SOME THEOREMS ON STABILITY OF DISCRETE CIRCULATORY SYSTEMS

by

Shyam N. Prasad*

Assistant Professor
Department of Civil Engineering
University of Mississippi
University, Mississippi

and

George Herrmann

Professor
Department of Civil Engineering
Northwestern University
Evanston, Illinois

This work was supported by the National Aeronautics and Space Administration under Grants NsG 605 and NGR 14-007-067.

---

* Formerly, Research Engineer, Department of Civil Engineering, Northwestern University, Evanston, Illinois.

Abstract

Several theorems pertaining to the stability of discrete, linear, circulatory (i.e., nonconservative) systems are established. One theorem states the condition under which static loss of stability cannot occur. The other theorems are associated with the destabilizing effect of velocity-dependent forces. The usefulness of the new theorems in stability analysis is indicated.

Introduction

In the recent past, increased attention has been paid by numerous investigators to the problem of stability of equilibrium of circulatory (i.e., nonconservative) elastic systems, as evidenced in a survey article [1].* The role played by velocity-dependent forces in such problems has been recognized to be especially intriguing. Ziegler [2] was first to indicate that linear viscous damping may have a destabilizing effect in such systems, i.e., the critical value of the parameter associated with the externally applied circulatory loading when even slight damping is present may be smaller than the corresponding value obtained in the absence of any damping. This discovery supplied the impetus for further studies of this effect [3-13], but it appears that certain of its features can be brought into a still broader framework.

In the present study we establish several stability theorems pertaining to a discrete, linear, elastic system with N degrees of freedom for static loss of stability (divergence) and for dynamic loss of stability (flutter) in the presence of sufficiently small velocity-dependent forces of any physical origin. The stability theorems, then, are shown to lead to several conclusions of some generality. In particular, the existence of the destabilizing effect of all sufficiently small velocity-dependent forces is brought into sharper focus, thus extending the results of Nemat-Nasser and Herrmann [8].

The new theorems contain considerable specific information concerning the behavior of systems under study and are thus of more immediate value

---

* Numbers in brackets designate References at end of paper.

than the general Routh-Hurwitz criterion.

The System

In the following, a holonomic, autonomous, linear, dynamic system with N degrees of freedom is described by generalized coordinates q_j and generalized velocities $\dot{q}_j = dq_j/dt$, $j = 1,2,\ldots N$. The system is subjected to a set of generalized forces, $Q_j = Q_j(F)$, $j = 1,2,\ldots N$, which are defined as functions of a real, finite parameter F. This parameter ($0 \leq F < \infty$) is associated with the magnitude of the externally applied loadings; $Q_j = 0$ for $F = 0$. Let

$$q_j = \dot{q}_j = 0, \; j = 1,2,\ldots N$$

be the equilibrium state of the system. The kinetic energy T and the potential (strain) energy V, which are assumed to be positive definite, are given by the following bilinear expressions:

$$T = \frac{1}{2} \sum_{j,k=1}^{N} M_{jk}\dot{q}_j\dot{q}_k, \qquad V = \frac{1}{2} \sum_{j,k=1}^{N} c_{jk}q_jq_k$$

The equations of motion are then given by

$$M_{jk}\ddot{q}_k + c_{jk}q_k = Q_j, \; j,k = 1,2,\ldots N \tag{1}$$

where the summation convention on all repeated indices is implied and will be employed in the sequel. The generalized forces Q_j are assumed to be linear, homogeneous functions of the generalized coordinates and are given by

$$Q_j = k_{jk}q_k, \; j,k = 1,2,\ldots N$$

where $[k_{jk}]$ is a nonsymmetric matrix which vanishes identically for $F = 0$.

If small velocity-dependent forces of order ϵ are present, the equations of motion (1) will become

$$M_{jk}\ddot{q}_k + \epsilon G_{jk}\dot{q}_k + c_{jk}q_k = Q_j, \quad j,k = 1,2,\ldots N \tag{2}$$

A Theorem on Static Loss of Stability (Divergence)

It is known that system (1) may lose stability by either flutter (oscillations with increasing amplitudes), or divergence (buckling: an adjacent equilibrium configuration is attained) [12]. In the present study we are primarily concerned with the effects of small velocity-dependent forces on the loss of stability by flutter, and may want to seek conditions which prevent divergence. In fact, for a very special class of the matrix $[k_{jk}]$ it is possible to show that buckling never occurs. For several problems in the field of aeroelasticity, especially flutter of elastic panels, the matrix $[k_{jk}]$ assumes a skew-symmetric form arising from the aerodynamic forces. For this particular system we may state the following theorem:

<u>Theorem I</u>. If the matrix $[k_{jk}]$ is skew-symmetric, system (1) does not lose stability by divergence.

<u>Proof</u>. The loss of stability by divergence is characterized by the vanishing of the following determinant [12]:

$$\Delta' = |c_{jk} - k_{jk}|$$

Clearly, then, if Δ' does not change sign the only possible static position is the initial configuration, $q_j \equiv 0$. The potential energy V was assumed to be positive-definite quadratic form in q_j so that $\det|c_{jk}|$ is always a positive quantity. We may also write

$$V = \frac{1}{2}\sum_{j,k=1}^{N} c_{jk}q_jq_k = \frac{1}{2}\sum_{j,k=1}^{N} c_{jk}q_jq_k - \frac{1}{2}\sum_{j,k=1}^{N} k_{jk}q_jq_k$$

because $k_{jk} = 0$, $j = k$ and $k_{jk} = -k_{kj}$, $j \neq k$. For V to be positive-definite, one of the requirements is

$$\Delta' = |c_{jk} - k_{jk}| > 0$$

Thus the determinant Δ' cannot vanish. This concludes the proof.

The result just established could be useful in the theory of determinants and may be generalized to take the following form:

"If a_{jk} are the real elements of a determinant such that there exists a positive-definite quadratic form $\sum_{j,k=1}^{N} a_{jk}x_j x_k$ (a_{jk} need not be symmetric), then $\det|a_{jk} + b_{jk}| > 0$, where b_{jk} are the real elements of any skew-symmetric determinant."

<u>Theorems on Dynamic Loss of Stability (Flutter)</u>

Let us assume solutions of (1) and (2) in the form $q_k = A_k e^{i\omega t}$, $i = (-1)^{\frac{1}{2}}$, and obtain the following:

$$\omega^2 M_{jk}A_k - (c_{jk} - k_{jk})A_k = 0 \tag{3}$$

$$\omega^2 M_{jk}A_k - i\epsilon\omega G_{jk}A_k - (c_{jk} - k_{jk})A_k = 0 \tag{4}$$

Equations (3) and (4) lead to the following frequency equations:

$$\det|a_{jk}| = \Delta(a_{jk}) = 0 \tag{5}$$

$$\det|a_{jk} - i\epsilon\omega G_{jk}| = 0 \tag{6}$$

where

$$a_{jk} = \omega^2 M_{jk} - c_{jk} + k_{jk}$$

without any restriction on the structure of k_{ij}.

For $F = 0$, i.e., $k_{ij} = 0$, equation (5) yields the natural frequencies of free vibration of the system which, in the following, are assumed to be distinct. As F increases from zero to a certain finite value, say F_e, equation (5) yields at least one double nonzero root. Such a possibility is indeed realizable under the condition of k_{jk} being nonsymmetric. We will not undertake the task of establishing necessary conditions, but merely assume that coalescence of two frequencies occurs such that beyond this value of $F = F_e$ equation (5) yields a pair of complex conjugate roots. Therefore, the system will oscillate with an exponentially increasing amplitude, i.e., flutter. We shall refer to F_e as the critical load for the system (1).

Equation (6) may be expanded in powers of ϵ as follows:

$$\Delta_1 = |a_{jk} - i\epsilon\omega G_{jk}|$$

$$= \Delta(a_{jk}) - (i\epsilon)\omega \frac{\partial\Delta}{\partial a_{jk}} G_{jk} + \frac{1}{2}(i\epsilon)^2\omega^2 \frac{\partial^2\Delta}{\partial a_{jk}\partial a_{\ell m}} G_{jk}G_{\ell m} + \ldots = 0$$

If, in the above expansion, we neglect terms containing $O(\epsilon^2)$ and higher, we are left with

$$\Delta_1 = \Delta(a_{jk}) - i\epsilon\omega \frac{\partial\Delta}{\partial a_{jk}} G_{jk} = 0 \tag{7}$$

which is an approximate form of the frequency equation of system (2).

Before we proceed further it is essential to estimate the roots of (7) in terms of the roots of (5). The determinant $\Delta(a_{jk})$ gives rise to a polynomial in ω^2 of degree N, and $\frac{\partial\Delta}{\partial a_{jk}} G_{jk}$ is a polynomial in ω^2 of degree N-1. Let us substitute the following for clarity of later exposition:

$$\Delta(a_{jk}) = P_N(\omega^2) = p_N\omega^{2N} + p_{N-1}\omega^{2(N-1)} + \ldots + p_0$$

$$\frac{\partial\Delta}{\partial a_{jk}} G_{jk} = R_{N-1}(\omega^2) = r_{N-1}\omega^{2(N-1)} + \ldots + r_0$$

and obtain from (7)

$$\Delta_1 = P_N(\omega^2) - i\epsilon\omega R_{N-1}(\omega^2) = 0 \tag{8}$$

There are several ways of estimating the roots of (8), but the method explained below has been found particularly useful; it is based on the following theorem:

<u>Theorem II</u>. Let α_j' be the roots of (8), and $\pm\alpha_j$, $j = 1,2,\ldots N$, those of $P_N(\omega^2) = 0$. Then, $\alpha_j' = \pm\alpha_j + i\beta_j$, where β_j is obtained from the following expression:

$$\beta_j = \frac{1}{2p_N} \frac{\epsilon R_{N-1}(\alpha_j^2)}{\prod_{\substack{k=1 \\ k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2)} \tag{9}$$

<u>Proof</u>. From the theory of equations we may write (8) as follows:

$$\Delta_1(\omega) = P_N(\omega^2) - i\epsilon\omega R_{N-1}(\omega^2) = p_N \prod_{k=1}^{2N} (\omega - \alpha_k') = 0 \tag{10}$$

If we substitute $\omega = \alpha_j$ in (10) and note that $\alpha_j' = \pm\alpha_j + i\beta_j$, we obtain

$$-i\epsilon\alpha_j R_{N-1}(\alpha_j^2) = -i\beta_j p_N \prod_{\substack{k=1 \\ k\neq j}}^{N} (\alpha_j - \alpha_k - i\beta_k) \prod_{k=1}^{N} (\alpha_j + \alpha_k - i\beta_k)$$

We also know that β_j are of the order of magnitude $O(\epsilon)$. Therefore, to be consistent, the above expression may be approximated as

$$\epsilon\alpha_j R_{N-1}(\alpha_j^2) = \beta_j(2\alpha_j)p_N \prod_{\substack{k=1 \\ k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2)$$

Had we substituted for $\omega = -\alpha_j$ in the above, we would have derived an identical result and, therefore

$$\beta_j = \frac{1}{2p_N} \frac{\epsilon R_{N-1}(\alpha_j^2)}{\prod_{\substack{k=1 \\ k \neq j}}^{N} (\alpha_j^2 - \alpha_k^2)}$$

It is also of interest to study higher order effects in ϵ, especially the quadratic terms. The above technique may be used to derive explicit expressions of any order of magnitude and we establish several such relations in the appendix.

For distinct, real roots of $P_N(\omega^2) = 0$, we note an important property of the expression for β_j. If we arrange α_j in descending order of magnitude, $\alpha_N > \alpha_{N-1} > \ldots \alpha_1 > 0$, we find that the denominator of the right-hand side of (9) alternates sign, starting with a positive quantity when j takes on the values of N, N-1, ... (p_N, being the discriminant of the kinetic energy, is a positive quantity.). $R_{N-1}(\omega^2) = 0$ will have N-1 roots $\pm \alpha_j''$, and if we assume that these roots are real and distinct such that $\alpha_{N-1}'' > \alpha_{N-2}'' > \ldots$ and the coefficient of the highest power in R_{N-1} is a positive quantity, then β_j obtained from (9) will always be positive if the inequality $\alpha_N > \alpha_{N-1}'' > \alpha_{N-1} > \ldots > \alpha_1'' > \alpha_1$ holds. For $\beta_j > 0$, we immediately conclude that system (2) is stable. It is also obvious that if any one of the above requirements is violated, at least one member of the set β_j will be a negative quantity. Therefore, the system will oscillate with increasing amplitude (flutter). The above is a set of necessary and sufficient conditions for the system to be stable, and, as we will discuss later, they are indeed fulfilled for certain types of the matrix $[G_{jk}]$. Hence, the following theorem may be stated:

<u>Theorem III</u>. For $F < F_e$, a necessary and sufficient condition for the system (2) to be stable is that the coefficient of the highest power of the polynomial R_{N-1} be positive and its roots separate those of P_N.

It may be worthwhile to point out the essential differences between the above stability theorem and the Routh-Hurwitz criterion. The Routh-Hurwitz criterion sets down the necessary and sufficient conditions that all the roots of a polynomial lie in the left half of the complex plane. These conditions have very complex dependence on the coefficients of the polynomial and thus are not capable of yielding any further information without carrying out extensive calculations. The readers are referred to the classical treatise by Routh [14] and a recent work by Gantmacher [15] for a comprehensive discussion of this criterion. By contrast, the stability theorem just established is sufficiently specific to permit one to draw several conclusions regarding the effect of velocity-dependent forces as illustrated below.

As a consequence of the established stability theorem, there arises an interesting special case. For $F \equiv 0$, the polynomial P_N, by virtue of the statement of the problem, possesses nonzero, distinct, real roots, and it is entirely possible that one or several of the requirements stated in Theorem III are violated by the polynomial R_{N-1}. A familiar example is that of a system with negative damping so that some of β_j obtained from (9) are negative quantities. In this special situation the critical load of the system (2) is zero.

If F_d is a nonvanishing value of the critical load of system (2), then we may state the following corollary:

<u>Corollary</u>. For all sufficiently small velocity-dependent forces,

$$0 \leq F_d \leq F_e.$$

The proof of the above corollary is quite elementary if one bears in mind the fact that, by the requirement of Theorem III, every root of R_{N-1} is bounded by the two adjacent roots of P_N when $F = 0$. P_N, regarded as a function of F, yields at least two roots which approach each other, and, when $F = F_e$, complete coalescence occurs. In this interval, P_N and R_{N-1} will have at least one common root, say when $F = F_d$. As soon as $F > F_d$, therefore, at least one of β_j will be negative, which indicates oscillation with increasing amplitude. When $F > F_e$, equation (9) will yield complex numbers for β_j in conjugate pairs whose real parts will be negative quantities. This concludes the proof (we indeed included $F_d \equiv 0$ as a critical load, as discussed earlier.).

So far, we have not indicated the physical origin of velocity-dependent forces. In view of the fact that the theorems established above may have applications in various branches of engineering science, it is not desirable to assign any definite form to $[G_{jk}]$. In the field of elastic stability, possible origins may be those associated with viscous damping or gyroscopic effects. In the case of viscous damping the matrix $[G_{jk}]$ assumes a positive-definite form. For this case, as was shown by Routh [14], the requirements of the stability Theorem III are indeed met when $F = 0$, so that the inequality $\beta_j > 0$ is satisfied. The destabilizing effect of viscous damping, therefore, follows from the corollary stated above. This special result was first obtained by Nemat-Nasser and Herrmann [8].

Appendix

The determinant Δ_1, when expanded in powers of ϵ, may be written in the following form:

$$\Delta_1 = \Delta(a_{jk}) + (-i\epsilon)\omega \frac{\partial\Delta}{\partial a_{jk}} G_{jk} + \frac{1}{2}(-i\epsilon)^2\omega^2 \frac{\partial^2\Delta}{\partial a_{jk}\partial a_{\ell m}} G_{jk}G_{\ell m} + \ldots = 0$$

If we neglect orders of $O(\epsilon^3)$ and higher, the following approximate expression is obtained:

$$\Delta_1 = P_N(\omega^2) - i\epsilon\omega R_{N-1}(\omega^2) - \frac{1}{2}\epsilon^2\omega^2 S_{N-2}(\omega^2) = 0 \quad (11)$$

where, in addition to the designations defined before, the following substitution has been made:

$$\frac{\partial^2\Delta}{\partial a_{jk}\partial a_{\ell m}} G_{jk}G_{\ell m} = s_{N-2}\omega^{2(N-2)} + \ldots + s_0 = S_{N-2}(\omega^2)$$

Let α_j' be the roots of (11) and $\pm\alpha_j$, $j = 1,2,\ldots N$, those of $P_N(\omega^2) = 0$. Then we may write

$$\alpha_j' = \pm\alpha_j + i\epsilon\beta_j \pm \epsilon^2\theta_j$$

to the accuracy of ϵ^2. Note that β_j and θ_j will be of the same order of magnitude as α_j. To obtain expressions for β_j and θ_j we follow the same procedure as before, and we obtain

$$P_N(\omega^2) - i\epsilon\omega R_{N-1}(\omega^2) - \frac{1}{2}\epsilon^2\omega^2 S_{N-2}(\omega^2) = p_N \prod_{j=1}^{2N} (\omega - \alpha_j')$$

If, in the above, we let $\omega = \alpha_j$, we have

$$-i\epsilon\alpha_j R_{N-1}(\alpha_j^2) - \frac{1}{2}\epsilon^2\alpha_j^2 S_{N-2}(\alpha_j^2) = p_N \prod_{k=1}^{2N} (\alpha_j - \alpha_k')$$

The right-hand side of the above expression may, after some calculations, be written in the following form to the accuracy of ϵ^2:

$$\prod_{k=1}^{2N} (\alpha_j - \alpha_k') = - i\epsilon\beta_j(2\alpha_j) \prod_{\substack{k=1\\k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2)$$

$$- \epsilon^2\left\{4\alpha_j\beta_j \sum_{\substack{k=1\\k\neq j}}^{N} \beta_k \prod_{\substack{\ell=1\\\ell\neq k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2) + 2\alpha_j\theta_j \prod_{\substack{k=1\\k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2)\right\}$$

Therefore, equating the coefficients of like powers of ϵ, we have

$$\beta_j = \frac{1}{2p_N} \frac{R_{N-1}(\alpha_j^2)}{\prod_{\substack{k=1\\k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2)} \tag{12}$$

$$\theta_j = \frac{\frac{1}{2}\alpha_j S_{N-2}(\alpha_j^2) - 4p_N\beta_j \sum_{\substack{k=1\\k\neq j}}^{N} \beta_k \prod_{\substack{\ell=1\\\ell\neq k\neq j}}^{N} (\alpha_j^2 - \alpha_\ell^2)}{2p_N \prod_{\substack{k=1\\k\neq j}}^{N} (\alpha_j^2 - \alpha_k^2)} \tag{13}$$

Note that by substituting $\omega = -\alpha_j$ and carrying out the above procedure, the expression on the right-hand side of (13) will be the same, but with a negative sign in front of it.

References

1. Herrmann, G., "Stability of Equilibrium of Elastic Systems Subjected to Nonconservative Forces," Appl. Mech. Rev., 20, No. 2, 1967, pp. 103-108.

2. Ziegler, H., "Die Stabilitätskriterien der Elastomechanik," Ingen.-Arch., 20, 1, 1952, pp. 49-56.

3. Bolotin, V. V., Nonconservative Problems of the Theory of Elastic Stability, Moscow, 1961; English translation published by Pergamon Press, Inc., New York, New York, 1963.

4. Herrmann, G. and Jong, I. C., "On the Destabilizing Effect of Damping in Nonconservative Elastic Systems," J. Appl. Mech., 32, 3, 1965, pp. 592-597.

5. Herrmann, G. and Jong, I. C., "On Nonconservative Stability Problems of Elastic Systems with Slight Damping," J. Appl. Mech., 33, 1, 1966, pp. 125-133.

6. Bottema, O., "On the Stability of the Equilibrium of a Linear Mechanical System," ZAMP, 6, 2, 1955, pp. 97-104.

7. Leipholz, H., "Über den Einfluss der Dämpfung bei nichtkonservativen Stabilitätsproblemen elastischer Stäbe," Ingen.-Arch., 33, 5, 1964, pp. 308-321.

8. Nemat-Nasser, S. and Herrmann, G., "Some General Considerations Concerning the Destabilizing Effect in Nonconservative Systems," ZAMP, 17, 2, 1966, pp. 305-313; also Northwestern University Structural Mechanics Laboratory Technical Report No. 65-2, June 1965.

9. Nemat-Nasser, S., Prasad, S. N. and Herrmann, G., "Destabilizing Effect of Velocity-Dependent Forces in Nonconservative Continuous Systems," AIAA J., 4, 7, 1966, pp. 1276-1280; also Northwestern University Structural Mechanics Laboratory Technical Report No. 65-4, December 1965.

10. Prasad, S. N. and Herrmann, G., "Complex Treatment of a Class of Nonconservative Stability Problems," Northwestern University Structural Mechanics Laboratory Technical Report No. 67-1, January 1967. To be published.

11. Nemat-Nasser, S., "On the Stability of Equilibrium of Nonconservative Systems with Slight Damping," J. Appl. Mech., 34, 2, 1967, pp. 344-348.

12. Herrmann, G. and Nemat-Nasser, S., "Energy Considerations in the Analysis of Stability of Nonconservative Structural Systems," Proc. Int. Conf. on Dynamic Stability of Structures, ed. G. Herrmann, Pergamon Press, 1967, pp. 297-308; also Northwestern University Structural Mechanics Laboratory Technical Report No. 65-5, December 1965.

13. Prasad, S. N., Nemat-Nasser, S. and Herrmann, G., "Bending-Torsional Flutter of a Swept Wing in a High-Density, Low-Speed Flow," AIAA J., 5, 2, 1967, pp. 316-321.

14. Routh, E. J., Advanced Dynamics of a System of Rigid Bodies, Dover Publications, Inc., New York, New York, 1955, pp. 221-231, 238-239.

15. Gantmacher, F. R., Matrix Theory, Vols. I and II, Chelsea, New York, New York, 1959.