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ANALYTICAL INVESTIGATION OF THE MIXING OF TWO PARALLEL STREAMS OF DISSIMILAR FLUIDS

by Richard L. Baker and Herbert Weinstein

Prepared by
ILLINOIS INSTITUTE OF TECHNOLOGY
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for

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ABSTRACT

The free jet mixing of parallel streams of different fluids is of fundamental importance and interest. The degree of mixing depends on the regime of flow in the mixing region, i.e., laminar or turbulent, and the velocity and density ratios.

In this work, similarity solutions were obtained for laminar and turbulent mixing of two parallel incompressible streams. The solutions apply to both similar and dissimilar fluids in the two streams with any velocity or density ratio and arbitrary laminar or turbulent Schmidt numbers.

A solution is numerical in nature and a set of solutions are presented in tabular form for a spectrum of density, velocity and Schmidt numbers.

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LIST OF SYMBOLS

Symbol	Definition
b	Width of mixing region for velocity
c	Constant of proportionality, i.e., $b = cx$
c_i	Amplification or damping factor
c_r	Velocity of propagation of disturbance
C	Molar density
$\mathcal{D}$	Diffusivity
d	Width of mixing region for velocity
f	Dependent variable for laminar similarity solution
F	Dependent variable for turbulent similarity solution
g	Acceleration of gravity
J	Richardson number
J_i	Molar flux of component i
K	$1/Sc$
K'	$1/Sc^{(t)}$
l	Prandtl's mixing length
P	Pressure or power input to sensor
p	Fluctuating power
R	Reynolds number
t	Time
u	x-component of velocity
U	Reference velocity
v	y-component of velocity
W	Dependent variable $= \int f d\eta$
x	Rectangular coordinate

Symbol	Definition
y	Rectangular coordinate
α	Wave number
$1/\beta$	Reference length for density
ϵ	Eddy viscosity
ξ	Independent variable in turbulent similarity solution
η	Independent variable in laminar similarity solution
λ	Constant related to velocity ratio, i.e., $\lambda = (u_1 - u_2)/(u_1 + u_2)$
Γ	Constant related to density ratio, i.e., $\Gamma = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$
χ_1	Empirical constant
φ	Amplitude function
μ	Viscosity
ν	Kinematic viscosity
ρ	Density
τ_{xy}	Apparent turbulent shearing stress
Nu	Nusselt number
Pr	Prandtl number
Ri	Richardson number
Sc	Schmidt number

Subscripts

A	Component A
B	Component B
i	Component i
1	Refers to lighter fluid

Symbol	Definition
2	Refers to heavier component
x	x-component
y	y-component
	Superscripts
—	Time average
*	Molar average
'	Fluctuating component
(t)	Turbulent

FOREWORD

Research related to advanced nuclear rocket propulsion is described herein. This work was performed under NASA Grant NsG-694 with Mr. Maynard F. Taylor, Nuclear Systems Division, NASA Lewis Research Center as Technical Manager.

I. INTRODUCTION

The behavior of adjacent free jets has been of interest for many years. Tollmein[1] in 1926 first considered analytically the mixing of two parallel streams of fluid. The hydrodynamic stability of this type of flow was discussed by Lord Rayleigh[2] as early as 1880.

The problem may be divided into two categories. In the first category the two streams are of the same fluid and in the second, the two streams are considered to be dissimilar fluids. Flows in the first category are referred to as being homogeneous, those in the second, are referred to as being heterogeneous.

The homogeneous case is related to studies of jet engines and rockets exhausts, while interest in the heterogeneous case has been mainly in the fields of meteorology and oceanography where so called density stratified flows commonly occur. Recently, the concept of a gaseous core nuclear rocket has stimulated new interest in the free jet mixing of co-flowing streams of dissimilar fluids. A better understanding of the fundamental problem of free jet mixing, i.e., mixing which takes place in the absence of solid boundaries, is needed for all these problems.

The object of this work is to obtain similarity solutions for the laminar and turbulent mixing of two dissimilar streams. In both cases the nature of the solution is numerical rather than closed form and tables of calculated results are presented covering the whole spectrum of parameter variation. The results are discussed in conjunction with experimental results in Ref. 67.

The analytical treatment applies only to the similar region of the flow field but no assumptions are made to limit density ratio or Schmidt number in either the laminar or turbulent case.

II. BACKGROUND

II-1 Homogeneous Case

The problem of laminar mixing in a half-jet is usually considered in connection with a stability analysis. The turbulent mixing problem is considered independently. In certain instances the solutions of the laminar and the turbulent mixing problems are related.

Most analytical investigations of either the laminar or the burbulent problem consider the initial velocity profiles to be shown in Fig.II-1.1.

Each stream is assumed to have a plug flow velocity distribution and no account is taken of boundary layer development on the plate separating the two streams. However, there have been some papers given which do take the initial boundary layers into consideration. The main advantage of analytically investigating an idealized velocity profile such as that shown above is that, while sacrificing some generality, it allows similar solutions to be obtained which are convenient to use in stability analyses and contain only one parameter to be determined experimentally in the case of turbulent mixing. Whereas, those investigations which do consider the initial boundary layers result in non-similar solutions with two or three parameters. The initial boundary layer(s) do in the real case definitely affect the flow in the mixing region a short distance downstream from the separating plate.

Among those investigations of the turbulent mixing problem which are based on the idealized initial velocity profiles, the main difference in analyses has been in the expression chosen to represent the eddy viscosity in the mixing region. Several theories of so-called free turbulence have been proposed. Prandtl's mixing length theory,[37] Taylor's free turbulence theory,[38] Reichardt's theory of turbulent mixing[39] and Prandtl's exchange coefficient theory[40] are most widely used and successful. These are discussed in some detail in Abramovich's work on turbulent jets.[41] Taylors model of turbulence assumes that tangential stresses in turbulent flows are caused by vorticity transfer and not by momentum transfer as in Prandtl's mixing length hypothesis. However, the expression obtained for the turbulent shearing stress is the same for both models with only a difference in the numerical value of the mixing length.

Reichardt's theory of turbulent mixing results in the reduction of the equations of motion to the form of the generalized heat conduction equation. It is not widely applicable.

The most commonly used of these theories are those of Prandtl.[42] The mixing length theory of Prandtl was used by Prandtl and by Tollmien[2] in their investigations of the turbulent half-jet problem and Prandtl's exchange coefficient theory was used by Goertler[9] in his analysis of the same problem.

Prandtl[42] used his concept of the mixing length to represent the apparent shearing stress of turbulent momentum interchange according to the relation

$$\tau_{xy} = \rho \ell^2 \left|\frac{du}{dy}\right| \frac{du}{dy} \qquad \text{II-1.1}$$

where ℓ is the mixing length.

Prandtl considered the velocity profile shown in Figure II-1.1 to exist at time t equal to zero. With the assumptions

$$u = u(y,t), \quad v = 0$$

and using his mixing length concept to represent the turbulent shearing stress, he wrote the momentum equation as

$$\frac{\partial u}{\partial t} = \ell^2 \left|\frac{\partial u}{\partial y}\right| \frac{\partial^2 u}{\partial y^2} \qquad \text{II-1.2}$$

By assuming the mixing length ℓ to be proportional to the width of the mixing region b, where $b = b(t)$, and introducing a new independent variable

$$\theta = y/b$$

and a new dependent variable

$$f(\theta) \sim u$$

Prandtl obtained the solution

$$u(y,t) = 1/2(u_1+u_2) + 1/2(u_1-u_2)\left[\frac{3}{2}\left(\frac{y}{b}\right) - \frac{1}{2}\left(\frac{y}{b}\right)^3\right] \qquad \text{II-1.3}$$

with

$$b = \frac{3}{2}\beta^2(u_1-u_2)t$$

the quantity

$$\beta = \ell/b$$

is the only empirical constant to be obtained from experimental data. The velocity in the mixing region does not approach the free stream velocity asymptotically. At

$$y = \pm b$$

the velocity reaches the free stream velocity with a discontinuity in

$$\frac{d^2u}{dy^2}$$

Tollmien also considered the velocity profile shown in Figure II-1.1. At $x = 0$, two parallel streams meet. Stream 1 has constant velocity u_1 and stream 2 constant velocity u_2 with

$$u_1 > u_2$$

Downstream a mixing region is formed in which the discontinuity in velocity is mapped out. Tollmien's analysis was for the case of turbulent mixing with

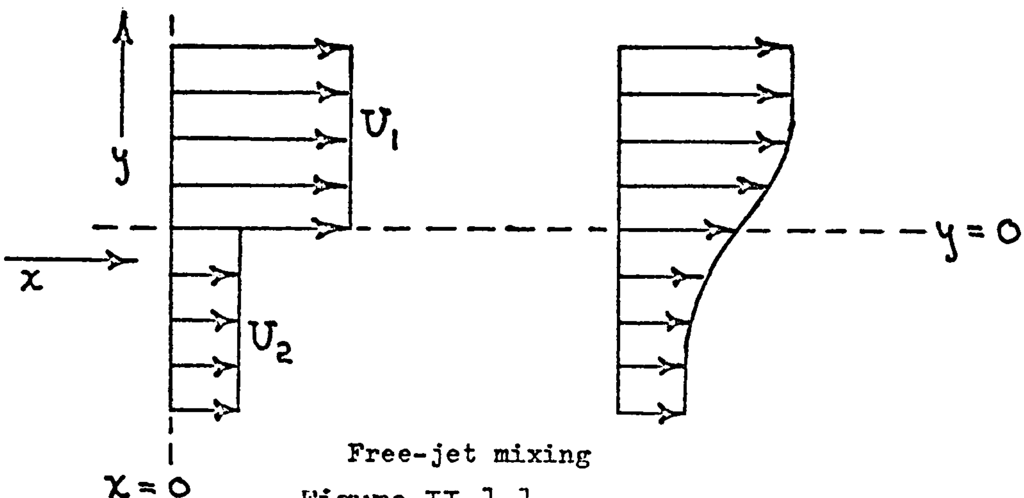
$$u_2 = 0$$

i.e., a half-jet.

For the two dimensional incompressible mixing region, the equations of continuity and momentum may be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{II-1.4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial \tau_{xy}}{\partial y} \qquad \text{II-1.5}$$



Free-jet mixing

Figure II-1.1

Tollmien[1] integrated the equation of continuity by using a stream function and defined the stream function in terms of the new variable

$$\theta = y/x$$

i.e.

$$\Psi = x\,F(\theta) \qquad \text{II-1.6}$$

He used Prandtl's mixing length hypothesis for the turbulent shearing stress in the mixing region and assumed that the mixing length is constant across each cross section and increases linearly with x, i.e., $\ell = cx$. Tollmien combined equations II-1.6, II-1.1 and II-1.5 to obtain the following differential equation

$$FF'' + 2c^2\,F''\,F''' = 0 \qquad \text{II-1.7}$$

He pointed out that this is solved by

$$F'' = 0 \qquad \text{II-1.8}$$

or by

$$F + 2c^2F''' = 0 \qquad \text{II-1.9}$$

He stated that equation II-1.9 applied between the limits

θ_1 and θ_2

and equation II-1.8 applied outside these limits. Tollmien's solution was

$$F = C_1e^{-\theta} + C_2e^{\theta/2}\cos\frac{\sqrt{3}}{2}\theta + C_3e^{\theta/2}\sin\frac{\sqrt{3}}{2}\theta \qquad \text{II-1.10}$$

He then applied the following five boundary conditions to solve for the constants C_1, C_2, C_3, θ_1 and θ_2.

$$F'(\theta_1) = 1$$
$$F''(\theta_1) = 0$$
$$F'(\theta_2) = 0$$
$$F''(\theta_2) = 0$$
$$F(\theta_1) = \theta_1 \qquad \text{II-1.11}$$

Tollmien's solution agrees reasonably well with the experimental data of Albertson[14] and Liepmann and Laufer.[12] The main disadvantage of his solution lies in the fact that F''', $\frac{d^2u}{dy^2}$, is discontinuous at θ_1 and θ_2 and the velocity in the mixing region does not asymptotically approach the free stream values. It will be recalled that Prandtl's solution displayed these same characteristics. Schlichting points out that this is a general property of all solutions based on Prandtl's mixing length hypothesis and calls this an esthetical deficiency of the hypothesis.

Kuethe[5] extended Tollmien's analysis to the case where

$$u_2 \neq 0$$

His analysis proceeded in a manner analogous to that of Tollmien except in the application of the fifth boundary condition. Tollmien's fifth boundary condition was

$$F(\theta_1) = \theta_1$$

which is equivalent to stating that the transverse velocity is equal to zero. Kuethe's fifth boundary was

$$u(\theta_1)v(\theta_1) + u(\theta_2)v(\theta_2) = 0 \qquad \text{II-1.12}$$

or

$$m[-F(\theta_2-\theta_1)+\theta_2 F'(\theta_2-\theta_1)] = F(\theta_1)-\theta_1 F'(\theta_1) \qquad \text{II-1.13}$$

where

$$m = u_2/u_1$$

Kuethe stated that this condition was suggested by von Karman and corresponds to the assumption that no external forces are acting on the total fluid system perpendicular to the main flow. He pointed out that neither the width of the mixing region nor the u-velocity profiles are affected by the use of this condition. However, the v-velocity profiles and θ_1 and θ_2 are affected an appreciable amount. Velocity profiles for m = 0, 0.5 and 0.8 were given.

Goertler[4] also considered the velocity profile shown in Figure II-1.1. He used the equations of continuity and momentum as given by equations II-1.4 and II-1.5 and defined the turbulent shearing stress by

$$\tau_{xy} = \rho\, \epsilon \frac{\partial u}{\partial y} \qquad \text{II-1.14}$$

In this expression ϵ is the eddy viscosity or the turbulent exchange coefficient and is given by

$$\epsilon = \chi_1 b(u_2-u_1) \qquad \text{II-1.15}$$

This is Prandtl's exchange coefficient hypothesis. In this expression, b denotes the width of the mixing region and χ_1 is an empirical constant. Goertler assumed that the mixing region spreads linearly with x, i.e., $b = cx$ and that the exchange coefficient does not vary across a cross section. Substituting equations II-1.15 and II-1.14 into equation II-1.5 and introducing a stream function

$$\psi = xUF(\xi)$$

Goertler obtained the following differential equation.

$$F''' + 2\sigma FF'' = 0 \qquad \text{II-1.16}$$

In this equation

$$\sigma = 1/2\ (\chi_1 c\lambda)^{-1/2} \qquad \text{II-1.17}$$

and

$$\xi = \sigma\ y/x$$

Goertler applied the following boundary conditions

$$\sigma F'(\infty) = u/U = 1 + \lambda$$
$$\sigma F'(o) = u/U = 1 \qquad \text{II-1.18}$$
$$\sigma F'(-\infty) = u/U = 1 - \lambda$$

where

$$U = 1/2\ (u_1 + u_2)$$

and

$$\lambda = u_1 - u_2/u_1 + u_2$$

He solved equation II-1.16 by assuming a power series expansion of the form

$$\sigma F(\xi) = F_o(\xi) + \lambda F_1(\xi) + \lambda^2 F_2(\xi) + \ldots \qquad \text{II-1.19}$$

Substituting this expression into equation II-1.16 together with $F_o = \xi$ and equating coefficients of λ^n, he obtained differential equations for F_1, F_2, etc. The equation for F_1 is

$$F_1''' + 2\xi F_1'' = 0 \qquad \text{II-1.20}$$

The solution of this equation with the boundary conditions II-1.18 is

$$F_1'(\xi) = \operatorname{erf}\xi = \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-Z^2} dZ \qquad \text{II-1.21}$$

or in terms of the velocity u

$$u = \frac{u_1+u_2}{2} \left[1 + \frac{u_1-u_2}{u_1+u_2} \operatorname{erf} \xi\right] \qquad \text{II-1.22}$$

This solution is the first order approximation to the exact solution.

Goertler's velocity profile approaches the free stream velocities asymptotically and there are no discontinuities in any of the velocity derivatives as in the case of Prandtl's and Tollmein's solutions. However, the independent variable in Goertler's solution is

$$\xi = \sigma y/x$$

This means that the x-axis should be the line $\xi = 0$. Then, according to Goertler's second boundary condition

$$\sigma F'(o) = \frac{u}{U} = 1$$

along the x-axis or

$$u = 1/2(u_1+u_2)$$

This is clearly not true from the experimental data of Albertson[14] and Liepmann and Laufer.[12]

Abramovich[41] points out that if Goertler's theoretical velocity profile is displaced so that the boundary of the jet ($y = 0$) passes along the line $\xi = 0.3$, then this profile passes close to the experimental points. He states that this means that Goertler's theory requires two experimental constants σ and ξ_o, whereas Tollmien's theory is made to correspond with the data with the aid of only one experimental constant.

Yen[38] clarified the discrepancy discussed above by pointing out that Goertler's theoretical velocity profile is correctly placed with respect to the boundary of the jet by applying von Karman's boundary condition, as discussed in connection with the work of Kuethe, instead of the condition

$$\sigma F'(o) = 1$$

However, an error in Yen's analysis invalidated his conclusions except for the case of

$$u_2 = 0$$

This will be discussed in greater detail in a later section.

Lessen,[3] Chiarulli[6] and Lin[7] have solved this same problem for the case of laminar mixing in connection with their stability analyses. The differential equation to be solved is

$$f''' + 1/2ff'' = 0 \qquad \text{II-1.23}$$

where

$$f = f(\eta) = \frac{u}{U}$$

and

$$\eta = y\sqrt{\frac{U}{vx}}$$

The boundary conditions are

$$f'(\infty) = 1$$
$$f'(o) = 0.5 \qquad \text{II-1.24}$$
$$f'(-\infty) = 0$$

Lin[7] points out that equation II-1.23 together with boundary conditions II-1.24 may be converted to Goertler's equation for turbulent mixing, equation II-1.16, together with boundary conditions II-1.18 by use of the following relations

$$f'(\eta) = \frac{\sigma F'(\xi)}{1+\lambda}$$
$$\eta = 2\xi\sqrt{1+\lambda} \qquad \text{II-1.25}$$

Thus the laminar solution and the turbulent solution of Goertler are related through a transformation of variables. Chiarulli and Lin solved equation II-1.23 for $\lambda = 0.2$, 0.4, 0.6, 0.8 and 1.0. Errors in the solution of Goertler were pointed out and corrected. However, Goertler's solution method was used. As previously mentioned, Lessen used the method of analytic continuation to numerically integrate equation II-1.23 and obtain the solution.

Crane[44] has extended Goertler's analysis to the case of compressible flow with temperature difference. The width of the mixing region was shown to depend upon the difference of the stagnation enthalpies of the two streams and on the Mach numbers of the flow.

The effect of the initial boundary layer development on the plate separating the two streams on the flow profiles in the mixing region has been considered by Torda, et. al.,[45] Chapman and Korst[46] and by Ackermann.[47]

Torda, et. al. considered the turbulent,incompressible, symmetric mixing of two parallel streams; i.e., both free streams have the same velocity. The von Karman intergral concept was applied to the momentum and energy equations to evaluate the thickness of the mixing region and the velocity distribution in it. Velocity profiles were presented for three downstream positions. Also, curves showing the increase in width of the mixing region with distance downstream were presented. These curves showed a considerable amount of curvature in the region immediately behind the plate as opposed to straight lines obtained from the analyses of Tollmien, Kuethe and Goertler. It was pointed out that this is in qualitative agreement with the experiments of Leipmann and Laufer.

Chapman and Korst considered the problem of free-jet mixing with the initial velocity distribution given by power law, power series and broken line representation.

A momentum integral method was applied to the linearized equations of motion to obtain a solution. Multi-parameter presentations for the velocity distribution in the mixing region were given.

The laminar incompressible mixing of two streams of different velocities with consideration of the initial boundary layers has been considered by Ackermann.

II-2 Heterogeneous Case

Here also the problems of laminar and of turbulent mixing are generally considered separately. The initial velocity profiles are again assumed to be as shown in Figure II-1.1. However, now the slower moving stream is assumed to have a higher density than the faster moving stream. The buoyancy forces due to the density variation are usually neglected.

The laminar mixing problem has been considered by Pai, Keulegan,[48], Lock[49] and Potter.[50] Pai considered the flow to be compressible. He used a stream function to integrate the continuity equation. Then, using the distance along the jet axis and the stream function as independent variables, he reduced the diffusion equation and the equation of motion to the form of the generalized heat conduction equation. These equations were then solved simultaneously by step-wise numerical procedure.

The analyses of Keulegan, Lock and Potter are all for the case in which the two streams are immiscible, i.e., there is no molecular diffusion. Keulegan considered a velocity profile such as that shown in Figure II-1.1 with

$$u_2 = 0$$

The viscosities and densities of the two streams were assumed to be not the same. By writing the continuity equation and the equation of motion for each fluid and solving them simultaneously, he determined the velocity distribution in the laminar boundary layers at the interface, the thickness of the layers and the stress at the interface.

Lock[49] independently considered the same problem. He showed that the solutions depend only on the ratio

$$u_2/u_1$$

of the velocities of the two streams and the product

$$\rho'\mu'$$

where

$$\rho' = \frac{\rho_2}{\rho_1} \quad \text{and} \quad \mu' = \frac{\mu_2}{\mu_1}$$

His results are in general agreement with those of Keulegan.[43]

Potter[45] extended Keulegan's analysis to the case where both fluids are moving.

The turbulent mixing problem has been investigated by Szablewski[8,9,10] and by Pai.[11] Szablewski in a series of articles considered the mixing of parallel streams of different temperatures. This problem is obviously closely related to the one in which the two streams are of different chemical composition. The special case in which the two streams have nearly the same velocity but widely differing temperatures was first considered. This assumption results in a simplification of the differential equations. Velocity and temperature distributions were given for various temperature ratios.

The problem of the turbulent mixing of parallel streams with unrestricted differences in velocity and density was considered by Szablewski[9] in a later paper. In this case the problem was simplified by linearizing the equation of motion. Prandtl's exchange coefficient hypothesis was used and the ratio of eddy viscosity to eddy conductivity was assumed to be 1/2, i.e., a turbulent Prandtl number of 1/2. Velocity and temperature distributions were again given for various temperature ratios. His analysis showed that as the density differences between the two streams is increased, the mixing region becomes displaced in the direction of the less dense stream.

Pai[11] considered the two dimensional incompressible jet mixing of two different gases. From the equations of continuity, motion and diffusion he obtained the following generalization of equation II-1.16 given by Goertler[4] for the homogeneous case:

$$F''' + 2\sigma F F'' = \frac{2A'\sigma F''^2}{1 - A'\sigma F'} \qquad \text{II-2.1}$$

In this equation, F has the same meaning as that given by Goertler. A' is given by

$$A' = \frac{\Gamma}{\Gamma+\lambda}$$

where

$$\Gamma = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} \quad \text{and} \quad \lambda = \frac{u_1 - u_2}{\mu_1 + \mu_2}$$

Obviously, if $\rho_1 = \rho_2$ then $A' = 0$ and the equation reduces to Goertler's equation for the homogeneous case. This equation was apparently first derived by Hu[61] for the case of turbulent mixing of immiscible fluids. Pai considered the case in which the density difference is much less than the velocity difference, i.e.

$$\Gamma \ll \lambda$$

He concluded that the first order effect of density differences of the two gases on the velocity distribution is small.

III. ANALYTICAL DEVELOPMENT

So-called "similar" solutions are desirable for several reasons. First of all, they allow a partial differential equation to be reduced by combination of variables to an ordinary differential equation. This factor is especially important in the case in which a set of partial differential equations may be reduced to one or more ordinary differential equations. Another desirable feature of similar solutions is that by their very nature they afford a straight forward and simple method of correlating experimental data. Finally, if a similar solution can be obtained, then by properly defining a reference length, the independent variable from the similar solution becomes the independent variable in the Orr-Sommerfield equation of hydrodynamic stability.

III-1 Homogeneous Laminar Mixing

Only the case of isothermal, incompressible mixing will be discussed here. For this case, as given by Schlichting,[60] the equation of continuity may be written

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{III-1.1}$$

The Navier-Stokes equations are given by

x-direction $\rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}] = -\frac{\partial P}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$

y-direction $\rho[\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}] = -\frac{\partial P}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})$

III-1.2

With the usual boundary layer assumptions that

$$\left|\frac{\partial(\;)}{\partial x}\right| << \left|\frac{\partial(\;)}{\partial y}\right|$$

$$|v| << |u| \qquad \text{III-1.3}$$

the Navier-Stokes equations and the continuity equation may be reduced to "Prandtl's boundary layer equations".

$$\rho[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}] = -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{III-1.4}$$

Assuming steady state flow and neglecting the variation of pressure in the flow direction, which is very small in the case of free jet mixing, these equations may be further reduced to give

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{III-1.5}$$

The boundary conditions are

$$u(x, \infty) = u_1$$

$$u(x, -\infty) = u_2 \qquad \text{III-1.6}$$

It can be shown that the application of von Karman's third necessary boundary condition does not affect the similarity solution obtained from equations III -1.5 with boundary conditions III-1.6

The equation of continuity may be integrated by introducing a stream function ψ such that

$$u = \frac{\partial\psi}{\psi y} , \quad v = -\frac{\partial\psi}{\partial x} \qquad \text{III-1.7}$$

Equations III -1.5 may then be reduced to

$$\left[\frac{\partial\psi}{\partial y} \left(\frac{\partial^2\psi}{\partial x \partial y}\right) - \frac{\partial\psi}{\partial x} \left(\frac{\partial^2\psi}{\partial y^2}\right)\right] = \nu \frac{\partial^3\psi}{\partial y^3} \qquad \text{III-1.8}$$

with the boundary conditions

$$\frac{\partial\psi(x, \infty)}{\partial y} = u_1, \quad \frac{\partial\psi(x, -\infty)}{\partial y} = u_2 \qquad \text{III-1.9}$$

To determine if a similar solution of equation III -1.8 with boundary conditions III-1.9 is possible, an affine transformation is applied. A new set of variables ψ', x' and y' are defined by the following relations

$$\psi' = a\psi'$$

$$x = rx'$$

$$y = sy' \qquad \text{III-1.10}$$

If these relationships are substituted into equation III-1.8, and it is required that the new equation be of the same form as the previous equation, then the following relation must be true

$$r/sa = 1 \qquad \text{III-1.11}$$

Substituting equations III-1.10 into the boundary conditions given by equations III -1.12 gives

$$\frac{a}{s}\frac{\partial\psi'(x,\infty)}{\partial y} = u_1$$

$$\frac{a}{s}\frac{\partial\psi'(x,-\infty)}{\partial y} = u_2 \qquad \text{III-1.12}$$

If these boundary conditions are to be invariant under the affine transformation, then

$$a/s = 1 \qquad \text{III-1.13}$$

Combining equations III-1.13 and III-1.11 gives

$$r/sa = r/s^2 = 1 \qquad \text{III -1.14}$$

or

$$\sqrt{r}/s = 1 \qquad \text{III -1.15}$$

Combining equations III-1.15 and III-1.10

$$s = y/y' = \sqrt{r} = \sqrt{x}/\sqrt{x'} \qquad \text{III-1.16}$$

or

$$y/\sqrt{x} = y'/\sqrt{x'} \qquad \text{III-1.17}$$

Thus, the quantity $y/\sqrt{x}$ is invariant under the affine transformation and a new independent variable η may be defined as

$$\eta = Cy/\sqrt{x} \qquad \text{III-1.18}$$

This may be put into dimensionless form by defining the constant C by

$$C = \sqrt{U/\nu} \qquad \text{III-1.19}$$

where U is a reference velocity and ν is the kinematic viscosity in equation III-1.8. A new dimensionless independent variable is then given by

$$\eta = y\sqrt{U/\nu x} \qquad \text{III-1.20}$$

Combining equations III-1.13, III-1.15, and III-1.10

$$a = \psi/\psi' = \sqrt{r} = \sqrt{x}/\sqrt{x'} \qquad \text{III-1.21}$$

or

$$\psi/\sqrt{x} = \psi'/\sqrt{x'} \qquad \text{III-1.22}$$

Thus, the quantity $\psi/\sqrt{x}$ is also invariant under the affine transformation and a new dimensionless dependent variable may be defined by

$$f(\eta) = \psi/\sqrt{U\nu x} \qquad \text{III-1.23}$$

Using equations III-1.23 and III-1.20, equation III-1.8 may be transformed from a partial differential equation with dependent variable ψ and independent variables x and y into an ordinary differential equation with dependent variable f and independent variable η. Performing the indicated algebraic manipulations, equation III-1.8

becomes

$$f''' + 1/2\ ff'' = 0 \qquad \text{III-1.24}$$

The boundary conditions obtained from equation III-1.12 are

$$f'(\infty) = \frac{u_1}{U}$$

$$f'(-\infty) = \frac{u_2}{U} \qquad \text{III-1.25}$$

Equation III-1.24 with boundary conditions III-1.25 is the same as equation II -1.23 with boundary conditions II-1.24. Various solutions of this equation are discussed in Section II -1. The solution method of Lessen[4] will be described in greater detail here, since the method is generally applicable and will be used in connection with the heterogeneous problem. The method is sometimes referred to as "analytic continuation".

Equation III-1.24 may be written

$$f''' = -1/2\ ff'' \qquad \text{III-1.24}$$

This expression may then be easily differentiated to obtain expressions for the fourth, fifth, and higher derivatives.

Expanding the function f in a Taylor series gives

$$f(\eta+w) = f(\eta) + f'(\eta)w + f''(\eta)w^2/2 + f'''(\eta)w^3/6 + \\ + f^{IV}(\eta)w^4/24 + f^V(\eta)w^5/120 + \ldots \qquad \text{III-1.26}$$

Similar expressions may be written to represent the first derivative f' and the second derivative f''. Now, if f, f' and f'' are known for some large negative value of the independent variable η, then f^{III}, f^{IV}, f^V etc. may be calculated for that value of η from equation III-1.24

and the expressions obtained by differentiating equation III-1.24. Equation III-1.26 and similar expressions for f' and f" may then be used to calculate f, f' and f" at $\eta + w$. By repeating this procedure, equation III-1.24 may be integrated over the entire interval of η.

To obtain the values of f, f' and f" for large negative η, an asymptotic solution of equation III-1.24 is obtained which is valid for large negative values of η. The manner in which the asymptotic solution is obtained depends upon whether $u_2 = 0$, i.e., a half-jet; or both streams are in motion.

If $u_2 = 0$, then from the second of equation III-1.25, the following is true

$$\eta \to -\infty$$
$$f'(\eta) \to 0$$
$$f(\eta) \to -S \qquad \text{III-1.27}$$

Substituting the last of these relations into equation III-1.24, it can be easily shown that

$$f'(\eta) \to k_1 \, e^{1/2\, S\eta}, \quad \eta \to -\infty \qquad \text{III-1.28}$$

This is then the desired form of an asymptotic expression for f' good for large negative values of η. From this, expressions for f and f" may also be obtained. However, before these expressions may be used, the values of the constants k_1 and S must be determined. Lessen used the following technique:

He first defined a new dependent variable q and a new independent variable x by the following relations

$$q(x) = \frac{1}{S} f(\eta)$$

$$x = S\eta \qquad \text{III-1.29}$$

Equation III-1.24 is transformed to

$$q''' + \frac{1}{2} qq'' = 0 \qquad \text{III-1.30}$$

If u_1 is used as the reference velocity U, then the boundary conditions from equations III-1.25 become

$$f'(\infty) = 1$$

$$f'(-\infty) = 0 \qquad \text{III-1.31}$$

The relationships expressed by equations III-1.27 become

$$x \to -\infty$$

$$q'(x) \to 0$$

$$q(x) \to -1 \qquad \text{III-1.32}$$

Differentiating the first of equations III-1.29.

$$q'(x) = \frac{1}{S} f'(\eta) \frac{1}{S}$$

or

$$S = \sqrt{f'(\eta)/q'(\infty)} \qquad \text{III-1.33}$$

Substituting the first of equations III-1.31.

$$S = 1/\sqrt{q'(\infty)} \qquad \text{III-1.34}$$

From equation III-1.28, it follows that for large negative values of x, q may be represented by

$$q(x) = B_0 + B_1\, e^{1/2x} + B_2\, e^{x} + B_3\, e^{3/2x} + \ldots$$

III-1.35

Substituting this relationship into equation III-1.30, equating coefficients of $e^{\frac{n}{2}x}$, n = 1, 2, 3 and recalling that $q(-\infty) = -1$; it can be easily shown that

$$B_0 = -1$$
$$B_1 = 1$$
$$B_2 = -1/4$$
$$B_3 = 5/72$$

III-1.36

Equation III-1.35 may be differentiated to obtain expressions for q' and q".

Using these expressions to represent q, q' and q" for some large negative value of x, equation III-1.30 may be integrated by the method of analytic continuation. Once the integration has been carried to large positive values of x, S may be determined from equation III-1.34.

The solution to equation III-1.24 is then obtained by transforming back using the first of equations III-1.29. Finally, the asymptotic form of $f(\eta)$, good for large negative values of η, is

$$f(\eta) = T_0 + T_1 e^{1/2S\eta} + T_2 e^{S\eta} + T_3 e^{3/2S\eta} + \ldots$$

III-1.37

where

$$T_0 = -S$$

$$T_1 = S$$

$$T_2 = -S/4$$

$$T_3 = 5S/72 \qquad \text{III-1.38}$$

If $u_2 \neq 0$, then the method just described is not applicable since $f(\eta)$ becomes infinite for large negative values of η. In this case, the transformation

$$f(\eta) = \int w(\eta)d\eta \qquad \text{III-1.39}$$

is used. Equation III-1.24 becomes

$$w''' w' - w''^2 + \frac{1}{2} ww'^2 = 0 \qquad \text{III-1.40}$$

with boundary conditions

$$w(\infty) = \frac{u_1}{U}$$

$$w(-\infty) = \frac{u_2}{U} \qquad \text{III-1.41}$$

if u_1 is again used as the reference velocity U, then for large negative values of η

$$w(\eta) \to u_2/u_1 = \sigma_1 \qquad \text{III-1.42}$$

Equation III-1.40 becomes

$$w''' w' - w''^2 + \frac{1}{2} \sigma_1 w'^2 = 0, \ \eta \to -\infty \qquad \text{III-1.43}$$

It can be shown that a solution of this equation is given by

$$w = \sigma_1 + \frac{2C_1}{\sqrt{\sigma_1}} [\text{erfc } (-\sqrt{\sigma_1} \tfrac{\eta}{2})] \qquad \text{III-1.44}$$

For large negative values of η, w, w' and w'' may then be represented by

$$w = \sigma_1 - \frac{2C_1}{\sqrt{\pi}\,\sigma_1}\left[\frac{1}{X} - \frac{1}{2X^3} + \frac{1.3}{2^2X^5} - \frac{1\cdot3\cdot5}{2^3X^7}\right] e^{-X^2} \qquad \text{III-1.45}$$

$$w' = \frac{2C_1}{\sqrt{\pi}} e^{-X^2} \qquad \text{III-1.46}$$

$$w'' = -\frac{\sigma_1 C_1}{\sqrt{\pi}} \eta\, e^{-X^2} \qquad \text{III-1.47}$$

where

$$X = \sqrt{\sigma_1}\,\frac{\eta}{2} \qquad \text{III-1.48}$$

Using these expressions, the method of analytic continuation may be applied to equation III-1.40. However, in this case the solution becomes trial and error because the constant C_1 can not be easily evaluated. Thus, C_1 is assumed and equation III-1.40 is integrated to large positive values of η. The correct value of C_1, is that one for which

$$w(\infty) = 1 \qquad \text{III-1.49}$$

The second boundary condition is satisfied by the asymptotic expression for w, i.e., equation III-1.44. Both of these cases were discussed by Lessen. However, he obtained a solution for the case in which $u_2 = 0$ only.

III-2 Homogeneous Turbulent Mixing

Following the steps outlined in section III-1, the Navier-Stokes equations and the equation of continuity are reduced to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{III-1.5}$$

The following development parallels that given by Schlichting. Turbulent flow is usually described mathematically by separating it into a mean motion and a fluctuating or eddy motion. Thus, the instantaneous velocity components u and v in the x and y direction respectively are written

$$u = \bar{u} + u'$$

$$v = \bar{v} + v' \qquad \text{III-2.1}$$

where $\bar{u}$ and $\bar{v}$ are the time-average velocity components and u' and v' are the fluctuating components. Introducing equations III-2.1 into equations III-1.5 and time averaging

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} + \overline{u' \frac{\partial u'}{\partial x}} + \overline{v' \frac{\partial u'}{\partial y}} = \nu \frac{\partial^2 \bar{u}}{\partial y^2}$$

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \qquad \text{III-2.2}$$

With the aid of the second of equations III-2.2, this may be written as

$$\rho[\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y}] + \frac{\partial}{\partial x} (\rho\overline{u'^2}) + \frac{\partial}{\partial y} (\rho\overline{u'v'}) = \mu \frac{\partial^2 \bar{u}}{\partial y^2}$$

III-2.3

Neglecting the term containing the molecular viscosity this becomes

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y} + \frac{1}{\rho} \frac{\partial \sigma_x}{\partial x} \qquad \text{III-2.4}$$

where

$$\sigma_x = -\rho \overline{u'^2}$$

and

$$\tau_{xy} = -\rho \overline{u'v'}$$

These quantities are "apparent" stresses due to turbulent flow or Reynolds stresses. The term containing σ_x is usually neglected in accordance with the boundary layer assumptions. The equations to be solved are then

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} = \frac{1}{\rho} \frac{\partial \tau_{xy}}{\partial y} \qquad \text{III-2.5}$$

Integrating the continuity equation using a stream function and substituting equations II-1.14 and II-1.15, Goertler[9] reduced these equations to

$$\frac{\partial \psi}{\partial y} \left(\frac{\partial^2 \psi}{\partial y \partial x}\right) - \frac{\partial \psi}{\partial x} \left(\frac{\partial^2 \psi}{\partial y^2}\right) = \chi_1 cx \, (u_1 - u_2) \frac{\partial^3 \psi}{\partial y^3} \qquad \text{III-2.6}$$

Applying an affine transformation it can be shown that y/x and ψ/x are invariant under the transformation. It is thus convenient to define a new dimensionless independent variable ξ by

$$\xi = \sigma \frac{y}{x}$$ III-2.7

where

$$\sigma = \frac{1}{2} (\chi_1 c\lambda)^{-1/2}$$

and

$$\lambda = \frac{u_1 - u_2}{u_1 + u_2}$$

A new dimensionless dependent variable $F(\xi)$ is defined by

$$F(\xi) = \psi/U\, x$$ III-2.8

Using equations III-2.8 and III-2.7, equation III-2.6 may be transformed to

$$F''' + 2\sigma F F'' = 0$$ III-2.9

The boundary conditions are

$$\sigma F'(\infty) = \frac{u_1}{U} = 1 + \lambda$$

$$\sigma F'(-\infty) = \frac{u_2}{U} = 1 - \lambda$$ III-2.10

In this case the reference velocity U is given by

$$U = \frac{1}{2} (u_1 + u_2)$$

It will be recalled from section II-1 that Lin pointed out that equation III-2.9 with boundary conditions III-2.10 may be converted to equation III-1.24 by use of the relations

$$f'(\eta) = \frac{\sigma F'(\xi)}{1 + \lambda}$$

$$\eta = 2\sqrt{1+\lambda}\; \xi$$ II-1.25

The important relationship between the solution of the homogeneous turbulent mixing problem by the method of Goertler and the solution of the homogeneous laminar mixing problem is thus re-emphasized here.

III-3 Heterogeneous Laminar Mixing

This problem may also be directly associated with a hydrodynamic stability analysis of the mixing region flow. However, at the present time no similar solution for the velocity and density profiles in the mixing region has been obtained without certain simplifying assumptions (see section II-2).

The equation of continuity for this case must be written as

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad \text{III-3.1}$$

because

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0$$

due to molecular diffusion. To obtain the velocity, and density profiles in the mixing region, equation III-3.1 must be solved simultaneously with the species diffusion equation and the equations of motion. An affine transformation to these equations was not found. Therefore, a different approach was necessary to obtain a similar solution.

The equation of continuity may be written in terms of the molar density C and the molar average velocities

u^* and v^* as $\frac{\partial(Cu^*)}{\partial x} + \frac{\partial(Cv^*)}{\partial y} = 0$ III-3.2

Rewriting, this equation becomes

$$C\left[\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y}\right] + u^* \frac{\partial C}{\partial x} + v^* \frac{\partial C}{\partial y} = 0 \qquad \text{III-3.2}$$

If it is assumed that the pressure and temperature of the system remain constant, then the molar density C is also a constant and the last two terms in equation III-3.2 vanish. The equation of continuity in terms of molar average velocities is then given by

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0 \qquad \text{III-3.3}$$

Next, the diffusion equation must be written in terms of u^*, v^* and ρ. The equation of continuity of component A may be written

$$\frac{\partial N_{Ax}}{\partial x} + \frac{\partial N_{Ay}}{\partial y} = 0 \qquad \text{III.-3.4}$$

where N_A is the molar flux of component A.

The molas concentration of component A C_A, the mole fraction of component A X_A and the velocity components of species i u_i and v_i are related by

$$N_{Ax} = C_A u_A$$
$$N_{Ay} = C_A v_A \qquad \text{III-3.5}$$

$$N_{Ax} = x_A(N_{Ax} + N_{Ay}) - \mathcal{D} \frac{\partial C_A}{\partial x}$$
$$N_{Ay} = x_B(N_{Ay} + N_{By}) - \mathcal{D} \frac{\partial C_A}{\partial y} \qquad \text{III-3.6}$$

where $\mathcal{D}$ is the molecular diffusivity.

$$N_{Ax} + N_{Bx} = C_A u_A + C_B u_B$$

$$N_{Ay} + N_{By} = C_A v_A + C_B v_B \qquad \text{III-3.7}$$

Combining equations III-3.3 through III-3.7 and neglecting diffusion in the direction of bulk flow, the equation of continuity of component A becomes

$$u^* \frac{\partial C_A}{\partial x} + v^* \frac{\partial C_A}{\partial y} = \mathscr{D} \frac{\partial^2 C_A}{\partial y^2} \qquad \text{III-3.8}$$

The relationship between C_A and ρ must now be established. Defining the mass concentration of components A and B by ρ_A and ρ_B respectively, the total density ρ is given by

$$\rho = \rho_A + \rho_B \qquad \text{III-3.9}$$

or

$$\rho = C_A M_A + C_B M_B \qquad \text{III-3.10}$$

where M_A and M_B are the molecular weights of components A and B. Since $C_A + C_B = C$ and C is a constant,

$$\frac{\partial C_A}{\partial x_i} = - \frac{\partial C_B}{\partial x_i} \qquad \text{III-3.11}$$

Combining equations III-3.11 and III-3.10

$$\frac{\partial \rho}{\partial x} = (M_A - M_B) \frac{\partial C_A}{\partial x}$$

$$\frac{\partial \rho}{\partial y} = (M_A - M_B) \frac{\partial C_A}{\partial y} \qquad \text{III-3.12}$$

Equations III-3.12 may now be substituted into equation III-3.8 to give the desired form of the diffusion equation.

$$u^* \frac{\partial \rho}{\partial x} + v^* \frac{\partial \rho}{\partial y} = \mathscr{D} \frac{\partial^2 \rho}{\partial y^2} \qquad \text{III-3.13}$$

Before the momentum equation can be written in terms of u^*, v^* and ρ, the relationship between the mass average velocity v and the molar average velocity v^* must be established.

The mass average velocity v is by definition given by

$$v = \frac{\sum_{i=1}^{n} \rho_i v_i}{\sum_{i=1}^{n} \rho_i} = \frac{\rho_A n_A + \rho_B v_B}{\rho} = w_A v_A + w_B v_B \qquad \text{III-3.14}$$

where w_A and w_B are the weight fractions of A and B respectively. Since $w_A + w_B = 1$, the difference between v and v^* may be written

$$v - v^* = w_A v_A + w_B v_B - (w_A + w_B)\, v^* \qquad \text{III -3.15}$$

or

$$v - v^* = w_A(v_A - v^*) + w_B(v_B - v^*) \qquad \text{III-3.15}$$

The molar flux J_i^* relative to the molar average velocity v^* is given by

$$J_i^* = C_i(v_i - v^*) = -\mathscr{D} \frac{\partial C_i}{\partial y} \qquad \text{III-3.16}$$

Using this relationship to substitute for $v_A - v^*$ and $v_B - v^*$ in equation III-3.15, the difference between v and v^* becomes

$$v - v^* = -\frac{w_A}{C_A} \mathscr{D} \frac{\partial C_A}{\partial y} - \frac{w_B}{C_B} \mathscr{D} \frac{\partial C_B}{\partial y} \qquad \text{III-3.17}$$

or

$$v - v^* = -\frac{\mathcal{D}}{\rho}(M_A - M_B)\frac{\partial C_A}{\partial y} \qquad \text{III-3.18}$$

Finally, substituting from the second of equations III-3.12

$$v - v^* = -\frac{\mathcal{D}}{\rho}\frac{\partial \rho}{\partial y} \qquad \text{III-3.19}$$

Similarly,

$$u - u^* = -\frac{\mathcal{D}}{\rho}\frac{\partial \rho}{\partial x} \qquad \text{III-3.20}$$

The x-component of the momentum equations may be written

$$\frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho u v)}{\partial y} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \qquad \text{III-3.21}$$

Here the usual boundary layer assumptions have been made and the pressure drop in the flow direction has been neglected. The body force term in the y-component of the momentum equations has been neglected thus eliminating that equation on the basis of the boundary layer assumptions. When the body force term was included in the analysis, a similar solution could not be found.

Equation III-3.21 is written in terms of mass average velocities. The derived relationships between u and u* and v and v* will now be used to rewrite this equation in terms of molar average velocities. Rewriting equation III-3.20

$$u - u^* = -\frac{\mathcal{D}}{\rho}\frac{\partial \rho}{\partial x} \qquad \text{III-3.20}$$

However, $\frac{\partial \rho}{\partial x} << \frac{\partial \rho}{\partial y}$ and therefore $u \approx u^*$. Thus in accordance with the boundary layer assumptions, it will be assumed that

$$u = u^*$$ III-3.22

Substituting equations III-3.22 and III-3.19 into equation III-3.21 and simplifying by use of equation III-3.3

$$u^* \frac{\partial(\rho u^*)}{\partial x} + v^* \frac{\partial(\rho u^*)}{\partial y} = \frac{\partial}{\partial y}\left[\nu\rho \frac{\partial u^*}{\partial y} + \mathcal{D} u^* \frac{\partial \rho}{\partial y}\right]$$ III-3.23

or

$$u^* \frac{\partial(\rho u^*)}{\partial x} + v^* \frac{\partial(\rho u^*)}{\partial y} = \nu \frac{\partial}{\partial y}\left[\rho \frac{\partial u^*}{\partial y} + K u^* \frac{\partial \rho}{\partial y}\right]$$ III-3.24

where

$$K = 1/Sc$$ III-3.25

and Sc is the Schmidt number defined by

$$Sc = \nu/\mathcal{D}$$ III-3.26

At this point it is convenient to divide the further discussion into two cases. The first case considered will be for a Schmidt number of unity and the second case will be for an arbitrary Schmidt number.

Case 1 Sc = 1

If the Schmidt number is unity then equation III-3.24 may be written as

$$u^* \frac{\partial(\rho u^*)}{\partial x} + v^* \frac{\partial(\rho u^*)}{\partial y} = \nu \frac{\partial^2(\rho u^*)}{\partial y^2}$$ III-3.27

The equations which are to be solved simultaneously to obtain the velocity and density profiles in the mixing region are

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0 \qquad \text{III-3.3}$$

$$u^* \frac{\partial \rho}{\partial x} + v^* \frac{\partial \rho}{\partial y} = \mathfrak{D} \frac{\partial^2 \rho}{\partial y^2} \qquad \text{III-3.13}$$

$$u^* \frac{\partial(\rho u^*)}{\partial x} + v^* \frac{\partial(\rho u^*)}{\partial y} = \nu \frac{\partial^2(\rho u^*)}{\partial y^2} \qquad \text{III-3.27}$$

The boundary conditions are

$$u(x, \infty) = u_1$$

$$u(x, -\infty) = u_2$$

$$\rho(x, \infty) = \rho_1$$

$$\rho(x, -\infty) = \rho_2 \qquad \text{III-3.28}$$

After once again integrating the continuity equation using a stream function, the search for a similar solution is carried out by applying an affine transformation to the remaining equations and the boundary conditions III-3.28. The quantities $y/\sqrt{x}$ and $\psi/\sqrt{x}$ are found to be invariant under the transformation. Thus new independent and dependent variables may be defined exactly the same as for the homogeneous laminar case described in section III-1

i.e.

$$\eta = y\sqrt{U/\nu x} \qquad \text{III-1.20}$$

$$f(\eta) = \psi/\sqrt{U\nu x} \qquad \text{III-1.23}$$

It is also found that

$$\rho = \rho(\eta) \qquad \text{III-3.29}$$

Using these three relationships equation III-3.27, rewritten in terms of the stream function ψ, may be transformed to

$$\frac{-f}{2} = \frac{[\rho f']''}{[\rho f']'} \qquad \text{III-3.30}$$

Equation III-3.13, rewritten in terms of the stream function ψ, transforms to

$$-\frac{f}{2} = \frac{\rho''}{\rho'} \qquad \text{III-3.31}$$

Combining equations III-3.31 and III-3.30

$$\frac{\rho''}{\rho'} = \frac{[\rho f']''}{[\rho f']'} \qquad \text{III-3.32}$$

Integrating,

$$\ln \rho' = \ln[\rho f'] + \ln A \qquad \text{III-3.33}$$

Simplifying,

$$\frac{\rho'}{\rho} = \frac{Af''}{1-Af'} \qquad \text{III-3.34}$$

Integrating again,

$$\ln \rho = -\ln[1-Af'] + \ln B \qquad \text{III-3.35}$$

Simplifying,

$$\rho = \frac{B}{1-Af'} \qquad \text{III-3.36}$$

Differentiating this relationship to find ρ' and ρ'' and substituting these relationships into equation III-3.31 gives

$$f''' + \frac{1}{2} ff'' = \frac{-2Af''^2}{1-Af'} \qquad \text{III-3.37}$$

The boundary conditions for equations III-3.37 and III-3.36 are from equations III-3.28

$$f'(\infty) = \frac{u_1}{U} = 1$$

$$f'(-\infty) = \frac{u_2}{U}$$

$$\rho(\infty) = \rho_1$$

$$\rho(-\infty) = \rho_2 \qquad \text{III-3.38}$$

In this case the reference velocity U is again taken as u_1. Substituting these relationships into equation III-3.36, gives two equations in the two unknowns A and B. Solving these equations gives

$$A = \frac{\Gamma(1+\lambda)}{\Gamma+\lambda}$$

$$B = \rho_1 \left[\frac{\lambda(1-\Gamma)}{\Gamma + \lambda} \right] \qquad \text{III-3.39}$$

where

$$\lambda = \frac{u_1-u_2}{u_1+u_2}$$

$$\Gamma = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}$$

Thus, if the two streams have the same density, then Γ and A are equal to zero and equation III-3.37 reduces to the homogeneous equation III-1.24.

It is proposed to solve equation III-3.37 using the method of analytic continuation. Equation III-3.37 may be rewritten in the form

$$f''' = -\frac{1}{2} ff'' - \frac{2Af''^2}{1-Af'} \qquad \text{III-3.40}$$

This equation may then be differentiated to obtain expressions for the fourth, fifth, and higher derivatives.

Asymptotic expressions for f, f', and f'' valid for large negative values of η must once again be obtained in order to start the numerical integration. Here, just as in the case of homogeneous laminar mixing, the manner in which these asymptotic expressions are obtained depends upon whether $u_2 = 0$ or $u_2 \neq 0$.

If $u_2 = 0$, then once again from the boundary conditions III-3.38

$$\eta \to -\infty$$
$$f'(\eta) \to 0$$
$$f(\eta) \to -S \qquad \text{III-3.41}$$

Then for large negative values of η, equation III-3.37 may be written

$$\frac{f'''}{f''} - \frac{1}{2} S = \frac{-2Af''}{1-Af'}, \quad \eta \to -\infty \qquad \text{III-3.42}$$

But $f'' \to 0$ for $\eta \to -\infty$, therefore

$$f'(\eta) \to k_1 e^{1/2\, S\eta}, \quad \eta \to -\infty \qquad \text{III-3.43}$$

To determine the constants k_1 and S, the same technique as that employed by Lessen[4] in the homogeneous case will be used. The new dependent and independent variables are

$$q(x) = \frac{1}{S} f(\eta) \qquad \text{III-1.29}$$

$$x = S\eta \qquad \text{III-1.29}$$

Equation III-3.37 becomes

$$q''' + \frac{1}{2} qq'' = \frac{-2Aq''}{Q-Aq'} \qquad \text{III-3.44}$$

where

$$Q = 1/S^2$$

It is again assumed that q may be represented by

$$q(x) = B_0 + B_1 e^{1/2x} + B_2 e^x + B_3 e^{3/2x} + \ldots \qquad \text{III-3.45}$$

Substituting this relationship into equation III-3.44, equating coefficients of $e^{\frac{n}{2}x}$, n = 1, 2, 3 and recalling that $q(-\infty) = -1$, it can be shown that

$$B_0 = -1$$

$$B_1 = 1$$

$$B_2 = -(A + Q)/4Q \qquad \text{III-3.46}$$

$$B_2 = (6A^2 + 13QA + 5Q^2)/72Q^2$$

Since A = o if the densities of the two streams are the same, these relationships reduce to equations III-1.36 for the homogeneous case.

Equation III-3.45 may be differentiated to obtain expressions for q' and q" for some large negative value of x. Equation III -3.44 may then be integrated by analytic continuation. However, in this case the value of the constant S may not be obtained directly since the quantity $Q = 1/S^2$ appears in the asymptotic expressions for q, q' and q". Therefore, it is necessary in this case to assume a value for S and then carry out the integration to large positive values of x. S may then be calculated from equation III-1.34. This procedure is repeated until the assumed value of S agrees with that calculated from equation III-1.34.

In this case the asymptotic form of $f(\eta)$ for large negative values of η is given by

$$f(\eta) = T_0 + T_1 e^{1/2 S\eta} T_2 e^{S\eta} + T_3 e^{3/2 S\eta} + \ldots \qquad \text{III-3.47}$$

where

$$T_0 = -S$$

$$T_1 = S$$

$$T_2 = B_2 S$$

$$T_3 = B_3 S \qquad \text{III-3.48}$$

If $u_2 \neq o$, then the transformation $f(\eta) = \int w(\eta) d\eta$ is again used. Equation III-3.37 becomes

$$w''' \; w' - w''^2 + \frac{1}{2} ww'^2 = \frac{-2Aw''w'^2}{1-Aw} - \frac{2A^2 w'^4}{(1-Aw)^2} \qquad \text{III-3.49}$$

In the limit for large negative values of η, the terms on the right hand side of this equation vanish and thus to a first order of approximation the asymptotic solution of equation III-3.49 is given by equation III-1.44, i.e.,

$$w = \sigma_1 + \frac{2C_1}{\sqrt{\sigma_1}} [\text{erfc}(-\sqrt{\sigma_1} \frac{\eta}{2})] \qquad \text{III-1.44}$$

For large negative values of η, w, w', and w" may represented by equations III-1.45, III-1.46 and III-1.47. The solution is again by trial and error to determine that value of C_1 for which the boundary condition at infinity is satisfied, i.e.,

$$f'(\infty) = w(\infty) = 1$$

If $-\Gamma = \lambda$, then from the first of equations III-3.39, the constant A becomes infinite. In this case equations III-3.37 and III-3.49 may be rewritten as

$$f''' + \frac{1}{2} ff'' = \frac{2f''^2}{f'} \qquad \text{III-3.50}$$

and

$$w''' \; w' - w''^2 + \frac{1}{2} ww'^2 = \frac{2w'' w'^2}{w} - \frac{2w'^4}{w^2} \qquad \text{III-3.51}$$

From the definitions of λ and Γ, it follows that if $-\Gamma = \lambda$, then

$$\rho_1 u_1 = \rho_2 u_2$$

i.e., the momentum of the two streams is the same.

Case 2 Arbitrary Schmidt Number*

The equations to be solved simultaneously in this case are

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} = 0 \qquad \text{III-3.3}$$

$$u^* \frac{\partial \rho}{\partial x} + v^* \frac{\partial \rho}{\partial y} = \mathscr{D} \frac{\partial^2 \rho}{\partial y^2} \qquad \text{III-3.13}$$

$$u^* \frac{\partial(\rho u^*)}{\partial x} + v^* \frac{\partial(\rho u^*)}{\partial y} = \nu \frac{\partial}{\partial y} \left[\rho \frac{\partial u^*}{\partial y} + K\, u^* \frac{\partial \rho}{\partial y}\right] \qquad \text{III-3.27}$$

The boundary conditions are the same as for Case 1. Once again a stream function is used to integrate the equation of continuity. The application of an affine transformation

* The author is indebted to Professor L. N. Tao for pointing out the exact differential form used in this case.

then results in the definition of the same new independent and dependent values as in Case 1. Thus, using equations III-1.20, III-1.23 and III-3.29 equation III-3.13, rewritten in terms of the stream function ψ, transforms to

$$-\frac{f}{2K} = \frac{\rho''}{\rho'} \qquad \text{III-3.52}$$

Equation III-3.24, when rewritten in terms of the stream function ψ, transforms to

$$-\frac{f}{2}\,[\rho f'' + f'\rho] = (\rho f'')' + K\,(f'\rho')' \qquad \text{III-3.53}$$

Equations III-3.51 and III-3.50 may be combined to give

$$\rho'\rho f''' + f''\,[(1+K)\,\rho'^2 - K\rho\rho''] = 0 \qquad \text{III-3.54}$$

or

$$\left[\frac{\rho^{1+K} f''}{\rho'^K}\right]' = 0 \qquad \text{III-3.55}$$

Integrating

$$\frac{\rho^{1+K} f''}{\rho'^K} = C_1 \qquad \text{III-3.56}$$

or

$$\frac{\rho'^K}{\rho^{1+K}} = C_2 f'' \qquad \text{III-3.57}$$

Taking the Kth root of both sides

$$\frac{\rho'}{\rho^{1+1/K}} = C_3 f''^{1/K} \qquad \text{III-3.58}$$

$$-K\rho^{-1/K} = C_4 + C_3 \int (f'')^{1/K} d\eta \qquad \text{III-3.59}$$

or

$$\rho^{1/K} = \left[\frac{C_4}{-K} + \frac{C_3}{-K} \int (f'')^{1/K} d\eta \right]^{-1} \qquad \text{III-3.60}$$

Equation III-3.52 may be rewritten as

$$f''' + f'' \left[(1+K) \frac{\rho'}{\rho} - K \frac{\rho''}{\rho'} \right] = 0 \qquad \text{III-3.61}$$

Multiplying equation III-3.58 by $\rho^{1/K}$ gives

$$\frac{\rho'}{\rho} = C_3 \, (\rho f'')^{1/K} \qquad \text{III-3.62}$$

Substituting for ρ'/ρ and ρ''/ρ' according to equations III-3.62 and III-3.52, equation III-3.61 becomes

$$f''' + f''\left[(1+K) C_3 ([\rho f'')^{1/K} - K \left(- \frac{f}{2K}\right) \right] = 0 \qquad \text{III-3.63}$$

or

$$f''' + \frac{1}{2} f f'' = -(1+K) \, C_3 \rho^{1/K} f''^{\frac{K+1}{K}} \qquad \text{III-3.64}$$

Substituting for $\rho^{1/K}$ from equation III-3.60 this becomes

$$f''' + \frac{1}{2} f f'' = \frac{1+K}{K} C_3 \left[C_4 + C_3 \int (f'')^{1/K} d\eta \right]^{-1} f''^{\frac{K+1}{K}}$$

III-3.65

or

$$f''' + \frac{1}{2} f f'' = (1+Sc) \left[C_5 + \int (f'')^{1/K} d\eta \right]^{-1} f''^{1+Sc}$$

III-3.66

where

$$C_5 = C_4/C_3$$

The boundary conditions are

$$f'(\infty) = 1$$

$$f'(-\infty) = \frac{u_2}{U} = \frac{1-\lambda}{1+\lambda}$$

$$\rho(\infty) = \rho_1$$

$$\rho(-\infty) = \rho_2 \qquad \text{III-3.67}$$

Applying these boundary conditions to equation III-3.60,

$$\rho_1^{1/K} = \left[\frac{C_4}{-K} + \frac{C_3}{-K} \int_{-\infty}^{\infty} (f'')^{1/K} d\eta \right]^{-1}$$

$$\rho_2^{1/K} = \left[\frac{C_4}{-K} + \frac{C_3}{-K} \int_{-\infty}^{\infty} (f'')^{1/K} d\eta \right]^{-1} \qquad \text{III-3.68}$$

These equations may be solved simultaneously to give

$$-C_4 = \frac{\rho_2^{-Sc}}{Sc} \qquad \text{III-3.69}$$

and

$$-C_3 = \frac{\rho_1^{-Sc} - \rho_2^{-Sc}}{Sc \int_{-\infty}^{\infty} (f'')^{Sc} d\eta} \qquad \text{III-3.70}$$

Thus,

$$C_5 = \frac{C_4}{C_3} = \frac{\rho_1^{Sc}}{\rho_2^{Sc} - \rho_1^{Sc}} \int_{-\infty}^{\infty} (f'')^{Sc} d\eta \qquad \text{III-3.71}$$

The equation to be solved to obtain the velocity distribution is

$$f''' + \frac{1}{2} ff'' = (1 + Sc)\left[C_5 + \int_{-\infty}^{\eta} (f'')^{Sc} d\eta\right]^{-1} f''^{1+Sc}$$

III-3.72

When solving this equation by the method of analytic continuation, the cases of $u_2 = o$ and $u_2 \neq o$ must once again be differentiated. Only the case for which $u_2 \neq o$ will be considered here. Using the transformation

$f(\eta) = \int w(\eta)\, d\eta$, equation III-3.72 becomes

$$w''' \; w' - w''^2 + \frac{1}{2} ww'^2 = \frac{Sc(1 + Sc)\, w'^{Sc + 1} \; w''}{[C_5 + \int_{-\infty}^{\eta} (w')^{Sc} \, d\eta \,]}$$

$$- \frac{(1 + Sc)\, w'^{2\, Sc + 1}}{[C_5 + \int_{-\infty}^{\eta} (w')^{Sc} \, d\eta \,]^2} \qquad \text{III-3.73}$$

Here again, in the limit for large negative values of η, the terms on the right hand side of this equation vanish. Thus, the asymptotic forms for w, w' and w" are again given by equations III-1.45, III-1.46 and III-1.47. However, these asymptotic forms contain the unknown constant C_1. Thus, the solution is again by trial and error. However, in this case an additional complicating factor is present. The constant C_5 contains an integral from minus infinity to plus infinity. The author was unable to evaluate C_5 until equation III-3.71 had been integrated to large positive values of η. This means that both the constants C_5 and C_1 had to be assumed. Equation III-3.73 may then be integrated to large positive values of η. The correct values of C_5 and C_1 are those for which the boundary condition at infinity is satisfied and the value of C_5 calculated from equation III-3.61 agrees with the assumed value. Once the proper values of C_5 and C_1 have been determined and equation III-3.73 has been integrated, it follows from equation III-3.60 that the density distribution is given by

$$\rho = [-Sc\ (C_4 + C_3 \int_{-\infty}^{\eta} (w')^{Sc}\ d\eta)]^{-\frac{1}{Sc}} \qquad \text{III-3.74}$$

The terms on the right hand side of equation III-3.73 contain the factor

$$\int_{-\infty}^{n} (w')^{Sc}\ d\eta$$

The asymptotic value of this term for large negative η may be obtained by substituting for w' from equation III-1.46 and performing the indicated integration to obtain a complementary error function. The integration of this term is then carried out step by step along with the integration of equation III-3.73.

III-4 Heterogeneous Turbulent Mixing

If molecular diffusion is neglected, then $u = u^*$ and $v = v^*$. Thus, neglecting molecular diffusion and the effect of molecular viscosity, it follows from equations III-3.3, III-3.13 and III-3.23 that the equations of continuity, diffusion and momentum may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{III-4.1}$$

$$u\frac{\partial \rho}{\partial x} + v\,\frac{\partial \rho}{\partial y} = 0 \qquad \text{III-4.2}$$

$$u\,\frac{\partial(\rho u)}{\partial x} + v\,\frac{\partial(\rho u)}{\partial y} = 0 \qquad \text{III-4.3}$$

The additional assumptions associated with these equations are outlined in section III-3. The velocity components and the density are assumed to be composed of an average plus a fluctuating component. Thus,

$$u = \bar{u} + u'$$
$$v = \bar{v} + v'$$
$$\rho = \bar{\rho} + \rho' \qquad \text{III-4.4}$$

Introducing these equations into equation III-4.1 and time averaging, the continuity equation becomes

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \quad \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \qquad \text{III-4.5}$$

Similarly, the diffusion equation, with the boundary layer assumptions, becomes

$$\bar{u} \frac{\partial \bar{\rho}}{\partial x} + \bar{v} \frac{\partial \bar{\rho}}{\partial y} = \frac{\partial J_y^{(t)}}{\partial y} \qquad \text{III-4.6}$$

where $J_y^{(t)}$ is the turbulent mass flux in the y-direction defined by

$$J_y^{(t)} = - \overline{\rho' v'} \qquad \text{III-4.7}$$

And the momentum equation, with the boundary layer assumptions, becomes

$$\bar{u} \frac{\partial (\bar{\rho}\, \bar{u})}{\partial x} + \bar{v} \frac{\partial (\bar{\rho}\, \bar{u})}{\partial y} = \frac{\partial}{\partial y} [\, \bar{u}\, J_y^{(t)} + \tau_{xy}] \qquad \text{III-4.8}$$

Substituting equations II-1.14 for the turbulent shearing stress τ_{xy} and assuming that

$$J_y^{(t)} = \mathcal{D}^{(t)} \frac{\partial \bar{\rho}}{\partial y} \qquad \text{III-4.9}$$

where $\mathcal{D}^{(t)}$ is the eddy diffusivity, equation III-4.6 becomes

$$\bar{u} \frac{\partial \bar{\rho}}{\partial x} + \bar{v} \frac{\partial \bar{\rho}}{\partial y} = \mathcal{D}^{(t)} \frac{\partial^2 \bar{\rho}}{\partial y^2} \qquad \text{III-4.10}$$

and equation III-4.8 becomes

$$\bar{u} \frac{\partial(\bar{\rho}\,\bar{u})}{\partial x} + \bar{v} \frac{\partial(\bar{\rho}\,\bar{u})}{\partial y} = \frac{\partial}{\partial y} [\bar{u}\, \mathcal{D}^{(t)} \frac{\partial \bar{\rho}}{\partial y} + \bar{\rho}\, \epsilon \frac{\partial \bar{u}}{\partial y}]$$

III-4.11

or

$$\bar{u} \frac{\partial(\overline{\rho u})}{\partial x} + \bar{v} \frac{\partial(\bar{\rho}\,\bar{u})}{\partial y} = \epsilon \frac{\partial}{\partial y} [K'\bar{u} \frac{\partial \bar{\rho}}{\partial y} + \bar{\rho} \frac{\partial \bar{u}}{\partial y}]$$

III-4.12

where

$$K' = 1/Sc^{(t)} \qquad \text{III-4.13}$$

and $Sc^{(t)}$ is the turbulent Schmidt number defined by

$$Sc^{(t)} = \epsilon/\mathcal{D}^{(t)} \qquad \text{III-4.14}$$

Thus, the equations to be solved simultaneously are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad \text{III-4.5}$$

$$\bar{u} \frac{\partial \bar{\rho}}{\partial x} + \bar{v} \frac{\partial \bar{\rho}}{\partial y} = \epsilon K' \frac{\partial^2 \bar{\rho}}{\partial y^2} \qquad \text{III-4.15}$$

$$\bar{u} \frac{\partial(\bar{\rho}\,\bar{u})}{\partial x} + \bar{v} \frac{\partial(\bar{\rho}\,\bar{u})}{\partial y} = \epsilon \frac{\partial}{\partial y} [K'\, \bar{u} \frac{\partial \bar{\rho}}{\partial y} + \bar{\rho} \frac{\partial \bar{u}}{\partial y}]$$

III-4.12

The eddy viscosity ϵ will be assumed to be represented by Prandtl's exchange coefficient hypothesis, i.e.,

$$\epsilon = \chi_1 b(u_1 - u_2) \qquad \text{II-1.15}$$

with $b = c\,x$. Once again the cases of a Schmidt number of unity and an arbitrary Schmidt number must be differentiated.

<u>Case 1</u> <u>Sc = 1</u>

Integrating the continuity equation using a stream function and applying an affine transformation, it can again be shown that y/x and ψ/x are invariant under the transformation. Thus, a new independent variable is

$$\xi = \sigma \frac{y}{x} \qquad \text{III-2.7}$$

where

$$\sigma = \tfrac{1}{2}/\sqrt{\chi_1 c\lambda}$$

and a new dependent variable is

$$F(\xi) = \psi/Ux \qquad \text{III-2.8}$$

where

$$U = \tfrac{1}{2}(u_1 + u_2)$$

Also,

$$\rho = \rho(\xi) \qquad \text{III-4.16}$$

In terms of these new variables, equation III-4.15 becomes

$$-2\sigma F = \frac{\rho''}{\rho'} \qquad \text{III-4.17}$$

and equation III-4.12 becomes

$$-2\sigma F = \frac{[\rho F']''}{[\rho F']'} \qquad \text{III-4.18}$$

Solving equations III-4.18 and III-4.17 simultaneously, as in section III-3, gives

$$\rho = \frac{B'}{1-A'\sigma F'} \qquad \text{III-4.19}$$

$$F''' + 2\sigma FF'' = \frac{-2A'\sigma F''^2}{1-A'\sigma F'} \qquad \text{III-4.20}$$

The development given in this section is similar to that given by Pai.[11] Equation III-4.20 was given by Pai, however, it was apparently first derived by Hu.[61] Pai's development was used as a guide for the laminar similarity solution developed in section III-2.

The boundary conditions are

$$\sigma F'(\infty) = \frac{u_1}{U} = 1 + \lambda$$

$$\sigma F'(-\infty) = \frac{u_2}{U} = 1 - \lambda$$

$$\rho(\infty) = \rho_1$$

$$\rho(-\infty) = \rho_2 \qquad \text{III-4.21}$$

From these boundary conditions and equation III-4.19 the constants A' and B' are given by

$$A' = \frac{\Gamma}{\Gamma + \lambda}$$

$$B' = \rho_1 \left[\frac{\lambda(1-\Gamma)}{\Gamma + \lambda}\right] \qquad \text{III-4.22}$$

By using the following relationships

$$f'(\eta) = \frac{\sigma F'(\xi)}{1+\lambda}$$

$$\eta = 2\sqrt{1+\lambda}\quad \xi$$

$$A' = \frac{A}{1+\lambda} \qquad \text{III-4.23}$$

equations III-4.19 and III-4.20 with boundary conditions III-4.21 may be converted to equations III-3.36 and III-3.37 with boundary conditions III-3.38. Thus, here also a simple relationship exists between the solution of the heterogeneous turbulent mixing problem using Prandtl's exchange coefficient hypothesis and the solution of the heterogeneous laminar mixing problem.

Case 2 Arbitrary Schmidt Number

A similarity analysis of equations III-4.5, III-4.15 and III-4.12 yields

$$F''' + 2\sigma FF'' = (1+Sc^{(t)})[C_5' + \int_{-\infty}^{\xi} (\sigma F'')^{Sc^{(t)}} d\xi]^{-1}(\sigma F'')^{1+Sc} \qquad \text{III-4.24}$$

$$\rho = [-Sc^{(t)}(C_4'+C_3' \int_{-\infty}^{\xi}(\sigma F'')^{Sc^{(t)}} d\xi]^{-1/Sc^{(t)}} \qquad \text{III-4.25}$$

where

$$-C_3' = \frac{\rho_1^{-Sc^{(t)}} - \rho_2^{-Sc^{(t)}}}{Sc^{(t)} \int_{-\infty}^{\infty} (\sigma F'')^{Sc^{(t)}} d\xi} \qquad \text{III-4.26}$$

$$-C_4' = \frac{\rho_2^{-Sc^{(t)}}}{Sc^{(t)}} \qquad \text{III-4.27}$$

$$C_5' = \frac{C_4'}{C_3'} = \frac{\rho_1^{Sc(t)}}{\rho_2^{Sc(t)} - \rho_1^{Sc(t)}} \int_{-\infty}^{\infty} (\sigma F'')^{Sc(t)} d\xi$$

III-4.28

The boundary conditions are given by equations III-4.21.

By using the following relationships

$$f'(\eta) = \frac{\sigma F'(\xi)}{1 + \lambda}$$

$$\eta = 2\sqrt{1+\lambda}\ \xi$$

$$C_3' = C_3/[2^{-Sc(t)+1} (1+\lambda)^{-\frac{3}{2} Sc(t) + \frac{1}{2}}]$$

$$C_4' = C_4$$

$$C_5' = [2^{-Sc(t)+1} (1+\lambda)^{-\frac{3}{2} Sc(t) + \frac{1}{2}}]$$

III-4.29

equations III-4.24 and III-4.25 with boundary conditions III-4.21 may be converted to equations III-3.72 and III-3.74 with boundary conditions III-3.67. Thus, once again a simple relationship exists between the turbulent and laminar solutions.

III-5 The Indeterminateness of the Third Boundary Condition

In each of the four previous sections of this chapter, the ordinary differential equation from which the velocity distribution was to be determined was of the third order. However, in each instance only two boundary conditions on the velocity were specified. A third applicable boundary condition is not readily apparent.

Most previous investigators[1,6,4] have arbitrarily specified this third condition as

$$f'(o) = \frac{u}{U} = \frac{1}{2}\left[1 + \frac{u_2}{u_1}\right] \qquad \text{III-5.1}$$

for the laminar case with $U = u_1$, or

$$F'(o) = \frac{u}{U} = 1 \qquad \text{III-5.2}$$

for the turbulent mixing case with $U = \frac{1}{2}(u_1 + u_2)$.

Physically, these boundary conditions specify that the line $y^* = 0$ passes through the points where the velocity is the arithmetic average of the two free stream velocities. This means that the line $y^* = 0$ is not the x-axis and the rectangular coordinate y and y^* are not one and the same. In other words, the functions $f(\eta)$ and $F(\xi)$ are written in terms of the new independent variables η^* and ξ^*, where

$$\eta^* = y^* \sqrt{U/\nu x} \qquad \text{III-5.3}$$

and

$$\xi^* = \sigma \frac{y^*}{x} \qquad \text{III-5.4}$$

The first two boundary conditions remain the same when written in terms of the new independent variables.

In the laminar case, the line $y^* = 0$ is a parabola given by

$$\frac{y}{\sqrt{x}} = k_2$$

where k_2 is a constant. Since $y^* = 0$ along this line, the relationship between y^* and the rectangular coordinate y is given by

$$y^* = y - k_2 \sqrt{x} \qquad \text{III-5.5}$$

Combining equations III-5.5 and III-5.3, the relationship between η^* and η is given by

$$\eta^* = (y - k_2\sqrt{x}) \sqrt{u/\nu x} = \eta - k_2 \sqrt{\frac{U}{\nu}} = \eta - \eta_o \qquad \text{III-5.6}$$

The constant k_2 must be determined experimentally.

Similarly, in the turbulent mixing case, the line $y^* = o$ is a straight line given by

$$y = mx \qquad \text{III-5.7}$$

Since $y^* = o$ along this line, y^* and y are related by

$$y^* = y - mx \qquad \text{III-5.8}$$

Combining equations III-5.8 and III-5.4,

$$\xi^* = \sigma(y-mx)/x = \xi - \sigma m = \xi - \xi_o \qquad \text{III-5.9}$$

In this case the constants σ and m must both be determined experimentally.

In summation, if the third boundary condition is arbitrarily specified, as in equations III-5.1 and III-5.2, then the laminar solution contains one empirical constant to be determined experimentally and the turbulent solution contains two empirical constants to be determined experimentally.

The remaining discussion in this section will be restricted to the laminar case. However, because of the relationship between the laminar and turbulent solutions, the discussion applies equally as well to the turbulent case.

An alternate form of the third boundary condition was discussed by Lessen[3] and by Crane.[44] The stream function ψ, as defined in section III-1, is given by

$$\psi = \sqrt{U\nu x}\ f(\eta) \qquad \text{III-1.23}$$

The boundary between the fluid of both streams must be a streamline and also must pass through the point x = 0, y = 0. From equation III-1.23, it can then be seen that $\psi = 0$ is the boundary streamline. Lessen and Crane suggested that the third boundary condition be specified as

$$\psi(\eta_o) = 0 \qquad \text{III-5.10}$$

Combining equations III-5.10 and III-1.23, the third boundary condition is given by

$$f(\eta_o) = 0 \qquad \text{III-5.11}$$

If the function f is written in terms a new independent variable $\eta^* = \eta - \eta_o$, then the third boundary condition becomes

$$f(o) = 0 \qquad \text{III-5.12}$$

The boundary streamline is given by

$$\frac{y}{\sqrt{x}} = \eta_o \sqrt{\frac{\nu}{U}} \qquad \text{III-5.13}$$

To determine the position of this streamline in the x-y plane, the value of η_o must be known. Crane suggested that η_o be determined experimentally.

von Karman[5] proposed that the third boundary condition be given by

$$u_1 v_1 + u_2 v_2 = 0 \qquad \text{III-5.14}$$

where v_1 and v_2 are the v-component of velocity for $\eta \to \infty$ and $\eta \to -\infty$ respectively. Physically, this boundary conditions specifies that no external forces are acting on the total fluid system perpendicular to the main flow, i.e., $u_1 v_1$ represents the transfer of x-momentum in the y-direction for $\eta \to \infty$ and $u_2 v_2$ represents the transfer of x-momentum in the y-direction for $\eta \to -\infty$. If there is to be no net external force acting on the total fluid system perpendicular to the main flow, then the sum of these two terms must be zero.

Yen[43] has shown that this boundary condition may be used to determine the value of η_0 and thus locate the $\psi = 0$ stream line in the x-y plane. The three boundary conditions are

$$f'(\infty) = u_1/U = 1$$

$$f'(-\infty) = u_2/U_1 = 1-\lambda/1+\lambda \qquad \text{III-5.15}$$

and

$$u_1 v_1 + u_2 v_2 = 0 \qquad \text{III-5.14}$$

In addition to these conditions, it is known that along the streamline $\psi = 0$, $\eta = \eta_0$ or

$$f(o) = 0 \qquad \text{III-5.12}$$

If η^* is the independent variable, then from equations III-5.15 for $\eta^* \to \infty$

$$f(\eta^*) \to \eta^* - \alpha, \quad \eta^* \to \infty \qquad \text{III-5.16}$$

and for $\eta^* \to -\infty$

$$f(\eta^*) \to \left(\frac{1-\lambda}{1+\lambda}\right) \eta^* - \beta, \quad \eta^* \to -\infty \qquad \text{III-5.17}$$

The stream function ψ is given by

$$\psi(x,y) = \sqrt{U\nu x}\; f(\eta^*) = \sqrt{U\nu x}\; f(\eta-\eta_o) \qquad \text{III-5.18}$$

Thus, since $v = -\partial\psi/\partial x$, v is given by

$$v = \frac{1}{2} \sqrt{U\nu/x}\; [-f(\eta^*) - \eta f'(\eta^*)] \qquad \text{III-5.19}$$

It then follows from equations III -5.16, III-5.17 and III-5.15 that for $\eta^* \to \infty$

$$v \to \frac{1}{2} \sqrt{U\nu/x}\; [\alpha + \eta_o], \quad \eta \to \infty \qquad \text{III-5.20}$$

and for $\eta \to -\infty$

$$v \to \frac{1}{2} \sqrt{U\nu/x}\; [\beta + \eta_o\left(\frac{1-\lambda}{1+\lambda}\right)], \quad \eta \to -\infty \qquad \text{III-5.21}$$

Finally, combining equations III-5.20, III-5.21 and III-5.15, equation III-5.14 becomes

$$[1][\alpha + \eta_o] + \left[\frac{1-\lambda}{1+\lambda}\right] [\beta + \eta_o \left(\frac{1-\lambda}{1+\lambda}\right)] = 0 \qquad \text{III-5.22}$$

or

$$\frac{1+\lambda}{1-\lambda} = \frac{\beta_o + \eta_o \left(\frac{1-\lambda}{1+\lambda}\right)}{-[\alpha+\eta_o]} \qquad \text{III-5.23}$$

Instead of equation III-5.23, Yen obtained the condition

$$\frac{1+\lambda}{1-\lambda} = \frac{\beta + \eta_o(1-\lambda)}{\alpha + \eta_o(1+\lambda)} \qquad \text{III-5.24}$$

The form of this equation is slightly different because Yen applied the boundary conditions

$$f'(\infty) = 1 + \lambda$$
$$f'(-\infty) = 1 - \lambda \qquad \text{III-5.25}$$

instead of equations III-5.15. However, the denominator of the term on the right hand side of equation III-5.23 is preceeded by a minus sign, whereas there is no minus sign in equation III-5.24.

Yen obtained equation III-5.24 by considering the momentum equilibrium in the y-direction for fluid within a control surface in the x-y plane. Performing an integration from minus infinity to plus infinity, Yen wrote one of the terms as

$$\{(1+\lambda)[\alpha+\eta_o(1+\lambda)] - (1-\lambda)[\beta + \eta_o(1-\lambda)]\}\ \eta \Big|_{\eta = -\infty}^{\infty} \qquad \text{III-5.26}$$

He then stated that in order to insure convergence of the integral, the terms inside the outer brackets in equation III-5.26 must vanish or equation III-5.24 must be satisfied.

Equation III-5.26 should have been written as

$$\{(1+\lambda)[\alpha + \eta_o(1+\lambda)]\}\ \eta \Big|^{\eta = \infty} - \{(1-\lambda)[\beta + \eta_o(1-\lambda)]\}\ \eta \Big|^{\eta = -\infty} \qquad \text{III-5.27}$$

or

$$\{(1+\lambda)[\alpha + \eta_o(1+\lambda)] + (1-\lambda)[\beta + \eta_o(1-\lambda)]\}\ \eta \Big|^{\eta = \infty} \qquad \text{III-5.28}$$

then, in order to insure convergence of the integral, the terms inside the outer brackets must vanish or

$$\frac{1 + \lambda}{1 - \lambda} = \frac{\beta + \eta_o(1-\lambda)}{-[\alpha + \eta_o(1+\lambda)]} \qquad \text{III-5.29}$$

Thus, Yen's condition as stated by equation III -5.24 is wrong. However, if $u_2 = 0$, i.e., $\lambda = 1$, then from either equation III-5.29 or equation III-5.24,

$$\alpha + \eta_o (1+\lambda) = 0 \qquad \text{III-5.30}$$

Therefore, Yen's condition is valid for the case of $\lambda = 1$.

Yen concluded from the numerical data of Lock that for $\lambda = 1$, the mixing region deflects toward the stationary fluid whereas for $\lambda = 0.5$ the interface deflects toward the higher velocity stream. The second conclusion is invalidated by the error in equation III -5.24 (see section IV-1-1).

A complete set of boundary conditions for the homogeneous laminar mixing problem is given by equations III-5.14, III-5.15 and III -5.12. The application of these boundary conditions leads to equation III -5.23 from which the value of η_o may be calculated without experimental data. For the case of heterogeneous laminar mixing, equation III-5.14 is logically extended to

$$\rho_1 u_1 v_1 + \rho_2 u_2 v_2 = 0 \qquad \text{III-5.31}$$

The constant η_o is then calculated from the relationship

$$\rho_1 [1][\alpha + \eta_o] + \rho_2 [\tfrac{1-\lambda}{1+\lambda}][\beta + \eta_o(\tfrac{1-\lambda}{1+\lambda})] = 0 \qquad \text{III-5.32}$$

or

$$\frac{\rho_1(1+\lambda)}{\rho_2(1-\lambda)} = \frac{\beta + \eta_0 \left(\frac{1-\lambda}{1+\lambda}\right)}{-[\alpha + \eta_0]} \qquad \text{III -5.33}$$

With these boundary conditions, the laminar mixing problem is completely determined and the turbulent mixing problem contains only the empirical constant σ to be determined experimentally.

IV. CALCULATION PROCEDURES

The solutions of the differential equations given in Chapter III were obtained basically by the method of analytic continuation. The procedures that were used to obtain the final solution for each special case are outlined in this chapter. There is, or course, no need to describe the solutions of the laminar and the turbulent problems separately since only the solution of the laminar problem was actually calculated. The solution for the turbulent problem was obtained by transformation of variables in each case. Most of the calculations were done on the IIT IBM 7040 computer.

When integrating an equation by the method of analytic continuation, it is necessary to specify the initial value of the independent variable and the increment size for the numerical integration. The initial value of the independent variable in each case was specified as that value for which the velocity, as calculated from the asymptotic expression for f' was identically equal to the asymptotic value of the velocity in the first four or five decimal places out of eight. The increment size was halved until the results no longer changed.

Section IV-4 is included to describe the method of determining the value of the constant σ from experimental data.

IV-1 Similarity, Homogeneous Case

IV-1-1 $u_2 = 0$

A schematic diagram of the calculation procedure is shown in Figure IV-1-1.1. First, equation III-1.30 was integrated by the method of analytic continuation using equation III-1.35 to represent the solution for a large negative value of x. Next, the value of S was calculated from equation III-1.34. Equation III-1.24 was then integrated by analytic continuation using equation III-1.37 to represent the solution for a large negative value of η.

Equation III-5.12 requires that

$$f(\eta^*) = 0, \quad \eta^* = 0 \qquad \text{III-5.12}$$

The independent variable in the solution obtained as outlined above was designated $\hat{\eta}$. To determine the relationship between $\hat{\eta}$ and η^*, the tabulated values of $f(\hat{\eta})$ were examined. That value of $\hat{\eta}$ for which $f(\hat{\eta}) = 0$ was designated $\eta^* = 0$ in accordance with equation III-5.12 above. Thus, η^* is equal to $\hat{\eta}$ plus or minus a constant.

The values of α and β were next calculated from equations III-5.16 and III-5.17. The value of η_0 was then calculated from equation III-5.23.

Knowing the value of η_0, the v-velocity profiles were then calculated from equation III-5.19 recalling that $\eta^* = \eta - \eta_0$.

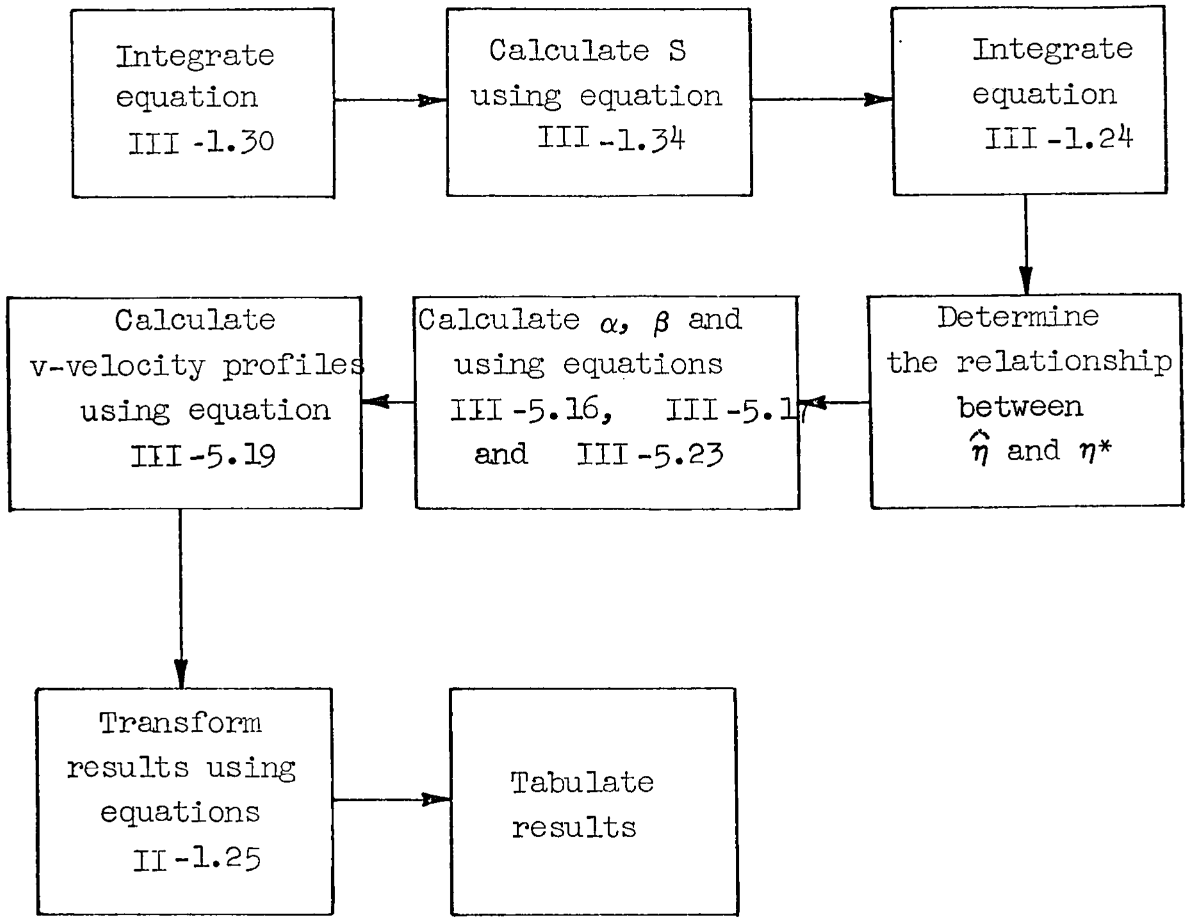


Fig.IV-1-1.1. Calculation Procedure
Homogeneous Case, $u_2 = 0$

IV-1-2 $u_2 \neq 0$

A schematic diagram of the calculation procedure is shown in FigureIV-1-2.1. Equation III-1.40 was integrated by the method of analytic continuation using equations III-1.45, III-1.46 and III-1.47 to represent w, w' and w" for a large negative value of η. Since these equations contain the unknown a constant C_1 it was first necessary to assume the value of C_1. Equation III-1.40 was then integrated to a large positive value of η and the boundary condition given by equation III-1.44 was applied. This process was repeated, assuming new values of C_1 until the boundary condition given by equation III-1.44 was satisfied.

Next, the function f was calculated by integrating w, i.e., $f = \int w d\eta$. It follows from equation III-1.25 that if f = 0, then f''' also must be zero. Since $w'' = f'''$, the location of the point where f = 0 was fixed at that point where w" = 0. The values of f were then obtained for plus and minus values of η^* by integrating forward and backward from the point where $f(\eta^*) = 0$, $\eta^* = 0$.

The values of α and β were then calculated from equations III-5.16 and III -5.17 and η_0 was calculated from equation III-5.23. Next, the v-velocity profiles were calculated using equation III-5.19 and the turbulent solution was obtained by transformation of variables using equations II-1.25. The laminar and turbulent solutions were then tabulated.

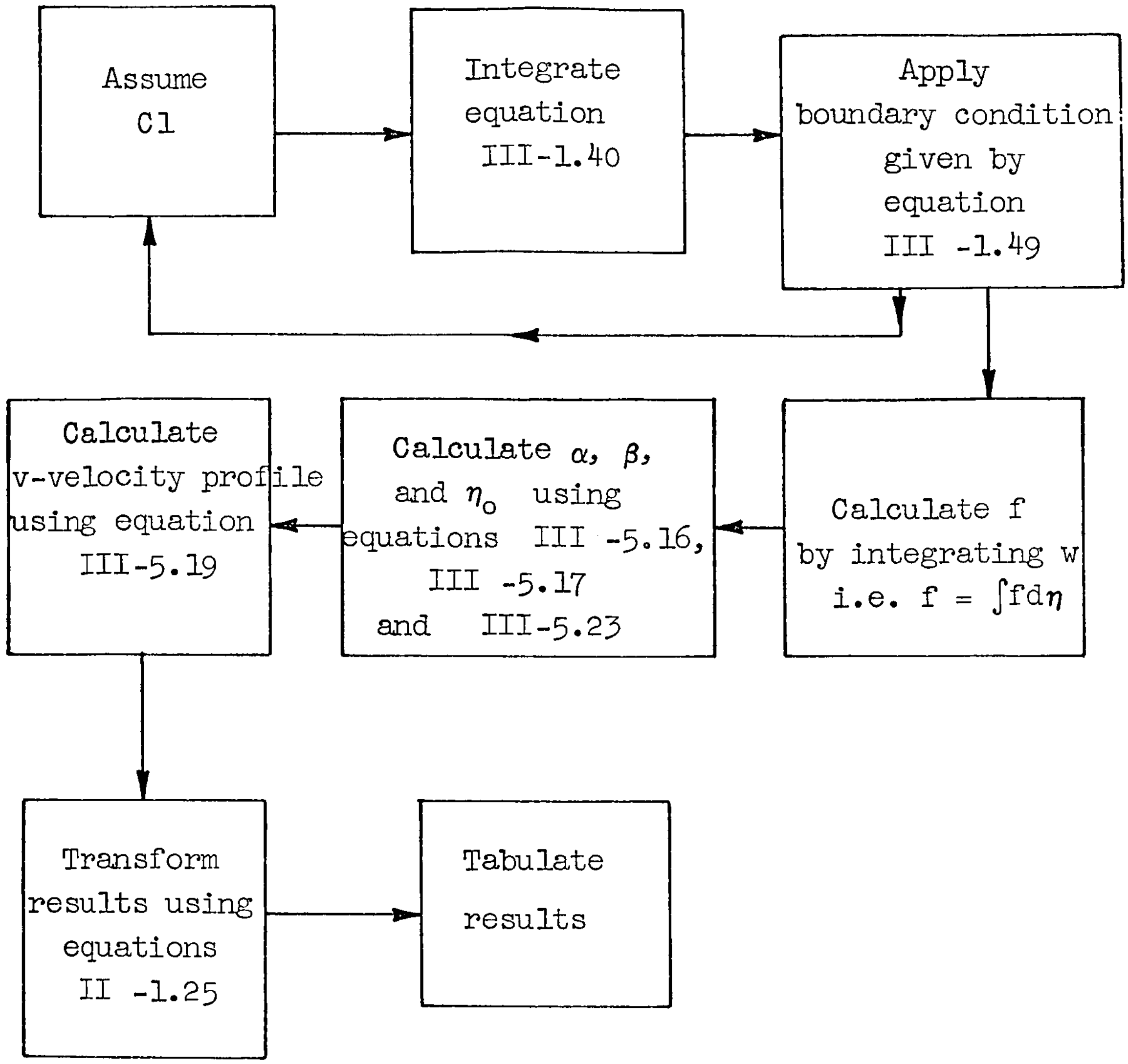


Fig.IV-1-2.1. Calculation Procedure
Homogeneous Case, $u_2 \neq 0$

IV-2 Similarity, Heterogeneous Case

IV-2-1 $Sc \doteq 1.0$, $u_2 = 0$

A schematic diagram of the calculation procedure is shown in FigureIV-2-1.1. Equation III-3.44 was integrated by analytic continuation using equation III-3.45 to represent the solution for a large negative value of . Since equations III-3.44 and III-3.45 contain the unknown constant $Q = 1/S^2$, it was first necessary to assume a value of this constant. After integrating equation III-3.44 to a large positive value of x, the value of S was calculated from equation III-1.34. This procedure was repeated until the assumed and calculated values were the same. Equation III-3.37 was then integrated using equation III-3.47 to represent the solution for large negative values of η. The remaining procedure for this case parallels that described in sectionIV-1-1 for the homogeneous case.

IV-2-2 $Sc = 1.0$, $u_2 \neq 0$

A schematic diagram of the calculation procedure is shown in Figure IV2-2.1. Equation III-3.49 was integrated by analytic continuation using equations III-1.45, III-1.46 and III-1.47 to represent w, w' and w" for a large negative value of η. As described in section IV-1-2, the constant C_1 was obtained by trial and error procedure.

The function f was then calculated by integrating w. It follows from equation III-3.37 that if $f = 0$, then

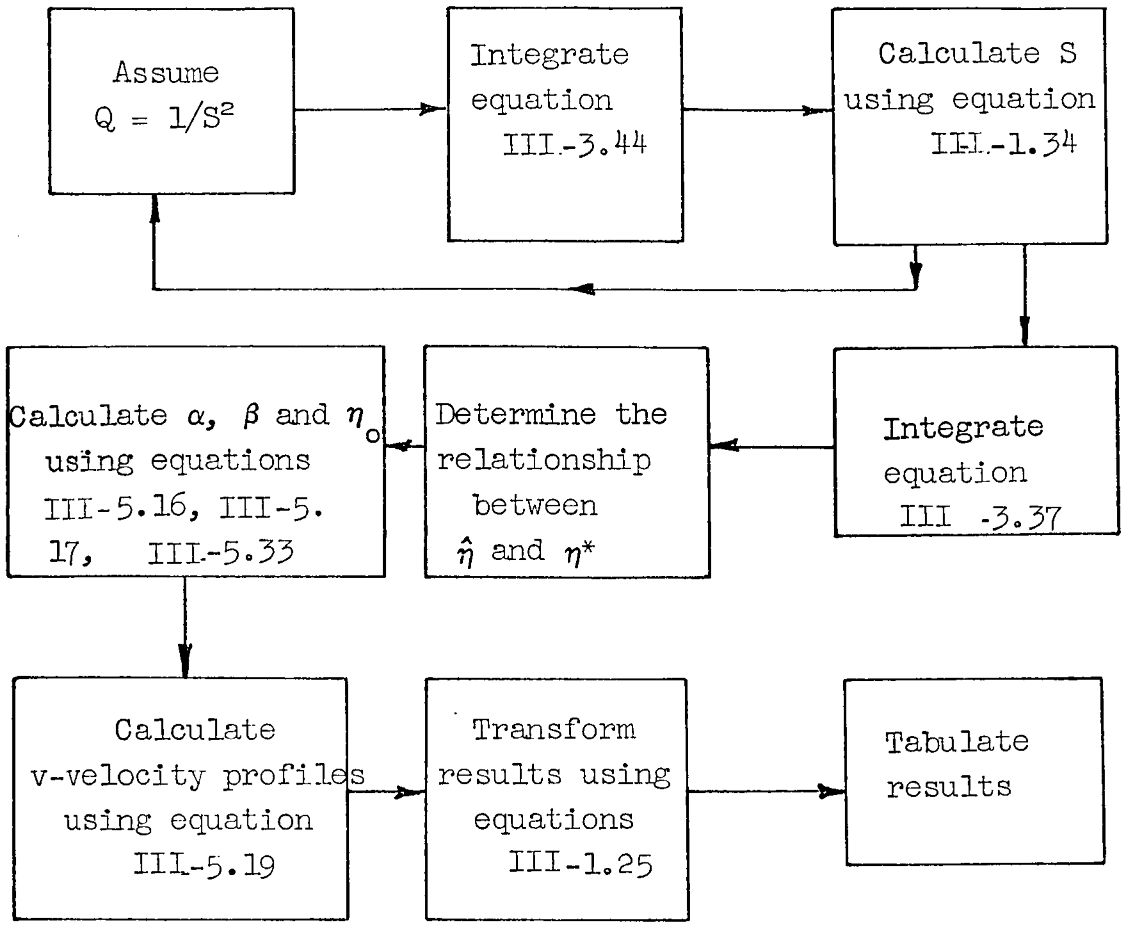


Fig.IV-2-1.1 Calculation Procedure
Heterogeneous Case Sc = 1.0, u_2 = 0

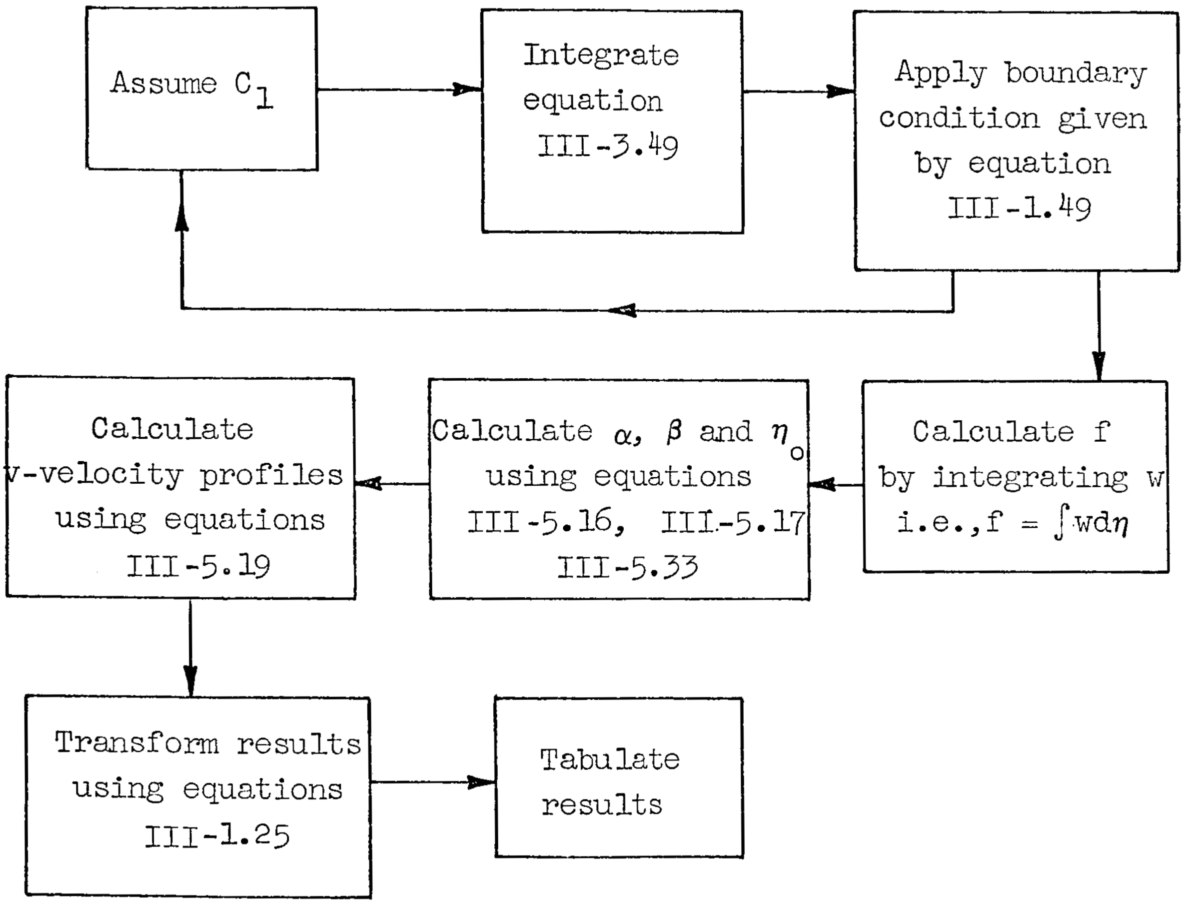


Fig. IV-2-2.1. Calculation Procedure
Heterogeneous Case $Sc = 1.0$, $u_2 \neq 0$

$$-f''' - \frac{2Af''^2}{1-Af'} = 0 \qquad \text{IV-2-2.1}$$

or

$$-w'' - \frac{2Aw'^2}{1-Aw} = 0 \qquad \text{IV-2-2.2}$$

The location of the point where $f = 0$ was fixed as that point where equation IV-2-2.2 above was satisfied. The values of f for plus and minus values of η^* were then obtained by integrating forward and backward from the point where $f(\eta^*) = 0$, $\eta^* = 0$. The remaining procedure for this case parallels that described in section IV-1-2 for the homogeneous case.

IV-2-3 $Sc \neq 1.0$, $u_2 \neq 0$

A schematic diagram of the calculation procedure is shown in Figure IV-2-3.1. Equation III-3.73 was integrated by analytic continuation using equations III-1.45, III-1.46 and III-1.47 to represent w, w' and w" for a large negative value of η. In this case, it was necessary to assume the values of the constants C_1 and C_5. Equation III-3.73 was then integrated to a large positive value of η. The boundary condition given by equation III-1.49 was then applied. Keeping the assumed value of C_5 constant, new values of C_1 were assumed until the boundary condition given by equation III-1.49 was satisfied. The value of the constant C_5 calculated from equation III-3.71 was then compared with the assumed value of C_5. This procedure was repeated until the boundary condition given by

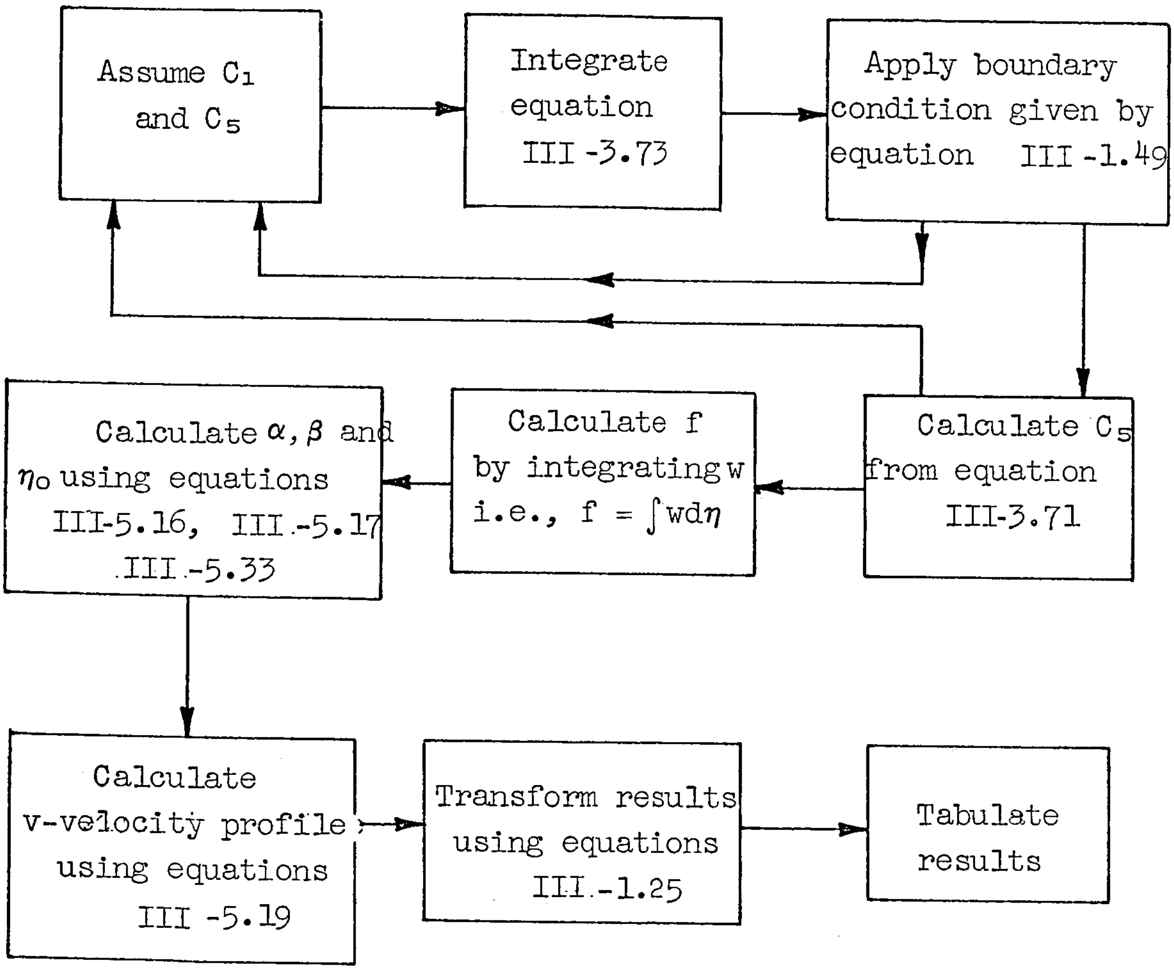


Fig.IV-2-3.1 Calculation Procedure
Heterogeneous Case Sc $\neq$ 1.0, $u_2 \neq 0$

equation III-1.49 was satisfied and the value of C_5 calculated from equation III-3.71 agreed with the assumed value.

The function f was then calculated by integrating w. It follows from equation III-3.72 that if f = 0, then

$$-f''' + \frac{(1+Sc)\, f''^{\,1+Sc}}{C_5 + \int_{-\infty}^{\eta} (f'')^{Sc}\, d\eta} = 0 \qquad \text{IV-2-3.1}$$

or

$$-w'' + \frac{(1+Sc)\, w'^{\,1+Sc}}{C_5 + \int_{-\infty}^{\eta} (w')^{Sc}\, d\eta} = 0 \qquad \text{IV-2-3.2}$$

The location of the point where f = 0 was fixed as that point where equation IV-2-3.2 above was satisfied. The values of f for plus and minus values of η^* were then obtained by integrating forward and backward from the point where $f(\eta) = 0$, $\eta^* = 0$.

The remaining procedure for this case parallels that described in section IV-1-2 for the homogeneous case. The procedure outlined in this section was followed for one case of a Schmidt number of unity in addition to those cases for which $Sc \neq 1.0$. If $Sc = 1.0$, the value of the constant C_5 can be calculated by analytically integrating equation III-3.71. The values of the constant C_5 calculated from the numerical integration and the analytic integration of equation III-3.71 agreed exactly in the first five decimal places.

IV-3 Determination of σ

The similarity solution of the turbulent mixing problem contains only the constant σ to be determined experimentally. (See section III -2). The method of determining σ from experimental data is as follows: For a particular value of λ, the numerical solution by the method of analytic continuation provides the u-velocity profile as a function of the similarity variable ξ. Since this velocity profile approaches the free stream velocities asymptotically, it is first necessary to suitably define the boundary layer thickness. The boundary layer thickness is defined here as that distance through which the u-velocity component changes from $u_1 + .05\ (u_1 - u_2)$ to $u_2 - .05\ (u_1 - u_2)$. From the numerical solution, the change $\Delta\xi$ in the independent variable ξ corresponding to the above mentioned change in the u-velocity component can be determined. Next, from the experimental data the change Δy in the rectangular coordinate y corresponding to the same change in the u-velocity component can be determined at a particular value of the downstream coordinate x . The constant σ can then be calculated from the relationship

$$\Delta \xi = \sigma \frac{\Delta y}{x} \qquad \text{IV-3.1}$$

It will be recalled that σ is given by

$$\sigma = \frac{1}{2} \sqrt{\chi_1 c \lambda} \qquad \text{IV-3.2}$$

where χ_1 is a dimensionless constant, λ is given by

$$\lambda = \frac{u_1 - u_2}{u_1 + u_2} \qquad \text{IV-3.3}$$

and c is the constant of proportionality in the relationship expressing the linear increase of the width of the mixing region b with downstream distance x, i.e.

$$b = cx \qquad \text{IV-3.4}$$

Thus, if the width of the mixing region is defined as in the previous paragraph, then equations IV-3.1 and IV-3.1 may be combined to give

$$c = \frac{\Delta \xi}{\sigma} \qquad \text{IV-3.5}$$

Therefore, the constants c and x_1 may be easily calculated once the value of σ has been determined.

APPENDICES

APPENDIX A

NUMERICAL RESULTS, HOMOGENEOUS CASE

A1. Tabulated Results

Table A1.1.- λ = 0.2

η	ξ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-4.897	-2.236	0.667	0.028
-4.297	-1.962	0.669	0.025
-3.697	-1.688	0.673	0.019
-3.097	-1.414	0.680	0.007
-2.497	-1.140	0.695	-0.012
-1.897	-0.866	0.719	-0.036
-1.297	-0.592	0.755	-0.061
-0.697	-0.318	0.800	-0.082
-0.097	-0.044	0.851	-0.091
0	0	0.859	--
+0.103	+0.047	0.868	-0.091
0.703	0.321	0.915	-0.082
1.303	0.595	0.952	-0.065
1.903	0.869	0.976	-0.048
2.503	1.143	0.996	-0.034
3.103	1.417	0.996	-0.024
3.703	1.691	0.999	-0.022
4.103	1.873	1.0	-0.021

Table A1.2, $\lambda = 0.4$

η	ξ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-6.184	-2.614	0.429	0.108
-5.584	-2.360	0.431	0.105
-4.984	-2.106	0.434	0.099
-4.384	-1.853	0.440	0.087
-3.784	-1.599	0.451	0.068
-3.184	-1.346	0.471	0.038
-2.584	-1.092	0.503	0.000
-1.984	-0.839	0.549	-0.045
-1.384	-0.585	0.612	-0.089
-0.784	-0.331	0.688	-0.123
-0.184	-0.078	0.769	-0.140
0	0	0.794	--
+0.16	+0.007	0.796	-0.141
0.616	0.260	0.869	-0.132
1.216	0.514	0.926	-0.110
1.816	0.768	0.963	-0.085
2.416	1.021	0.984	-0.067
3.016	1.275	0.994	-0.056
3.616	1.528	0.998	-0.051
4.216	1.782	1.0	-0.048

Table A1.3, λ = 0.6

η	ξ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-7.809	-3.087	0.250	0.215
-7.009	-2.778	0.252	0.212
-6.609	-2.612	0.253	--
-6.209	-2.454	0.254	0.205
-5.809	-2.295	0.257	--
-5.409	-.2.109	0.261	0.190
-5.009	-1.980	0.266	--
-4.609	-1.820	0.274	0.164
-4.209		0.285	--
-3.809	-1.505	0.300	0.120
-3.409	-1.397	0.321	--
-3.009	-1.189	0.347	0.058
-2.609	-1.031	0.380	--
-2.209	-0.873	0.422	0.018
-1.809	-0.715	0.471	--
-1.409	-0.557	0.527	0.093
-1.009	-0.399	0.590	--
-0.609	-0.241	0.656	-0.144
-0.209	-0.083	0.723	--
0	0	0.756	--
+0.191	+0.075	0.786	-0.155
0.591	0.234	0.843	--
0.991	0.392	0.891	0.131
1.391	0.550	0.929	--
1.791	0.708	0.956	0.094
2.191	0.866	0.975	--
2.591	1.024	0.986	0.071
2.991	1.182	0.993	--
3.391	1.340	0.997	0.059
3.791	1.498	0.999	--
4.191	1.656	0.999	0.056
4.391	1.736	1.0	--

Table A1.4, $\lambda = 0.8$

η	ξ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-9.986	-3.720	0.111	0.326
-9.586	-3.571	0.112	--
-9.186	-3.422	0.112	0.324
-8.786	-3.273	0.112	--
-8.386	-3.124	0.113	0.321
-7.986	-2.975	0.114	--
-7.586	-2.826	0.115	0.315
-7.186	-2.677	0.117	--
-6.786	-2.528	0.119	0.304
-6.386	-2.379	0.123	
-5.986	-2.230	0.127	0.285
-5.586	-2.081	0.133	--
-5.186	-1.932	0.142	0.255
-4.786	-1.783	0.153	
-4.386	-1.634	0.168	0.209
-3.986	-1.485	0.186	--
-3.586	-1.336	0.211	0.149
-3.186	-1.187	0.241	--
-2.786	-1.038	0.278	0.066
-2.386	-0.889	0.324	--
-1.986	-0.740	0.377	0.021
-1.586	-0.591	0.439	--
-1.186	-0.442	0.507	-0.097
-0.786	-0.293	0.580	--
-0.386	-0.144	0.655	-0.140
0	0	0.725	--
0.414	0.154	0.795	--
0.814	0.303	0.853	-0.127
1.214	0.452	0.901	--
1.614	0.601	0.937	-0.091
2.014	0.750	0.962	--
2.414	0.899	0.979	-0.060
2.814	1.048	0.989	--
3.214	1.197	0.995	-0.044
3.614	1.346	0.998	--
4.014	1.495	0.999	-0.039
4.414	1.645	1.0	-0.037

Table A1.5, $\lambda = 1.0$

η	ξ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-9.376	-3.315	0.004	0.4215
-8.976	-3.174	0.005	0.4178
-8.576	-3.032	0.006	0.4131
-8.176	-2.891	0.009	0.4074
-7.776	-2.750	0.011	0.4004
-7.376	-2.608	0.014	0.3921
-6.976	-2.467	0.018	--
-6.576	-2.325	0.024	0.3703
-6.176	-2.184	0.030	--
-5.776	-2.042	0.038	0.3376
-5.376	-1.901	0.049	--
-4.976	-1.760	0.062	0.2931
-4.576	-1.618	0.078	--
-4.176	-1.477	0.099	0.2337
-3.776	-1.335	0.124	--
-3.376	-1.194	0.156	0.1583
-2.976	-1.052	0.194	--
-2.576	-0.911	0.239	0.0613
-2.176	-0.769	0.239	--
-1.776	-0.628	0.354	-0.0166
-1.376	-0.487	0.423	--
-0.976	-0.345	0.499	-0.0863
-0.576	-0.204	0.578	-0.1180
-0.176	-0.062	0.657	--
0	0	0.691	--
0.224	0.079	0.733	-0.1182
0.624	0.221	0.802	--
1.024	0.362	0.861	-0.0909
1.424	0.504	0.907	--
1.824	0.645	0.942	-0.0510
2.224	0.786	0.966	-0.0344
2.624	0.928	0.981	-0.0208
3.024	1.069	0.991	-0.0123
3.424	1.211	0.995	-0.0057
3.824	1.352	0.998	-0.0033
4.224	1.494	0.999	-0.0010
4.624	1.635	1.00	-0.0007

A2. Calculated Constants

Table A2.1

λ	α	β	η_o	ξ_o
0.2	0.18973	0.22382	-0.23465	-0.10710
0.4	0.32339	0.45013	-0.43619	-0.18431
0.6	0.41790	0.68622	-0.55488	-0.21934
0.8	0.48400	0.93367	-0.58057	-0.21638
1.0	0.53007	--	-0.53007	-0.18743

APPENDIX B

NUMERICAL RESULTS, HETEROGENEOUS CASE

B1. Tabulated Results, Sc = 1.0

Table B1.1, $\Gamma = -0.2$, $\lambda = 1.0$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-14.157	-5.005	1.000	0.000	0.378
-13.757	-4.864	1.000	0.001	--
-13.357	-4.722	1.000	0.001	0.377
-12.959	-4.581	1.000	0.001	00
-12.557	-4.440	0.999	0.001	0.375
-12.159	-4.298	0.999	0.001	--
-11.757	-4.157	0.999	0.002	0.373
-11.357	-4.015	0.999	0.002	--
-10.957	-3.874	0.999	0.003	0.369
-10.557	-3.732	0.998	0.003	--
-10.157	-3.591	0.998	0.004	0.364
-9.757	-3.450	0.997	0.005	--
-9.357	-3.308	0.997	0.006	0.356
-8.957	-3.167	0.996	0.008	0.346
-8.557	-3.025	0.995	0.010	--
-8.157	-2.884	0.994	0.012	--
-7.957	-2.742	0.993	0.015	0.331
-6.957	-2.460	0.989	0.023	0.310
-6.557	-2.318	0.986	0.028	--
-6.157	-2.177	0.983	0.035	0.282
-5.757	-2.035	0.979	0.043	--
-5.357	-1.894	0.974	0.053	0.245
-4.957	-1.753	0.968	0.066	--

Table B1.1 (Continued)

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-4.551	-1.611	0.961	0.081	0.197
-4.157	-1.470	0.953	0.100	--
-3.757	-1.378	0.942	0.120	0.136
-3.357	-1.187	0.930	0.150	--
-2.957	-1.045	0.916	0.183	0.065
-2.557	-0.904	0.900	0.223	--
-2.157	0.763	0.881	0.270	-0.013
-1.757	-0.621	0.860	0.324	--
-1.357	-0.480	0.838	0.387	-0.085
-0.957	-0.338	0.815	0.457	--
-0.557	-0.197	0.790	0.533	-0.134
-0.157	-0.056	0.766	0.612	--
+0.043	+0.015	0.754	0.652	-0.145
0.443	0.157	0.733	0.729	--
0.843	0.298	0.714	0.800	-0.122
1.243	0.439	0.699	0.861	--
1.643	0.581	0.687	0.909	0.085
2.043	0.722	0.679	0.945	--
2.443	0.864	0.674	0.969	0.034
2.843	1.005	0.670	0.983	--
3.243	1.147	0.669	0.992	-0.011
3.643	1.288	0.668	0.996	--
4.043	1.429	0.667	0.998	-0.003
4.643	1.642	0.667	1.000	-0.001

Table B1.2, $\Gamma = -1/3$, $\lambda = 1/3$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-5.978	-2.589	0.999	0.500	+0.000
-5.578	-2.416	0.998	0.501	--
-5.178	-2.242	0.997	0.502	-0.002
-4.778	-2.069	0.994	0.503	--
-4.378	-1.896	0.990	0.505	-0.010
-3.978	-1.723	0.984	0.508	--
-3.578	-1.549	0.975	0.513	-0.023
-3.178	-1.376	0.962	0.520	--
-2.778	-1.203	0.944	0.530	-0.046
-2.378	-1.030	0.921	0.543	--
-1.978	-0.857	0.891	0.561	-0.78
-1.578	-0.683	0.856	0.584	--
-1.178	-0.510	0.815	0.613	-0.113
-0.778	-0.337	0.771	0.649	--
-0.378	-0.164	0.724	0.690	-0.139
-0.78	-0.077	0.701	0.713	--
0	0	--	0.734	--
+0.022	0.010	0.678	0.737	-0.142
0.422	0.183	0.635	0.787	--
0.822	0.356	0.598	0.836	-0.125
1.222	0.529	0.567	0.882	--
1.622	0.702	0.543	0.921	-0.082
2.022	0.876	0.526	0.950	--
2.422	1.049	0.515	0.971	-0.039
2.822	1.222	0.508	0.985	--
3.222	1.395	0.504	0.992	-0.014
4.022	1.742	0.501	0.999	-0.005
4.622	2.002	0.500	1.000	-0.003

Table B1.3, $\Gamma = -1/3$, $\lambda = 0.6$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-8.288	-3.275	0.999	0.250	0.098
-7.888	-3.118	0.999	0.251	--
-7.488	-2.959	0.998	0.251	0.096
-7.088	-2.801	0.997	0.252	--
-6.688	-2.643	0.996	0.253	0.091
-6.288	-2.485	0.994	0.255	--
-5.888	-2.237	0.991	0.257	0.082
-5.488	-2.169	0.987	0.260	--
-5.088	-2.011	0.981	0.264	0.065
-4.688	-1.853	0.973	0.271	--
-4.288	-1.695	0.963	0.279	0.038
-3.888	-1.537	0.950	0.290	--
-3.488	-1.378	0.933	0.304	0.000
-3.088	-1.220	0.912	0.323	--
-2.688	-1.062	0.886	0.346	-0.051
-2.888	-0.904	0.857	0.375	--
-1.888	-0.746	0.823	0.411	-0.109
-1.488	-0.588	0.786	0.454	--
-1.088	-0.430	0.746	0.505	-0.163
-0.688	-0.272	0.706	0.563	--
-0.288	-0.114	0.665	0.627	-0.196
-0.088	-0.035	0.646	0.661	--
0.00	0	--	0.676	--
0.112	-0.044	0.628	0.695	-0.198
0.512	0.202	0.596	0.762	--
0.912	0.360	0.566	0.825	-0.173
1.312	0.519	0.544	0.880	--
1.712	0.677	0.527	0.923	-0.123
2.112	0.835	0.516	0.954	--
2.512	0.993	0.509	0.974	-0.081
2.912	1.151	0.504	0.967	--
3.312	1.309	0.502	0.994	-0.060
3.712	1.467	0.501	0.997	--
4.112	1.625	0.500	0.999	-0.052
4.512	1.783	0.500	1.000	--

Table B1.4, $\Gamma = -1/3$, $\lambda = 0.8$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-10.767	-4.011	1.000	0.111	0.211
-10.367	-3.862	0.999	0.112	--
-9.967	-3.713	0.999	0.112	0.209
-9.567	-3.564	0.999	0.112	--
-9.167	-3.415	0.998	0.113	0.207
-8.767	-3.266	0.997	0.113	--
-8.367	-3.117	0.996	0.114	0.202
-7.967	-2.968	0.995	0.115	--
-7.567	-2.819	0.993	0.117	0.193
-7.167	-2.270	0.991	0.119	--
-6.767	-2.521	0.988	0.122	0.180
-6.637	-2.372	0.984	0.126	--
-5.967	-2.223	0.979	0.130	0.160
-5.567	-2.074	0.972	0.137	--
-5.167	-1.925	0.964	0.144	0.132
-4.767	-1.776	0.954	0.154	--
-4.367	-1.627	0.941	0.167	0.092
-3.967	-1.478	0.926	0.183	--
-3.567	-1.329	0.907	0.202	0.040
-3.167	-1.180	0.885	0.226	--
-2.767	-1.031	0.860	0.255	-0.023
-2.367	-0.882	0.832	0.291	--
-1.967	-0.733	0.800	0.334	-0.091
-1.567	-0.584	0.765	0.384	--
-1.167	-0.435	0.729	0.442	-0.154
-0.767	-0.286	0.691	0.508	--
-0.367	-0.137	0.655	0.580	-0.193
-0.137	-0.062	0.637	0.618	--
0.000	0.000	--	0.649	--
0.033	0.012	0.620	0.656	-0.197
0.433	0.161	0.589	0.731	--
0.833	0.310	0.563	0.801	-0.175
1.233	0.459	0.542	0.862	--
1.633	0.608	0.527	0.910	-0.126
2.033	0.757	0.516	0.946	--
2.433	0.906	0.509	0.970	-0.082
2.833	1.055	0.504	0.984	--
3.233	1.205	0.502	0.992	-0.058
3.633	1.359	0.501	0.997	--
4.033	1.503	0.500	0.999	-0.050
4.433	1.652	0.500	1.000	--

Table B1.5, $\Gamma = -0.4$, $\lambda = 1.0$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-10.976	-3.881	0.994	0.004	0.304
-10.376	-3.668	0.993	0.006	0.297
-9.776	-3.456	0.990	0.007	--
-9.376	-3.315	0.988	0.009	0.288
08.976	-3.173	0.986	0.011	--
-8.576	-3.032	0.983	0.013	0.275
-8.176	-2.891	0.980	0.016	--
-7.776	-2.749	0.976	0.019	0.258
-7.376	-2.608	0.971	0.022	--
-6.976	-2.466	0.965	0.027	0.237
-6.576	-2.325	0.959	0.032	--
-6.176	-2.184	0.951	0.039	0.209
-5.776	-2.042	0.941	0.047	--
-5.376	-1.901	0.930	0.056	0.174
-4.976	-1.759	0.917	0.068	--
-4.576	-1.618	0.902	0.081	0.130
-4.176	-1.476	0.885	0.098	--
-3.776	-1.335	0.864	0.118	0.077
-3.376	-1.194	0.841	0.141	--
-2.976	-1.052	0.815	0.170	0.015
-2.576	-0.911	0.786	0.204	--
-2.176	-0.769	0.755	0.244	0.052
-1.776	-0.628	0.720	0.291	--
-1.376	-0.486	0.684	0.347	0.116
-0.976	-0.345	0.647	0.410	--
-0.576	-0.204	0.609	0.481	0.162
-0.176	-0.062	0.573	0.559	--
0	0	--	0.594	--
0.024	+0.008	0.556	0.599	-0.173
0.424	0.150	0.525	0.680	--
0.824	0.291	0.497	0.758	-0.149
1.224	0.433	0.475	0.827	--
1.624	0.574	0.459	0.885	-0.095
2.024	0.716	0.447	0.929	--
2.424	0.857	0.439	0.959	-0.044
2.824	0.998	0.434	0.978	--
3.224	1.140	0.431	0.989	-0.014
3.624	1.281	0.430	0.995	--
4.024	1.423	0.429	0.998	-0.003
4.424	1.564	0.429	0.999	--
4.824	1.706	0.429	1.000	0

Table B1.6, $\Gamma = -0.6$, $\lambda = 1/3$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-5.719	-2.476	0.997	0.500	-0.048
-5.319	-2.303	0.995	0.501	--
-4.917	-2.130	0.991	0.501	-0.051
-4.519	-1.957	0.986	0.502	--
-4.119	-1.784	0.977	0.504	-0.056
-3.719	-1.610	0.963	0.506	--
-3.319	-1.437	0.945	0.510	-0.065
-2.919	-1.264	0.919	0.515	--
-2.519	-1.091	0.886	0.521	-0.080
-2.119	-0.918	0.845	0.531	--
-1.719	-0.744	0.795	0.543	-0.099
-1.319	-0.571	0.738	0.559	--
-0.919	-0.398	0.675	0.580	-0.121
-0.519	-0.225	0.609	0.607	--
-0.119	-0.052	0.543	0.640	-0.134
0.081	0.035	0.510	+0.660	-0.134
0.481	0.208	0.450	0.702	--
0.881	0.381	0.396	0.754	-0.114
1.281	0.555	0.352	0.807	--
1.681	0.728	0.317	0.859	0.056
2.081	0.901	0.291	0.905	--
2.481	1.074	0.274	0.941	+0.016
2.881	1.248	0.263	0.967	--
3.281	1.421	0.257	0.983	0.067
3.681	1.594	0.253	0.992	--
4.081	1.767	0.251	0.996	0.087
4.481	1.940	0.251	0.998	--
4.881	2.113	0.250	0.999	0.092
5.081	2.201	0.250	1.000	--

Table B1.7, $\Gamma = -0.6$, $\lambda = 0.6$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-8.366	-3.306	0.998	0.250	-0.002
-7.966	-3.149	0.997	0.251	--
-7.566	-2.990	0.995	0.251	-0.004
-7.166	-2.832	0.993	0.252	--
-6.766	-2.674	0.989	0.253	-0.008
-6.366	-2.516	0.984	0.254	--
-5.966	-2.358	0.977	0.256	-0.016
-5.566	-2.200	0.968	0.258	--
-5.166	-2.042	0.956	0.261	-0.029
-4.766	-1.884	0.941	0.266	--
-4.366	-1.727	0.921	0.271	-0.048
-3.966	-1.568	0.896	0.279	--
-3.566	-1.409	0.867	0.288	-0.073
-3.166	-1.251	0.832	0.301	--
-2.766	-1.093	0.791	0.316	-0.107
-2.366	-0.935	0.746	0.335	--
-1.966	-0.771	0.696	0.359	-0.146
-1.566	-0.619	0.642	0.389	--
-1.166	-0.461	0.587	0.426	-0.188
-0.766	-0.303	0.531	0.471	--
-0.366	-0.145	0.477	0.524	-0.217
				--
0.034	+0.013	0.427	+0.586	-0.221
0.434	0.172	+0.382	0.654	--
0.834	0.330	0.344	0.726	-0.196
1.236	0.488	0.314	0.797	--
1.634	0.646	0.291	0.860	-0.132
2.034	0.804	0.295	0.911	--
2.434	0.962	0.261	0.948	-0.063
2.834	1.120	0.257	0.972	--
3.234	1.278	0.254	0.986	-0.021
3.634	1.436	0.250	0.994	--
4.034	1.594	0.251	0.997	-0.006
4.434	1.753	0.250	0.999	--
4.134	1.911	0.250	1.000	-0.001

Table B1.8, $\Gamma = -0.6$, $\lambda = 0.8$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-11.622	-4.330	0.999	0.111	+0.105
-11.221	-4.181	0.998	0.112	--
-10.822	-4.032	0.998	0.112	0.104
-10.422	-3.883	0.996	0.112	--
-10.022	-3.734	0.996	0.112	0.101
-9.622	-3.585	0.994	0.113	--
-9.222	-3.436	0.992	0.114	0.098
-8.822	-3.287	0.989	0.114	--
-8.422	-3.138	0.986	0.115	0.091
-8.022	-2.989	0.981	0.117	--
-7.622	-2.840	0.976	0.119	0.082
-7.222	-2.691	0.969	0.121	--
-6.822	-2.542	0.960	0.123	0.069
-6.422	-2.393	0.949	0.127	--
-6.022	-2.244	0.936	0.131	0.050
-5.622	-2.095	0.921	0.137	--
-5.222	-1.946	0.902	0.143	0.025
-4.822	-1.797	0.880	0.151	--
-4.422	-1.648	0.855	0.161	-0.008
-4.022	-1.498	0.826	0.171	--
-3.622	-1.349	0.793	0.186	-0.048
-3.222	-1.200	0.756	0.207	--
-2.822	-1.051	0.715	0.229	-0.096
-2.422	-0.902	0.671	0.256	--
-2.022	-0.753	0.625	0.289	-0.150

Table B1.8 (Continued)

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-1.622	-0.604	0.577	0.329	--
-1.222	-0.455	0.528	0.376	-0.202
-0.822	-0.306	0.480	0.432	--
-0.422	-0.157	0.434	0.497	-0.238
-0.022	-0.008	0.392	0.570	--
0	0	--	0.575	--
+0.178	+0.066	0.373	0.609	-0.243
0.578	0.215	0.339	0.689	--
0.978	0.364	0.311	0.766	-0.209
1.378	0.513	0.290	0.836	--
1.778	0.662	0.275	0.894	-0.145
2.178	0.811	0.264	0.936	--
2.578	0.960	0.258	0.965	-0.089
2.978	1.110	0.254	0.982	--
3.378	1.259	0.252	0.991	-0.060
3.778	1.408	0.251	0.996	--
4.178	1.557	0.250	0.998	-0.050
4.378	1.631	0.250	0.999	--
4.778	1.780	0.250	1.000	-0.049

Table B1.9, $\Gamma = -0.6$, $\lambda = 1.0$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-15.267	-5.398	0.996	0.001	0.254
-14.867	-5.256	0.995	0.002	--
-14.467	-5.115	0.994	0.002	0.251
-14.067	-4.973	0.993	0.002	--
-13.667	-4.832	0.992	0.003	0.248
-13.267	-4.691	0.991	0.003	--
-12,867	-4.549	0.990	0.003	0.244
-12.467	-4.408	0.988	0.004	--
-12.067	-4.266	0.986	0.005	0.238
-11.667	-4.125	0.984	0.005	--
-11.267	-3.983	0.982	0.006	0.232
-10.867	-3.842	0.979	0.007	--
-10.467	-3.701	0.975	0.008	0.223
-10.067	-3.559	0.971	0.010	--
-9.667	-3.418	0.967	0.011	0.213
-9.267	-3.276	0.960	0.013	--
-8.867	-3.135	0.956	0.015	0.200
-8.467	-2.994	0.949	0.018	--
-8.067	-2.852	0.941	0.021	0.183
-7.667	-2.711	0.932	0.024	--
-7.267	-2.569	0.921	0.028	0.163
-6.867	-2.428	0.909	0.033	--
-6.467	-2.286	0.896	0.039	0.138
-6.067	-2.145	0.880	0.045	--
-5.667	-2.004	0.863	0.053	0.108

Table B1.9 (Continued)

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-5.267	-1.862	0.843	0.062	--
-4.867	-1.721	0.821	0.073	0.072
-4.467	-1.379	0.796	0.085	--
-4.067	-1.438	0.769	0.100	0.028
-3.667	-1.296	0.739	0.118	--
03,267	-1.155	0.705	0.139	-0.022
-2.867	-1.014	0.670	0.164	--
-2.467	-0.872	0.631	0.195	-0.078
-2.067	-0.731	0.591	0.230	--
-1.667	-0.589	0.550	0.273	-0.135
-1.267	-0.448	0.507	0.324	--
-0.867	-0.307	0.465	0.383	-0.183
-0.467	-0.165	0.425	0.451	--
-0.067	-0.024	0.387	0.527	-0.206
0	0	--	0.541	--
+0.333	0.118	0.354	0.609	--
0.733	0.259	0.325	0.693	-0.187
1.133	0.401	0.301	0.773	--
1.533	0.542	0.283	0.843	-0.127
1.933	0.683	0.270	0.899	--
2.333	0.825	0.262	0.940	-0.062
2.733	0.966	0.256	0.963	--
3.133	1.108	0.253	0.983	-0.021
3.533	1.249	0.251	0.992	--
3.933	1.391	0.250	0.996	-0.005
4.333	1.532	0.250	0.999	--
4.733	1.673	0.250	0.999	-0.002
4.933	1.744	0.250	1.000	--

Table B1.10, $\Gamma = -0.75$, $\lambda = 1/3$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-5.385	-2.332	0.995	0.500	-0.062
-4.985	-2.159	0.991	0.501	--
-4.585	-1.986	0.985	0.501	-0.064
-4.185	-1.812	0.975	0.502	--
-3.785	-1.639	0.961	0.503	-0.068
-3.385	-1.466	0.941	0.505	--
-2.985	-1.293	0.914	0.505	-0.075
-2.585	-1.119	0.878	0.512	--
-2.185	-0.946	0.834	0.517	-0.084
-1.785	-0.773	0.780	0.523	--
-1.385	-0.600	0.719	0.533	-0.097
-0.985	-0.427	0.650	0.545	--
-0.585	-0.253	0.577	0.561	-0.098
-0.185	-0.080	0.503	0.582	--
+0.015	-0.006	0.467	0.585	-0.113
+0.415	0.180	0.398	0.626	--
+0.815	0.353	0.335	0.665	-0.100
1.215	0.526	0.232	0.713	--
1.615	0.700	0.238	0.767	-0.045
2.015	0.873	0.205	0.824	--
2.415	1.046	0.181	0.878	+0.050
2.815	1.219	0.165	0.923	--
3.215	1.392	0.155	0.956	0.143
3.615	1.565	0.149	0.977	--
4.015	1.739	0.146	0.989	0.194
4.415	1.912	0.144	0.995	--
4.815	2.085	0.143	0.998	0.210
5.015	2.172	0.143	0.999	--
5.415	2.345	0.143	1.000	0.214

Table B1.11, $\Gamma = -0.75$, $\lambda = 0.6$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-8.396	-3.319	0.996	0.250	-0.061
-7.996	-3.160	0.995	0.251	--
-7.596	-3.002	0.992	0.251	-0.062
-7.196	-2.844	0.988	0.252	--
-6.796	-2.686	0.983	0.052	-0.067
-6.396	-2.528	0.975	0.053	--
-5.996	-2.370	0.965	0.254	0.072
-5.596	-2.212	0.953	0.256	--
-5.196	-2.054	0.936	0.259	-0.081
-4.796	-1.896	0.915	0.262	--
-4.396	-1.738	0.890	0.265	-0.093
-3.996	-1.579	0.860	0.270	--
-3.596	$1.421	0.823	0.277	-0.111
-3.196	-1.263	0.782	0.285	--
-2.796	-1.105	0.735	0.285	-0.134
-2.396	-0.947	0.684	0.308	--
-1.996	-0.789	0.628	0.324	-0.161
-1.596	-0.631	0.570	0.344	--
-1.196	-0.473	0.510	0.370	-0.190
-0.796	-0.315	0.451	0.402	--
-0.396	-0.157	0.395	0.442	-0.212
-0.196	-0.077	0.368	0.465	--
0	0	--	0.490	--
+0.004	0.002	0.342	0.491	-0.215
0.404	0.160	0.295	0.549	--
0.804	0.318	0.254	0.617	-0.194
1.204	0.476	0.220	0.692	--
1.604	0.634	0.194	0.768	-0.122
2.004	0.792	0.175	0.838	--
2.404	0.950	0.162	0.897	-0.022
2.804	1.108	0.153	0.940	--
3.204	1.266	0.148	0.968	+0.055
3.604	1.424	0.146	0.984	--
4.004	1.583	0.144	0.993	+0.091
4.404	1.741	0.143	0.997	--
4.804	1.899	0.143	0.999	0.101
5.204	2.057	0.143	1.000	0.103

Table B1.12, $\Gamma = -0.75$, $\lambda = 0.75$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-10.452	-3.950	0.996	0.143	-0.003
-9.652	-3.648	0.992	0.144	-0.005
-8.852	-3.345	0.985	0.145	-0.008
-7.052	-3.043	0.973	0.147	-0.014
-7.252	-2.741	0.955	0.150	-0.022
-6.452	-2.438	0.928	0.154	-0.033
-5.652	-2.136	0.888	0.161	-0.049
-4.852	-1.834	0.835	0.171	-0.069
-4.052	-1.531	0.766	0.187	-0.095
-3.252	-1.229	0.681	0.210	-0.120
-2.452	-0.927	0.584	0.245	-0.164
-1.652	-0.624	0.480	0.298	-0.205
-0.852	-0.322	0.379	0.377	-0.241
-0.052	-0.020	0.290	0.493	-0.261
+0.148	+0.056	0.270	0.528	-0.260
0.948	0.358	0.208	0.687	-0.227
1.748	0.661	0.170	0.842	-0.069
2.548	0.963	0.152	0.942	-0.069
3.348	1.265	0.145	0.985	-0.018
4.148	1.568	0.143	0.997	-0.005
4.948	1.870	0.143	1.000	-0.001

Table B1.13, $\Gamma = -0.75$, $\lambda = 1.0$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-17.451	-6.170	0.991	0.001	0.198
-17.051	-6.028	0.990	0.002	--
-16.651	-5.887	0.989	0.002	0.195
-16.251	-5.746	0.988	0.002	--
-15.851	-5.604	0.986	0.002	0.193
-15.451	-5.463	0.984	0.003	--
-15.051	-5.321	0.982	0.003	0.189
-14.651	-5.180	0.980	0.003	--
-14.251	-5.038	0.978	0.004	0.185
-13.851	-4.897	0.975	0.004	--
-13.451	-4.756	0.972	0.005	0.180
-13.051	-4.614	0.968	0.005	--
-12.651	-4.473	0.965	0.006	0.174
-12.251	-4.331	0.960	0.007	--
-11.851	-4.190	0.955	0.008	0.167
-11.451	-4.049	0.950	0.009	--
-11.051	-3.907	0.944	0.010	0.158
-10.651	-3.761	0.937	0.011	--
-10.251	-3.624	0.929	0.013	0.148
-9.851	-3.483	0.921	0.014	--
-9.451	-3.341	0.911	0.016	0.135
-9.051	-3.200	0.901	0.018	--
-8.651	-3.059	0.889	0.021	0.121
-8.251	-2.917	0.876	0.024	--
-7.851	-2.776	0.861	0.027	0.103
-7.451	-2.634	0.845	0.031	--
-7.051	-2.493	0.828	0.035	0.083
-6.651	-2.351	0.808	0.040	--
-6.251	-2.210	0.787	0.045	0.058
-5.851	-2.069	0.763	0.052	--

Table B1.13 (Continued)

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-5.451	-1.927	0.738	0.059	0.029
-5.051	-1.786	0.711	0.068	--
-4.651	-1.644	0.681	0.078	-0.004
-4.251	-1.503	0.649	0.090	--
-3.851	-1.362	0.615	0.104	-0.044
-3.451	-1.220	0.579	0.121	--
-3.051	-1.079	0.541	0.141	-0.088
-2.651	-0.937	0.502	0.165	--
-2.251	-0.796	0.463	0.194	-0.137
-1.851	-0.654	0.422	0.228	--
-1.451	-0.513	0.383	0.269	-0.186
-1.051	-0.372	0.343	0.319	--
-0.651	-0.230	0.306	0.377	-0.025
-0.051	-0.018	0.256	0.484	-0.238
0	0	--	0.494	--
+0.149	+0.053	0.241	0.524	-0.237
0.549	0.194	0.215	0.608	--
0.949	0.336	0.193	0.695	-0.203
1.349	0.477	0.176	0.778	--
1.749	0.618	0.164	0.850	-0.131
2.149	0.760	0.155	0.906	--
2.549	0.901	0.150	0.946	-0.060
2.949	1.043	0.146	0.972	--
3.349	1.184	0.145	0.986	-0.019
3.747	1.325	0.144	0.994	--
4.149	1.467	0.143	0.997	-0.004
4.549	1.508	0.143	0.999	--
4.949	1.750	0.143	1.000	-0.001

B2. Tabulated Results, Sc = 0.75 and 0.5

Table B2.1, Sc = 0.75, $\Gamma = -0.6$, $\lambda = 1/3$

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-5.846	-2.531	0.991	0.500	-0.010
-5.046	-2.185	0.977	0.502	--
-4.246	-1.839	0.949	0.505	-0.019
-3.446	-1.492	0.898	0.512	--
-2.646	-1.146	0.818	0.528	-0.052
-1.846	-0.799	0.711	0.558	--
-1.046	-0.453	0.591	0.610	-0.113
-0.246	-0.107	0.475	0.689	-0.135
0.00	0.00	--	0.719	--
+0.154	+0.067	0.425	0.737	-0.136
0.954	0.413	0.346	0.838	--
1.754	0.760	0.296	0.922	-0.064
2.554	1.106	0.268	0.972	--
3.354	1.452	0.256	0.993	+0.007
4.157	1.800	0.252	0.999	--
4.957	2.146	0.250	1.000	+0.018

Table B2.2, Sc = 0.75, Γ = -0.6, λ = 0.6

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-8.297	-3.279	0.993	0.250	0.068
-7.497	-2.963	0.986	0.251	--
-6.697	-2.647	0.973	0.253	0.061
-5.897	-2.331	0.950	0.256	--
-5.097	-2.015	0.914	0.263	0.038
-4.297	-1.698	0.861	0.276	--
-3.497	-1.382	0.789	0.300	-0.022
-2.697	-1.066	0.700	0.339	--
-1.897	-0.750	0.600	0.403	-0.127
-1.097	-0.434	0.500	0.497	--
-0.297	-0.117	0.411	0.622	-0.216
0	0	--	0.673	--
+0.103	0.041	0.374	0.691	-0.218
0.903	0.357	0.317	0.823	--
1.703	0.673	0.281	0.922	-0.143
2.503	0.989	0.262	0.974	--
3.303	1.306	0.254	0.994	-0.078
3.703	1.464	0.252	0.997	--
4.503	1.780	0.250	1.000	-0.070

Table B2.3, Sc = 0.75, Γ = -0.6, λ = 0.8

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-10.877	-4.053	0.994	0.111	0.181
-10.077	-3.754	0.990	0.112	--
-9.277	-3.456	0.984	0.113	0.177
-8.477	-3.158	0.973	0.114	--
-7.677	-2.860	0.958	0.116	0.165
-6.877	-2.562	0.936	0.121	--
-6.077	-2.264	0.904	0.128	0.136
-5.277	-1.966	0.851	0.141	--
-4.477	-1.668	0.805	0.161	0.073
-3.677	-1.370	0.735	0.193	--
-2.877	-1.072	0.654	0.243	-0.037
-2.077	-0.774	0.566	0.319	--
-1.277	-0.476	0.479	0.426	-0.172
-0.477	-0.178	0.401	0.565	--
-0.077	-0.029	0.368	0.642	-0.225
+0.323	0.120	0.339	0.718	-0.222
1.123	0.418	0.296	0.853	--
1.923	0.715	0.270	0.941	-0.138
2.723	1.015	0.258	0.982	--
3.523	1.313	0.252	0.996	-0.087
3.923	1.462	0.251	0.998	--
4.723	1.760	0.250	1.000	-0.082

Table B2.4, Sc = 0.75, Γ = -0.75, λ = 0.75

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-10.047	-3.780	0.984	0.143	0.084
-9.247	-3.479	0.973	0.144	--
-8.447	-3.178	0.956	0.145	0.078
-7.647	-2.877	0.930	0.148	--
-6.847	-2.576	0.893	0.152	0.058
-6.047	-2.275	0.842	0.159	--
-5.247	-1.974	0.775	0.170	0.016
-4.447	-1.673	0.694	0.190	--
-3.647	-1.372	0.601	0.221	-0.066
-2.847	-1.071	0.502	0.270	--
-2.047	-0.770	0.405	0.345	-0.195
-1.247	-0.469	0.318	0.455	--
-0.447	-0.168	0.250	0.597	-0.308
-0.047	-0.018	0.223	0.674	--
0.00	0.0	--	0.680	--
0.353	+0.133	0.201	0.751	-0.312
1.153	0.434	0.171	0.879	--
1.953	0.735	0.154	0.956	-0.235
2.753	1.036	0.147	0.988	--
3.553	1.337	0.144	0.998	-0.199
4.353	1.638	0.143	1.000	-0.198

Table B2.5, Sc = 0.5, Γ = -0.6, λ = 1/3

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-6.000	-2.598	0.968	0.500	0.020
-5.2	-2.252	0.940	0.502	--
-4.4	-1.905	0.894	0.505	0.009
-3.6	-1.559	0.828	0.514	--
-2.8	-1.212	0.742	0.534	-0.034
-2.0	-0.866	0.643	0.573	--
-1.2	-0.520	0.542	0.637	-0.118
-0.4	-0.173	0.451	0.724	-0.147
-0.0	0.00	0.412	0.774	--
0.4	+0.173	0.378	0.823	-0.148
1.2	0.520	0.325	0.908	--
2.0	0.866	0.290	0.963	-0.073
3.8	1.212	0.269	0.989	--
3.6	1.559	0.258	0.997	-0.045
4.4	1.905	0.253	1.000	-0.043

Table B2.6, Sc = 0.5, Γ = -0.6, λ = 0.6

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-8.206	-3.243	0.970	0.250	0.124
-7.406	-2.927	0.952	0.251	--
-6.606	-2.611	0.924	0.253	0.116
-5.806	-2.295	0.886	0.258	--
-5.006	-1.979	0.834	0.267	0.085
-4.206	-1.662	0.769	0.285	--
-3.406	-1.346	0.692	0.318	0.003
-2.606	-1.030	0.609	0.374	--
-1.806	-0.714	0.525	0.459	-0.136
-1.006	-0.398	0.447	0.575	--
-0.206	-0.081	0.382	0.709	-0.230
0	0	--	0.742	--
+0.194	+0.077	0.354	0.774	-0.231
0.994	0.393	0.311	0.885	--
1.794	0.709	0.283	0.954	-0.169
2.594	1.025	0.265	0.986	--
3.394	1.342	0.256	0.997	-0.130
4.194	1.658	0.252	1.000	-0.126

Table B2.7, Sc = 0.5, Γ = -0.6, λ = 0.8

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma V}{u_1}$
-10.338	-3.852	0.966	0.111	0.146
-9.538	-3.554	0.954	0.112	--
-8.738	-3.256	0.937	0.113	0.241
-7.938	-2.958	0.915	0.115	--
-7.138	-2.659	0.885	0.118	0.226
-6.338	-2.361	0.847	0.125	--
-5.538	-2.063	0.799	0.137	0.184
-4.738	-1.765	0.743	0.157	--
-3.938	-1.467	0.679	0.190	0.091
-3.138	-1.169	0.609	0.244	--
-2.338	-0.871	0.536	0.325	-0.061
-1.538	-0.573	0.466	0.439	--
-0.738	-0.275	0.403	0.580	-0.202
+0.062	0.023	0.351	0.729	-0.221
0.862	0.321	0.311	0.857	--
1.662	0.619	0.283	0.941	-0.242
2.462	0.917	0.267	0.981	--
3.662	1.364	0.255	0.998	--
4.462	1.662	0.252	1.000	-0.112

Table B2.8, Sc = 0.5, Γ = -0.75, λ = 0.75

η	ξ	ρ	$\frac{u}{u_1}$	$\frac{\sigma v}{u_1}$
-9.662	-3.652	0.937	0.143	0.188
-8.862	-3.349	0.916	0.144	--
-8.062	-3.096	0.884	0.145	0.177
-7.263	-2.745	0.842	0.148	--
-6.462	-2.443	0.788	0.159	0.155
-5.662	-2.140	0.724	0.163	--
-4.862	-1.838	0.649	0.181	0.100
-4.062	-1.535	0.568	0.211	--
-3.262	-1.233	0.984	0.259	-0.014
-2.462	-0.931	0.403	0.335	--
-1.662	-0.628	0.331	0.443	-0.172
-0.862	-0.326	0.270	0.580	--
-0.462	-0.175	0.245	0.655	-0.248
-0.062	-0.023	0.223	0.728	-0.252
0.738	0.279	0.190	0.856	--
1.538	0.581	0.168	0.941	-0.203
2.338	0.884	0.155	0.981	--
3.138	1.186	0.198	0.996	-0.148
3.538	1.337	0.146	0.998	--
4.338	1.640	0.144	1.000	-0.143

B3. Calculated Constants

Table B3.1, Sc = 1.0

Γ	λ	α	β	η_o	ξ_o
-0.2	1.0	0.71613	--	-0.71613	-0.25344
-1/3	1/3	0.46572	0.23299	-0.46581	-0.20197
-1/3	0.6	0.70102	0.45902	-0.82713	-0.32697
-1/3	0.8	0.81388	0.67465	-0.94054	-0.35055
-0.4	1.0	0.97986	--	-0.97986	-0.34648
-0.6	1/3	0.71372	0.13329	-0.49015	-0.21253
-0.6	0.6	1.11705	0.28007	-1.11705	-0.44183
-0.6	0.8	1.31924	0.44697	-1.44646	-0.53910
-0.6	1.0	1.41089	--	-1.41089	-0.49889
-0.75	1/3	0.95102	0.08065	-0.44897	-0.19446
-0.75	0.6	1.55956	0.17502	-1.29797	-0.51309
-0.75	0.75	1.82426	0.26044	-1.82411	-0.68951
-0.8	1.0	2.01403	--	-2.01403	-0.71216
		Sc = 0.75			
-0.6	1/3	0.50299	0.20387	-0.45537	-0.18745
-0.6	0.6	0.75568	0.40829	-0.93118	-0.36809
-0.6	0.8	0.87141	0.61171	-1.08948	-0.40605
-0.75	0.75	1.04683	0.72331	-1.54893	-0.58548
		Sc = 0.5			
-0.6	1/3	0.38069	0.28055	-0.47090	-0.20418
-0.6	0.6	0.56176	0.53789	-0.87972	-0.34775
-0.6	0.8	0.64492	0.76907	-0.94030	-0.35045
-0.75	0.75	0.69503	0.64307	-1.17084	-0.44258

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