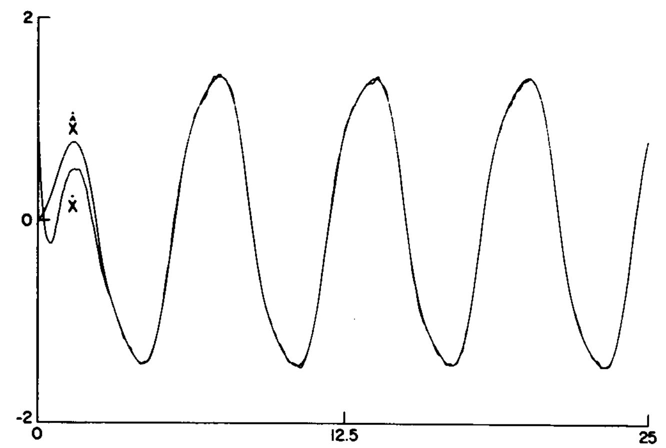

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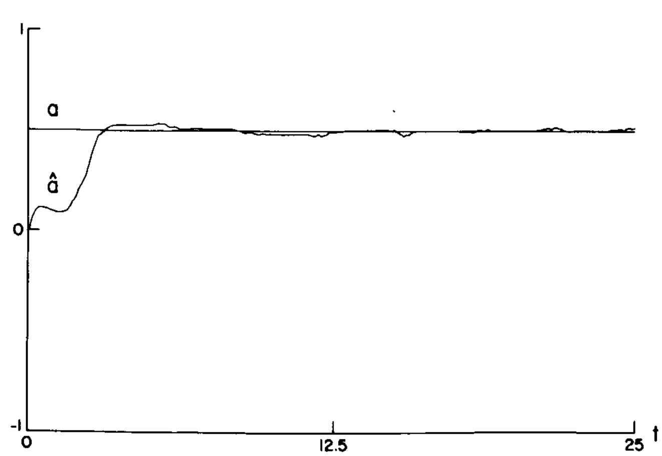
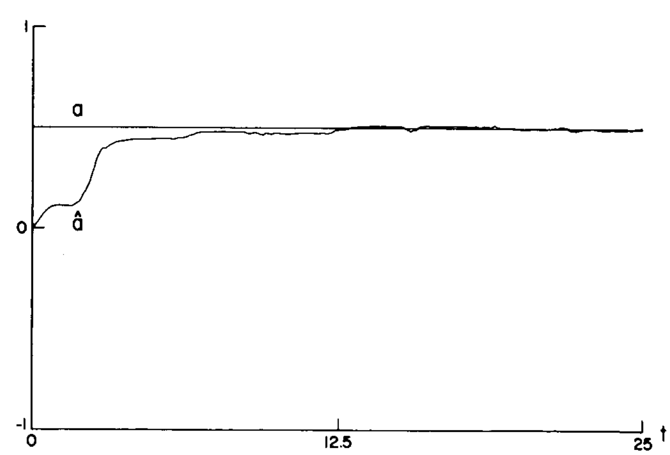


Fig. 6.3



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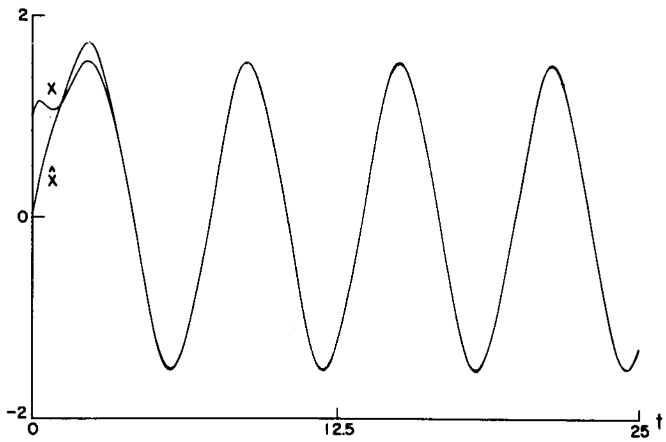

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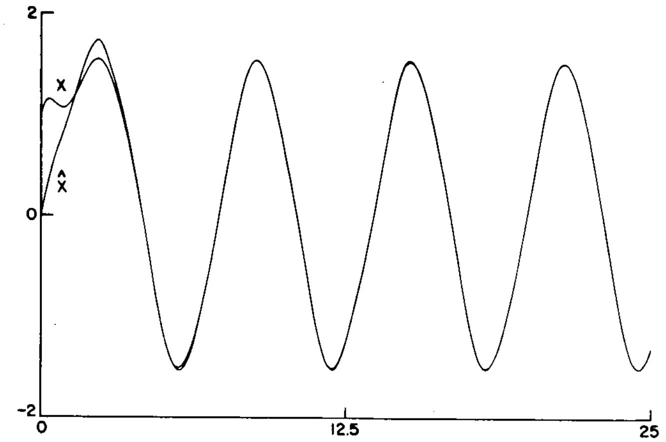

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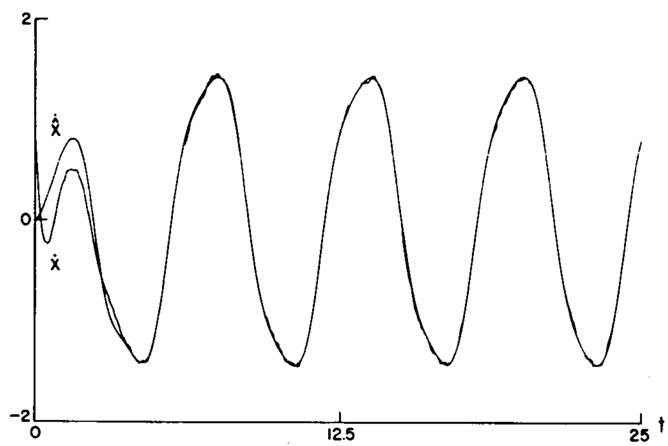

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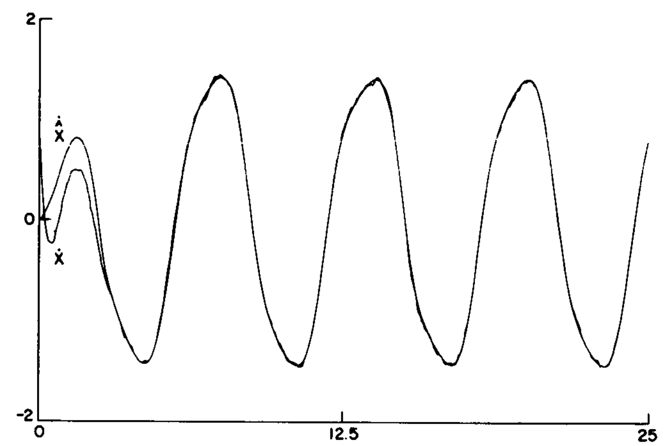

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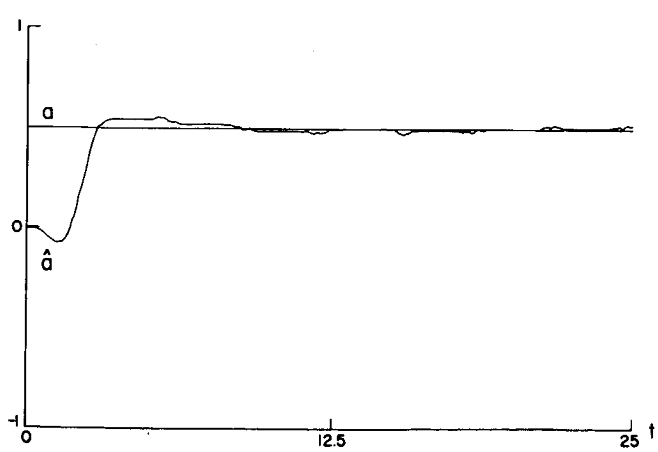

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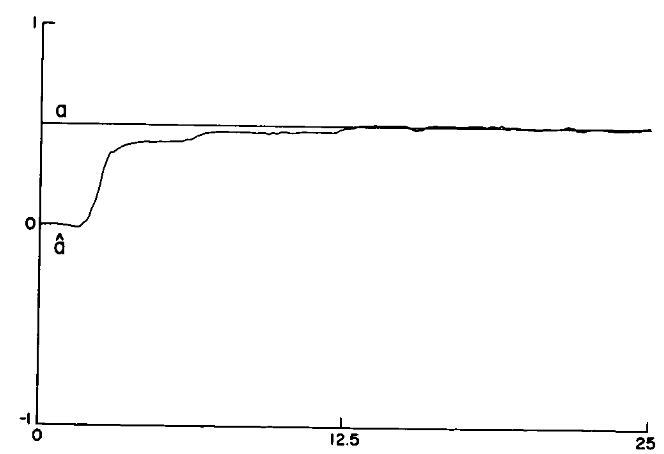

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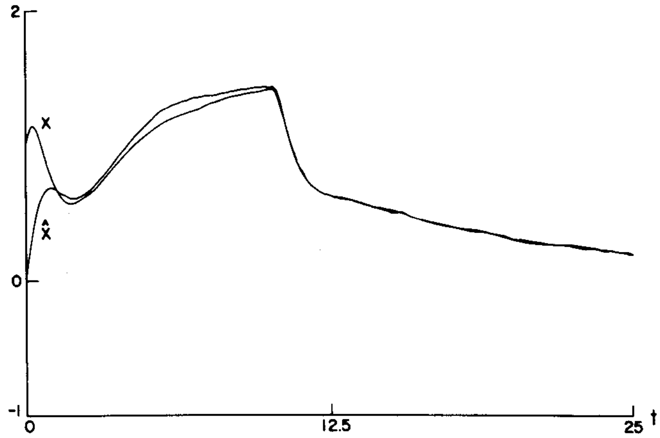

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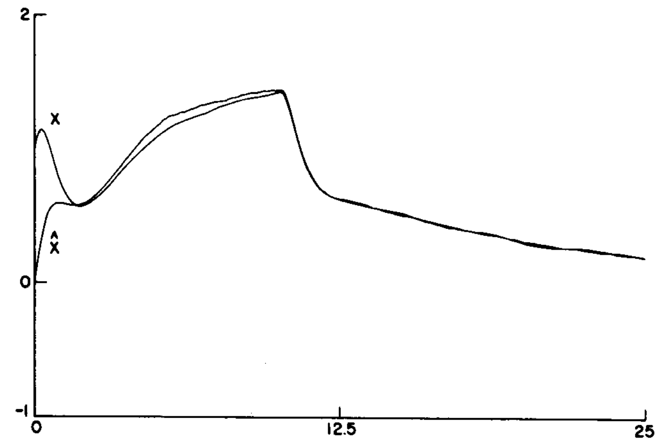

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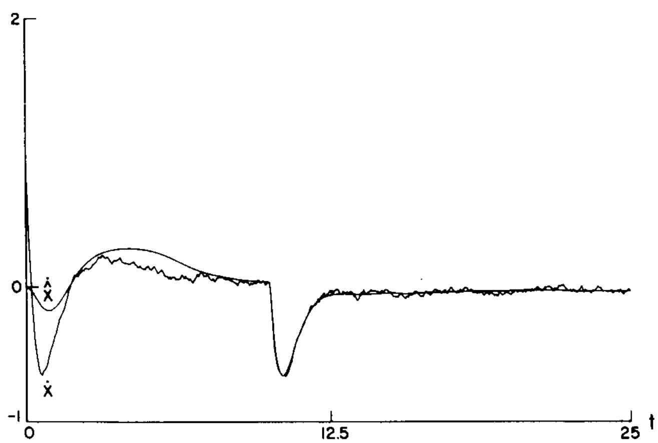

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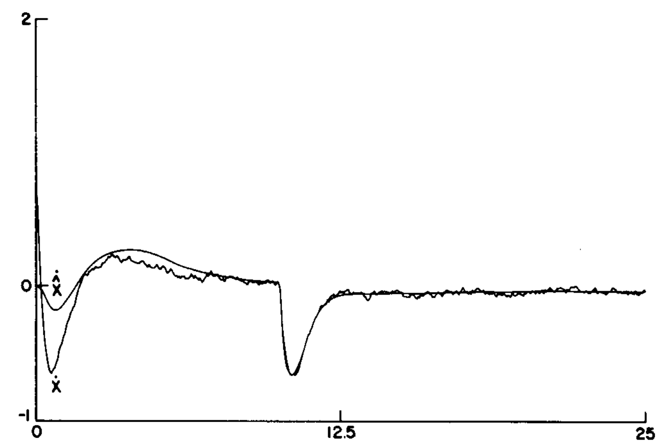

Fig. 11.2

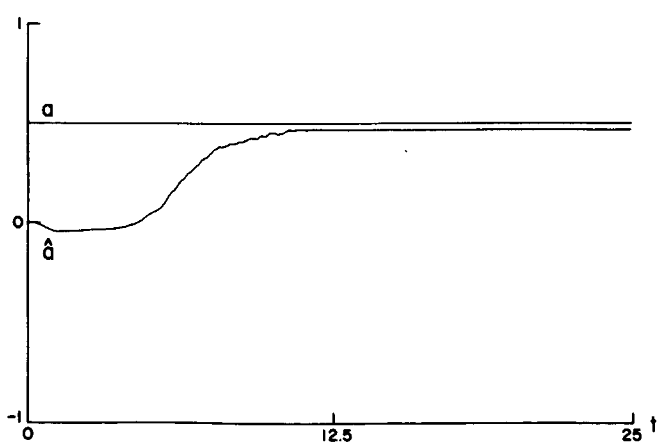

Fig. 10.3

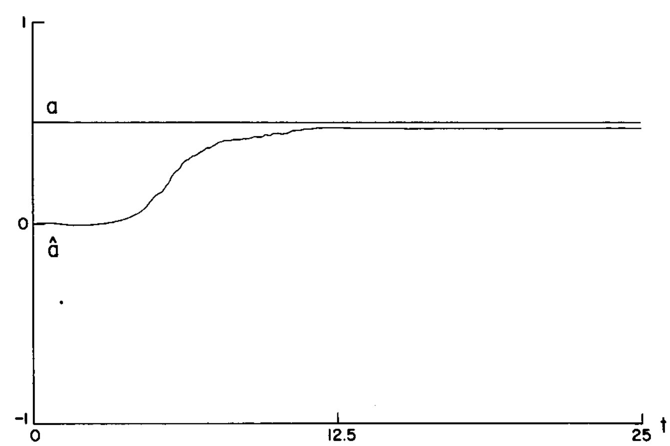

Fig. 11.3

CHAPTER 3

MINIMUM-ENERGY CONTROL OF ELECTRIC PROPULSION VEHICLES

3.1. Introduction.

This chapter presents the analytical results obtained in the determination of the minimum-energy controller for a class of electric propulsion vehicles. The purpose of the controller is the accomplishment of various control actions for the vehicle while minimizing the net energy flow from a rechargeable battery. The control voltage is applied to the armature circuit of a d-c motor with fixed but reversible field excitation which produces the propulsive force.

The complete mathematical description of the plant is obtained by using Lagrange's energy methods from classical mechanics and making the following assumptions:

1. The motion of the vehicle takes place on a terrain whose profile can be approximated by straight lines, each of which is inclined at a particular constant slope angle with respect to the horizontal direction.

2. The disturbance torque which appears in the plant equation remains constant during the transient phase of the control. This is a reasonable assumption if the time interval during the transient phase is much smaller than the shortest time taken for the vehicle to pass over a portion of the terrain with constant slope.

3. The air resistance and viscous friction in the system are proportional to the speed of the vehicle. This is correct as long as the speed remains below a certain threshold value.

4. The speed of the vehicle is proportional to the speed of the electric drive motor.

5. The effective system moment of inertia is constant.

6. The motor armature inductance is negligible. This assumption is satisfactory for small motors but is inaccurate for large machines.

7. The motor operation is unsaturated which means that the armature flux is proportional to the armature current.

The plant equation is

$$\dot{x}(t) = -\left(\frac{f_e}{j_e} + \frac{k_t k_b}{j_e r_a}\right) x(t) - \frac{1}{j_e} v(t) + \frac{k_t}{j_e r_a} u(t) \tag{3.1}$$

where $x(t)$, $\dot{x}(t)$ = Angular speed and angular acceleration of the motor

$u(t)$ = Armature control voltage

$v(t)$ = Disturbance torque

j_e, f_e = Effective inertia and damping coefficients respectively

r_a = Armature resistance

k_t, k_b = Motor torque and back emf constants respectively

The performance index to be minimized for this problem is selected to be

$$E(u) = \int_0^T \left(\frac{1}{r_a} u^2(t) - \frac{k_b}{r_a} u(t)\, x(t)\right) dt \tag{3.2}$$

(see Appendix for the derivation of Eq.(3.1) and Eq.(3.2)).

The integrand of Eq.(3.2) represents the electrical power which can flow from the battery into the motor circuit or from the motor circuit into the battery over some intervals of time during the controlling process. Therefore energy transfer takes place into and from the battery. In the transient phase of the control process, $0 \le t \le T$, the nature of electrical energy transfer may be best understood in reference to the following cases:

1. During the speed-control, whenever a larger positive disturbance torque is applied to the system, it is required to increase the control voltage in order to keep the vehicle speed constant. Since the control voltage is larger than the back emf voltage, the energy transfer takes place from the battery into the motor circuit.

2. During the speed-setting, if the disturbance torque is positive and it is required to increase the speed, the control voltage must be increased in order to obtain the desired speed. Thus, once more the energy transfer is from the battery into the motor circuit.

3. During the speed-control of the vehicle if a large negative disturbance torque is applied to the system, the control voltage must be reduced in magnitude in order to maintain the speed constant. Over the interval of time in which back emf voltage is greater than the control voltage the energy transfer takes place from the motor circuit into the battery. The motor current reverses its polarity in the armature circuit and its magnitude is controlled in such a way that the speed of the vehicle is brought back to its desired value at $t = T$. In some cases of speed-control at low speeds with large negative disturbance torques applied to the system, mechanical brakes may be used to supplement the controller effort.

4. During the speed setting of the vehicle under positive disturbance torques, if it is desired to reduce the speed of the vehicle by reducing the control voltage; the back emf voltage becomes greater than the control voltage. Therefore the energy transfer is from the motor circuit into the battery. This condition exists until the control voltage exceeds the back emf voltage in order to supply the necessary motor drive torque corresponding to new desired speed. The energy transfer is now from the battery into the motor circuit.

The above conditions indicate that in general, $E(u)$, is a measure of net flow of energy from the battery into the motor circuit during the transient phase of the control.

The set of boundary conditions to be satisfied by the state variable $x(t)$ for three cases of the control action are given below:

Case 1. The speed-control.

$$x(0) = x(T) = \alpha \tag{3.3}$$

$$v(0-) \neq v(0+) = v(T) = \beta$$

Case 2. The speed-setting.

$$x(0) = x_0 \tag{3.4}$$

$$x(T) = \alpha$$

$$v(0-) = v(0+) = v(T) = \beta$$

Case 3. The combination of Case 1. and Case 2.

$$x(0) = x_0 \tag{3.5}$$

$$x(T) = \alpha$$

$$v(0-) \neq v(0+) = v(T) = \beta$$

3.2. The Statement of the Control Problem.

Given the linear time-invariant system (3.1), the performance index (3.2), a terminal time T, and no constraints on the control $u(t)$ determine the control $u(t)$ which satisfies the set of boundary conditions:

Eq.(3.3) for speed-control

Eq.(3.4) for speed-setting

Eq.(3.5) for speed-control and speed-setting

and minimizes the performance index given by Eq.(3.2). It is important to note that for $t \geq T$, it is required to maintain the vehicle speed constant at its terminal value until a disturbance such as a new speed-setting or a new disturbance torque comes into the system. It is assumed that the state variable $x(t)$ and the disturbance torque $v(t)$ can be measured exactly by suitable instrumentation.

3.3. The Optimal Solution of the Control Problem.

The Hamiltonian function H for this problem is given by the equation

$$H = \frac{1}{r_a} u^2(t) - \frac{k_b}{r_a} u(t)\, x(t) + \lambda(t) \left(-\left(\frac{f_e}{j_e} + \frac{k_t k_b}{j_e r_a}\right) x(t) - \frac{1}{j_e} v(t) + \frac{k_t}{j_e r_a} u(t) \right) \quad (3.6)$$

where $\lambda(t)$ = Lagrange multiplier.

Pontriagin's minimum principle is used to determine the canonic equations as follows:

The optimal control is found by satisfying the necessary conditions

$$\frac{\partial H}{\partial u(t)} = 0 \quad (3.7)$$

$$\frac{\partial^2 H}{\partial u^2(t)} > 0 \quad (3.8)$$

Hence using (3.7) and (3.8) in (3.6) give

$$\frac{\partial H}{\partial u(t)} = \frac{2}{r_a} u(t) - \frac{k_b}{r_a} x(t) + \lambda(t) \frac{k_t}{j_e r_a} = 0 \tag{3.9}$$

since

$$\frac{\partial^2 H}{\partial u^2(t)} = \frac{2}{r_a} > 0 \qquad \text{for} \qquad r_a > 0 \tag{3.10}$$

From (3.9)

$$u^*(t) = \frac{k_b}{r_a} x(t) - \frac{k_t}{2j_e} \lambda(t) \qquad 0 \le t \le T \tag{3.11}$$

minimizes (3.6).

It is seen from (3.11) $u^*(t)$ is a linear function of $x(t)$ and $\lambda(t)$ and is unique. Therefore $u^*(t)$ is the optimal solution. This is because the system is linear and the performance index is quadratic. Substituting (3.11) into (3.6) gives

$$H^* = - \frac{k_b^2}{4r_a} x^2(t) - \left(\frac{f_e}{j_e} + \frac{k_t k_b}{2j_e r_a}\right) x(t)\lambda(t) - \frac{k_t^2}{4j_e^2 r_a} \lambda^2(t) - \frac{1}{j_e} v(t)\lambda(t) \tag{3.12}$$

Using (3.12) in

$$\dot{x}(t) = \frac{\partial H^*}{\partial \lambda(t)} \tag{3.13}$$

$$\dot{\lambda}(t) = - \frac{\partial H^*}{\partial x(t)}$$

gives the canonic equations

$$\dot{x}(t) = - ax(t) - b\lambda(t) - \frac{1}{j_e} v(t) \tag{3.14}$$

$$\dot{\lambda}(t) = cx(t) + a\lambda(t)$$

where

$$a = \left(\frac{f_e}{j_e} + \frac{k_t k_b}{2 j_e r_a} \right) \qquad a > 0$$

$$b = \frac{k_t^{\,2}}{2 j_e^{\,2} r_a} \qquad b > 0$$

$$c = \frac{k_b^{\,2}}{2 r_a} \qquad c > 0$$

For convenience substitute

$$x^*(t) = x(t) - \alpha \tag{3.15}$$

$$\beta = v(t)$$

in (3.14) to obtain

$$\dot{x}^*(t) = - ax^*(t) - b\lambda(t) - \frac{1}{j_e} \beta - a\alpha$$

(3.16)

$$\dot{\lambda}(t) = cx^*(t) + a\lambda(t) + c\alpha$$

A. Open-Loop Solution.

Write (3.16) in matrix form

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{\lambda}(t) \end{bmatrix} = \begin{bmatrix} -a & -b \\ c & a \end{bmatrix} \begin{bmatrix} x^*(t) \\ \lambda(t) \end{bmatrix} + \begin{bmatrix} -\frac{1}{j_e}\beta - a\alpha \\ c\alpha \end{bmatrix} \tag{3.17}$$

The general solution of (3.17) is given by

$$\begin{bmatrix} x^*(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} + \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix} \tag{3.18}$$

Let

$$z(t) = \begin{bmatrix} x^*(t) \\ \lambda(t) \end{bmatrix}$$

$$A = \begin{bmatrix} -a & -b \\ c & a \end{bmatrix}$$

$$r = \begin{bmatrix} -\frac{1}{j_e} p - a\alpha \\ c\alpha \end{bmatrix}$$

Eq.(3.17) becomes

$$\dot{z}(t) = Az(t) + r \tag{3.19}$$

Let

$$\Phi(t) = \begin{bmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{21}(t) & Q_{22}(t) \end{bmatrix}$$

$$k = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

$$p(t) = \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

Eq.(3.18) becomes

$$z(t) = \Phi(t)\ k + p(t) \qquad (3.20)$$

In Eq.(3.20) $\Phi(t)$ is the fundamental matrix solution of (3.19) which satisfies

$$\dot{\Phi}(t) = A\Phi(t) \quad \text{with boundary condition} \quad \Phi(0) = I \qquad (3.21)$$

where I = Identity matrix, and $p(t)$ is the particular vector solution of (3.19) which satisfies

$$\dot{p}(t) = Ap(t) + r \quad \text{with boundary condition} \quad p(0) = 0 \qquad (3.22)$$

The fundamental matrix is determined from Eq.(3.21) as follows: since A is time-invariant applying Laplace transform to both sides of Eq.(3.21) yields

$$s\Phi(s) - \Phi(0) = A\Phi(s)$$

since $\Phi(0) = I$

$$\Phi(s) = (sI-A)^{-1} \tag{3.23}$$

Here

$$(sI-A)^{-1} = \begin{bmatrix} \dfrac{(s-a)}{(s-\sqrt{d})(s+\sqrt{d})} & \dfrac{-b}{(s-\sqrt{d})(s+\sqrt{d})} \\ \dfrac{c}{(s-\sqrt{d})(s+\sqrt{d})} & \dfrac{(s+a)}{(s-\sqrt{d})(s+\sqrt{d})} \end{bmatrix} \tag{3.24}$$

where $d = a^2 - bc$

Inverting (3.24) gives

$$\Phi(t) = \begin{bmatrix} \operatorname{Cosh}\sqrt{d}\ t - \dfrac{a}{\sqrt{d}} \operatorname{Sinh}\sqrt{d}\ t & -\dfrac{b}{\sqrt{d}} \operatorname{Sinh}\sqrt{d}\ t \\ \dfrac{c}{\sqrt{d}} \operatorname{Sinh}\sqrt{d}\ t & \operatorname{Cosh}\sqrt{d}\ t + \dfrac{a}{\sqrt{d}} \operatorname{Sinh}\sqrt{d}\ t \end{bmatrix} \tag{3.25}$$

Note in Eq.(3.25)

$$\Phi(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \text{Identity matrix}$$

The particular vector solution $p(t)$ of (3.19) is computed as follows: Taking the Laplace transform of both sides of Eq.(3.22)

$$sp(s) - p(0) = Ap(s) + r(s)$$

since $p(0) = 0$

$$p(s) = (sI-A)^{-1}\, r(s) \tag{3.26}$$

Inverting (3.26) by using convolution integral gives

$$p(t) = \int_0^t \Phi(t-\tau)\, r(\tau)\, d\tau \tag{3.27}$$

Hence substituting (3.25) and $r(\tau) = \begin{bmatrix} -\frac{1}{j_e}\beta - a\alpha \\ c\alpha \end{bmatrix}$ into (3.27) and carrying out the integration gives

$$p(t) = \begin{bmatrix} \left(-\frac{1}{j_e}\, p - a\alpha\right)\left\{\frac{1}{\sqrt{d}}\,\text{Sinh}\sqrt{d}\;t + \frac{a}{d}\,(1-\text{Cosh}\sqrt{d}\;t)\right\} + (c\alpha)\left\{\frac{b}{d}\,(1-\text{Cosh}\sqrt{d}\;t)\right\} \\ \\ \left(-\frac{1}{j_e}\,\beta - a\alpha\right)\left\{\frac{c}{d}\,(\text{Cosh}\sqrt{d}\;t-1)\right\} + (c\alpha)\left\{\frac{1}{\sqrt{d}}\,\text{Sinh}\sqrt{d}\;t + \frac{a}{d}\,(\text{Cosh}\sqrt{d}\;t-1)\right\} \end{bmatrix} \tag{3.28}$$

Note in Eq.(3.28) $\quad p(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ = Null vector.

Substituting $\Phi(t)$ and $p(t)$ into (3.20) gives the general solution $z(t)$. The constant vector k is determined in such a way that the two boundary conditions on $z(t)$ are satisfied for a particular control action.

For speed-control,

$$x^*(0) = 0$$

$$x^*(T) = 0$$

Hence

$$k_1 = 0 \tag{3.29}$$

$$k_2 = \left(-\frac{1}{j_e}\beta - a\alpha\right)\left\{\frac{1}{b} + \frac{a}{b\sqrt{d}}\left(\frac{1 - \text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\right\} + \frac{c}{\sqrt{d}}\left(\frac{1 - \text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\alpha$$

Substituting (3.29) into (3.20) determines $x^*(t)$ and $\lambda(t)$. Since from (3.15) $x(t) = x^*(t) + \alpha$, the optimal trajectory is given by

$$x(t) = \alpha\left\{\text{Cosh}\sqrt{d}\ t + \left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\} - \beta\left\{\frac{a}{dj_e}(1-\text{Cosh}\sqrt{d}\ t) - \frac{a}{dj_e}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\} \tag{3.30}$$

The Lagrance multiplier $\lambda(t)$ is given by

$$\lambda(t) = \alpha\left\{-\frac{a}{b}\text{Cosh}\sqrt{d}\ t - \frac{\sqrt{d}}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Cosh}\sqrt{d}\ t - \frac{\sqrt{d}}{b}\text{Sinh}\sqrt{d}\ t - \frac{a}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\} - \beta\left\{\frac{a^2}{bdj_e}\left(\text{Cosh}\sqrt{d}\ t-1\right) + \frac{1}{bj_e} + \frac{a}{bj_e\sqrt{A}}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Cosh}\sqrt{d}\ t + \frac{a}{bj_e\sqrt{d}}\text{Sinh}\sqrt{d}\ t + \frac{a^2}{bj_e d}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\} \tag{3.31}$$

Using (3.30) and (3.31) in (3.11) gives the expression for the optimal control function $u^*(t)$

$$u^*(t) = \alpha\left\{\left(\frac{k_b}{2} + \frac{k_t}{2j_e}\frac{a}{b}\right)\text{Cosh}\sqrt{d}\ t + \left(\frac{k_b}{2} + \frac{k_t}{2j_e}\frac{a}{b}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right.$$

$$\left. + \frac{k_t}{2j_e}\frac{\sqrt{d}}{b}\text{Sinh}\sqrt{d}\ t + \frac{k_t}{2j_e}\frac{\sqrt{d}}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Cosh}\sqrt{d}\ t\right\} -$$

$$\beta\left\{\left(\frac{k_b}{2j_e\sqrt{d}}\frac{a}{} + \frac{a^2k_t}{2j_e^2 bd}\right)\left(1-\text{Cosh}\sqrt{d}\ t\right) - \left(\frac{k_b a}{2j_e d} + \frac{k_t a^2}{2j_e^2 bd}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t - \right.$$

$$\left.\frac{k_t a}{2j_e^2 b\sqrt{d}}\text{Sinh}\sqrt{d}\ t - \frac{k_t a}{2b\sqrt{d}\ j_e^2}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Cosh}\sqrt{d}\ t - \frac{k_b}{2j_e^2}\frac{1}{b}\right\} \qquad (3.32)$$

For speed-setting,

$$x^*(0) \neq 0 = \gamma \qquad \text{or} \qquad x(0) \neq \alpha = x(T)$$

$$x^*(T) = 0$$

Hence $\quad k_1 = \gamma$

$$k_2 = \gamma\left\{\frac{\sqrt{d}}{b}\frac{\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T} - \frac{a}{b}\right\} - \left(\frac{1}{j_e}\beta + a\alpha\right)\left\{\frac{1}{b} + \frac{a}{b\sqrt{d}}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\right\}$$

$$+\left(\alpha\frac{c}{\sqrt{d}}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right) \qquad (3.33)$$

Substituting (3.33) into (3.20) and using $x(t) = x^*(t) + \alpha$, the following results are obtained:

$$x(t) = \gamma\left\{\text{Cosh}\sqrt{d}\ t - \left(\frac{\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\} + \alpha\left\{\text{Cosh}\sqrt{d}\ t + \left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\}$$

$$- \beta\left\{\frac{a}{j_e d}\left(1-\text{Cosh}\sqrt{d}\ t\right) - \frac{a}{j_e d}\left(\frac{1-\text{Cosh}\sqrt{d}\ T}{\text{Sinh}\sqrt{d}\ T}\right)\text{Sinh}\sqrt{d}\ t\right\} \qquad (3.34)$$

$$\lambda(t) = \gamma\left\{-\frac{\sqrt{d}}{b}\,\text{Sinh}\sqrt{d}\;t + \left(\frac{\sqrt{d}}{b}\,\frac{\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Cosh}\sqrt{d}\;t - \frac{a}{b}\,\text{Cosh}\sqrt{d}\;t + \left(\frac{a}{b}\,\frac{\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Sinh}\sqrt{d}\;t\right\}$$

$$+\,\alpha\left\{-\frac{\sqrt{d}}{b}\,\text{Sinh}\sqrt{d}\;t - \frac{a}{b}\,\text{Cosh}\sqrt{d}\;t - \frac{\sqrt{d}}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Cosh}\sqrt{d}\;t - \frac{a}{b}\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\right.$$

$$\left.\text{Sinh}\sqrt{d}\;t\right\} - \beta\left\{-\frac{a^2}{j_e bd}\,(1-\text{Cosh}\sqrt{d}\;t) + \frac{1}{bj_e} + \frac{a}{bj_e\sqrt{d}}\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Cosh}\sqrt{d}\;t\right.$$

$$\left.+\,\frac{a}{bj_e\sqrt{d}}\,\text{Sinh}\sqrt{d}\;t + \frac{a^2}{j_e bd}\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Sinh}\sqrt{d}\;t\right\} \qquad (3.35)$$

Using (3.34) and (3.35) in (3.11) gives

$$u^*(t) = \gamma\left\{\left(\frac{k_b}{2} + \frac{k_t}{2j_e}\,\frac{a}{b}\right)\text{Cosh}\sqrt{d}\;t - \left(\frac{k_b}{2} + \frac{k_t}{2j_e}\,\frac{a}{b}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Sinh}\sqrt{d}\;t\right.$$

$$\left.+\,\frac{k_t\sqrt{d}}{2j_e b}\,\text{Sinh}\sqrt{d}\;t - \frac{k_t}{2j_e}\,\frac{\sqrt{d}}{b}\,\frac{\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\,\text{Cosh}\sqrt{d}\;t\right\} + \alpha\left\{\left(\frac{k_b}{2} + \frac{k_t}{2j_e}\,\frac{a}{b}\right)\text{Cosh}\sqrt{d}\;t\right.$$

$$+\left(\frac{k_b}{2} + \frac{k_t}{2j_e}\,\frac{a}{b}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Sinh}\sqrt{d}\;t + \frac{k_t\sqrt{d}}{2j_e b}\,\text{Sinh}\sqrt{d}\;t + \frac{k_t\sqrt{d}}{2j_e b}\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)$$

$$\left.\text{Cosh}\sqrt{d}\;t\right\} - \beta\left\{\left(\frac{k_b}{2j_e}\,\frac{a}{d} + \frac{k_t\,a^2}{2j_e^2 bA}\right)(1-\text{Cosh}\sqrt{d}\;t) - \left(\frac{k_b}{j_e}\,\frac{a}{A} + \frac{k_t a^2}{2j_e^2 bA}\right)\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\right.$$

$$\left.\text{Sinh}\sqrt{d}\;t - \frac{k_t a}{2j_e^2 b\sqrt{d}}\,\text{Sinh}\sqrt{d}\;t - \frac{k_t a}{2j_e^2 b\sqrt{d}}\left(\frac{1-\text{Cosh}\sqrt{d}\;T}{\text{Sinh}\sqrt{d}\;T}\right)\text{Cosh}\sqrt{d}\;t - \frac{k_t}{2j_e^2 b}\right\} \qquad (3.36)$$

Note that if $\gamma = 0$ in (3.36) Eqs.(3.30), (3.31) and (3.32) agree with (3.34), (3.35) and (3.36) respectively.

For speed-control and speed-setting the results are the same as given by (3.34), (3.35), (3.36) except that

$$v(0-) \neq v(0+) = v(T) = \beta \qquad \text{i.e.} \qquad \beta\text{'s are different.}$$

B. Closed-Loop Solution.

Recall

$$\dot{x}(t) = -ax^*(t) - b\lambda(t) - \frac{1}{j_e}\beta - a\alpha$$
$$\dot{\lambda}(t) = cx^*(t) + a\lambda(t) + c\alpha \qquad (3.17)$$

Equation (3.17) is a two-point boundary value problem. It can be converted to an initial value problem by making a linear transformation of the type

$$x^*(t) = m(t)\lambda + n(t) \qquad (3.37)$$

Substitute (3.37) into (3.17) and collecting equal powers of λ gives

$$\lambda^{(1)}(t)\{\dot{m}(t) + cm^2(t) + 2am(t) + b\} + \lambda^{(0)}(t)\{\dot{n}(t) + (cm(t)+a)\,n(t) + \alpha c\,m(t) + a\alpha + \frac{1}{j_e}\beta\} = 0 \qquad (3.38)$$

where $\lambda^{(1)}(t) = \lambda(t)$ = First power of $\lambda(t)$

$\lambda^{(0)}(t) = 1$ = Zeroth power of $\lambda(t)$

Since Eq.(3.38) is valid for all values of $\lambda(t)$, the conditions are:

$$\dot{m}(t) + cm^2(t) + 2am(t) + b = 0 \qquad (3.39)$$

$$\dot{n}(t) + (cm(t)+a)\,n(t) + \alpha c\,m(t) + a\alpha + \frac{1}{j_e}\beta = 0 \qquad (3.40)$$

Using $x^*(T) = m(T)\ \lambda + n(T) = 0$ (3.41)

gives the two initial conditions

$$m(T) = 0 \quad , \quad n(T) = 0$$

which are required to solve (3.39) and (3.40).

Consider first Eq.(3.39) which represents a first order nonlinear differential equation of the Riccati type. Substitute $\tau = T - t$ in (3.39) to obtain

$$\dot{m}(\tau) - cm^2(\tau) - 2am(\tau) - b = 0 \tag{3.42}$$

The boundary condition for (3.42) is $m(0) = 0$.

Let $$m(\tau) = -\frac{1}{c}\frac{\dot{w}(\tau)}{w(\tau)} \tag{3.43}$$

Substitute (3.43) to (3.42) gives the second-order linear differential equation with constant coefficients as shown below:

$$\ddot{w}(\tau) - 2a\dot{w}(\tau) + bcw(\tau) = 0 \tag{3.44}$$

The general solution of (3.44) is given by

$$w(\tau) = c_1 e^{\mu_1 \tau} + c_2 e^{\mu_2 \tau} \tag{3.45}$$

where c_1, c_2 = Two arbitrary constants to be determined

$$\mu_1 = a + \sqrt{a^2 - bc} = a + \sqrt{d}$$
$$\mu_2 = a - \sqrt{a^2 - bc} = a - \sqrt{d}$$

Substituting (3.45) into (3.43) gives

$$m(\tau) = \frac{1}{c}\left(\frac{c_1\mu_1 e^{\mu_1\tau} + c_2\mu_2 e^{\mu_2\tau}}{c_1 e^{\mu_1\tau} + c_2 e^{\mu_2\tau}}\right) \qquad (3.46)$$

since $m(0) = 0$.

From (3.46) $c_1\mu_1 + c_2\mu_2 = 0$

Hence

$$c_2 = - c_1 \frac{\mu_1}{\mu_2} \qquad (3.47)$$

Substituting (3.47) into (3.46) yields

$$m(\tau) = - b\left(\frac{\mathrm{Sinh}\sqrt{d}\ \tau}{a\ \mathrm{Sinh}\sqrt{d}\ \tau - \sqrt{d}\ \mathrm{Cosh}\sqrt{d}\ \tau}\right) \qquad (3.48)$$

Consider now Eq.(3.40) which represents a first order time-varying linear differential equation with a constant forcing function. Substitute $\tau = T - t$ in (3.40) to obtain

$$\dot{n}(\tau) - (cm(\tau)+a)\ n(\tau) - \alpha cm(\tau) - a\alpha - \frac{1}{j_e}\beta = 0 \qquad (3.49)$$

The boundary condition for (3.49) is $n(0) = 0$. The general solution of (3.49) is given by

$$n(\tau) = n(0)e^{\int_0^\tau (cm(\ell)+a)d\ell} + e^{\int_0^\tau (cm(\ell)+a)d\ell} \int_0^\tau \left(c\alpha m(\ell)+a\alpha + \frac{\beta}{j_e}\right) e^{-\int_0^\ell (cm(g)+a)dg}\, d\ell \qquad (3.50)$$

Since $n(0) = 0$ Eq.(3.50) becomes

$$n(\tau) = e^{\int_0^{\tau}(cm(\ell)+a)d\ell} \int_0^{\tau}\left(c\alpha m(\ell) + a\alpha + \frac{\beta}{j_e}\right) e^{-\int_0^{\ell}(cm(g) + a)dg} d\ell \tag{3.51}$$

Substituting for $m(\tau)$ from Eq.(3.48) into (3.51) and using a change of variable of the form

$$\sqrt{d}\ \text{Cosh}\sqrt{d}\ \ell - a\ \text{Sinh}\sqrt{d}\ \ell = y$$

in the transformed equation and integrating, the following equation is obtained:

$$n(\tau) = \frac{\alpha\sqrt{d}}{\left(\sqrt{d}\ \text{Cosh}\sqrt{d}\ \tau - a\ \text{Sinh}\sqrt{d}\ \tau\right)}\left(\frac{a}{\sqrt{d}}\ \text{Sinh}\sqrt{d}\ \tau - \text{Cosh}\sqrt{d}\ \tau + 1\right) +$$

$$+ \frac{\beta}{j_e\left(\sqrt{d}\ \text{Cosh}\sqrt{d}\ \tau - a\ \text{Sinh}\sqrt{d}\ \tau\right)}\left(\text{Sinh}\sqrt{d}\ \tau - \frac{a}{\sqrt{d}}\ \text{Cosh}\sqrt{d}\ \tau + \frac{a}{\sqrt{d}}\right) \tag{3.52}$$

Noting from (3.37)

$$x^*(\tau) = x(\tau) - \alpha = m(\tau)\lambda + n(\tau)$$

$$\lambda(\tau) = \frac{x(\tau) - \alpha - n(\tau)}{m(\tau)} \tag{3.53}$$

Substituting (3.53) into (3.11)

$$u^*(\tau) = x(\tau)\left\{\frac{k_b}{2} - \left(\frac{k_t}{2j_e}\right)\frac{1}{m(\tau)}\right\} + \left\{\left(\frac{k_t}{2j_e}\right)\frac{n(\tau)}{m(\tau)}\right\} + \left\{\left(\frac{k_t}{2j_e}\right)\frac{\alpha}{m(\tau)}\right\} \tag{3.54}$$

Now using (3.48) and (3.52) in (3.54) and noting that $\tau = T - t$ the closed-loop solution or feedback control law for this optimization problem is determined to be

$$u^*(t) = x(t)\left\{\frac{k_b}{2} + \frac{k_t a}{2j_e b}\right\} + \beta\left\{\frac{k_t}{2j_e^2 b}\right\} + \left(\frac{\alpha - x(t)\,\mathrm{Cosh}\sqrt{d}\,(\tau-t)}{\mathrm{Sinh}\sqrt{d}\,(\tau-t)}\right)\left\{\frac{k_t\sqrt{d}}{2j_e b}\right\} +$$

$$\beta\left(\frac{1 - \mathrm{Cosh}\sqrt{d}\,(T-t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)\left\{\frac{k_t\,a}{2j_e^2 b\sqrt{d}}\right\} \tag{3.55}$$

Note that in (3.55) the expression $\left(\frac{\alpha - x(t)\,\mathrm{Cosh}\sqrt{d}\,(T-t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)$ can be written as

$$\frac{\alpha - x(t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)} + x(t)\left(\frac{1 - \mathrm{Cosh}\sqrt{d}\,(T-t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right) \tag{3.56}$$

Therefore (3.55) takes the following form

$$u^*(t) = x(t)\left\{\frac{k_b}{2} + \frac{k_t a}{2j_e b}\right\} + x(t)\left(\frac{1 - \mathrm{Cosh}\sqrt{d}\,(T-t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)\left\{\frac{k_t\sqrt{d}}{2j_e b}\right\} + \beta\left\{\frac{k_t}{2j_e^2 b}\right\} +$$

$$\beta\left(\frac{1 - \mathrm{Cosh}\sqrt{d}\,(T-t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)\left\{\frac{k_t a}{2j_e^2 b\sqrt{d}}\right\} + \left(\frac{\alpha - x(t)}{\mathrm{Sinh}\sqrt{d}\,(T-t)}\right)\left\{\frac{k_t\sqrt{d}}{2j_e b}\right\} \tag{3.57}$$

Figure 3-1 shows the structure of the optimal feedback system given by (3.57). From Eq.(3.57) note at $t = T$

$$u^*(T) = x(T)\left\{\frac{k_b}{2} + \frac{k_t a}{2j_e b}\right\} + \beta\left\{\frac{k_t}{2j_e^2 b}\right\} + \left\{\frac{k_t}{2j_e b}\right\}\dot{x}(T) \tag{3.58}$$

Since using L'hospital's rule

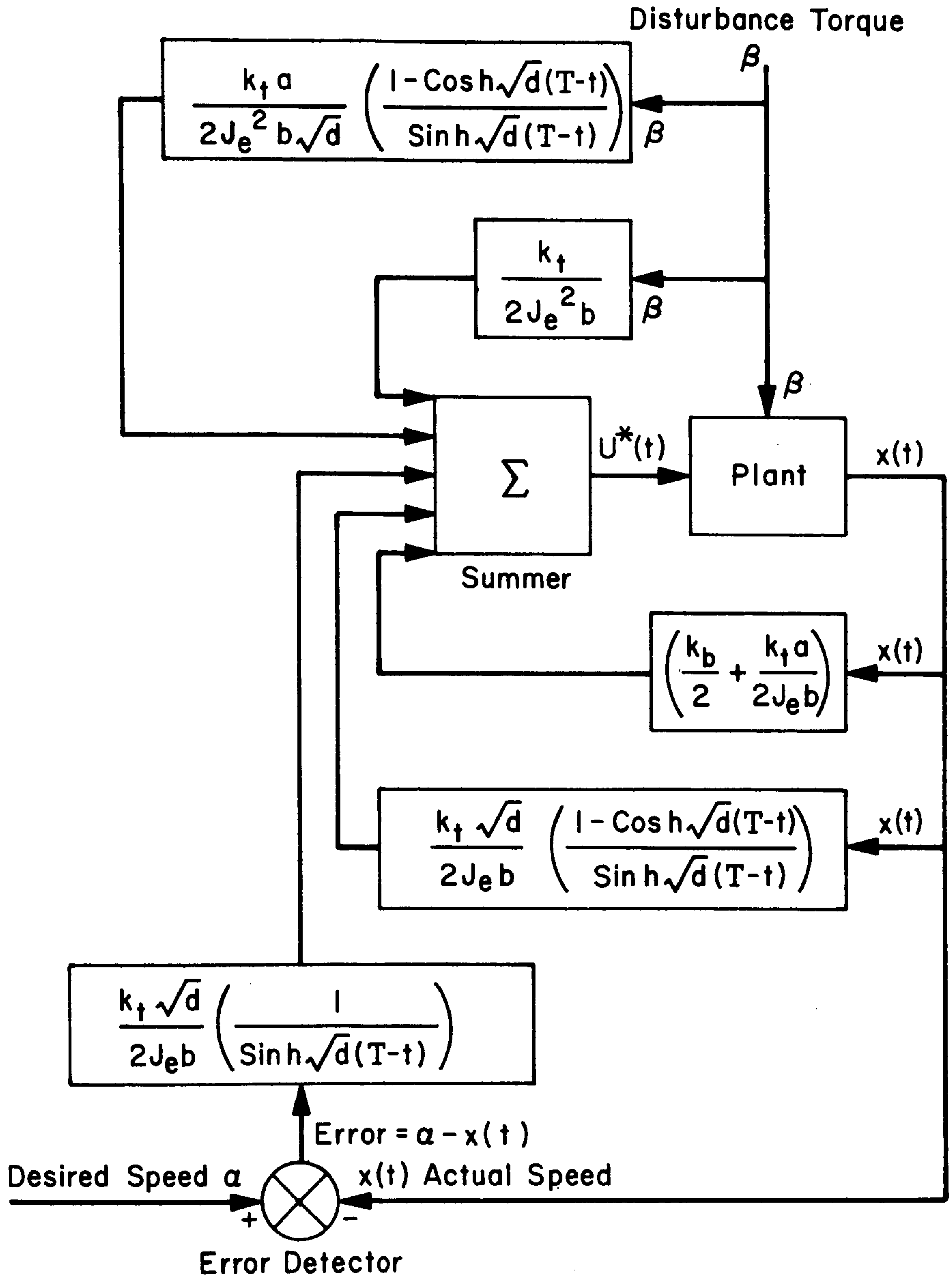

Fig. 3-1. The simulation of the optimal control system given by Eq.(3.57).

$$\lim_{t \to T} \left(\frac{1 - \cosh\sqrt{d}\,(T-t)}{\sinh\sqrt{d}\,(T-t)} \right) = \left(\frac{+\sqrt{d}\,\sinh\sqrt{d}\,(T-t)}{-\sqrt{d}\,\cosh\sqrt{d}\,(T-t)} \right)_{t=T} = 0$$

$$\lim_{t \to T} \left(\frac{\alpha - x(t)}{\sinh\sqrt{d}\,(T-t)} \right) = \left(\frac{-\dot{x}(t)}{-\sqrt{d}\,\cosh\sqrt{d}\,(T-t)} \right)_{t=T} = \frac{\dot{x}(T)}{\sqrt{d}}$$

If, however, $\frac{\dot{x}(T)}{\sqrt{d}} = 0$ in Eq.(3.58) for $t \geq T$ then

$$u^*(t) = x(T)\left\{ \frac{k_b}{2} + \frac{k_t a}{2j_e b} \right\} + \beta \left\{ \frac{k_t}{2j_e^{\,2} b} \right\} \quad t \geq T \tag{3.59}$$

Substituting (3.59) into (3.1) where $v(t) = \beta$ gives

$$\dot{x}(t) = 0 \qquad \text{for} \qquad t \geq T \tag{3.60}$$

If this requirement is met, the controller enters the steady-state phase of its operation. Therefore it is clear that some means must be incorporated into the structure of the optimal feedback controller shown in Figure 3-1, to turn the time varying feedback gains on at $t = 0$ and to turn them off at $t = T$.

In this analysis, it has been assumed that the value of T is known a priori. As shown in Fig. 3-2 the optimal control function $u^*(t)$ and hence the optimal trajectory $x(t)$ vary greatly with the particular choice of T. Figure 3-2 describes this dependence for the case of speed-control. Note that in the plot $x(t)$ vs T, $\dot{x}(t) = 0$ at $t = \frac{T}{2}$.

This salient feature holds for all α's and all β's. The proof follows from the use of Rolle's theorem which guarantees that $\dot{x}(t) = 0$ at some point t which satisfies the inequality $0 < t < T$. Using Eq.(3.30) to obtain

$$\dot{x}(t) = \frac{\alpha\sqrt{d} + \beta \frac{a}{j_e\sqrt{d}}}{\sinh\sqrt{d}\,T} \left(\cosh\sqrt{d}\,t - \cosh\sqrt{d}\,(T-t) \right) \tag{3.61}$$

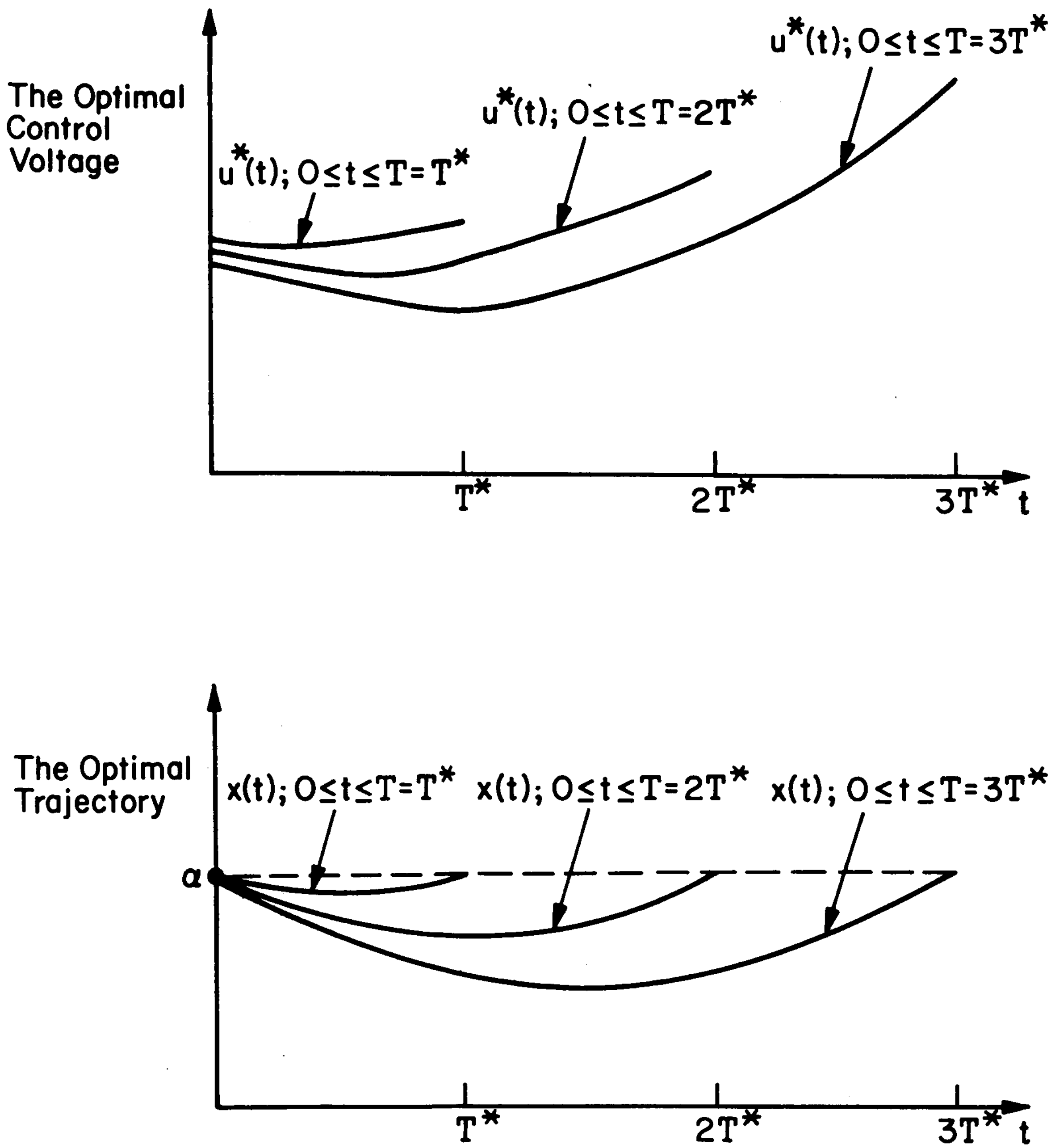

Fig. 3-2. The behaviour of the optimal control $u^*(t)$ and the optimal trajectory $x(t)$ for a particular set of α and β as terminal time T varies for the case of speed-control.

$\dot{x}(t) = 0$ in $0 < t < T$ if and only if

$$\text{Cosh}\sqrt{d}\ t = \text{Cosh}\sqrt{d}\ (T-t) \tag{3.62}$$

The solution of Eq.(3.62) is $t = \frac{T}{2}$. Since there is a time-delay between $u^*(t)$ and $x(t)$, $\dot{u}^*(t) = 0$ at a time t such that $0 < t < \frac{T}{2}$. Fig. 3-3 shows the relationships between $u^*(t)$, $x(t)$ and T for the case of speed-setting. Note that $\dot{u}^*(t) = 0$ at a time t which is less than the time t for which $\dot{x}(t) = 0$ in the interval $0 \leq t \leq T$.

From the observation of the above figures it is clear than the terminal time T must be selected to satisfy the following requirements:

1. The optimal control $u^*(t)$ and the optimal trajectory $x(t)$ remain within acceptable limits under all practical working conditions.

2. The desired value of $x(t)$ at $t = T$ is obtained as soon as possible in a given control action. This is in complete agreement with the assumption 2 in section 3.1

C. Practical Considerations.

As shown in Eq.(3.57) the feedback control law $u^*(t)$ requires the measurements of the speed $x(t)$, the disturbance torque β and the desired speed α. The speed $x(t)$ may be measured by a tachogenerator or a suitable electronic pulse counter which gives a voltage proportional to speed. The disturbance torque β is a function of sine of the slope angle of the terrain. Therefore, it is required to have a device which produces a voltage proportional to slope angle of the terrain. The desired speed setting α is obtained by means of a potentiometer connected to a voltage source. These analogous measurements are multiplied by time varying gains and added linearly to produce the optimal control $u^*(t)$.

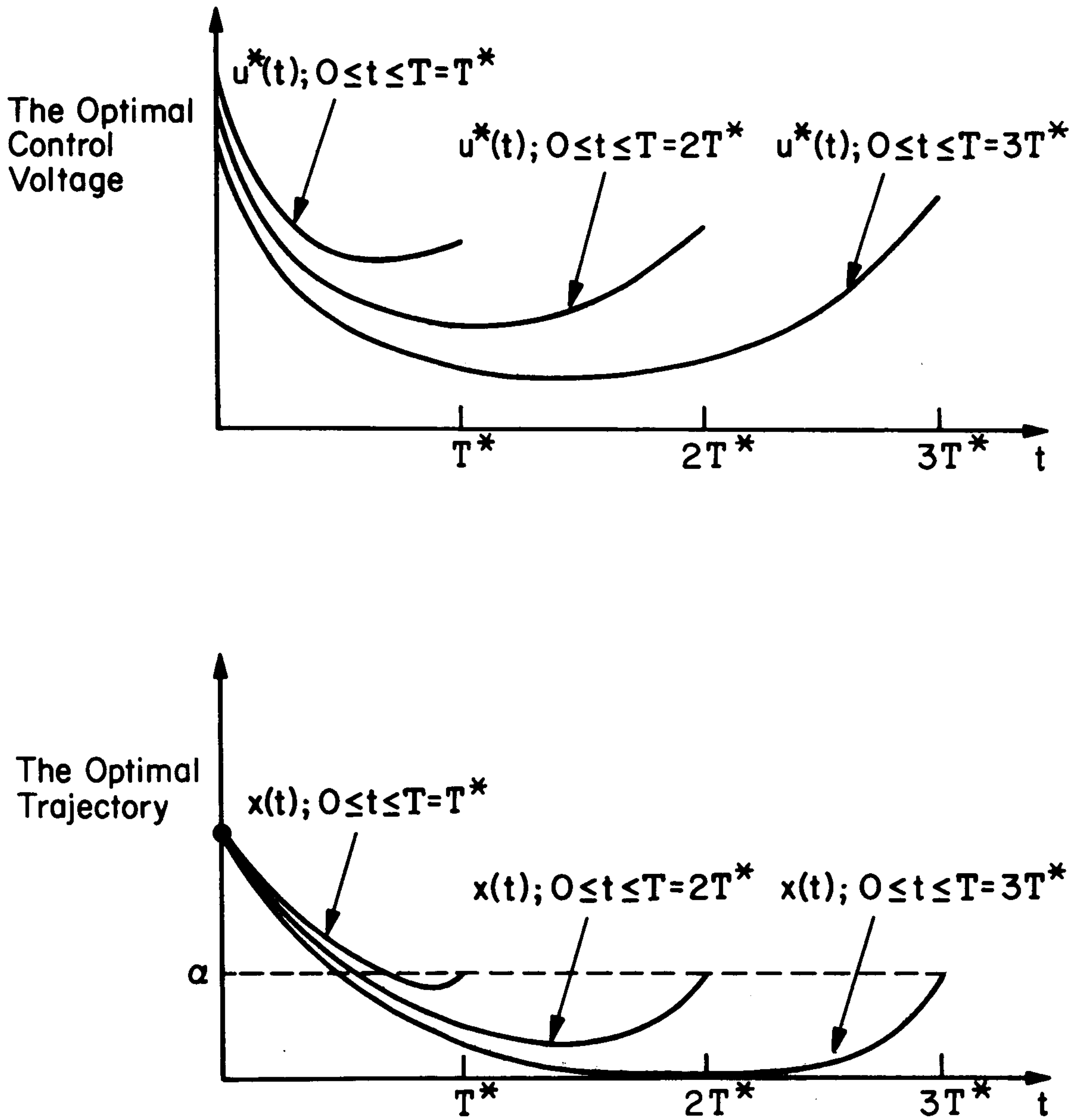

Fig. 3-3. The behaviour of the optimal control $u^*(t)$ and the optimal trajectory $x(t)$ for a particular set of α and β as terminal time T varies in the case of speed-setting.

When a disturbance comes into the system any error $(\alpha - \omega(t))$ may be used to activate an auxiliary electronic logic circuit (not shown in Fig. 3-1) which produces the time varying gains shown in Fig. 3-1.

For $t \geq T$ the logic circuit interrupts all output signals of time varying gains and hence $x(t) = \alpha$ for $t \geq T$. The optimum controller which is determined has the following characteristics:

1. In any control action, it brings the speed to its desired value at $t = T$. i.e. $x(T) = \alpha$

2. It minimizes the net flow of energy from the battery during the transient phase of the control.

3. With the help of an auxiliary electronic logic circuit it keeps the speed constant at its desired value for $t \geq T$ until a new disturbance comes into the system i.e. $\dot{x}(t) = 0$ for $t \geq T$.

The operation of the optimal controller is illustrated in Fig. 3-4 for three different control actions. The reason for jumps on the value of $u^*(t)$ at $t = T$ can be seen most easily in referring to Eq.(3.58). Here $u^*(T)$ contains a derivative term $\dot{x}(T)$ which is made zero for $t \geq T$ by the action of auxiliary electronic logic device. Thus if $\dot{x}(t)$ is negative as $t \to T$ then there is an upward jump in the value of $u^*(t)$ at $t = T$ and if $\dot{x}(t)$ is positive as $t \to T$ then there is a downward jump in the value of $u^*(t)$ at $t = T$.

D. Discussion of the Results.

The minimum-energy controller is determined to be a linear time-varying system. The engineering construction is costly but can be done with the help of present technology. Since the controller is required to perform as many times as the need arises during its lifetime, the initial and maintenance costs may be negligible in comparison with the economical savings which will be gained with the best possible use of the energy source. The asymptotic optimal solution

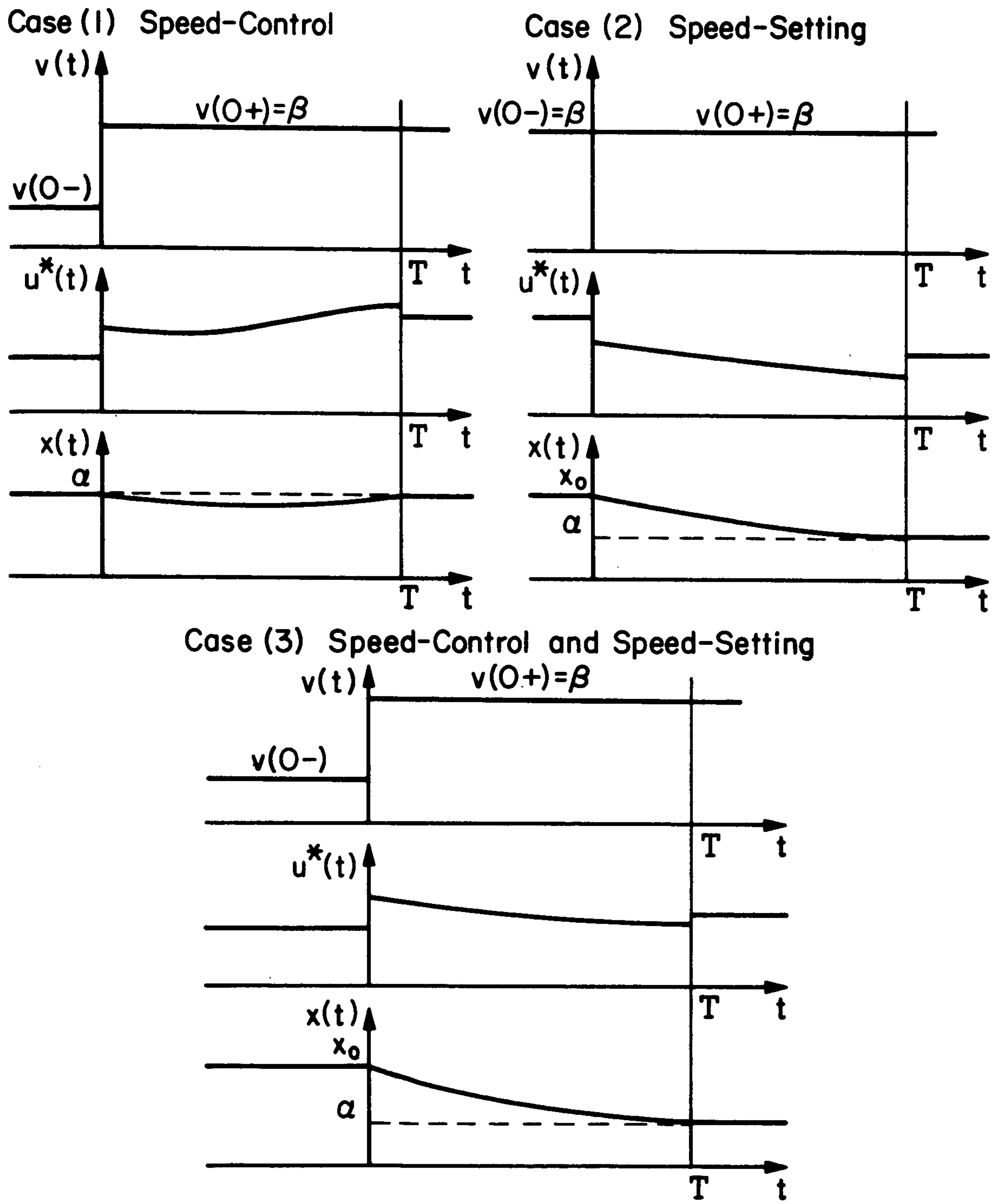

Fig. 3-4. The operation of the optimum feedback controller.

for this control problem does not exist as shown below:

Assume $T \to \infty$. From Eq.(3.48) and Eq.(3.52)

$$\lim_{T\to\infty} m(t) = \frac{-b}{a-\sqrt{d}} \tag{3.63}$$

$$\lim_{T\to\infty} n(t) = -\alpha + \frac{\beta}{j_e\sqrt{d}} \tag{3.64}$$

Substituting (3.63) and (3.64) into (3.54) gives

$$u^*(t) = x(t)\left\{\frac{k_b}{2} + \frac{k_t}{2j_e}\frac{a}{b} - \frac{k_t\sqrt{d}}{2j_e b}\right\} + \beta\left\{\frac{k_t}{2j_e^2 b} - \frac{k_t a}{2j_e^2 b\sqrt{d}}\right\} \tag{3.65}$$

Using (3.65) in (3.1) gives the following first order linear differential equation with constant coefficients:

$$\dot{x}(t) + \sqrt{d}\, x(t) = -\frac{a}{\sqrt{d}\, j_e}\beta \tag{3.66}$$

The solution of (3.66) is given by

$$x(t) = x(0)\, e^{-\sqrt{d}\, t} - \beta\frac{a}{dj_e}\left(1 - e^{-\sqrt{d}\, t}\right) \tag{3.67}$$

From (3.67) it is clear that at steady state

$$x(t) = -\beta\frac{a}{dj_e} \tag{3.68}$$

clearly, the terminal condition on $x(t)$ is not satisfied.

The reason for this can be seen from Fig. 3-2. Namely as $T \to \infty$ the control voltage remains very small until the terminal point is reached and becomes infinite as the terminal point is reached. Therefore if β is positive the steady state speed is negative and if β is negative the steady state speed is positive.

The performance of such a controller is not satisfactory from an engineering point of view.

Note in the above analysis no constraints are put on the control variable $u^*(t)$ and the state variable $x(t)$. This of course is not the case in practice. However, if these constraints are included in the formulation of the optimal control problem, mathematical complications arise and the solutions are no longer linear.

E. Future Work.

Future work involves the accomplishment of the following:

1. The determination of the optimal solutions for all cases considered above by taking motor inductance into account.

2. The formulation of the optimization problem for a traction motor whose dynamical behaviour is described by a first order nonlinear differential equation.

3. The formulation of the optimization problem for a dc motor with armature and field controls.

F. Appendix.

The electrical energy input into the plant described by

$$j_e \dot{x}(t) + f_e x(t) = k_t i_a - v(t) \tag{3.69}$$

$$\ell_a i_a(t) + r_a i_a(t) = u(t) - k_b x(t) \tag{3.70}$$

is given by

$$E(u) = \int_0^T u(t) i_a(t) dt \tag{3.71}$$

where i_a = Armature current. ℓ_a = Armature inductance.

If $\ell_a = 0$. From (3.70)

$$i_a = \frac{u(t) - k_b\, x(t)}{r_a} \tag{3.72}$$

Substituting (3.72) into (3.69) and (3.71) give the following set of equations:

$$\dot{x}(t) = -\left(\frac{f_e}{j_e} + \frac{k_t k_b}{j_e r_a}\right) x(t) - \frac{1}{j_e}\, v(t) + \frac{k_t}{j_e r_a}\, u(t) \tag{3.1}$$

and

$$E(u) = \int_0^T \left(\frac{1}{r_a}\, u^2(t) - \frac{k_b}{r_a}\, u(t)\, x(t)\right) dt \tag{3.2}$$

G. References.

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CHAPTER 4

SENSITIVITY CONSIDERATIONS IN THE DESIGN OF FEEDBACK CONTROLS

4.1. Introduction.

In optimum control theory, one assumes a mathematical model for the physical process and then determines either a control function or a control law that minimizes a well defined performance index, subject to various types of constraints. The implementation of the above control on the actual physical process may or may not produce the calculated optimum results. One of the reasons for this is usually the parameter variations in both the process and the controller. When design is carried out on the basis of an assumed mathematical model, it is important to know how the actual performance deviates from the theoretical results. How does the value of the performance index change for changes in process parameters, how do the trajectories change for changes in process parameters are some of the important questions that are assumed under the title of Sensitivity Analysis.

When a meaningful definition of sensitivity is made in relation to a given problem, methods of reducing sensitivity are known as minimum sensitivity design methods.

In the subsequent sections, these problems will be examined.

4.2. Review of Literature.

Let the dynamic process be described by the ordinary differential equation

$$\dot{\underline{x}} = \underline{f}(t,\underline{x},\underline{u},\underline{\alpha}) \tag{4.1}$$

$$\underline{x}(t_0) = \underline{c} \tag{4.2}$$

where $\underline{x}$: n x 1 state vector, $\underline{u}$: m x 1 control vector, $\underline{\alpha}$: p x 1 parameter vector and $\underline{c}$: n x 1 initial state vector.

Suppose it is desired to determine a control $\underline{u}^*(t,\underline{x})$, $t_0 \le t \le T$, such that the following performance index is minimized

$$I(\underline{u}) = \int_{t_0}^{T} g(t,\underline{x},\underline{u})\, dt \tag{4.3}$$

where $g(\cdot)$ is a scalar valued, non-negative function of its arguments.

By the well known procedures of optimum control theory, one can obtain the optimum feedback control and let it be denoted by

$$\underline{u}^* = \underline{\phi}(t,\underline{x},\underline{\alpha}_0) \tag{4.4}$$

where $\underline{\alpha}_0$ is the nominal value of the parameter vector. Using the control (4.4) in the actual system with the parameter vector $\underline{\alpha}$, the trajectory and the value of the performance are denoted by

$$\underline{x}(t,t_0,\underline{\alpha}_0,\underline{\alpha}) \quad \text{and} \quad I(\underline{c},\underline{\alpha},\underline{\alpha}_0) \tag{4.5}$$

respectively.

In actual practice, if $\underline{\alpha} = \underline{\alpha}_0$, then $I(\underline{c},\underline{\alpha}_0,\underline{\alpha}_0)$ will coincide with the theoretically computed minimum value and $\underline{x}(t,t_0,\underline{\alpha}_0,\underline{\alpha}_0)$ will coincide with the theoretically computed trajectory.

The literature consists mainly of examining (4.5) or variations of it. Dorato [1] considers the expression for $\frac{\partial I}{\partial \underline{\alpha}}$ which can be written as

$$\frac{\partial I}{\partial \underline{\alpha}} = \int_{t_0}^{T} \frac{\partial g}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial \underline{\alpha}}\, dt \tag{4.6}$$

where $\frac{\partial g}{\partial \underline{x}}$ is a row vector with components $\frac{\partial g}{\partial x_i}$ and $\frac{\partial \underline{x}}{\partial \underline{\alpha}}$ is a n x p matrix with elements $\frac{\partial x_i}{\partial p_j}$. The latter is evaluated by the usual variational formula

$$\left(\frac{d}{dt} \frac{\partial \underline{x}}{\partial \underline{\alpha}}\right) = \frac{\partial \underline{f}}{\partial \underline{x}}_{\alpha=\alpha_0} \frac{\partial \underline{x}}{\partial \underline{\alpha}} + \left[\frac{\partial \underline{f}}{\partial \underline{\alpha}}\right]_{\alpha=\alpha_0} \quad ; \quad \left.\frac{\partial \underline{x}}{\partial \underline{\alpha}}\right|_{t=t_0} = 0 \tag{4.7}$$

where $\frac{\partial \underline{f}}{\partial \underline{x}}$ is a n x n matrix with elements $\frac{\partial f_i}{\partial x_j}$ and $\frac{\partial \underline{f}}{\partial \underline{\alpha}}$ is a n x p matrix with elements $\frac{\partial f_i}{\partial \alpha_j}$.

For small variations of parameters from the nominal $\underline{\alpha}_0$, the change in the performance index is given by

$$\Delta I = \frac{\partial I}{\partial \underline{\alpha}} \cdot d\underline{\alpha} \tag{4.8}$$

where $d\underline{\alpha}$ is a p x 1 column vector with elements $\alpha_i - \alpha_{0_i}$. This allows the designer to determine the performance index sensitivity to small changes in the parameters from their nominal values. In any given situation, one should be careful in assigning meaning to either the <u>performance sensitivity vector</u> $\frac{\partial I}{\partial \underline{\alpha}}$ or its norm $\left\|\frac{\partial I}{\partial \underline{\alpha}}\right\|$.

Turning now to the so called <u>trajectory sensitivity</u>, the situation is as follows. Consider the system (4.1)-(4.4) and let it be desired to examine the variations of the trajectory $\underline{x}(t, t_0, \underline{\alpha}_0, \underline{\alpha})$ due to small changes in the parameter values. It is easy to see that the variation of $\underline{x}$ with respect to a parameter α_i obeys the linear differential equation

$$\frac{d}{dt} \frac{\partial \underline{x}}{\partial \alpha_i} = \left(\frac{\partial \underline{f}}{\partial \underline{x}} \frac{\partial \underline{x}}{\partial \alpha_i} + \left(\frac{\partial \underline{f}}{\partial \underline{u}}\right)\left(\frac{\partial \underline{\phi}}{\partial \underline{x}}\right)\left(\frac{\partial \underline{x}}{\partial \alpha_i}\right) + \left(\frac{\partial \underline{f}}{\partial \alpha_i}\right.\right. \tag{4.9}$$

with $\frac{\partial \underline{x}}{\partial \alpha_i}$ at $t = t_0$ and $\alpha_i = \alpha_{0i}$ is $\underline{0}$. (4.10)

For the nominal $\underline{\alpha}_0$, the various Jacobians in (4.9) can be evaluated along the trajectory corresponding to $\underline{\alpha}_0$ and thus (4.9) is a linear, time-varying differential equation.

$\frac{\partial \underline{x}}{\partial \alpha_i}$ are called the trajectory sensitivity functions by Tomovic [2]. Thus, changes in the trajectory to small changes in the parameter can be easily computed.

In order to use this approach to design less sensitive systems, Dougherty, et.al. [3] propose the following. Letting $\underline{S} \triangleq \frac{\partial \underline{x}}{\partial \alpha_i}$, (4.9) and (4.10) can be written

$$\dot{S} = A(t)\, \underline{S} + \underline{b}(t) \tag{4.11}$$

$$\underline{S}(t_0) = \underline{0} \tag{4.12}$$

This is adjoined to the plant equations (4.1) and (4.2) and a new performance index is defined as

$$I = \int_{t_0}^{T} [g(t,\underline{x},\underline{u}) + h(t,\underline{S})]\, dt \tag{4.13}$$

where $h(\cdot)$ penalizes the sensitivity terms. It is not possible to solve the above optimization problem directly and one must assume some form for the control law. For example, one may assume

$$\underline{u} = [K_1]\, \underline{x} + [K_2]\, \underline{S} \tag{4.14}$$

and then determine the elements of the matrices K_1 and K_2 to minimize (4.13).

Even if this can be done, to implement the control, one must generate $\underline{S}$ corresponding to the actual parameter value $\underline{\alpha}$ which is assumed unknown. The above authors get around this difficulty by computing $\underline{S}$ always for $\underline{\alpha} = \underline{\alpha}_0$ (known). In the examples they have presented, they have chosen to ignore the feedback from $\underline{S}$ completely. The results are not conclusive at the moment.

It seems that this approach will not be satisfactory in general because it depends so much on the calculations done on the basis of the nominal value, $\underline{\alpha}_0$, of the parameter and at best, it is a local theory. If one assumes that the actual parameters are in a region around $\underline{\alpha}_0$, the method may yield satisfactory results when this region is very small. Actual experiments have to be carried out for a given situation to assess the merits of this approach. The main motivation for this approach seems to be that it can be forced into the framework of the standard optimum control problem.

4.3. Discussion.

From what has been said, it is clear that one can compute the changes in the performance index and (or) changes in the trajectory (using a suitable norm) for small changes in the parameter values from their nominal ones [5]. Unfortunately, it is not easy to establish the "smallness" quantitatively except through experimentation on a given problem.

One could augment either some measure of the performance sensitivity vector or the trajectory sensitivity function to the primary performance index (4.3) and compute an approximate control law. As has been observed in the literature, this approach need not always result in an insensitive system. The extent to which parameter variations are tolerated must be determined by simulation. The biggest drawback of these methods seem to be that they are forced into the optimal control theory formulation in an artificial way and the merits of this must be determined by simulation experiments.

4.4. Min-Max Formulation.

In the approach of Dougherty, et.al. referred to earlier, it was seen that they assumed a feedback structure and then computed the free parameters in this structure to minimize an augmented performance index. Let us examine this problem in a different way.

Suppose we know that the parameter vector $\underline{\alpha}$ belongs to some set M. For example, we may have reason to believe that $m_1 \le \alpha_i \le M_i$. Similar bounds on the other parameters may be specified. Let us also suppose that a fixed feedback structure is chosen given by

$$\underline{u} = \underline{\psi}(\underline{y},\underline{b}) \tag{4.15}$$

where $\underline{y}$ is the output vector (measured) related to the state $\underline{x}$ via a known transformation (sensor dynamics for example) and $\underline{b}$ is a $r \times 1$ vector of adjustable controller parameters to be chosen optimally. The reasons for assuming the structure of feedback is fully discussed in Ref. 4.

Let us denote the minimum value of (4.3) when optimized subject to (4.1) (via the methods of optimum control) by

$$I(\underline{c},\underline{\alpha}) \tag{4.16}$$

Let us denote the minimum value of the performance index

$$\int_{t_0}^{T} g(t,\underline{x},\underline{\psi},(\underline{y},\underline{b}))\ dt \tag{4.17}$$

optimized subject to

$$\dot{\underline{x}} = \underline{f}(t,\underline{x},\underline{\psi}(\underline{y},\underline{b}),\underline{\alpha}) \tag{4.18}$$

$$\underline{x}(t_0) = \underline{c}$$

by

$$V(\underline{c},\underline{\alpha},\underline{b}) \tag{4.19}$$

In general, the minimizing value of $\underline{b}$ depends upon the initial condition $\underline{c}$. What is reasonable therefore is to match the performance index surface (4.19) with (4.16) in some suitable sense.

Since the performance index plays a crucial role in optimization and one is trying to find a control which minimizes the performance index, it is reasonable to ask if the performance index surface with a fixed controller configuration can be matched (in some suitably defined sense) to the performance index surface of the optimum system. With this motivation, one is naturally lead to find $\underline{b}$ according to the criterion

$$\underset{\underline{b}\in B}{\text{Min}}\ \underset{\underline{c}\in C}{\text{Max}}\ \underset{\underline{\alpha}\in M}{\text{Max}}\ \frac{V(\underline{c},\underline{\alpha},\underline{b}) - I(\underline{c},\underline{\alpha})}{I(\underline{c},\underline{\alpha})} \tag{4.20}$$

where B is the set of controller parameters and C is the set of allowable initial conditions.

In essence what this min-max operation does is to synthesize a controller that guards against the worst initial condition and the worst parameter vector. In actual practice these worst cases may not arise which merely means that the controller is designed in a very conservative way.

It must be mentioned that the operations involved in (4.20) are not easy from a computational point of view but this is the price one has to pay for a realistic approach to the problem. We shall illustrate below the details for a linear problem and give computational results.

Example.

Consider the second order linear system described by

$$\dot{x}_1 = x_2 \qquad x_1(0) = 1$$

$$\dot{x}_2 = -\alpha x_2 + u \qquad x_2(0) = 0$$

where $0.2 \le \alpha \le 2$. The set M is the segment $[0.2, 2]$ on the real line. In this problem, we shall assume that the initial conditions are precisely known.

Let the primary performance index be

$$I = \int_0^\infty (x_1^2 + x_2^2 + u^2)\, dt$$

The optimum feedback control and the optimum performance index are found to be

$$u(x_1, x_2, \alpha) = -x_1 + \left(\alpha - \sqrt{\alpha^2 + 3}\right) x_2 \qquad \text{and}$$

$$I(\alpha) = \sqrt{\alpha^2 + 3} \qquad \text{respectively.}$$

Suppose we fix the controller configuration (variations on this are possible) to be

$$u = -K_1 x_1 - K_2 x_2$$

and want to choose K_1 and K_2 in the Min Max sense. With this fixed controller configuration, the value of the performance index is given by

$$V(\alpha, K_1, K_2) = \frac{1 + K_1^2 + K_1(1 + K_2^2)}{2(K_2 + \alpha)} + \frac{(K_2 + \alpha)(1 + K_1^2)}{2K_1} - K_1 K_2$$

To illustrate the approach, let us set $K_1 = 1$ and merely find the optimum K_2 in the min max sense. The results are given below. K_2 was restricted to the range [0.2,2.0]

$K_{2\ opt} = 1.4$

$\alpha_{opt} = 2.0$

The value of the performance index is 0.0316. To get an idea of how the trajectories behave, we can compute the closed loop poles of the system when the feedback is $- x_1 - 1.4\ x_2$ and α assumes the values 0.2, 1 and 2.

α	Closed Loop Poles	I	V
0.2	$- 0.8 \pm j\ 0.6$	1.676	1.743
1	- 1.875 , - 0.525	2.000	2.000
2	- 3.05 , - 0.35	2.645	2.713

If the interest was more on the trajectories than the performance index, one could have determined a feedback gain that would minimize the maximum deviation of the controlled trajectory from the optimal one. Here, we choose to concentrate on the deviation of the performance index instead.

<u>min max design when $u = - Kx_1$</u>

If we fix the controller configuration as above and seek for a value of K (K in the range [0.2, 2.0]) in the min-max sense, computational results show that $K_{opt} = 0.2$. One could compute the value of K such that the trajectories of the controlled system deviate from those of the optimum in a min max sense. Experiments were conducted for $x_1(0) = 1$ and $x_2(0) = 1$.

For $u = - x_1 - Kx_2$, K in $[0.2, 2.0]$, the results were

$$K = 0.6$$

The location of closed loop poles and the values of I and V are summarized below.

α	Closed Loop Poles	I	V
0.2	$- 0.4 \pm j\, 0.9$	5.2	6.4
1.0	$- 0.8 \pm j\, 0.6$	5.0	5.1
2.0	$- 0.47, - 2.13$	5.29	5.3

For the control configuration $u = - Kx_1$, the optimum value of the feedback coefficient was 0.8. The closed loop poles, values of I and V for different values of α are summarized below.

α	Closed Loop Poles	I	V
0.2	$- 0.1 \pm j\, 0.88$	5.2	15.95
1.0	$- 0.5 \pm j\, 0.74$	5.0	5.79
2.0	$- 0.55, - 1.45$	5.29	5.47

4.5. Discussion.

In the above example, we assumed fixed initial conditions and thus eliminated the maximization over the initial condition set. It is not too difficult to include this maximization but the computational effort increases. To avoid multiple minimization and maximization operations, we may proceed as follows. Considering the above numerical example, we found that $K_{opt} = 1.4$ and $\alpha_{opt} = 2.0$. Now, we may use $\alpha = 2.0$ in the plant equations and find a new value of K which minimizes the maximum deviation of the performance index with respect to a given initial condition set. This may or may not yield a satisfactory solution because the min and max operations cannot be interchanged in general.

On the basis of the limited amount of digital experimentation, it is felt that a min-max design is feasible whether one chooses to design feedback controllers on the basis of performance index deviations or trajectory deviations. The computational time increases as the number of variables over which min-max is done increases but it is felt that this should not be a great disadvantage because the design is carried out off line.

A simulation experiment based on this design procedure should be conducted to assess the merits of this approach.

4.6. Min max Design Based on Performance Index and Trajectory Deviations.

In the earlier sections, we have seen an approach to the design of feedback controllers based on minimizing the maximum deviation of the performance index. While this design accomplishes what it is intended to accomplish, the resulting trajectories may vary quite a lot depending on the value of the unknown plant parameter. If one is interested in deviations on the trajectories, it is appropriate to consider a design based on the performance index

$$\underset{K}{\text{Min}}\ \underset{\alpha}{\text{Max}} \left[\frac{V - I}{I} + W \cdot \|x_S^* - x^*\|^2 \right] \tag{4.21}$$

where K refers to the fixed configuration feedback controller parameter, α refers to the plant parameter, $x_S^*(t)$ refers to the trajectory of the system with the fixed configuration controller (also called the specific optimum trajectory) and $x^*(t)$ refers to the system trajectory in the optimum controller. $\|\cdot\|^2$ is an appropriate norm and W is a relative non-negative weighting factor.

The first term in (4.21) corresponds to the deviation in the performance index and the second term corresponds to the deviation in the trajectories.

To illustrate this line of design approach, the following illustrative example is considered. The plant is governed by

$$\dot{x}_1 = x_2 \qquad x_1(0) = 1$$

$$\dot{x}_2 = -\alpha x_2 + u \qquad x_2(0) = 0$$

and the performance index is

$$\int_0^\infty (x_1^{\,2} + x_2^{\,2} + u^2)\, dt$$

Let α be in the range $[2,4]$.

The optimum control is

$$u^* = -x_1 + \left(\alpha - \sqrt{\alpha^2 + 3}\right) x_2$$

and the optimum closed loop system is

$$\ddot{x}_1 + \sqrt{3 + \alpha^2}\; \dot{x}_1 + x_1 = 0$$

which results in

$$x_1^{\,*}(t) = c_1\, e^{\lambda_1 t} + c_2\, e^{\lambda_2 t}$$

where

$$\lambda_1 = \frac{1}{2}\left[-\sqrt{3+\alpha^2} + \sqrt{\alpha^2 - 1}\right]$$

$$\lambda_2 = \frac{1}{2}\left[-\sqrt{3+\alpha^2} - \sqrt{\alpha^2 - 1}\right]$$

$$c_1 = \lambda_2/(\lambda_2-\lambda_1)$$

and $$c_2 = \lambda_1/(\lambda_1-\lambda_2)$$

Let the specific optimum controller be fixed as

$$u = -Kx_1$$

where K is to be chosen in the range $[0.1, 0.9]$.

With this controller, the specific optimum trajectory is given by

$$x_S(t) = A_1 e^{\mu_1 t} + A_2 e^{\mu_2 t}$$

where

$$\mu_1 = \frac{1}{2}\left[-\alpha + \sqrt{\alpha^2 - 4K}\right]$$

$$\mu_2 = \frac{1}{2}\left[-\alpha - \sqrt{\alpha^2 - 4K}\right]$$

$$A_1 = \mu_2/(\mu_2-\mu_1)$$

and $$A_2 = \mu_1/(\mu_1-\mu_2)$$

Now the problem is to choose the feedback parameter K according to the performance criterion

$$\underset{K}{\text{Min}}\ \underset{\alpha}{\text{Max}} \left[\frac{V - I}{I} + W \int_0^{\infty} (x_S(t) - x^*(t))^2 dt \right]$$

where V and I are defined as in Section 4.4.

A numerical experiment results in the values of $K = 0.8$ for $W = 0$ and $K = 0.9$ for $W = 2.0$. The closed loop poles are shown in the following table.

$W = 0$

α	Closed Loop Poles
2.0	- 0.55 , - 1.45
4.0	- 0.22 , - 3.78

$W = 2.0$

α	Closed Loop Poles
2.0	- 0.685 , - 1.315
4.0	- 0.24 , - 3.760

It may be noted that the location of the closed loop poles by themselves may not yield much information. A plot of the trajectory $x_S(t)$ will have to be examined to decide if an adequate design has been accomplished. In this example, if we plot the $x_S(t)$ trajectory for $W = 0$, $\alpha = 2.0$ and 4.0 and $W = 2.0$, $\alpha = 2.0$ and 4.0, we find the following. The trajectories for $\alpha = 2.0$ and $\alpha = 4.0$ are closer together when $W = 2.0$ than when $W = 0.0$ which is as it should be. But, the addition of the trajectory deviation term in the performance index has not produced dramatic results in this example. This is partly due to the particular nature of this example where the trajectories are always well damped exponentials for values of α between 2.0 and 4.0. Even though the results are not spectacular in this example, this method merits consideration in other problems where the min max design procedure is used.

4.7. References.

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