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A STOCHASTIC APPROXIMATION SCHEME WITH ACCELERATED CONVERGENCE PROPERTIES

By

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ABSTRACT

An examination of the first order steepest descent method for minimizing deterministic functions leads to the specification of sufficient conditions assuring the convergence with probability one of a new stochastic approximation algorithm. The algorithm is interpreted as a stochastic descent method using a constant, as opposed to a continually decreasing step length. The crucial convergence condition is that the variance of the gradient should be zero as well as a minimum at the solution point.

The rather narrow class of problems to which the new algorithm is applicable can be extended to include many useful problems by employing accelerator methods adapted from the field of Monte-Carlo techniques, both to satisfy the condition, and to accelerate convergence.

Simple illustrative examples demonstrating the effectiveness of the new algorithm are presented.

1. Introduction

The convergence conditions for the basic stochastic approximation algorithm originated by Robbins and Monro [1] are now well known. Important results are due to Blum [2], Dvoretzky [3], and Gladyshev [4]. The crucial condition for convergence with probability one is that the step size should progressively decrease in such a way that the stochastic variation in the correction term dies out.

Although the stochastic approximation algorithm can be given numerous interpretations, one particularly fruitful point of view is to interpret it as a stochastic descent method for finding the expected minimum of a stochastic function. In this light it is logical to investigate possible correspondences with deterministic descent algorithms. The deterministic counterpart of the Robbins-Monro algorithm, the first order steepest descent method, differs only in that a constant step length can be used, thereby resulting in a faster convergence for an equivalent problem. In this paper the term 'equivalent problem' is taken to mean that deterministic problem which is identical with some stochastic problem under consideration except for the absence of the primary random variable(s).

The present paper is concerned with developing a stochastic approximation algorithm whose rate of convergence is comparable to that of the deterministic steepest descent method for the equivalent problem. This algorithm uses constant

step lengths as opposed to the conventional schemes, and is shown to converge under the appropriate conditions with probability one.

2. Convergence of the Deterministic Algorithm

As an introduction it is useful to derive briefly sufficient convergence conditions for the deterministic steepest descent algorithm; it will be found in the next section that these are very similar to the stochastic convergence requirements. Consider the deterministic algorithm

$$u_{k+1} = u_k - \rho\, f(u_k) \tag{1}$$

for minimizing with respect to u some function H(u).

$f(u_k) \triangleq \dfrac{\partial H^T(u_k)}{\partial u_k}$ is assumed to exist everywhere; u_k is an n-vector. Let there be a unique solution point (i.e., a zero in f(u)) at u = 0. The algorithm converges to the solution point under the following conditions:

(a) $u_k^T f(u_k) \geq 0$, the inequality being strict except at the solution point; (2)

(b) $\|f(u_k)\|^2 \leq d\, u_k^T f(u_k)$; (3)

(c) $\rho < 2/d$. (4)

Proof:

Forming the inner product:

$$u_{k+1}^T u_{k+1} = u_k^T u_k - 2\rho\, u_k^T f(u_k) + \rho^2 f(u_k)^T f(u_k) .$$

Using the inequality of Eqn. (3),

$$u_{k+1}^T u_{k+1} \leq u_k^T u_k - \{2\rho - \rho^2 d\} u_k^T f(u_k) \; ; \tag{5}$$

and if Eqn. (4) is satisfied

$$u_{k+1}^T u_{k+1} \leq u_k^T u_k \; . \tag{6}$$

Hence if u_1 is finite, then u_k and $f(u_k)$ are finite for all $k > 1$. Taking the infinite sum of Eqn. (5)

$$u_\infty^T u_\infty \leq u_1^T u_1 - \{2\rho - \rho^2 d\} \sum_{k=1}^{\infty} u_k^T f(u_k) \; ;$$

hence $\sum_{k=1}^{\infty} u_k^T f(u_k) < \infty$, which must then imply that $u_k, f(u_k) \rightarrow 0$

and so the algorithm converges.

3. Convergence of the Stochastic Algorithm

In considering the stochastic approximation algorithm, "convergence" will be used in two senses. "Deterministic convergence" will be taken to mean the convergence of the algorithm in expectation; that is, $Eu_k \rightarrow 0$; $k \rightarrow \infty$. "Stochastic convergence" will be taken to mean the reduction of the variance of the correction term $\rho f(u_k)$ to zero as the solution point is reached. In this section it will be shown that very similar conditions hold on the one hand for the convergence of the deterministic algorithm given above; and on the other for the deterministic convergence of the stochastic algorithm to be described.

The problem is now to minimize with respect to u the function $EH(u, \xi)$ where ξ is a vector of random variables with zero expectation and finite variance. The stochastic descent algorithm is

$$u_{k+1} = u_k - \rho f(u_k) \quad , \tag{7}$$

where as before the step length ρ is a constant, but now

$$f(u_k) \triangleq \frac{\partial H(u_k, \xi_k)}{\partial u_k} \quad ,$$

which is a random variable. Suppose that all the relevant moments of $f(u_k)$ exist, and let the unique solution point be $u = 0$ at $Ef(u) = 0$. Then the algorithm converges to the solution point under the following conditions:

(a) $u_k^T E_\xi f(u_k) \geq 0$, the inequality being strict except at the solution point; (8)

(b) $E_\xi \| f(u_k) \|^2 \leq d u_k^T E_\xi f(u_k)$; (9)

(c) $\rho < 2/d$. (10)

Here E_ξ denotes expectation with respect to the current sample of the random variable ξ_k.

Proof:

Taking the total expectation (i.e., on all samples of ξ from ξ_1 through ξ_k) of the inner product form of Eqn. (7)

$$E u_{k+1}^T u_{k+1} = E u_k^T u_k - 2\rho E u_k f(u_k) + \rho^2 E \| f(u_k) \|^2 \, .$$

Using Eqn. (9) after taking its total expectation also,

$$E u_{k+1}^T u_{k+1} \leq E u_k^T u_k - \{2\rho - \rho^2 d\} E u_k^T f(u_k) \quad ; \tag{11}$$

and if Eqn. (10) is satisfied

$$E u_{k+1}^T u_{k+1} \leq E u_k^T u_k \quad . \tag{12}$$

Assuming the moments of u_1 to be finite then $u_k^T u_k$ is a supermartingale [5]. Again taking an infinite sum of Eqn. (11)

$$E u_\infty^T u_\infty \leq E u_1^T u_1 - \{2\rho - \rho^2 d\} \sum_{k=1}^{\infty} E u_k^T f(u_k) \quad ,$$

which implies that

$$\sum_{k=1}^{\infty} E u_k^T f(u_k) < \infty \quad . \tag{13}$$

Using the expectation of Eqn. (9), Eqn. (13) implies that

$$\sum_{k=1}^{\infty} E \| f(u_k) \|^2 < \infty \quad ;$$

and since it is universally true that

$$E \| f(u_k) \|^2 \geq \| E f(u_k) \|^2 \quad ,$$

then

$$\sum_{k=1}^{\infty} \| E f(u_k) \|^2 < \infty \quad .$$

Hence $Ef(u_k) \rightarrow 0$ (but not necessarily monotonically unless $Ef(u)$ is monotonic in u), and so $u_k \rightarrow 0$ as $k \rightarrow \infty$; and the stochastic algorithm converges in expectation.

The close similarity in the conditions for each of the two proofs is to be noted. Intuitively it may be expected that the rates of convergence (in the sense of deterministic convergence

in the stochastic case) will be substantially the same for both the stochastic problem and its equivalent deterministic problem.

Only a few further steps are required to show convergence with probability one under the same conditions.

Proof:

It has already been shown that

$$\sum_{k=1}^{\infty} E u_k^T f(u_k) < \infty \quad .$$

Hence over some subsequence of the sequence $\{k\}$ $u_k^T E_\xi f(u_k) \rightarrow 0$ with probability one. But $u_k^T u_k$ is a supermartingale, implying that $u_k^T u_k$ and hence also u_k converge with probability one to a random variable [5]. These two facts are sufficient for convergence of the algorithm with probability one to the solution point.

4. Interpretation of the Convergence Conditions

It is easily seen that the implication of the convergence condition of Eqn. (9) is that at the solution point not only is $Ef(u)$ zero, but so also is $Var\{f(u)\}$. This differs quite sharply from the corresponding condition for the original stochastic approximation scheme. As is given by Gladyshev for example [4] there is the looser requirement that $Var\{f(u)\}$ should have some finite minimum, not necessarily zero, at the solution point. At first sight this is a monumental difficulty, since very

few functions f(u), excluding even the linear case, satisfy this condition. Apparently the new method can only be applied to problems in which there is simple multiplicative noise.

Example 1:

Let $f(u) = u + u\xi$ where ξ has zero mean and unit variance. This satisfies the conditions given by Eqns. (8) through (10); the solution point is at $u = 0$. Figure 1 shows the results from the simulation of the descent algorithms over 25 iterations. Clearly the new algorithm (curve 2) is a considerable improvement over the original stochastic approximation scheme (curve 1) where in this case the step length ρ decreases as $1/k$ from its initial value. The solution of the equivalent deterministic problem by the steepest descent method is given for comparison (curve 3).

5. Application of Monte-Carlo Accelerator Techniques

The class of problems which can be solved effectively by use of the new stochastic descent algorithm can be considerably enlarged by borrowing accelerator techniques from the field of Monte-Carlo methods. To date there seems to have been little cross-fertilization between these two fields: the monograph of Hammersley and Handscomb [6] has been found very useful in the development of the work in this section.

The so-called naive Monte-Carlo method is itself a stochastic approximation scheme of the original type using a harmonically decreasing step length. Let ξ_i be a sequence of

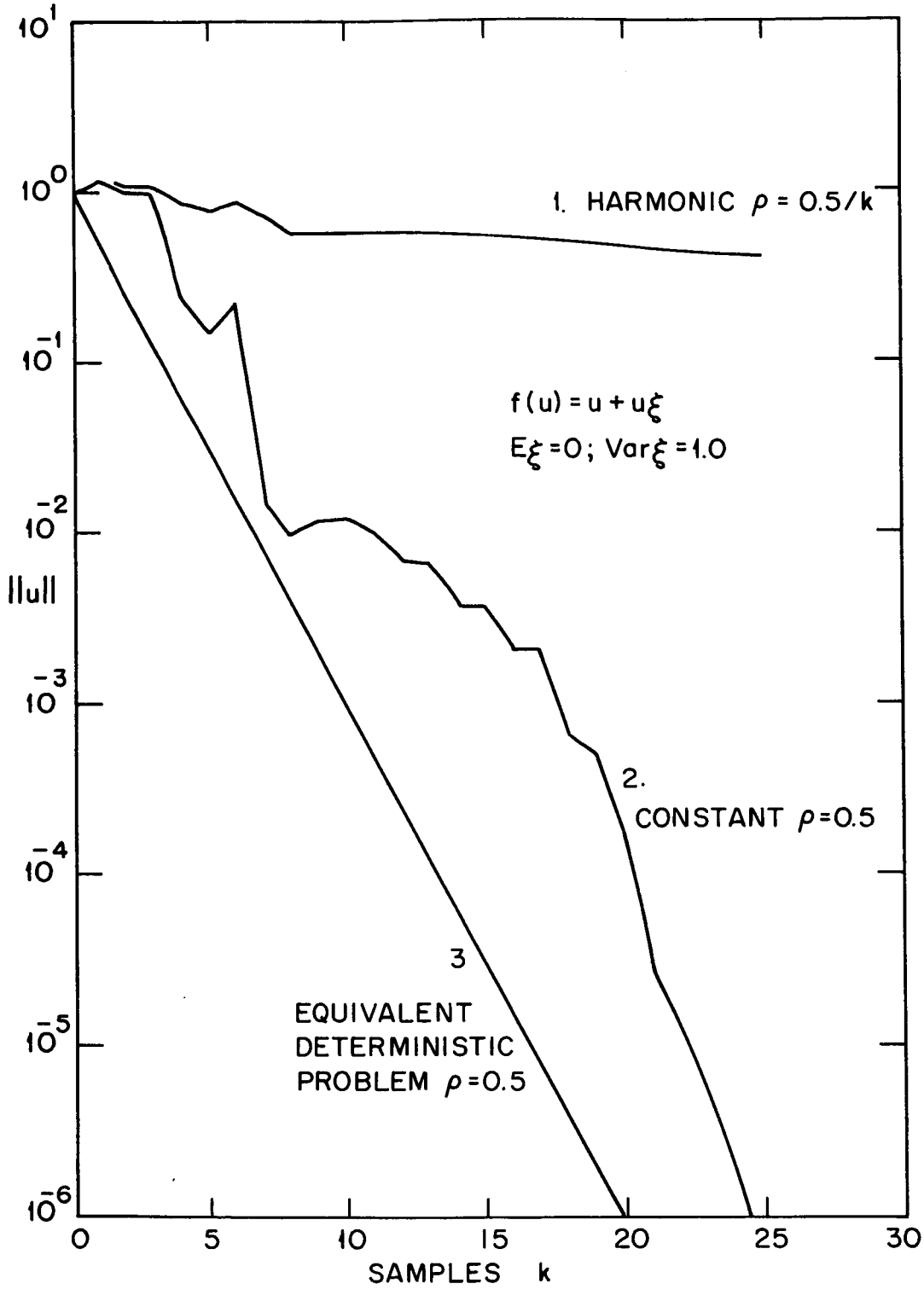

FIG. 1 MINIMIZATION OF A SIMPLE FUNCTION WITH PURELY MULTIPLICATIVE NOISE.

independent random samples from some probability distribution; let $g(\xi_i)$ be a sequence of random variates with expectation $\bar{g}$.

Then $u_k = \frac{1}{k} \sum_{i=1}^{k} g(\xi_i)$ is an unbiased estimator of $\bar{g}$. This summation can be written iteratively in the form:

$$u_k = u_{k-1} - \frac{1}{k}\{u_{k-1} - g(\xi_k)\} \quad ,$$

which is immediately recognizable.

It might be expected that Monte-Carlo accelerator techniques would be of considerable use in the present algorithm, in the same way that they are for the conventional stochastic approximation algorithm. On examination, two methods in particular are immediately applicable; the antithetic variate method, and the control variate method.

(a) The Antithetic Variate Method

The basic idea of the antithetic variate method is to use instead of the sample $g(\xi_i)$, a linear combination of two negatively correlated samples such that the expectation remains unchanged. If ξ_i is a sample from a symmetric distribution with zero mean, and $g(\xi_i)$ is predominantly odd in ξ_i, then the substitution

$$g^a(\xi_i) = \frac{1}{2}\{g(\xi_i) + g(-\xi_i)\}$$

is both simple and effective.

Consider now the application to the new stochastic approximation algorithm, replacing $f(u_k)$ by $f^a(u_k)$ in Eqn. (7) where

$$f^a(u_k) \triangleq \frac{1}{2}\left\{ \frac{\partial H(u_k, \xi_k)^T}{\partial u_k} + \frac{\partial H(u_k, -\xi_k)^T}{\partial u_k} \right\} \quad . \qquad (14)$$

Under the above assumption on the symmetry of the distribution for ξ_k, $Ef^a(u_k) = Ef(u_k)$ so that if the modified algorithm converges, it will do so to the solution point. If in addition

$$\frac{\partial H(\overset{*}{u}, \xi)}{\partial u} = -\frac{\partial H(\overset{*}{u}, -\xi)}{\partial u} \quad , \tag{15}$$

where $\overset{*}{u}$ represents the solution point $u = 0$, then exactly as in the equivalent deterministic problem, convergence to the solution point can be achieved over some range of ρ for functions $f(u)$ which do not increase too fast with u (i.e., Eqn. (9) can be satisfied over all the interesting range of u).

Example 2:

Let $f(u_k) = (1 + \xi_k^2)u_k + 0.1(u_k + \xi_k)^3$. For this example Eqn. (9) does not hold, so the new algorithm will not converge with probability one; the original stochastic approximation scheme with decreasing as $1/k$ does so however. On the other hand

$$f^a(u_k) = (1 + \xi_k^2)u_k + 0.1(u_k^3 + 3u_k\xi_k^2) \quad , \tag{16}$$

which satisfies Eqns. (9) and (14)*, so that the modified new algorithm will converge with probability one. Figure 2 shows the results of simulation of the various algorithms over 25 iterations. The modified new algorithm (curve 3) again converges

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* Actually in this example for any finite ρ there is a small finite probability of the algorithm diverging, but in practice this does not invalidate the scheme.

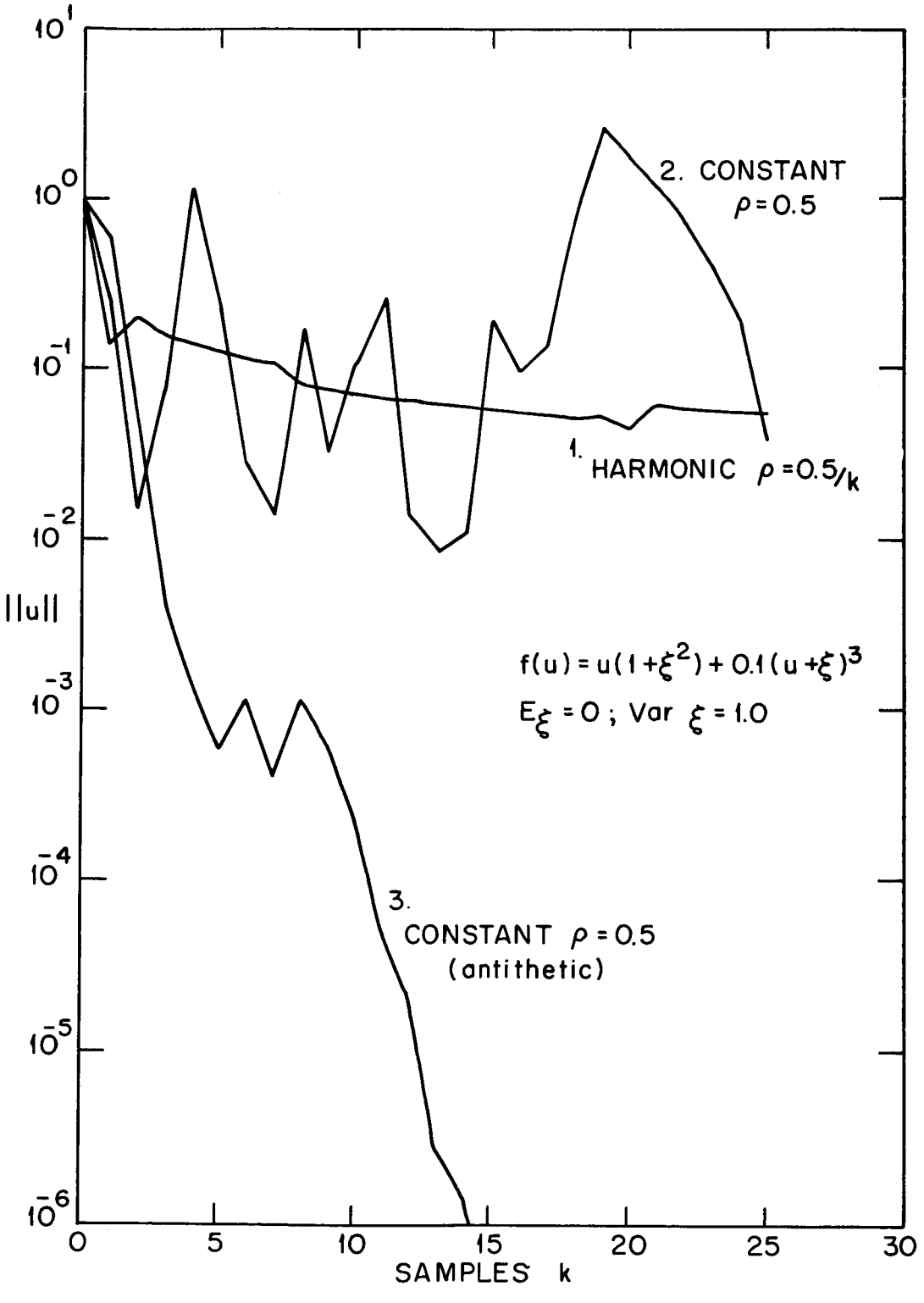

FIG. 2. MINIMIZATION OF A FUNCTION USING ANTITHETIC VARIATES.

considerably faster than the original stochastic approximation algorithm (curve 1). If the antithetic variate technique is not used, stochastic convergence is not assured (curve 2).

(b) The Control Variate Method

Essentially the control variate method splits the variate $g(\xi_i)$ into the sum of two parts: $g(\xi_i) = \tilde{g}(\xi_i) + g^c(\xi_i)$. The split is made so that the expectation of $\tilde{g}(\xi_i)$ may be found analytically. Let this expectation be $\hat{g}$; then an unbiased estimate of the expectation of $g(\xi_i)$, denoted by $\bar{g}$, is given by

$$u_k = \hat{g} + \frac{1}{k} \sum_{i=1}^{k} g^c(\xi_i) \quad ,$$

and this estimate will have a smaller variance than the estimate computed by the naive Monte-Carlo method for the same value of k.

Turning to the present descent method for solving $Ef(u) = 0$, a similar split can be made:

$$Ef(u) = E\tilde{f}(u) + Ef^c(u) \quad ,$$

where $E\tilde{f}(u) = \hat{f}(u)$ can be found analytically. Then in Eqn. (7) the sample $f(u_k)$ is replaced by

$$\hat{f}(u_k) + f^c(u_k) \quad . \tag{17}$$

Used by itself, the control variate method will not be able to widen the class of problems which the new stochastic descent algorithm can handle unless fortuitously ξ_k can be contrived only to enter multiplicatively into $f^c(u_k)$. However used in

conjunction with the antithetic variate method it will often improve convergence rates to a still higher degree (but still without introducing any bias).

Example 3:

As in the previous example $f(u_k) = (1 + \xi_k^2)u_k + 0.1(u_k + \xi_k)^3$. The antithetic variate sample $f^a(u_k)$ given by Eqn. (16) is split into the two parts:

$$\tilde{f}(u_k) = (1 + \xi_k^2)u_k \quad ; \quad f^c(u_k) = 0.1(u_k^3 + 3u_k\xi_k^2) \quad .$$

For this example, then, $\hat{f}(u_k) = 2u_k$ if ξ_k has unit variance. The descent algorithm is given by Eqn. (7), substituting $2u_k + f^c(u_k)$ for $f(u_k)$. Figure 3 shows the improvement obtainable by the use of a combination of the antithetic and control variate methods (curve 2) over the use of the antithetic variate method only (curve 1).

6. Conclusion

It has been shown at any rate for the simple and rather contrived examples given in the previous sections, that a constant step length algorithm converges considerably faster than the usual stochastic approximation algorithm. This is intuitively reasonable in general (except maybe for pathological problems) if the variance of the parameter u can be reduced fast enough. Hopefully this can be achieved by using Monte-Carlo accelerator techniques.

Further examination of deterministic descent methods is likely to lead to new algorithms for stochastic optimization. The

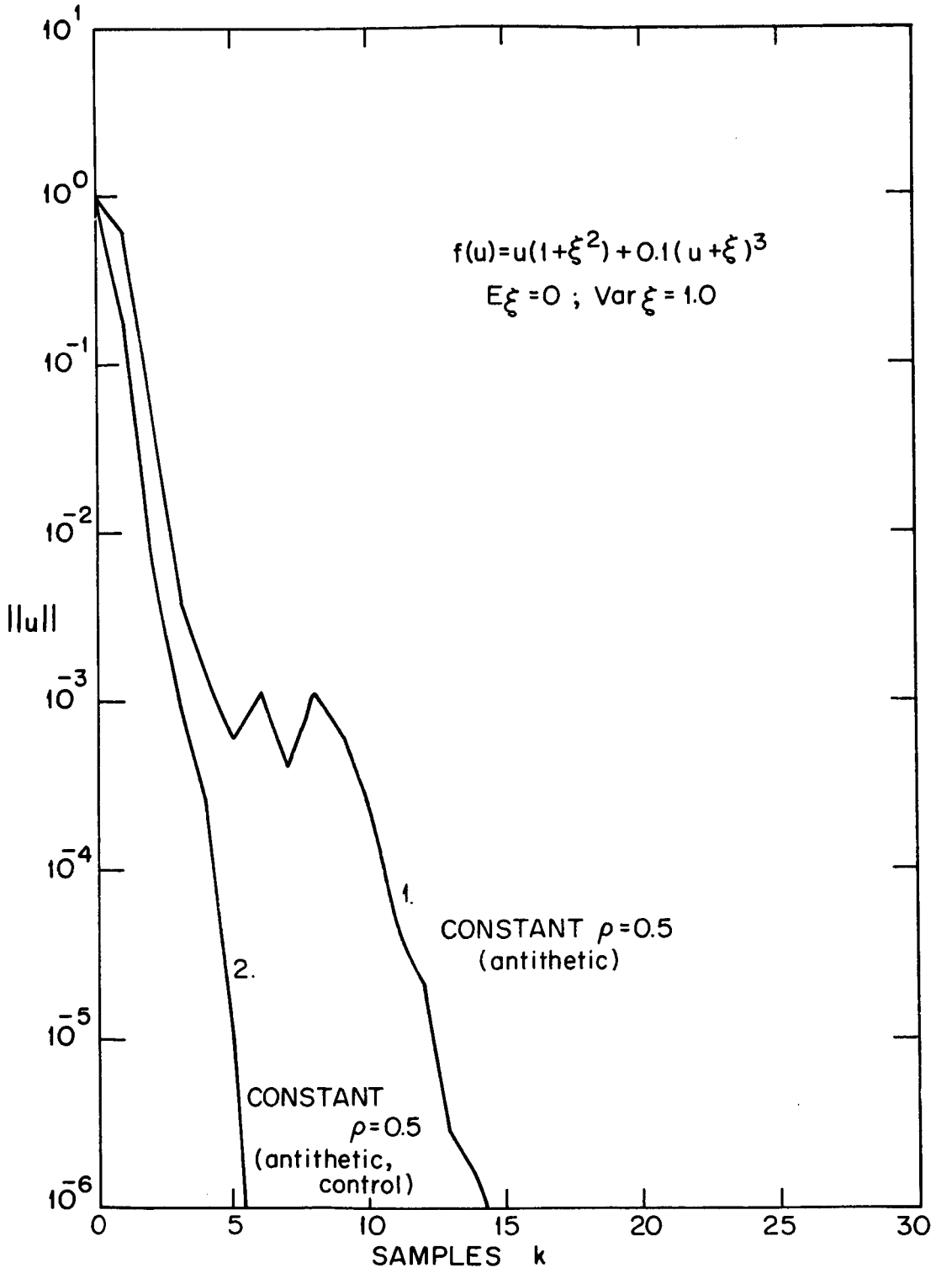

FIG. 3 INCREASE OF CONVERGENCE RATE BY CONTROL VARIATE METHOD.

ideal goal would be to create a range of algorithms so that most problems could be solved as quickly and efficiently from the computational point of view as their deterministic counterparts.

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13. ABSTRACT

An examination of the first order steepest descent method for minimizing deterministic functions leads to the specification of sufficient conditions assuring the convergence with probability one of a new stochastic approximation algorithm. The algorithm is interpreted as a stochastic descent method using a constant, as opposed to a continually decreasing step length. The crucial convergence condition is that the variance of the gradient should be zero as well as a minimum at the solution point.

The rather narrow class of problems to which the new algorithm is applicable can be extended to include many useful problems by employing accelerator methods adapted from the field of Monte-Carlo techniques, both to satisfy the condition, and to accelerate convergence.

Simple illustrative examples demonstrating the effectiveness of the new algorithm are presented.

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