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THE STUDY OF PARAMETER OPTIMIZATION IN VEHICLE-BORNE TRACKING SYSTEMS

by

George D. Cole
Richard Jacob
J. C. Chang
R. J. Polge

Final Technical Report

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PART I

A. INTRODUCTION

Preceding "Parameter Optimization" reports have been primarily concerned with methods of determining orbital parameters from various types of observational data, relations between errors in the parameters so determined and errors in the observational data, and to an extent, with the nature of the observational errors themselves. We now turn our attention to the related, but essentially independent, problem of describing the state of our information about the dynamical variables of the orbiting body when the most probable values of the orbital parameters and the uncertainties in these values have been determined using one of the methods previously described.

Knowledge of the analytical dependence of the uncertainties in these variables may be put to a variety of applications. It allows us to find out at what points in an orbit a particular variable is known most accurately (or least accurately), and by inference, at what point or points (and what type of) additional measurements would be most advantageous. It also permits us to establish quite arbitrary criteria for discriminating between sets of parameters (For example, if two sets of orbit parameters describing the same orbit are obtained by different techniques, we may decide which set will allow the most accurate determination of position or velocity at a specific point in the orbit, or for which set the maximum error in position or velocity will be least). The results of such a study are virtually independent of the methods used in determining the orbital parameters, and are thus a generally applicable extension of all previous "Parameter Optimization" reports.

In order that results of such general validity might be obtained in a brief and easily utilized form, and to facilitate computation, the following simplifying assumptions have been made:

(1) Only elliptical orbits of low eccentricity are considered, and analysis is limited to variables in the plane of the orbit.

(2) Errors in orbital elements are assumed to be purely statistical in nature. The classical elements are, in fact, treated as uncorrelated random variables, following an approximately Gaussian distribution.

(3) All perturbations on the classical equations of motion, which produce secular variations in the parameters, are neglected in determining the dynamical propagation of these statistical errors.

(4) The duration of time for which the propagation of errors is followed is less than, or of the order of, the period of the motion. (This is in keeping with our omission of secular variations in orbital parameters).

Our discussion is divided into three parts. In Part A, we reiterate in concise form a part of the work done in previous reports which is directly related to our present purpose. In particular, we develop, in a form especially suitable for small eccentricity, expressions for the variances of the classical orbital parameters obtained from a single position-velocity measurement. In Part B, expressions are derived, under the previously mentioned assumptions, for the propagation of the variances in the dynamical variables as a function of the angular position of the orbiting body, the classical parameters of the ellipse and the variances in these parameters. The variations of errors from initial values are found to be functions of the eccentricity only. Simplified forms of these expressions, in which the relative values of the parameter variances are estimated from the results of Part A, are also given. Part C is devoted to a summary of qualitative results, including illustration in graphical form of the behavior of the error propagation functions for a few values of ratios of parameter variances, and suggestions for further extension of this approach.

B. DEFINITION OF THE CLASSICAL ORBITAL PARAMETERS, AND THEIR DETERMINATION FROM A SINGLE POSITION-VELOCITY MEASUREMENT.

The equations for a body moving in an elliptical orbit may be written in a number of equivalent forms. In our present treatment, we shall adhere to the use of a polar coordinate system, with the origin located at the focus approximately coincident with the center of force. For the four parameters required to describe the in-plane motion of such a body, we take the eccentricity "e," semi-latus rectum "p," perihelion angle θ_p, and time of perifocal passage t_p. The equations of motion then leave the form:

$$r = \frac{p}{1 + e\cos(\theta - \theta_p)} \tag{1}$$

$$t = t_p + \sqrt{\frac{p^3}{K}} \int_{\theta_p}^{\theta} \frac{d\theta'}{[1 + e\cos(\theta' - \theta_p)]^2} \tag{2}$$

where $K = G M_e$. The velocity components are consequently given by:

$$\dot{r} = V_r = \frac{pe\sin(\theta - \theta_p)\,\dot{\theta}}{[1 + e\cos(\theta - \theta_p)]^2}$$

$$V_r = \sqrt{\frac{K}{p}}\; e\sin(\theta - \theta_p) \tag{3}$$

$$r\dot{\theta} = V_\theta = \sqrt{\frac{K}{p}}\,[1 + e\cos(\theta - \theta_p)] \tag{4}$$

If measurements of r, θ, V_r, V_θ are made at the same time $(t = o)$, and give results r_o, θ_o, V_{r_o}, V_{θ_o}, the classical parameters may be determined as functions of these measured quantities from equations (1) → (4). The results are

$$e = \sqrt{1 - \frac{2r_oV_{\theta_o}^2}{K} + \left(1 + \frac{V_{r_o}^2}{V_{\theta_o}^2}\right)\left(\frac{r_oV_{\theta_o}^2}{K}\right)^2} \tag{5}$$

$$p = \frac{V_o^2V_{\theta_o}^2}{K} \tag{6}$$

$$\theta_p = \theta_o - \cos^{-1}\left[\frac{1}{e}\left(\frac{r_oV_{\theta_o}^2}{K} - 1\right)\right] \tag{7}$$

$$t_p = \frac{r_o^3V_{\theta_o}^3}{K^2}\int_{\theta_o}^{\theta_p}\frac{d\theta'}{[1 + e\cos(\theta' - \theta_p)]^2} \tag{8}$$

If the results of the measurements (V_o, θ_o, V_{r_o}, V_{θ_o}) are subject to uncorrelated random errors of known standard deviation (σr_o, $\sigma\theta_o$, σV_{r_o}, σV_{θ_o}) the variances of the parameters are given, to a good approximation, (if the standard deviations of the measurements are sufficiently small) by, e.g.,

$$\sigma e^2 = \left(\frac{\partial e}{\partial r_o}\right)^2 \sigma r_o^2 + \left(\frac{\partial e}{\partial \theta_o}\right)^2 \sigma\theta_o^2 + \left(\frac{\partial e}{\partial V_r}\right)^2 \sigma V_{r_o}^2 + \left(\frac{\partial e}{\partial V_{\theta_o}}\right)^2 \sigma V_{\theta_o}^2 \tag{9}$$

with similar expressions for the other parameters. Before deriving expressions for the necessary partial derivatives, it is convenient to introduce the assumption of small eccentricity in Equation (8), which then becomes, to order e^2;

$$t_p \approx \frac{r_o^3V_{\theta_o}^3}{K^2}\left[(\theta_p - \theta_o)\left(1 + \frac{3}{2}e^2\right) + 2e\sin(\theta_o - \theta_p) - \frac{3e^2}{4}\sin 2(\theta_o - \theta_p)\right]. \tag{10}$$

* Note: σe^2, σr_o^2, $\sigma\theta_o^2$, etc. are to be read $(\sigma e)^2$, $(\sigma r_o)^2$, $(\sigma\theta_o)^2$, etc. throughout this report.

The sixteen partial derivatives may then be written as functions of e and $(\theta_o - \theta_p)$, a form particularly suitable for the small e approximation, as (putting $\theta_o - \theta_p = \alpha_o$)

$$\frac{\partial e}{\partial r_o} = \frac{1}{r_o}\left[e + \cos\alpha_o\right]$$

$$\frac{\partial e}{\partial \theta_o} = 0$$

$$\frac{\partial e}{\partial V_{r_o}} = \frac{1}{V_{r_o}}\left[e \sin^2\alpha_o\right] \tag{11}$$

$$\frac{\partial e}{\partial V_{\theta_o}} = \frac{1}{V_{\theta_o}}\left[e + 2\cos\alpha_o + e\cos^2\alpha_o\right];$$

$$\frac{\partial \theta_p}{\partial r_o} = \frac{1}{r_o}\left[\left(\frac{1}{e}\right)\sin\alpha_o\right]$$

$$\frac{\partial \theta_p}{\partial \theta_o} = 1$$

$$\frac{\partial \theta_p}{\partial V_{\theta_o}} = \frac{1}{V_{\theta_o}}\left[\left(\frac{1}{e}\right)\sin\alpha_o\right]\left[2 + e\cos\alpha_o\right] \tag{12}$$

$$\frac{\partial \theta_p}{\partial V_{r_o}} = 0 \quad ;$$

$$\frac{\partial p}{\partial r_o} = \frac{2p}{r_o}$$

$$\frac{\partial p}{\partial \theta_o} = 0$$

$$\frac{\partial p}{\partial V_{r_o}} = 0 \qquad (13)$$

$$\frac{\partial p}{\partial V_{\theta_o}} = \frac{2p}{V_{\theta_o}} \; ;$$

$$\frac{\partial t_p}{\partial r_o} \approx \frac{1}{r_o} \sqrt{\frac{p^3}{K}} \left[-3\,\alpha_o + \frac{1}{e} \sin \alpha_o \right]$$

$$\frac{\partial t_p}{\partial \theta_o} \approx 0$$

$$\frac{\partial t_p}{\partial V_{r_o}} \approx \frac{1}{V_{r_o}} \sqrt{\frac{p^3}{K}} \left[2\, e \sin^3 \alpha_o \right] \qquad (14)$$

$$\frac{\partial t_p}{\partial V_{\theta_o}} \approx \frac{1}{V_{\theta_o}} \sqrt{\frac{p^3}{K}} \left[-3\,\alpha_o + \sin \alpha_o \cos \alpha_o + \frac{2}{e} \sin \alpha_o \right]$$

(All of the above expressions, with the exception of those for partials of t_p, are, to this point, exact, and involve no approximation.)

Utilization of Equations (11) - (14) in Equation (9), and in similar expressions for the other parameter variances, gives (omitting terms of order e^2)

$$\sigma e^2 \approx \left[\cos^2\alpha_o + 2\, e \cos\alpha_o\right] \frac{\sigma r_o^{\ 2}}{r_o^{\ 2}} + \left[e^2 \sin^4 \alpha_o\right] \frac{\sigma V_{r_o}^{\ 2}}{V_{r_o}^{\ 2}} +$$

$$+ \left[4 \cos^2 \alpha_o + 4\, e \cos\alpha_o + 4\, e \cos^2 \alpha_o\right] \frac{\sigma V_{\theta_o}^{\ 2}}{V_{\theta_o}^{\ 2}} \tag{15}$$

$$\sigma\theta_p^{\ 2} \approx \left[\frac{1}{e^2} \sin^2 \alpha_o\right] \frac{\sigma r_o^{\ 2}}{r_o^{\ 2}} + \sigma\theta_o^{\ 2}$$

$$+ \left[\frac{4}{e^2} \sin^2 \alpha_o + \frac{4}{e} \sin^2 \alpha_o \cos\alpha_o + \sin^2 \alpha_o \cos^2 \alpha_o\right] \frac{\sigma V_{\theta_o}^{\ 2}}{V_{\theta_o}^{\ 2}} \tag{16}$$

$$\sigma p^2 \approx \left[4p^2\right] \frac{\sigma r_o^{\ 2}}{r_o^{\ 2}} + \left[4_p^{\ 2}\right] \frac{\sigma V_{\theta_o}^{\ 2}}{V_{\theta_o}^{\ 2}} \tag{17}$$

$$\sigma t_p^{\ 2} \approx \left(\frac{p^3}{K}\right) \left\{\left[-\frac{6\alpha_o}{e} \sin\alpha_o + \frac{1}{e^2} \sin^2 \alpha_o\right] \left(\frac{\sigma r_o}{r_o}\right)^2\right. \tag{18}$$

$$\left. + \left[-\frac{12\alpha_o}{e} \sin\alpha_o + \frac{4}{e^2} \sin^2 \alpha_o\right] \left(\frac{\sigma V_{\theta_o}}{V_{\theta_o}}\right)^2 \right\}.$$

The parameter variances may thus, in principle, be determined, to the approximation considered, from Equations (15) → (18). However, detailed knowledge of the relative magnitudes of the percentage standard deviations of the measured quantities, and their functional variations with α_o is required. The immediate utility of these equations is that it becomes apparent that, for this type measurement, σe^2 and $\frac{\sigma p^2}{p^2}$ will be of about the same order of magnitude, while (for small e) $\sigma\theta_p^{\ 2}$, $\sigma t_p^{\ 2}$ will be about $(\frac{1}{e})^2$ larger than the former quantities. In our subsequent development, this result is assumed to be generally valid, even when other sets of measurements may be used in determining the orbital parameters.

C. DEVELOPMENT OF EXPRESSIONS FOR PROPAGATION OF VARIANCES OF DYNAMICAL COORDINATES

We have shown in part A how the classical orbital parameters (e, p, θ_p, t_p) may be determined from a particular set of measurements, and have indicated also how the accuracy of these parameters may be determined if the accuracy of the original measurements is known. We shall now consider the complimentary question: given the most probable orbital parameters, and associated standard deviations for each at time $t = 0$, what standard deviations should be associated with the dynamical variables at a later time? If the probability density functions for the orbital parameters themselves is approximately Gaussian, we will have, in analogy with (9),

$$\sigma r^2 = \left(\frac{\partial r}{\partial e}\right)^2 \sigma e^2 + \left(\frac{\partial r}{\partial p}\right)^2 \sigma p^2 + \left(\frac{\partial r}{\partial \theta_p}\right)^2 \sigma \theta_p^{\,2} + \left(\frac{\partial r}{\partial t_p}\right)^2 \sigma t_p^{\,2} \; . \tag{19}$$

(It is perhaps worth emphasizing that the σr_o of part A differs from the σr of Equation (19) in that the former refers to the probability distribution for errors associated with measurement of the independent variable r_o, while the latter relates to deviations in the calculated value of r at a specific time t to the true value of the radial position of the satellite at this same time.)

Again, equations similar to Eq. (19) will hold for $\sigma\theta^2$, $\sigma V_r^{\,2}$, $\sigma V_\theta^{\,2}$. If the equations of motion are written in the form of Eqs. (1) and (2), however, care must be exercised in obtaining the partial derivatives appropriate for the assumption of "t" as the independent variable. (One obvious alternative not considered in detail is to choose θ as the independent variable, in which case σt, the variance in the predicted time of arrival of the satellite at a given angle θ, would be calculated rather than $\sigma\theta$). The correct expression for the $\frac{\partial r}{\partial e}$ appearing in Eq. (19), for example, is

$$\begin{aligned} \frac{\partial r}{\partial e} &= \left[\frac{\partial r}{\partial e}\right]_\theta + \left[\frac{\partial r}{\partial \theta}\right]_e \left[\frac{\partial \theta}{\partial e}\right] , \\ \frac{\partial r}{\partial e} &= \left[\frac{\partial r}{\partial e}\right]_\theta - \left[\frac{\partial r}{\partial \theta}\right]_e \left(\frac{\partial t}{\partial e}\right)\left(\frac{d\theta}{dt}\right). \end{aligned} \tag{20}$$

Similarly,

$$\frac{\partial r}{\partial \theta_p} = \left[\frac{\partial r}{\partial \theta_p} \right]_\theta + \left[\frac{\partial r}{\partial \theta} \right]_{\theta_p} \left[\frac{\partial \theta}{\partial \theta_p} \right]$$

$$= \left[\frac{\partial r}{\partial \theta_p} \right]_\theta + \left[\frac{\partial r}{\partial \theta} \right]_{\theta_p}$$

$$\frac{\partial r}{\partial \theta_p} = 0 \tag{21}$$

Generally speaking, the expressions obtained from differentiation of Eqs. (1) → (4) are quite complicated; however, and it is convenient to introduce the requirement of small eccentricity to simpify Eqs. (1) and (2) before the partial derivatives are taken. The results are (to order e^2)

$$r \approx p\left[1 - e \cos\alpha + \frac{e^2}{2} \cos^2 \alpha \right] \tag{22}$$

$$t \approx t_p + \sqrt{\frac{p^3}{K}} \left[\alpha \left(1 + \frac{3e^2}{2} \right) - 2 e \sin\alpha + \frac{3e^2}{4} \sin 2\alpha \right] \tag{23}$$

(where $\alpha = \theta - \theta_p$).

The sixteen partial derivatives required are then

$$\frac{\partial r}{\partial e} \approx -p\left[\cos\alpha - 2e + 3e^2\alpha\sin\alpha\right]$$

$$\frac{\partial r}{\partial p} \approx 1 - e\cos\alpha - \frac{3e\alpha}{2}\sin\alpha$$

$$\frac{\partial r}{\partial t_p} = -\sqrt{\frac{K}{p}}\, e\sin\alpha \tag{24}$$

$$\frac{\partial r}{\partial \theta_p} = 0$$

$$\frac{\partial \theta}{\partial e} \approx 2\sin\alpha - 3e\alpha + e\sin\alpha\cos\alpha - 6e^2\alpha\cos\alpha$$

$$\frac{\partial \theta}{\partial p} \approx -\left(\frac{3}{2p}\right)\left[\alpha + 2e\alpha\cos\alpha - 2e\sin\alpha\right]$$

$$\frac{\partial \theta}{\partial t_p} = -\sqrt{\frac{K}{p^3}}\left[1 + e\cos\alpha\right]^2 \tag{25}$$

$$\frac{\partial \theta}{\partial \theta_p} = 1$$

$$\frac{\partial V_r}{\partial e} \approx \sqrt{\frac{K}{p}} \left[\sin\alpha + 2\,e \sin\alpha\cos\alpha - 3\,e^2\alpha\cos\alpha \right]$$

$$\frac{\partial V_r}{\partial p} \approx -\frac{1}{2}\sqrt{\frac{K}{p^3}} \left[e\sin\alpha + 3\,e\alpha\cos\alpha + 6\,e^2\alpha\cos^2\alpha \right]$$

$$\frac{\partial V_r}{\partial t_p} = -\left(\frac{K}{p^2}\right) e\cos\alpha \left[1 + e\cos\alpha \right]^2 \tag{26}$$

$$\frac{\partial V_r}{\partial \theta_p} = 0$$

$$\frac{\partial V_\theta}{\partial e} \approx \sqrt{\frac{K}{p}} \left[\cos\alpha - 2\,e\sin^2\alpha + 3\,e^2\alpha\sin\alpha \right]$$

$$\frac{\partial V_\theta}{\partial p} \approx -\frac{1}{2}\sqrt{\frac{K}{p^3}} \left[1 + e\cos\alpha - 3\,e\alpha\sin\alpha - 6\,e^2\alpha\sin\alpha\cos\alpha \right]$$

$$\frac{\partial V_\theta}{\partial t_p} = \left(\frac{K}{p^2}\right) e\sin\alpha \left[1 + e\cos\alpha \right]^2 \tag{27}$$

$$\frac{\partial V_\theta}{\partial \theta_p} = 0$$

(Terms in e^2 have been omitted from the above expressions, except when e^2 occurs as a multiplier of a term linear in α; this is consistent with the fact that, by carrying terms through order e^2 in our expression for t, $\frac{\partial t}{\partial e}$, and hence all partials with respect to e, will be good to order e only.)

The variances computed from Eqs. (24) → (27), and from Eq. (9) and its counterparts for the other variables are (putting $e\sqrt{\frac{p^3}{K}}\,\sigma t = \sigma\tau$)

$$\frac{1}{p^2}\sigma r^2 \approx \left[\left(1-\frac{3e}{2}\alpha\sin\alpha\right)^2 - 2e\cos\alpha + 3e^2\sin\alpha\cos\alpha\right]\left(\frac{\sigma p}{p}\right)^2 +$$

$$+\left[\cos^2\alpha + 6e^2\alpha\sin\alpha\cos\alpha - 4e\cos\alpha\right]\sigma e^2$$

$$+\left[\sin^2\alpha\right]\sigma\tau^2 \tag{28}$$

$$\sigma\theta^2 \approx \left[\frac{9\alpha^2}{4} + 9e\alpha^2\cos\alpha\right]\left(\frac{\sigma p}{p}\right)^2 + \left[4\sin^2\alpha - 12e\alpha\sin\alpha\right]\sigma e^2 +$$

$$+\left[\frac{1}{e^2}+\frac{4}{e}\cos\alpha + 6\cos^2\alpha\right]\sigma\tau^2 + \frac{1}{e^2}\left(e\sigma\theta_p\right)^2, \tag{29}$$

$$\left(\frac{p}{K}\right) \sigma V_r^{\,2} = \left[\frac{3e^2}{2}\,\alpha \sin\alpha \cos\alpha + \frac{9e^2}{4}\,\alpha^2 \cos^2\alpha \right] \left(\frac{\sigma p}{p}\right)^2 +$$

$$+ \left[\sin^2\alpha + 4\,e \sin^2\alpha \cos\alpha - 6\,e^2 \alpha \cos\alpha \sin\alpha \right] \sigma e^2 +$$

$$+ \left[\cos^2\alpha + 4\,e \cos^3\alpha \right] \sigma\tau^2 \,, \tag{30}$$

$$\left(\frac{p}{K}\right) \sigma V_\theta^{\,2} = \frac{1}{4} \left[1 + 2\,e \cos\alpha - 6\,e\,\alpha \sin\alpha - 18\,e^2 \alpha \sin\alpha \cos\alpha + 9\,e^2 \alpha^2 \sin\alpha \right] \left(\frac{\sigma p}{p}\right)^2 +$$

$$+ \left[\cos^2\alpha + 6\,e^2 \alpha \sin\alpha \cos\alpha - 4\,e \sin^2\alpha \cos\alpha \right] \sigma e^2 +$$

$$+ \left[\sin^2\alpha + 4\,e \cos\alpha \sin^2\alpha \right] \sigma\tau^2 \,. \tag{31}$$

D. RESULTS AND COMMENTS

In Figures 1 → 4 and 1a → 4a, Eqs. (28) → (31) are plotted for eccentricities $e = 0.10$, 0.05, and 0.01, and for two different assumptions as to the relative magnitudes of the standard deviations in the orbital parameters. In obtaining Figures 1 → 4, we assume $\frac{\sigma p}{p} = \sigma e = e\sigma\theta_p = \sigma x$, while for Figures 1a → 4a, we again take $\sigma e = \sigma\tau = e\sigma\theta_p = \sigma x$, but $\frac{\sigma p}{p} = 2\sigma x$. These assumptions are suggested by Eqs. (15) → (18), and should be reasonably valid for illustrating some of the aspects of the behaviour of these functions.

A general feature of all the curves (except the σr^2 curve in Fig. 2) is a pronounced minimum somewhere in the second quadrant. The location of this minimum is plotted as a function of e for $\sigma\theta^2$ in Figures 5 and 5a.

For small eccentricity, $\sigma\theta$ is clearly an order magnitude greater than $\frac{\sigma r}{p}$ The magnitudes of the fluctuations in the variances for all the variables, and, for $\sigma\theta$, the magnitude of the variance at the peribelion as well, is strongly dependent on e. This fact suggests that an extension of this investigation to orbits of higher eccentricity might mean even more significant variations. The simplifications introduced for small e would, of course, then be invalid; however, the eccentric anomaly "E" as an independent parameter and expressing the equations of motion in the form $r = r(E)$, $\theta = \theta(E), t = t(E)$ leads to a vigorously valid expression for $t(E)$ in closed form. Through this approach, analysis valid for larger e and for longer periods of observation, but otherwise similar to that carried out here, might be possible.

Other possibilities for further investigation include extension of the analysis to three dimensions, and more sophisticated treatment of the parameter errors. (Utilizing other parameters, allowing non-gaussian distributions or correlations for the parameters, etc.) The former problem should be relatively straightforward, while the latter would necessarily involve more detailed knowledge of typical parameter uncertainties.

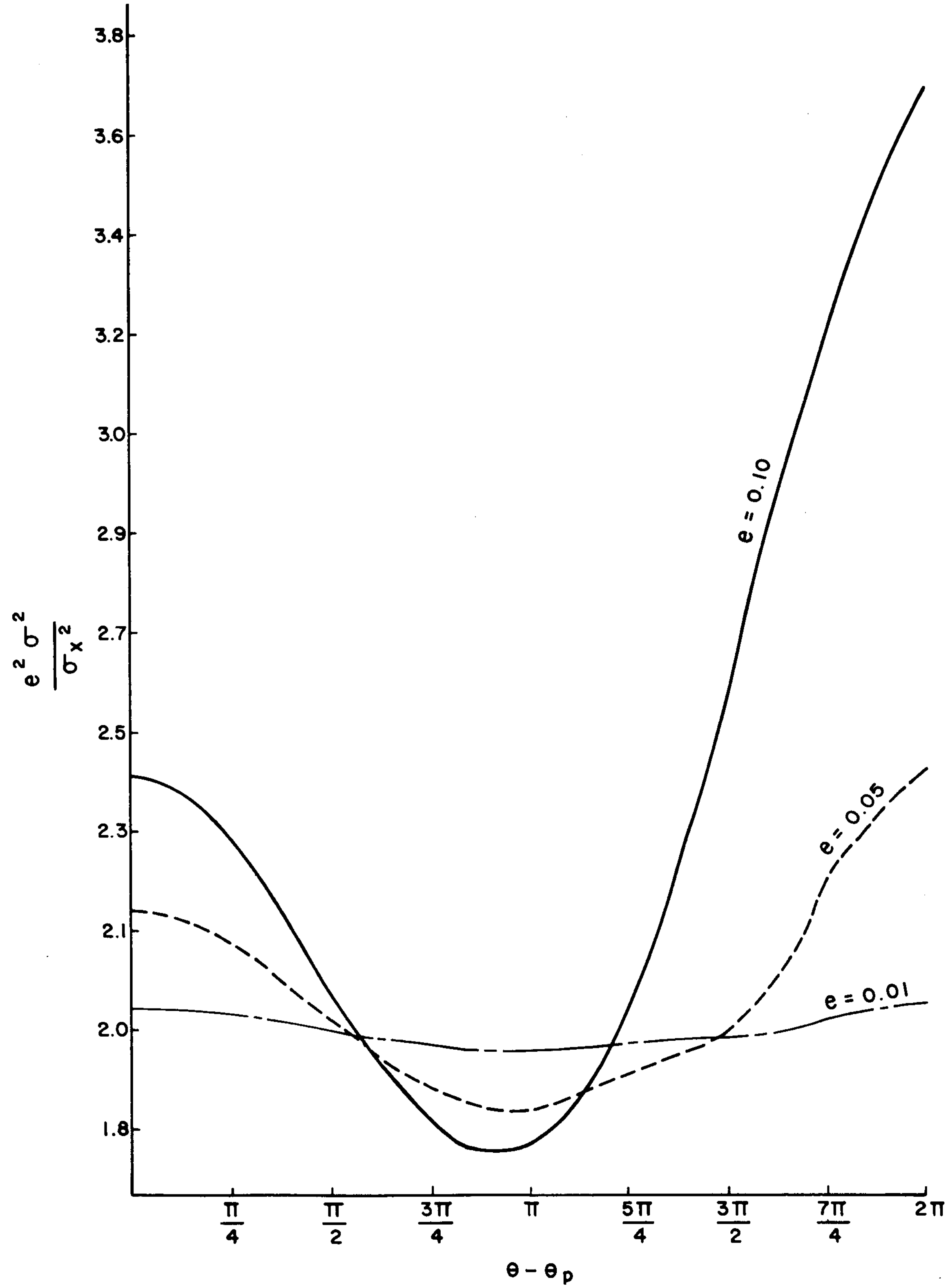

Figure 1: Propagation of $(\sigma_\theta)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma e)^2 = (\frac{\sigma p}{p})^2 = (\frac{\sigma p}{p})^2 = (e \sigma \theta_p)^2 = (\sigma \tau)^2 = (\sigma x)^2$).

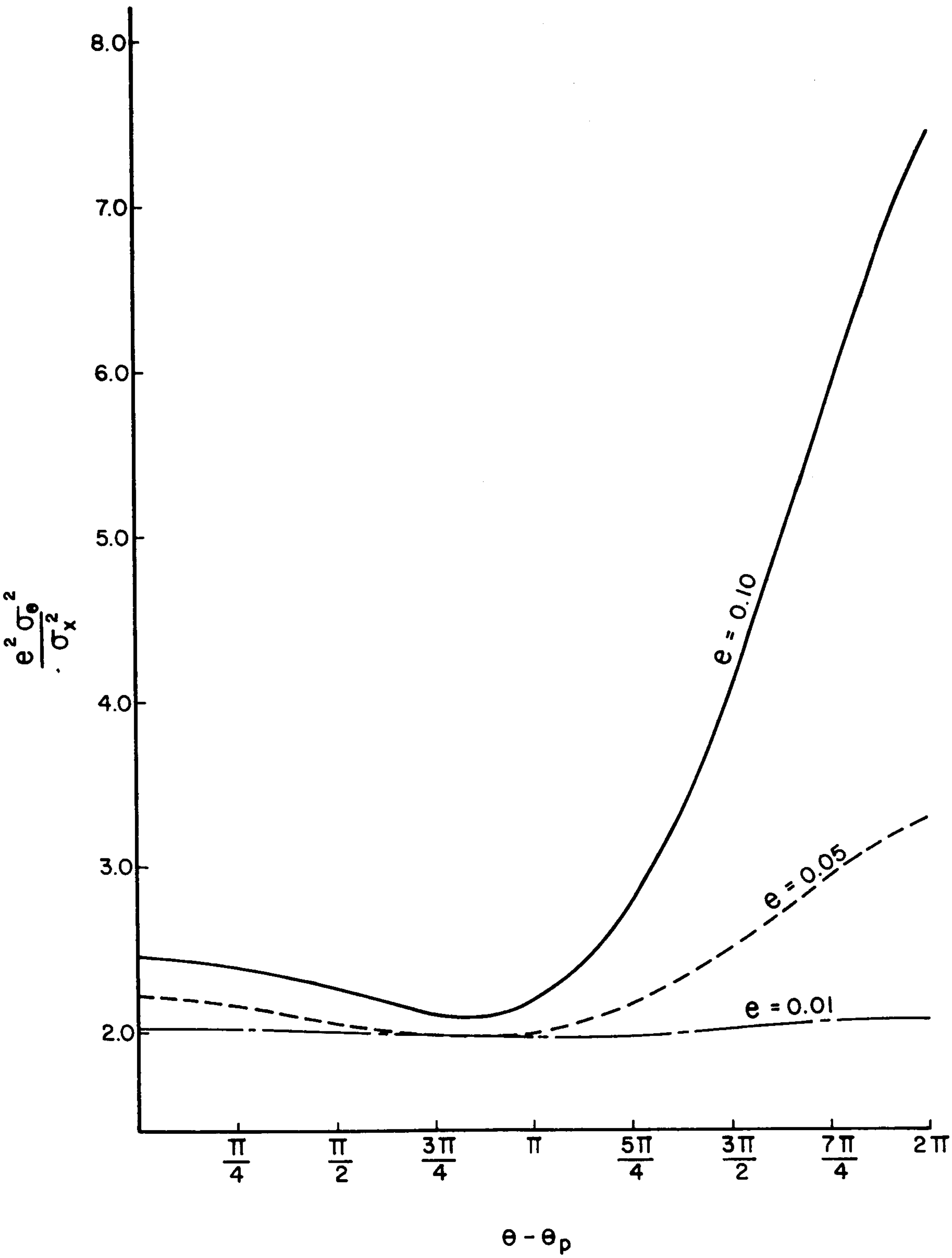

Figure 1a: Propagation of $(\sigma_\theta)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma x)^2 = (\sigma e)^2 = (\frac{\sigma p}{2p})^2 = (e\sigma\theta_p)^2 = (\sigma\tau)^2$).

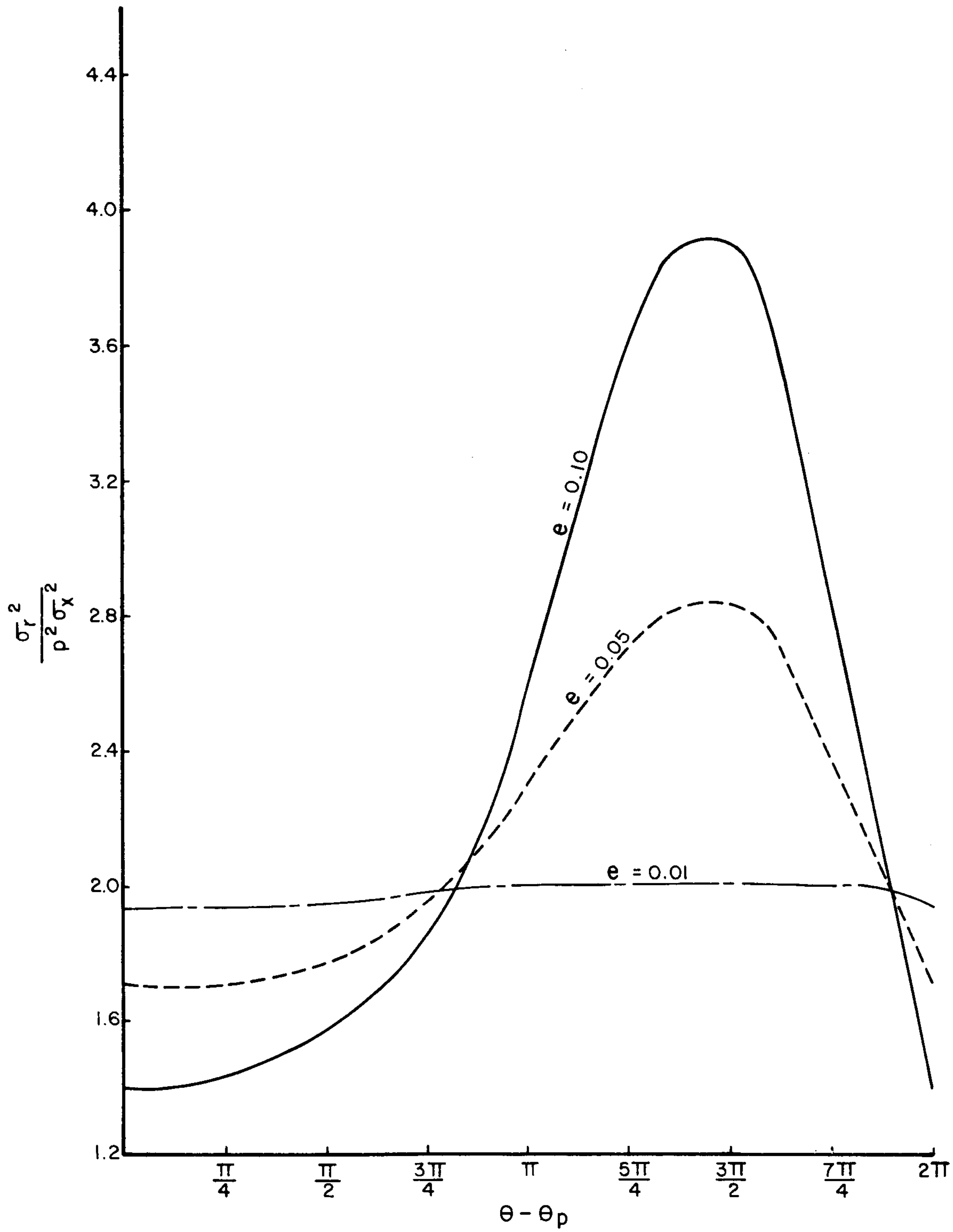

Figure 2: Propagation of $(\sigma r)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma e)^2 = (\frac{\sigma p}{p})^2 = (\frac{\sigma p}{p})^2 = (e\,\sigma\theta_p)^2 = (\sigma\tau)^2 = (\sigma x)^2$).

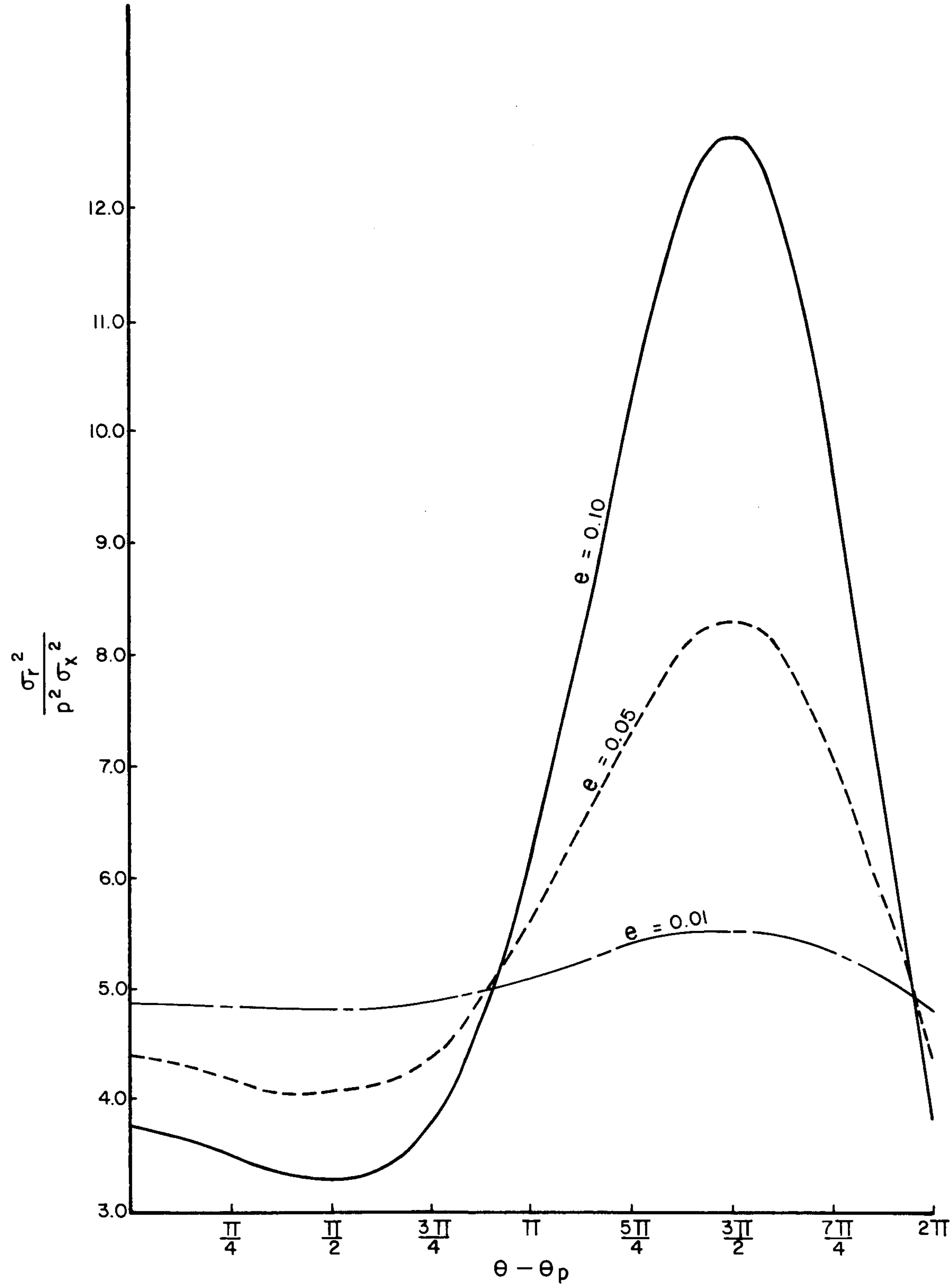

Figure 2a: Propagation of $(\sigma r)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma x)^2 = (\sigma e)^2 = (\frac{\sigma p}{2p})^2 = (e \sigma \theta_p)^2 = (\sigma \tau)^2$).

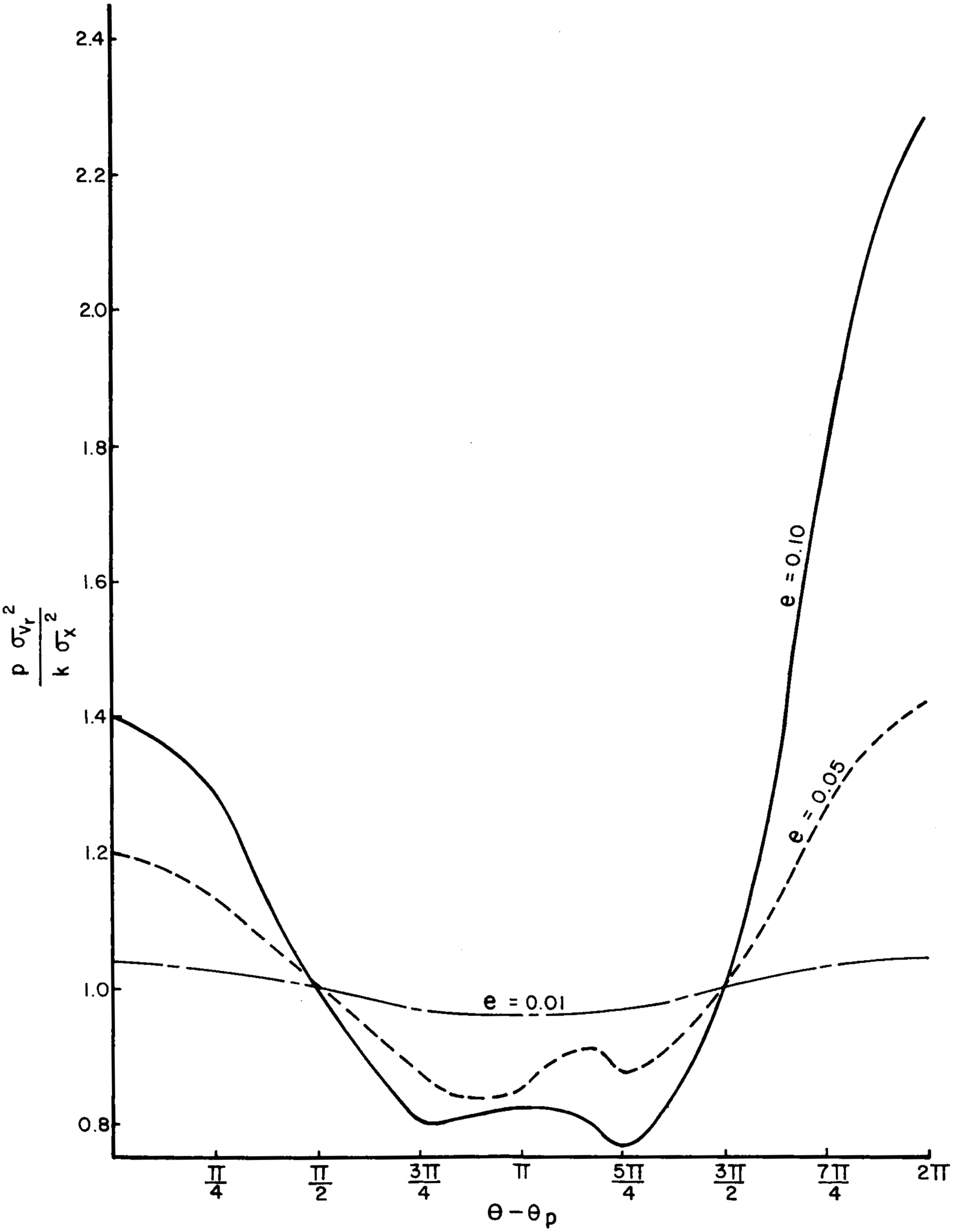

Figure 3: Propagation of $(\sigma v_r)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma e)^2 = (\frac{\sigma p}{p})^2 = (\frac{\sigma P}{P})^2 = (e\sigma\theta_p)^2 = (\sigma\tau)^2 = (\sigma x)^2$).

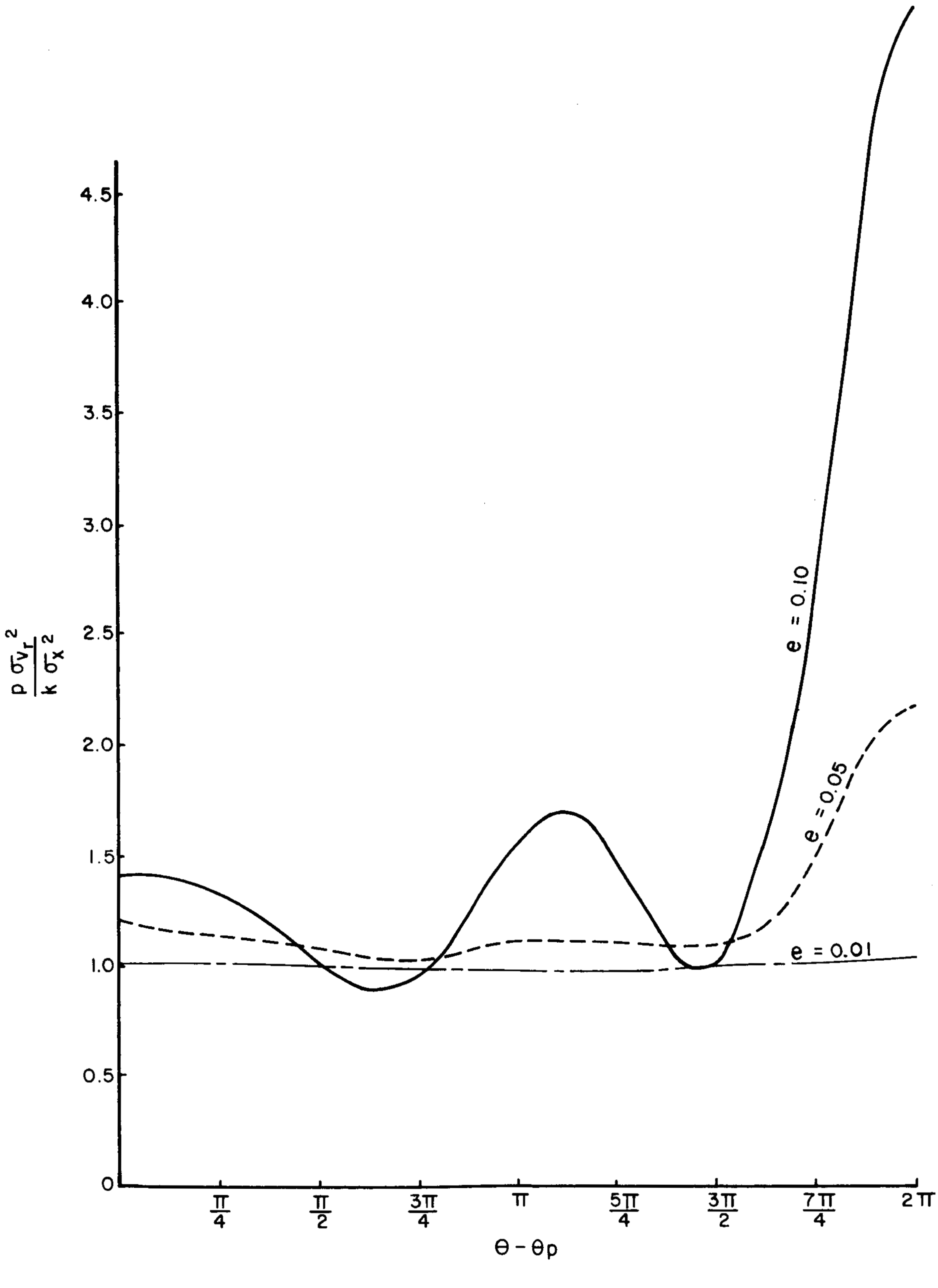

Figure 3a: Propagation of $(\sigma v_r)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma x)^2 = (\sigma e)^2 = (\frac{\sigma p}{2p})^2 = (e\,\sigma\theta_p)^2 = (\sigma\tau)^2$).

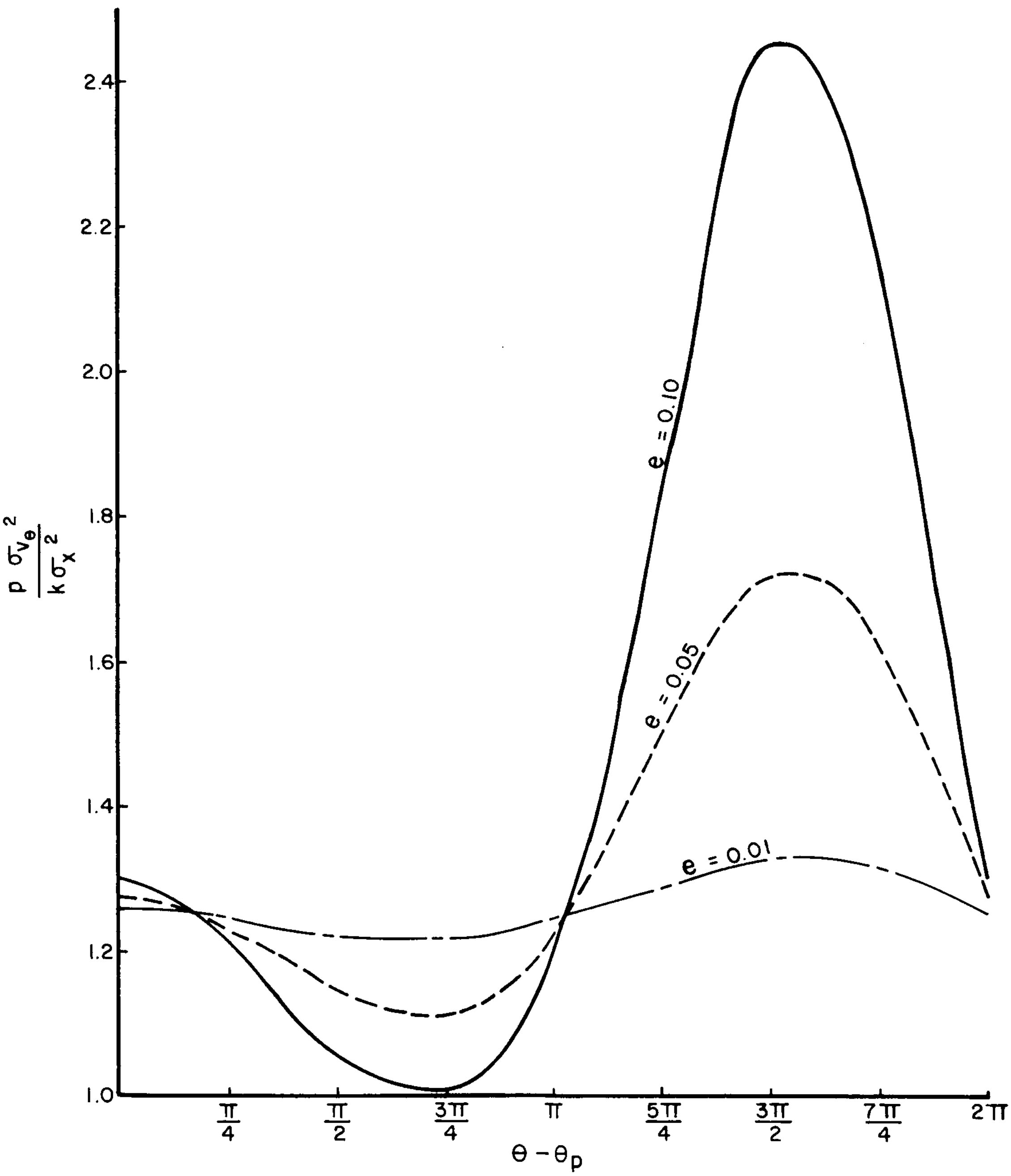

Figure 4: Propagation of $(\sigma v_\theta)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma e)^2 = (\frac{\sigma p}{p})^2 = (\frac{\sigma p}{p})^2 = (e\,\sigma\,\theta_p)^2 = (\sigma\tau)^2 = (\sigma x)^2$).

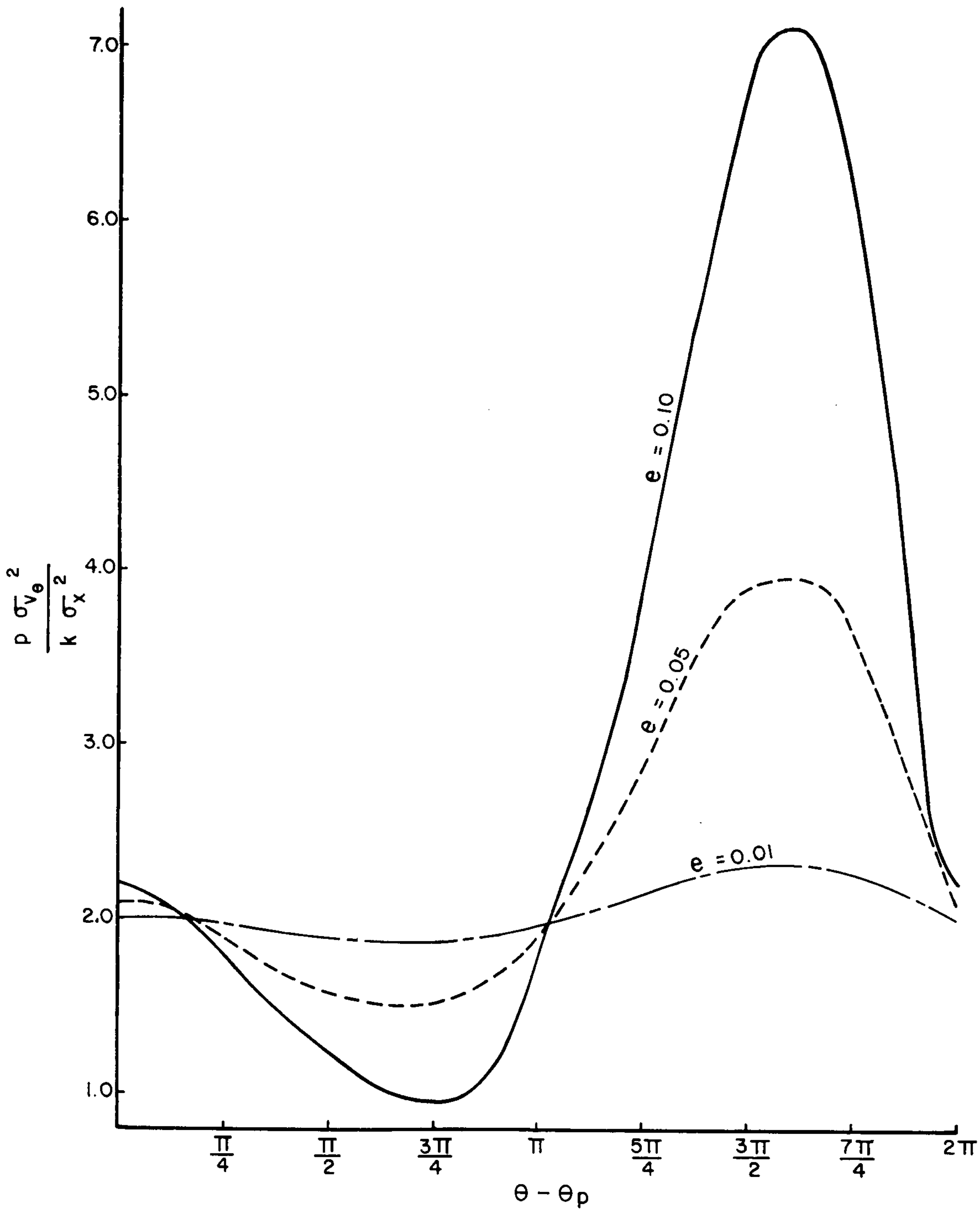

Figure 4a: Propagation of $(\sigma v_\theta)^2$ as a function of angular position of the orbiting body relative to perihelion angle θ_p (for $(\sigma x)^2 = (\sigma e)^2 = (\frac{\sigma p}{2p})^2 = (e \sigma \theta_p)^2 = (\sigma \tau)^2$).

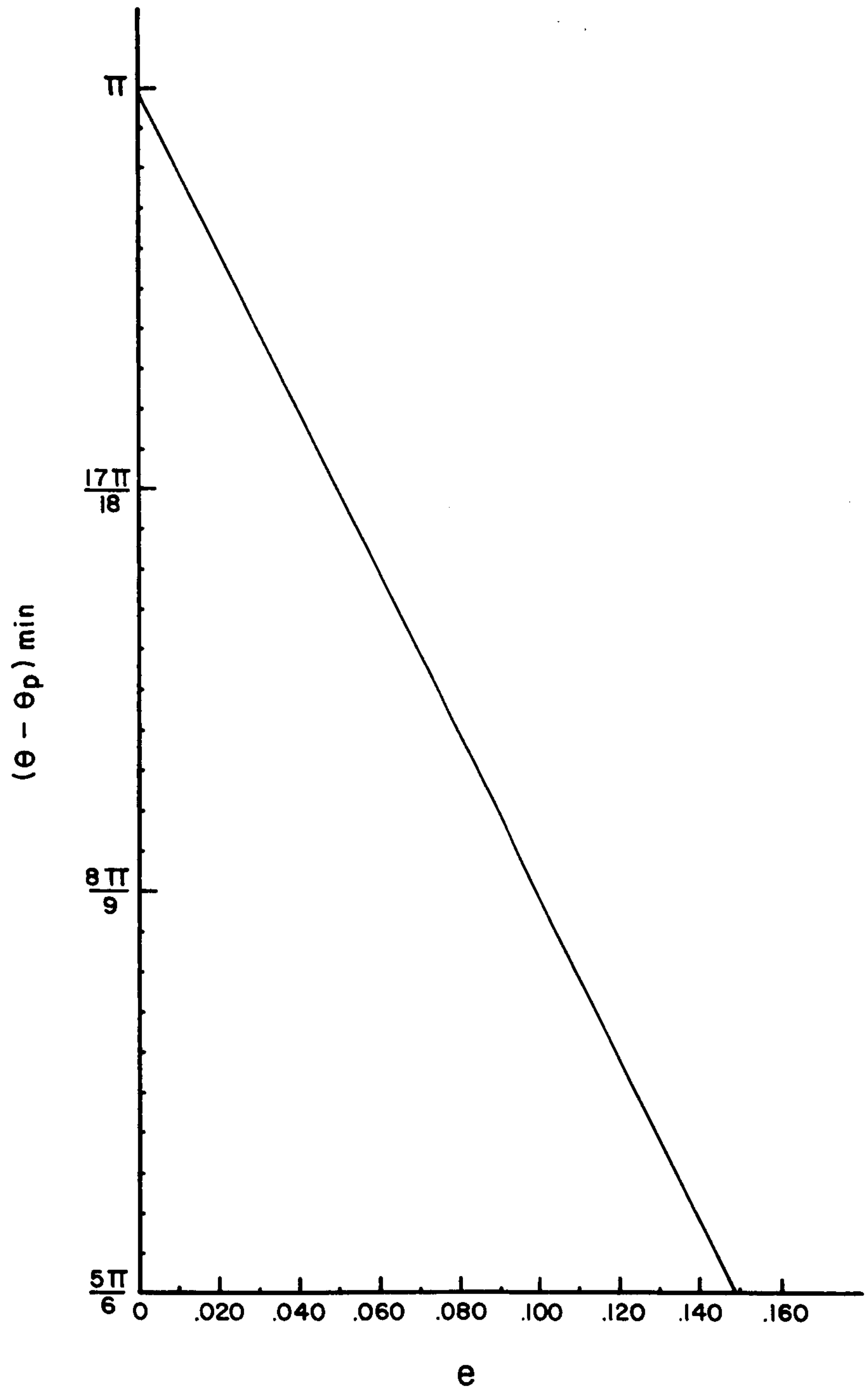

Figure 5: Angular location, α_{min}, of minimum in $(\sigma\theta)^2$ as a function of "e" for small "e" (for $(\sigma e)^2 = (\frac{\sigma p}{p})^2 = (e\,\sigma\,\theta_p)^2 = (\sigma\tau)^2 = (\sigma x)^2$).

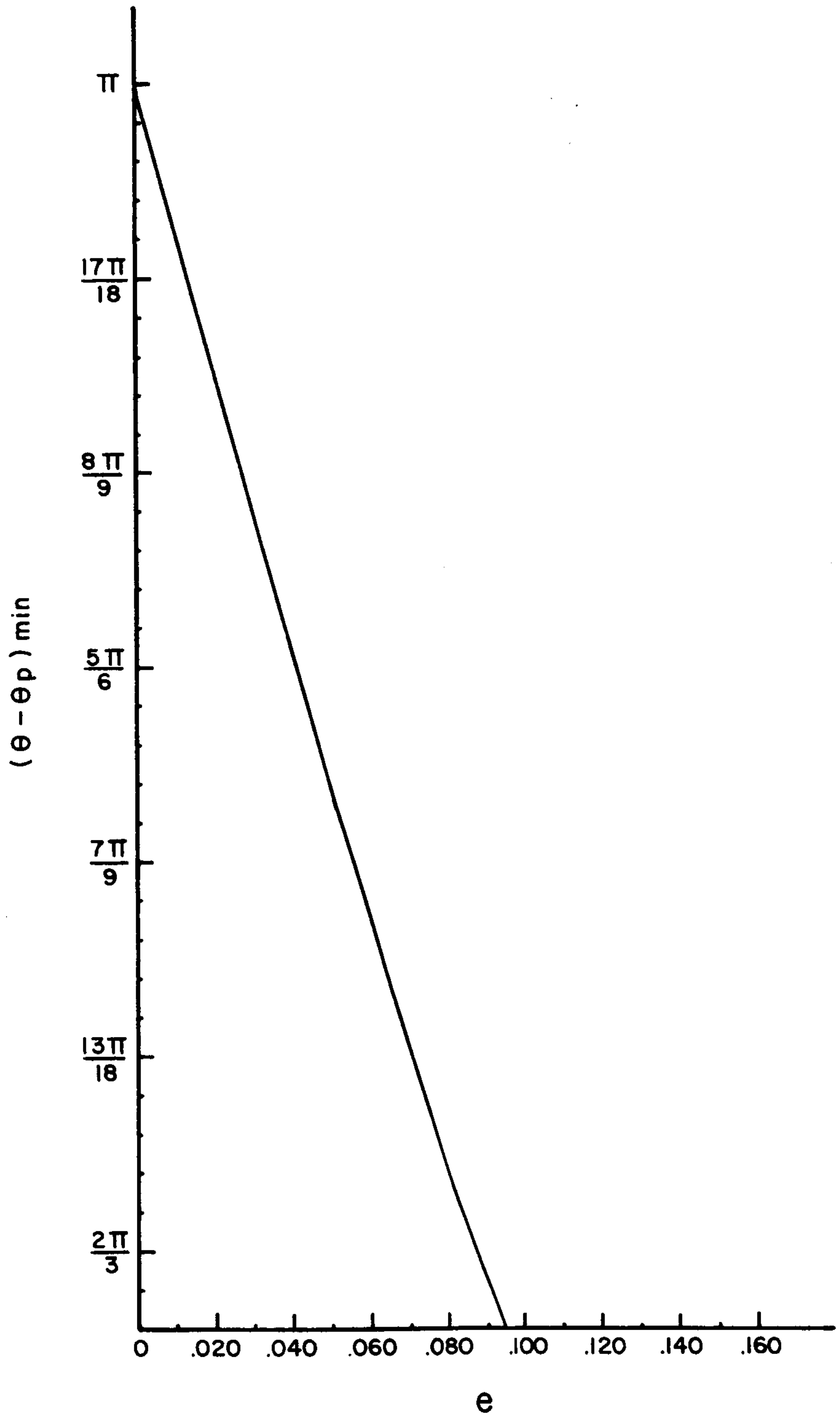

Figure 5a: Angular location, α_{min}, of minimum in $(\sigma\,\theta)^2$ as a function of "e" for small "e" (for $(\sigma e)^2 = (\frac{\sigma p}{2p})^2 = (e\,\sigma\,\theta_p)^2 = (\sigma\,\tau)^2 = (\sigma\,x)^2$).

PART II

A. INTRODUCTION

The object of this part of the study is to provide background information and procedures in the area of orbit determination from the data obtained by a radio tracking system to aid in defining the orbital parameters. The accuracy with which range and range-rate can be determined using specific station tracking systems has been studied in general. The following discussion is the methods and procedures for utilizing range and range-rate data from specific radio tracking systems to determine the desired orbital parameters.

The first approach of this study was the orbital parameters determination procedure by range measurement only. A systematic method of orbit determination which can correct the measurement error has been developed. Next we considered the orbital determination procedures using the data of range-rate measurement. It was found that for a minimum of three stations the orbital parameters cannot be determined by range-rate measurement alone.

The effectiveness of range-rate measurement for the orbit determination was found to be dependent upon the accuracy of range measurement. Finally, a general method of orbit determination is proposed by merely using four range-rate doppler tracking stations.

B. PROCEDURES OF ORBITAL PARAMETERS DETERMINATION BY RANGE MEASUREMENTS ONLY.

The range measurements r_s between a radio tracking station and a spaceship can be expressed in terms of a specific azimuth angle A and elevation angle E. The azimuth angle is the angle of the projection of r_s on the horizon plane of the tracking station measured with respect to the direction of south. The elevation angle is the angle between vector $\vec{r}$ and the horizon plane of the tracking station.

As shown in Figure 1, let us set up a rectangular coordinate system with the origin situated at the center of the tracking station, the $x_s - y_s$ plane coincides with the plane of the horizon and the x_s- axis points south. The relationship between x_s , y_s , z_s and r_s , E, A is:

$$
\begin{aligned}
x_s &= r_s \cos E \cos A \\
y_s &= r_s \cos E \sin A \\
z_s &= r_s \sin E
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
r_s &= \sqrt{x_s^2 + y_s^2 + z_s^2} \\
A &= \tan^{-1} \frac{y_s}{x_s} \\
E &= \tan^{-1} \frac{z_s}{\sqrt{x_s^2 + y_s^2}}
\end{aligned}
\tag{2}
$$

For a given measurement of range r_s with elevation angle E and azimuth angle A, we can find the corresponding x_s, y_s and z_s.

We now proceed to transform the rectangular station coordinate (x_s, y_s, z_s) into a geocentric coordinate system, (x_g, y_g, z_g). As shown in Figure 2, let the origin of the station coordinate system be a distance r_o from the geocenter. Furthermore, let the z_g-axis of the geocentric coordinate system coincide with the axis of rotation of the earth. The $x_g - y_g$ plane of the geocentric system coincides with the equatorial plane.

The transformation between x_s, y_s, z_s and x_g, y_g, z_g is

$$\begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} = M_1 \quad M_2 \quad M_3 \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix}, \tag{3}$$

where

$$M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \dfrac{z_s + r_o}{z_s} \end{bmatrix} \tag{4}$$

is the translation matrix.

$$M_2 = \begin{bmatrix} \cos(\frac{\pi}{2} - \beta) & 0 & \sin(\frac{\pi}{2} - \beta) \\ 0 & 1 & 0 \\ -\sin(\frac{\pi}{2} - \beta) & 0 & \cos(\frac{\pi}{2} - \beta) \end{bmatrix} = \begin{bmatrix} \sin\beta & 0 & \cos\beta \\ 0 & 1 & 0 \\ -\cos\beta & 0 & \sin\beta \end{bmatrix} \tag{5}$$

is the rotation matrix by keeping y_s unchanged and rotating the $x_s - z_s$ plane $(\frac{\pi}{2} - \beta)$ radians.

$$M_1 = \begin{bmatrix} \cos\lambda & -\sin\lambda & 0 \\ \sin\lambda & \cos\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6}$$

is the rotation matrix by keeping the z-axis stationary and rotating the x-y plane λ radians.

So we have:

$$\begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} x_s \\ y_s \\ z_s \end{bmatrix}, \tag{7}$$

where

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} = M_1\, M_2\, M_3$$

$$= \begin{bmatrix} \cos\lambda\sin\beta & -\sin\lambda & \cos\lambda\cos\beta\left(\frac{z_s + r_o}{z_s}\right) \\ \sin\lambda\sin\beta & \cos\lambda & \sin\lambda\cos\beta\left(\frac{z_s + r_o}{z_s}\right) \\ -\cos\beta & 0 & \sin\beta\left(\frac{z_s + r_o}{z_s}\right) \end{bmatrix}. \tag{8}$$

Therefore,

$$x_g = (\cos\lambda \ \ \sin\beta)\, x_s - (\sin\lambda)\, y_s + (\cos\lambda \cos\beta)\,(z_s + r_o)$$

$$y_g = (\sin\lambda \ \ \sin\beta)\, x_s + (\cos\lambda)\, y_s + (\sin\lambda \cos\beta)\,(z_s + r_o) \tag{9}$$

$$z_g = -(\cos\beta)\, x_s + (\sin\beta)\,(z_s + r_o) \quad .$$

In terms of the measured station parameters (range r_s, elevation angle E and azimuth angle A), the x_g, y_g, z_g coordinates for the geocentric system are:

$$x_g = r_s \cos E \ \cos A \ \cos\lambda \sin\beta - r_s \cos E \ \sin A \ \sin\lambda + (\cos\lambda \cos\beta)\,(r_s \sin E + r_o) \tag{10}$$

$$y_g = r_s \cos E \ \cos A \ \sin\lambda \sin\beta + r_s \cos E \ \sin A \cos\lambda + (\sin\lambda \cos\beta)\,(r_s \sin E + r_o) \tag{11}$$

$$z_g = - r_o \ \cos E \ \cos A \ \cos\beta + \sin\beta \ (r_s \sin E + r_o) \quad . \tag{12}$$

Now, consider the earth which is rotating with a constant angular velocity $\vec{\omega}$. As shown in Figure 3, take $\vec{\omega}$ in the same direction as the $+z_g$-axis. The relationship between the geocentric system (x_g, y_g, z_g) and the inertial geocentric system (x_i, y_i, z_i) is:

$$\begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_g \\ y_g \\ z_g \end{bmatrix} . \tag{13}$$

So we have:

$$\begin{aligned} x_i &= x_g \ \cos\omega t - y_g \ \sin\omega t \\ &= r_s \cos\omega t \ \ [\cos E \ \cos A \ \cos\lambda \sin\beta - \cos E \ \sin A \ \sin\lambda \\ &+ \cos\lambda \cos\beta \sin E \,] - r_s \sin\omega t \ \ [\cos E \ \cos A \ \sin\lambda \sin\beta \\ &+ \cos E \ \sin A \ \cos\lambda + \sin\lambda \cos\beta \ \sin E \,] \\ &+ r_o \ [\cos\lambda \ \cos\beta \ \cos\omega t - \sin\lambda \ \cos\beta \ \sin\omega t \,] \end{aligned} \tag{14}$$

$$y_i = x_g \sin \omega t + y_g \cos \omega t$$

$$= r_s \sin \omega t \; [\cos E \; \cos A \; \cos \lambda \; \sin \beta \; - \; \cos E \; \sin A \; \sin \lambda$$

$$+ \cos \lambda \; \cos \beta \; \sin E] \; + \; r_s \cos \omega t \; [\cos E \; \cos A \; \sin \lambda \; \sin \beta \tag{15}$$

$$+ \cos E \; \sin A \; \cos \lambda + \sin \lambda \cos \beta \; \sin E]$$

$$+ \; r_o \; [\cos \lambda \; \cos \beta \; \sin \omega t + \sin \lambda \; \cos \beta \; \cos \omega t]$$

$$z_i = z_g$$

$$= - r_o \cos E \; \cos A \; \cos \beta \; + r_s \sin \beta \; \sin E + r_o \sin \beta \tag{16}$$

x_i can be simplified as:

$$x_i = r_s \cos E \; \cos A \; \sin \beta \; [\cos \omega t \cos \lambda - \sin \omega t \sin \lambda]$$

$$- r_s \; \cos E \; \sin A \; [\cos \omega t \sin \lambda \; + \sin \omega t \; \cos \lambda]$$

$$+ r_s \; \cos \beta \sin E \; [\cos \omega t \; \cos \lambda - \sin \omega t \; \sin \lambda] \tag{17}$$

$$+ r_o \; \cos \beta \; [\cos \omega t \; \cos \lambda \; - \; \sin \omega t \; \sin \lambda]$$

$$= (r_s \; \cos E \; \cos A \; \sin \beta \; + r_s \cos \beta \sin E \; + r_o \; \cos \beta)$$

$$\cos (\omega t \; + \lambda) - r_s \cos E \; \sin A \; \sin (\omega t + \lambda) \; .$$

Likewise, y_i can be simplified as:

$$y_i = r_s \cos E \; \cos A \; \sin \beta \; [\sin \omega t \; \cos \lambda \; + \cos \omega t \; \sin \lambda]$$

$$+ r_s \cos E \; \sin A \; [\cos \omega t \; \cos \lambda - \sin \omega t \; \sin \lambda]$$

$$+ r_s \; \cos \beta \sin E \; [\sin \omega t \; \cos \lambda + \cos \omega t \; \sin \lambda] \tag{18}$$

$$+ r_o \; \cos \beta \; [\sin \omega t \; \cos \lambda + \cos \omega t \; \sin \lambda]$$

$$= (r_s \; \cos E \; \cos A \; \sin \beta + r_s \sin E \; \cos \beta + r_o \; \cos \beta)$$

$$\sin (\omega t + \lambda) + r_s \; \cos E \; \sin A \; \cos (\omega t + \lambda)$$

Therefore, the coordinates of the inertial geocentric system are:

$$x_i = (r_s \cos E \ \cos A \ \sin\beta + r_s \ \cos\beta \ \sin E + r_o \ \cos\beta) \cos(\omega t + \lambda) - r_s \cos E \ \sin A \ \sin(\omega t + \lambda) \tag{19}$$

$$y_i = (r_s \cos E \ \cos A \ \sin\beta + r_s \ \sin E \ \cos\beta + r_o \ \cos\beta) \sin(\omega t + \lambda) + r_s \ \cos E \ \sin A \ \cos(\omega t + \lambda) \tag{20}$$

$$z_i = r_s \ \sin\beta \sin E - r_o \ \cos E \ \cos A \ \cos\beta + r_o \sin\beta . \tag{21}$$

Now, let us find the intersecting angle ξ of the orbital plane and the inertial equatorial plane.

From each set of data (r_s, E, A) and information concerning the position (λ, β) of the tracking station, we can determine the corresponding position (x_i, y_i, z_i) of the space ship with respect to a certain time reference (t) of the geocentric inertial system.

Suppose the orbital plane ot the inertial geocentric system is:

$$a\,x_i + b\,y_i + c\,z_i = d . \tag{22}$$

With three sets of measured data, we can determine the corresponding three positions (x_{ij}, y_{ij}, z_{ij}, j = 1, 2, 3) of the space ship. So we have:

$$a\,x_{i1} + b\,y_{i1} + c\,z_{i1} = d \tag{23}$$

$$a\,x_{i2} + b\,y_{i2} + c\,z_{i2} = d \tag{24}$$

$$a\,x_{i3} + b\,y_{i3} + c\,z_{i3} = d . \tag{25}$$

Solving Eqs. (24), (25), and (26), we can find the coefficients a, b, and c expressed in terms of d. We have:

$$a = \frac{\begin{vmatrix} 1 & y_{i1} & z_{i1} \\ 1 & y_{i2} & z_{i2} \\ 1 & y_{i3} & z_{i3} \end{vmatrix}}{|M|} d \tag{26}$$

$$b = \frac{\begin{vmatrix} x_{i1} & 1 & z_{i1} \\ x_{i2} & 1 & z_{i2} \\ x_{i3} & 1 & z_{i3} \end{vmatrix}}{|M|} d \tag{27}$$

$$c = \frac{\begin{vmatrix} x_{i1} & y_{i1} & 1 \\ x_{i2} & y_{i2} & 1 \\ x_{i3} & y_{i3} & 1 \end{vmatrix}}{|M|} d , \tag{28}$$

$$\text{where} \quad \left| M \right| = \begin{vmatrix} x_{i1} & y_{i1} & z_{i1} \\ x_{i2} & y_{i2} & z_{i2} \\ x_{i3} & y_{i3} & z_{i3} \end{vmatrix} . \tag{29}$$

The orbital plane expressed in terms of the inertial geocentric data is:

$$\begin{aligned} &(y_{i2} z_{i3} - z_{i2} y_{i3} - y_{i1} z_{i3} + y_{i1} z_{i2} + z_{i1} y_{i3} - z_{i1} y_{i2}) \; x \\ &+ (x_{i1} z_{i3} - x_{i1} z_{i2} - x_{i2} z_{i3} + x_{i3} z_{i2} + z_{i1} x_{i2} - z_{i1} x_{i3}) \; y \\ &+ (x_{i1} y_{i2} - x_{i1} y_{i3} - y_{i1} x_{i2} + y_{i1} x_{i3} + x_{i2} y_{i3} - x_{i3} y_{i2}) \; z \\ &= x_{i1} y_{i2} z_{i3} - x_{i1} y_{i3} z_{i2} + x_{i2} y_{i3} z_{i1} - x_{i2} y_{i1} z_{i3} \\ &+ x_{i3} y_{i1} z_{i2} - x_{i3} y_{i2} z_{i1} \end{aligned} \tag{30}$$

Express the equation of the orbital plane as:

$$a_o x_i + b_o y_i + c_o z_i = d_o \quad , \tag{31}$$

where

$$a_o = (y_{i2} z_{i3} - z_{i2} y_{i3} - y_{i1} z_{i3} + y_{i1} z_{i2} + z_{i1} y_{i3} - z_{i1} y_{i2}) \tag{32}$$

$$b_o = x_{i1} z_{i3} - x_{i1} z_{i2} - x_{i2} z_{i3} + x_{i3} z_{i2} + z_{i1} x_{i2} - z_{i1} x_{i3} \tag{33}$$

$$c_o = x_{i1} y_{i2} - x_{i1} y_{i3} - y_{i1} x_{i2} + y_{i1} x_{i3} + x_{i2} y_{i3} - x_{i3} y_{i2} \tag{34}$$

$$d_o = x_{i1} y_{i2} z_{i3} - x_{i1} y_{i3} z_{i2} + x_{i2} y_{i3} z_{i1} - x_{i2} y_{i1} z_{i3}$$
$$+ x_{i3} y_{i1} z_{i2} - x_{i3} y_{i2} z_{i1} \quad . \tag{35}$$

The normal of the orbital plane can be expressed as:

$$\vec{n} = a_o \hat{x}_i + b_o \hat{y}_i + c_o \hat{z}_i \quad , \tag{36}$$

where $\hat{x}_i$, $\hat{y}_i$ and $\hat{z}_i$ represent the unit vectors in the directions of the inertial Cartesian coordinates.

To normalize $\vec{n}$ we have:

$$\hat{n} = \frac{a_o \hat{x}_i + b_o \hat{y}_i + c_o \hat{z}_i}{(a_o^2 + b_o^2 + c_o^2)^{1/2}} \quad . \tag{37}$$

We know that the unit normal vector of the equatorial plane can be expressed as:

$$\hat{n}_e = \hat{z}_i \quad . \tag{38}$$

As shown in Figure 4, the angle of intersection between the orbital plane and the equatorial plane (ξ) can be found from the following relation:

$$\hat{n} \cdot \hat{n}_e = \cos \xi = \frac{c_o}{(a_o^2 + b_o^2 + c_o^2)^{1/2}} \quad . \tag{39}$$

Therefore,

$$\xi = \cos^{-1} \frac{c_o}{(a_o^2 + b_o^2 + c_o^2)^{1/2}} \quad . \tag{40}$$

From Equations (19), (20), and (21), we can determine the position (x_i, y_i, z_i) of the spaceship with the data (r_s, E, A) furnished by a known station (λ, β). Using the same equations and a common time reference, we can determine three positions $(x_{ij}, y_{ij}, z_{ij},\ j = 1, 2, 3)$ of the spaceship at different times with the same or different known tracking stations. From the three sets of data $(x_{ij}, y_{ij}, z_{ij},\ j = 1, 2, 3)$, we can find the parameters a_o, b_o, c_o of the orbital plane by Equations (32), (33) and (34). Then, the intersecting angle ξ between the orbital plane and the equatorial plane can be determined by Equation (40). The direction $\hat{L}$ of the intersecting line of the two planes is:

$$\hat{L} = \hat{n}_e \times \hat{n}$$

$$= \frac{1}{(a_o^2 + b_o^2 + c_o^2)^{1/2}} \begin{vmatrix} \hat{x}_i & \hat{y}_i & \hat{z}_i \\ 0 & 0 & 1 \\ a_o & b_o & c_o \end{vmatrix}$$

$$= \frac{a_o \hat{y}_i - b_o \hat{x}_i}{(a_o^2 + b_o^2 + c_o^2)^{1/2}} \quad . \tag{41}$$

The angle η between the intersecting line $\hat{L}$ and the x_i-axis (See Figure 4.) is:

$$\eta = \tan^{-1} \left(\frac{b_o}{a_o}\right) \quad . \tag{42}$$

Now, we are ready to transform the data of the inertial geocentric system (x_i, y_i, z_i) into the orbital plane system (x_o, y_o, z_o). First, keep the z_i-axis fixed, then, turn the x_i-y_i plane η radians to coincide the x_i-axis with the intersecting line of the horizontal plane and the orbital plane. Let the coordinates of the orbital plane be x_o, y_o and z_o. Then, the transformed coordinates are:

$$\begin{bmatrix} x_o \\ y'_o \\ z_i \end{bmatrix} = \begin{bmatrix} \cos\eta & \sin\eta & 0 \\ -\sin\eta & \cos\eta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} . \tag{43}$$

Next, fix the x_o-axis and rotate the y'_o - z_i plane through an angle ξ. Then, the orbital coordinates become:

$$\begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\xi & \sin\xi \\ 0 & -\sin\xi & \cos\xi \end{bmatrix} \begin{bmatrix} x_o \\ y'_o \\ z_i \end{bmatrix} . \tag{44}$$

Substituting Equation (43) into Equation (44), we get:

$$\begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} = \begin{bmatrix} \cos\eta & \sin\eta & 0 \\ -\cos\xi\sin\eta & \cos\xi\cos\eta & \sin\xi \\ \sin\xi\sin\eta & -\sin\xi\cos\eta & \cos\xi \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} . \tag{45}$$

Therefore, the orbital coordinates are:

$$x_{oj} = (\cos\eta)\, x_{ij} + (\sin\eta)\, y_{ij} \tag{46}$$

$$y_{oj} = -(\cos\xi\sin\eta)\, x_{ij} + (\cos\xi\cos\eta)\, y_{ij} + (\sin\xi)\, z_{ij} \tag{47}$$

$$z_{oj} = (\sin\xi\sin\eta)\, x_{ij} - (\sin\xi\cos\eta)\, y_{ij} + (\cos\xi)\, z_{ij} , \tag{48}$$

where $j = 1, 2, 3$, and

$$x_{ij} = [r_s \cos E \cos A \sin \beta + r_s \cos \beta \sin E + r_o \cos \beta] \cos(\omega t_j + \lambda)$$
$$- r_s \cos E \sin A \sin(\omega t_j + \lambda) \tag{49}$$

$$y_{ij} = [r_s \cos E \cos A \sin \beta + r_s \sin E \cos \beta + r_o \cos \beta] \sin(\omega t_j + \lambda)$$
$$+ r_s \cos E \sin A \cos(\omega t_j + \lambda) \tag{50}$$

$$z_{ij} = r_s \sin \beta \sin E - r_o \cos E \cos A \cos \beta + r_o \sin \beta \ . \tag{51}$$

The transformed z_{0j} should be very small (if not zero) compared to the values of x_{0j} and y_{0j}. This fact suggests a method of checking the correctness of the measured data. In fact, if we are certain that only one of the parameters of the measured data of a specific station is in error, we can find the correct answer by minimizing z. Analytically minimizing z with respect to a certain parameter of a specific station may be difficult. But by using the modern high speed digital computer, it is easy to find the correct answer. Because, once one chooses the correct direction of changing the value of the specific parameter, the value of z will approach zero monotonically (to a certain degree of accuracy of course). The correct direction is not difficult to find. The degree of accuracy, of course, will be dependent upon the accuracy of the measured data and the station position accuracy. If we use the same three stations all the time for checking the orbit of a space vehicle, the station errors may be neglected. The timing error and the perturbations due to air drag and oblateness of earth, etc. have not been considered.

Now, we can proceed to use the orbital data (x_{0j}, y_{0j}) to determine the orbital parameters of the space vehicle. In the orbital plane ($x_0 - y_0$ plane), the equation of the orbit in polar coordinates is (See Figure 5):

$$r = \frac{p}{1 + e \cos(\theta - \theta_0)} \tag{52}$$

There are only three unknowns in Equation (52). Thus, the same three points used to determine the orbital plane can be used to determine the three unknowns e, p and θ_0.

The perigee of the observed space vehicle is at:

$$\theta_p = \theta_0 \tag{53}$$

$$r_p = \frac{p}{1 + e} \ . \tag{54}$$

The coordinates of apogee are:

$$\theta_a = (\pi + \theta_0) \tag{55}$$

$$r_a = \frac{p}{1 - e} \ . \tag{56}$$

For three sets of data (x_{01}, y_{01}), (x_{02}, y_{02}) and (x_{03}, y_{03}), we have:

$$r_i = (x_{0i}^2 + y_{0i}^2)^{1/2} \tag{57}$$

$$\theta_i = \tan^{-1}\left(\frac{y_{0i}}{x_{0i}}\right), \tag{58}$$

where $i = 1, 2, 3$.

Knowing (r_1, θ_1), (r_2, θ_2) and (r_3, θ_3), the angle θ_0 can be found by the following procedures:

Let,

$$r_i = \frac{p}{1 + e\cos(\theta_i - \theta_0)}, \quad i = 1, 2, 3. \tag{59}$$

Eliminating p we get the following two expressions:

$$r_1 [1 + e\cos(\theta_1 - \theta_0)] = r_2 [1 + e\cos(\theta_2 - \theta_0)] \tag{60}$$

$$r_1 [1 + e\cos(\theta_1 - \theta_0)] = r_3 [1 + e\cos(\theta_3 - \theta_0)] \tag{61}$$

By rearranging the above two equations, we get,

$$r_1 - r_2 = e\,[r_2\cos(\theta_2 - \theta_0) - r_1\cos(\theta_1 - \theta_0)] \tag{62}$$

$$r_1 - r_3 = e\,[r_3\cos(\theta_3 - \theta_0) - r_1\cos(\theta_1 - \theta_0)]. \tag{63}$$

By eliminating e we get,

$$\frac{r_1 - r_2}{r_1 - r_3} = \frac{[r_2\cos(\theta_2 - \theta_0) - r_1\cos(\theta_1 - \theta_0)]}{[r_3\cos(\theta_3 - \theta_0) - r_1\cos(\theta_1 - \theta_0)]}$$

$$= \frac{(r_2\cos\theta_2 - r_1\cos\theta_1)\cos\theta_0 + (r_2\sin\theta_2 - r_1\sin\theta_1)\sin\theta_0}{(r_3\cos\theta_3 - r_1\cos\theta_1)\cos\theta_0 + (r_3\sin\theta_3 - r_1\sin\theta_1)\sin\theta_0}. \tag{64}$$

By simplification:

$$\tan\theta_0 = \frac{[r_1(r_3\cos\theta_3 - r_2\cos\theta_2) + r_2(-r_3\cos\theta_3 + r_1\cos\theta_1) + r_3(r_2\cos\theta_2 - r_1\cos\theta_1)]}{[r_1(r_2\sin\theta_2 - r_3\sin\theta_3) + r_3(r_1\sin\theta_1 - r_2\sin\theta_2) + r_2(r_3\sin\theta_3 - r_1\sin\theta_1)]}. \tag{65}$$

Therefore,

$$\theta_0 = \tan^{-1}\left[\frac{r_1\,(r_3 \cos\theta_3 - r_2 \cos\theta_2) + r_2\,(r_1 \cos\theta_1 - r_3 \cos\theta_3) + r_3\,(r_2 \cos\theta_2 - r_1 \cos\theta_1)}{r_1\,(r_2 \sin\theta_2 - r_3 \sin\theta_3) + r_2\,(r_3 \sin\theta_3 - r_1 \sin\theta_1) + r_3\,(r_1 \sin\theta_1 - r_2 \sin\theta_2)}\right]. \quad (66)$$

Once θ_0 is found, the eccentricity can be found from the following equations:

$$e = \frac{r_1 - r_2}{[r_2 \cos(\theta_2 - \theta_0) - r_1 \cos(\theta_1 - \theta_0)]} \quad (67)$$

or

$$e = \frac{r_1 - r_3}{[r_3 \cos(\theta_3 - \theta_0) - r_1 \cos(\theta_1 - \theta_0)]}\,. \quad (68)$$

Finally the semi-latus rectum p of the orbit is:

$$p = r_i\,[1 + e \cos(\theta_i - \theta_0)] \quad (69)$$

where $i = 1, 2, 3.$

By measuring any three points of the orbit of the space vehicle, we can determine parameters p, e, and θ_0 from Equations (69), (68) and (66). The orbit is thus determined in the orbital plane. At the same time, the orientation of the orbital plane is also determined.

The perigee of the observed vehicle is:

$$r_p = \frac{r_i\,[1 + e \cos(\theta_i - \theta_0)]}{1 + e}, \quad i = 1, 2, 3. \quad (70)$$

The apogee is:

$$r_a = \frac{r_i\,[1 + e \cos(\theta_i - \theta_0)]}{1 - e}, \quad i = 1, 2, 3. \quad (71)$$

The parameters e and θ_0 can be found from previous equations.

C. ORBITAL PARAMETERS DETERMINATION BY RANGE AND RANGE RATE MEASUREMENTS.

Range rate measurements can give us the information concerning the speed of the space vehicle. With the knowledge of the correct speed of the space vehicle, the orbital parameters can then be determined. In order to obtain the correct information concerning the speed of the space vehicle, we have to know the relative position of the tracking station with respect to the orbital plane. As described in the previous chapter, the orbital plane can be determined from the range data obtained by the tracking stations. In other words, for a minimum of three tracking stations, the range rate information is useful only when the range information is available at the same time.

Suppose the coordinates of stations are (λ, β_i, r_o),

where $i = 1, 2, 3$

r_o = radius of earth

λ_i = geocentric station longitude

β_i = geocentric station latitude.

From Equation (9) of the previous chapter, the geocentric coordinates of the tracking stations are:

$$(x_g)_i = r_o \cos \lambda_i \cos \beta_i$$

$$(y_g)_i = r_o \sin \lambda_i \cos \beta_i$$

$$(z_g)_i = r_o \sin \beta_i \,. \tag{1}$$

Since

$$\begin{bmatrix} (x_i)_i \\ (y_i)_i \\ (z_i)_i \end{bmatrix} = \begin{bmatrix} \cos \omega t & -\sin \omega t & 0 \\ \sin \omega t & \cos \omega t & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} (x_g)_i \\ (y_g)_i \\ (z_g)_i \end{bmatrix}, \tag{2}$$

where $\vec{\omega}$ = constant angular velocity of the earth, then the coordinates of the tracking stations in inertial geocentric frame are:

$$(x_i)_j = r_o \cos \lambda_j \cos \beta_j \cos \omega t - r_o \sin \lambda_j \cos \beta_j \sin \omega t \tag{3}$$

$$(y_i)_j = r_o \cos \lambda_j \cos \beta_j \sin \omega t + r_o \sin \lambda_j \cos \beta_j \cos \omega t \tag{4}$$

$$(z_i)_j = r_o \sin \beta_j \quad . \tag{5}$$

We know that the vehicle is at position (x_{ij}, y_{ij}, z_{ij}). The line joining the station $[(x_i)_j, (y_i)_j, (z_i)_j]$ and the vehicle (x_{ij}, y_{ij}, z_{ij}) is:

$$\frac{x_i - x_{ij}}{x_{ij} - (x_i)_j} = \frac{y_i - y_{ij}}{y_{ij} - (y_i)_j} = \frac{z_i - z_{ij}}{z_{ij} - (z_i)_j} \quad . \tag{6}$$

Suppose the tangent line of the orbit at the point (x_{ij}, y_{ij}, z_{ij}) is:

$$\frac{x_i - x_{ij}}{m} = \frac{y_i - y_{ij}}{n} = \frac{z_i - z_{ij}}{l} \quad . \tag{7}$$

The condition that the line of Equation (7) is perpendicular to the radius vector extending from the origin to the space vehicle (x_{ij}, y_{ij}, z_{ij}) is:

$$m\, x_{ij} + n\, y_{ij} + l\, z_{ij} = 0 , \tag{8}$$

since the line of Equation (7) is also in the orbital plane,

$$a_o x_i + b_o y_i + c_o z_i = d_o \quad , \tag{9}$$

where

$$a_o = y_{i2} z_{i3} - z_{i2} y_{i3} + y_{i1} z_{i2} - z_{i3} y_{i1} + z_{i1} y_{i3} - z_{i1} y_{i2} \tag{10}$$

$$b_o = x_{i1} z_{i3} - x_{i1} z_{i2} + x_{i3} z_{i2} - x_{i2} z_{i3} + z_{i1} x_{i2} - z_{i1} x_{i3} \tag{11}$$

$$c_o = x_{i1} y_{i2} - x_{i1} y_{i3} + y_{i1} x_{i3} - y_{i1} x_{i2} + x_{i2} y_{i3} - x_{i3} y_{i2} \tag{12}$$

$$\begin{aligned} d_o = {} & x_{i1} y_{i2} z_{i3} - x_{i1} y_{i3} y_{i2} + x_{i2} y_{i3} z_{i1} - x_{i2} y_{i1} z_{i3} \\ & + x_{i3} y_{i1} z_{i2} - x_{i3} y_{i2} z_{i1} . \end{aligned} \tag{13}$$

From Equation (7), we can express y_i, z_i in terms of x_i and other parameters as follows:

$$y_i = \frac{n}{m} (x_i - x_{ij}) + y_{ij} \tag{14}$$

$$z_i = \frac{l}{m} (x_i - x_{ij}) + z_{ij} . \tag{15}$$

Substituting Equations (14) and (15) into Equation (9), we get:

$$a_o x_i + \frac{b_o n}{m} (x_i - x_{ij}) + b_o y_{ij} + \frac{c_o l}{m} (x_i - x_{ij}) + c_o z_{ij} = d_o . \tag{16}$$

The points (x_{ij}, y_{ij}, z_{ij}) are on the orbital plane. Therefore,

$$a_o x_{ij} + b_o y_{ij} + c_o z_{ij} = d_o \tag{17}$$

i.e.,

$$a_o x_{ij} = d_o - b_o y_{ij} - c_o z_{ij} . \tag{18}$$

Then, Equation (16) can be written as:

$$\begin{aligned} & a_o x_i + \frac{b_o n}{m} (x_i - x_{ij}) + \frac{c_o l}{m} (x_i - x_{ij}) \\ & = d_o - b_o y_{ij} - c_o z_{ij} = a_o x_{ij} . \end{aligned} \tag{19}$$

Equation (19) can be rearranged as:

$$(a_o + \frac{b_o n}{m} + \frac{c_o l}{m})(x_i - x_{ij}) = 0 \cdot \tag{20}$$

The term $(x_i - x_{ij})$ cannot always be zero. Therefore, we have:

$$a_o + \frac{b_o n}{m} + \frac{c_o l}{m} = 0. \tag{21}$$

That is:

$$a_o m + b_o n + c_o l = 0. \tag{22}$$

Multiply Equations (8) and (22) by a and x_{ij}, respectively, then take the difference. We get:

$$a_o n y_{ij} + a_o l z_{ij} - b_o n x_{ij} - c_o l x_{ij} = 0. \tag{23}$$

Thus,

$$n = \frac{(c_o x_{ij} - a_o z_{ij}) l}{a_o y_{ij} - b_o x_{ij}} \tag{24}$$

$$m = \frac{l}{x_{ij}} \left[\left(\frac{a_o z_{ij} - c_o x_{ij}}{a_o y_{ij} - b_o x_{ij}} \right) y_{ij} - z_{ij} \right] . \tag{25}$$

The direction components of the line of Equation (6) are:

$$m_1 : n_1 : l_1 = [x_{ij} - (x_i)_j] : [y_{ij} - (y_i)_j] : [z_{ij} - (z_i)_j]. \tag{26}$$

From Equations (24) and (25), we know that the direction components of the line of Equation (7) are:

$$m : n : l = \left[\left(\frac{a_o z_{ij} - c_o x_{ij}}{a_o y_{ij} - b_o x_{ij}}\right) \frac{y_{ij}}{x_{ij}} - \frac{z_{ij}}{x_{ij}}\right]$$

$$: \left[\frac{c_o x_{ij} - a_o z_{ij}}{a_o y_{ij} - b_o x_{ij}}\right] : 1 . \tag{27}$$

Therefore, the cosine of the angle between lines of Equations (6) and (7) is:

$$\cos k_i = \frac{m_1 m + n_1 n + l_1 l}{(m_1^2 + n_1^2 + l_1^2)^{1/2} (m^2 + n^2 + l^2)^{1/2}} . \tag{28}$$

That is,

$$\cos k = [\{ [x_{ij} - (x_i)_j][(\frac{a_o z_{ij} - c_o x_{ij}}{a_o y_{ij} - b_o x_{ij}}) \frac{y_{ij}}{x_{ij}} - \frac{z_{ij}}{x_{ij}}]$$

$$+ [y_{ij} - (y_i)_j] [\frac{c_o x_{ij} - a_o z_{ij}}{a_o y_{ij} - b_o x_{ij}}] + [z_{ij} - (z_i)_j]\} \Big/ \{ [x_{ij} - (x_i)_j]^2$$

$$+ [y_{ij} - (y_i)_j]^2 + [z_{ij} - (z_i)_j]^2 \}^{1/2} \{ [(\frac{a_o z_{ij} - c_o x_{ij}}{a_o y_{ij} - b_o x_{ij}}) \frac{y_{ij}}{x_{ij}} - \frac{z_{ij}}{x_{ij}}]^2$$

$$+ [\frac{c_o x_{ij} - a_o z_{ij}}{a_o y_{ij} - b_o x_{ij}}]^2 + l^2 \}^{1/2}] . \tag{29}$$

After calculating $\cos k_i$, the speed v_i of the space vehicle at (x_{ij}, y_{ij}, z_{ij}) can be obtained from the range rate measurement $\dot{r}_i$. The relation is simply:

$$v_i = \frac{\dot{r}_i}{\cos k_i} , \tag{30}$$

where $\dot{r}_i$ is the range rate measurement at the jth tracking station. Knowing the total velocity $\vec{v}_i$ at point r_i (x_{ij}, y_{ij}, z_{ij}), we can find the total energy of the spaceship as the sum of kinetic energy T and gravitational potential energy U. i.e.,

$$E_{Tot} = T + U$$

$$= \frac{1}{2} M_s \, v_i^2 - \frac{G \, M_e \, M_s}{r_i}, \tag{31}$$

where M_s = Mass of the spaceship

M_e = Mass of the earth

G = Universal gravitational constant.

We know the total energy can be expressed as:

$$E_{Tot} = \frac{G^2 \, M_e^2 \, M_s^2}{2 \, L^2} (e^2 - 1), \tag{32}$$

where

$$\vec{L} = M_s \, \vec{r}_i \times \vec{v}_i \, . \tag{33}$$

For a nearly circular orbit

$$L = M_s \, r_i \, v_i \, . \tag{34}$$

Thus, from Equation (32),

$$e = \sqrt{1 + \frac{2 \, E_{Tot} \, L^2}{M_s^2 \, M_e^2 \, G^2}} \quad . \tag{35}$$

The parameter p can then be found from:

$$p = \frac{L^2}{M_s^2 M_e G} , \tag{36}$$

Since,

$$r_i = \frac{p}{1 + e \cos(\theta_i - \theta_0)} , \tag{37}$$

then

$$\theta_0 = \theta_i - \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r_i} - 1\right)\right] \tag{38}$$

D. ORBITAL DETERMINATION BY RANGE-RATE MEASUREMENT ONLY.

This method employs four stations with Doppler radar equipment which measures only the range-rate of the spaceship. Each station observes an incoming frequency $f_i(t)$, where $i = 1, 2, 3, 4$ is an index specifying the station. Suppose f_o is the constant frequency emitted by the spaceship, and the electromagnetic wave propagates with the velocity of light c. Since the velocity of the spaceship is, in general, much less than the velocity of light, the radial velocity $V_{ri}(t)$ of the spaceship with respect to the ith station is:

$$V_{ri}(t) = c\left[1 - \frac{f_o}{f_i(t)}\right] \tag{1}$$

Equation (1) is based upon straight ray approach and the atmospheric refraction has been neglected. When the spaceship is approaching the ith station, the incoming frequency f_i will be higher than the constant frequency f_o, then the radial velocity $V_{ri}(t)$ is defined as positive. Therefore, if $R_i(t)$ is the instantaneous distance from the station to the satellite, we have,

$$V_{ri} = - \frac{d R_i(t)}{dt} \tag{2}$$

Thus,

$$R_i(t) = R_{i0} - \int_{t_o}^{t} V_{ri}(t)\ dt, \tag{3}$$

where R_{i0} represents the initial value of $R_i(t)$ at $t = t_o$. And the four initial distances R_{i0} are in general unknown.

Substituting (1) into (3), we get

$$R_i(t) = R_{i0} - \int_{t_o}^{t} c\left[1 - \frac{f_o}{f_i(t)}\right] dt$$

$$= R_{i0} - c(t - t_o) + c\, f_o \int_{t_o}^{t} \frac{dt}{f_i(t)}\ . \tag{4}$$

Let $x(t)$, $y(t)$, $z(t)$ be the instantaneous coordinates of the spaceship with respect to a given Cartesian coordinate system of reference and x_i, y_i, z_i be the coordinates of the i th station with respect to the same system of coordinates. Consider three of the four stations. We have, for $i = 1, 2, 3$:

$$(x(t) - x_1)^2 + (y(t) - y_1)^2 + (z(t) - z_1)^2 = R_1^2(t) \tag{5}$$

$$(x(t) - x_2)^2 + (y(t) - y_2)^2 + (z(t) - z_2)^2 = R_2^2(t) \tag{6}$$

$$(x(t) - x_3)^2 + (y(t) - y_3)^2 + (z(t) - z_3)^2 = R_3^2(t). \tag{7}$$

From Equations (5), (6) and (7), $x(t)$, $y(t)$ and $z(t)$ can be solved in terms of $R_1(t)$, $R_2(t)$ and $R_3(t)$. Since we know that

$$R_i(t) = R_{i0} - c(t - t_o) + c\, f_o \int_{t_o}^{t} \frac{dt}{f_i(t)},$$

then we have,

$$(x(t) - x_1)^2 + (y(t) - y_1)^2 + (z(t) - z_1)^2 = R_{10} - c(t - t_o) + c\, f_o \int_{t_o}^{t} \frac{dt}{f_1(t)} , \tag{8}$$

$$(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = R_{20} - c\,(t - t_o) + c\, f_o \int_{t_o}^{t} \frac{dt}{f_2(t)} , \tag{9}$$

$$(x - x_3)^2 + (y - y_3)^2 + (z - z_3)^2 = R_{30} - c(t - t_o) + c\, f_o \int_{t_o}^{t} \frac{dt}{f_3(t)} . \tag{10}$$

From Equations (8), (9) and (10), we know that x, y and z can be solved in terms of f_o, t, R_{10}, R_{20} and R_{30}. i.e.,

$$x(t) = x(t,\ f_o,\ R_{10},\ R_{20},\ R_{30}) \tag{11}$$

$$y(t) = y(t,\ f_o,\ R_{10},\ R_{20},\ R_{30}) \tag{12}$$

$$z(t) = z(t,\ f_o,\ R_{10},\ R_{20},\ R_{30}) . \tag{13}$$

Now, let x_4, y_4, z_4 be the coordinates of the fourth station in the chosen reference system. We have,

$$V_{R4} = - \frac{d\,R_4(t)}{dt} = c\,[\,1 - \frac{f_o}{f_4(t)}\,] . \tag{14}$$

Likewise, $R_4(t)$ can be expressed as:

$$R_4(t) = \sqrt{(x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2} \tag{15}$$

Substituting Equations (11), (12) and (13) into Equation (15), we know that $R_4(t)$ can be written as a function of f_o, t, R_{10}, R_{20} and R_{30}. i.e.,

$$R_4(t) = R_4(f_o, t, R_{10}, R_{20}, R_{30}) . \tag{16}$$

Now substitute Equation (15) into Equation (14). We get,

$$-\frac{dR_4(t)}{dt} = \frac{-\frac{1}{2}[2(x - x_4)\frac{dx}{dt} + 2(y - y_4)\frac{dy}{dt} + z(z - z_4)\frac{dz}{dt}]}{[(x - x_4)^2 + (y - y_4)^2 + (z - z_4)^2]^{1/2}}$$

$$= \frac{-1}{R_4(t)}[(x - x_4)\frac{dx}{dt} + (y - y_4)\frac{dy}{dt} + (z - z_4)\frac{dz}{dt}] . \tag{17}$$

Thus,

$$\frac{-1}{R_4(t)}[(x - x_4)\frac{dx}{dt} + (y - y_4)\frac{dy}{dt} + (z - z_4)\frac{dz}{dt}] = c[1 - \frac{f_o}{f_4(t)}]. \tag{18}$$

Note the right hand side of Equation (18) is a function of t and f_o while the left hand side of Equation (18) is a function of t, f_o, R_{10}, R_{20}, and R_{30}. At $t = t_1$, t_2, t_3 and t_4, Equation (18) can be split into four equations, namely:

$$\frac{-1}{R_4(t_i)}[[x(t_i, f_o, R_{10}, R_{20}, R_{30}) - x_4]\left.\frac{dx}{dt}\right|_{at\ t = t_i} + [y(t_i, f_o, R_{10}, R_{20}, R_{30}) - y_4]\left.\frac{dy}{dt}\right|_{at\ t = t_i}$$

$$+ [z(t_i, f_o, R_{10}, R_{20}, R_{30}) - z_4]\left.\frac{dz}{dt}\right|_{at\ t = t_i}] = c[1 - \frac{f_o}{f_4(t_i)}] , \tag{19}$$

where $i = 1, 2, 3, 4.$

Solve Equation (19), and we find the four unknowns (i.e., f_o, R_{10}, R_{20}, and R_{30}). Once we know the value of f_o, R_{10}, R_{20}, and R_{30}, the position of the spaceship can be determined from Equations (11), (12) and (13).

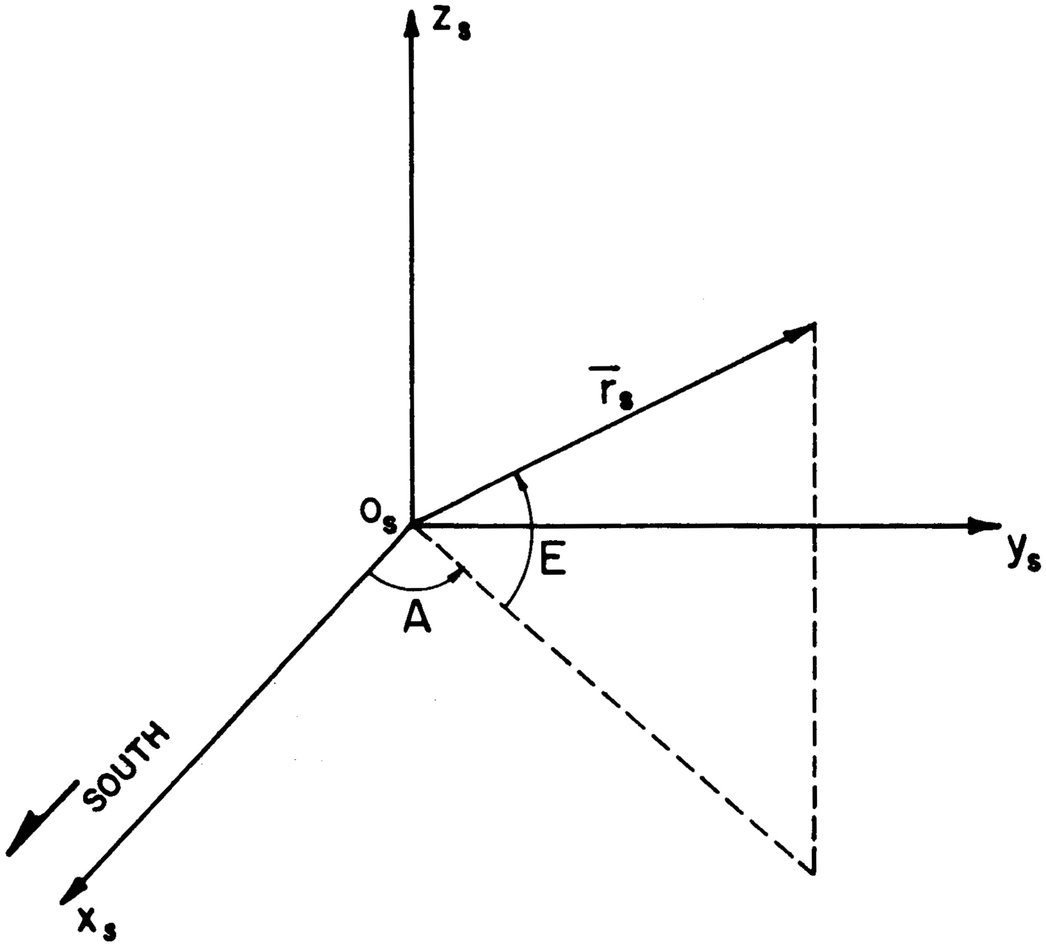

FIGURE 1. Coordinate systems at the tracking station.

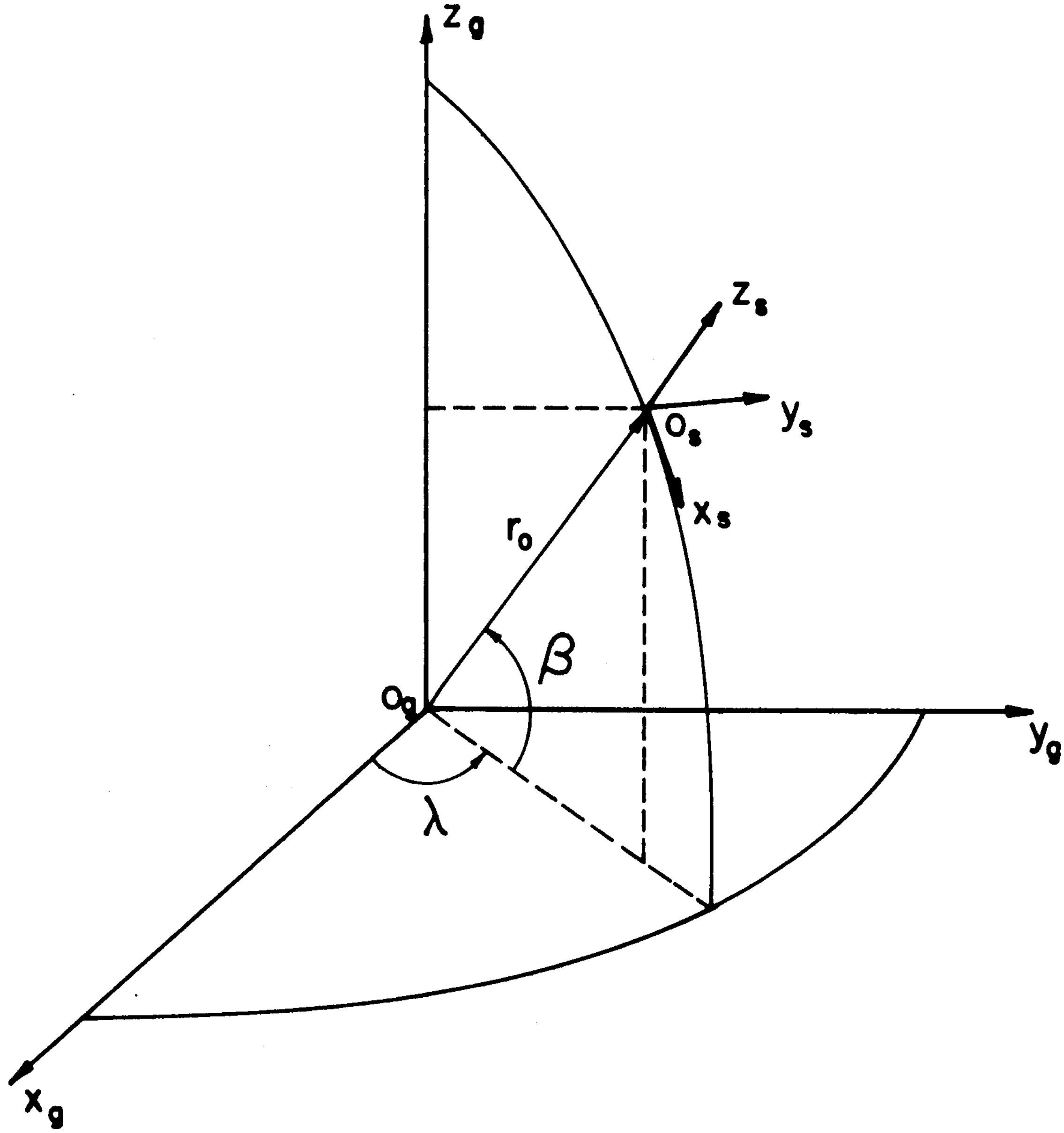

FIGURE 2. Transformation from the rectangular station coordinate system into the geocentric coordinate system.

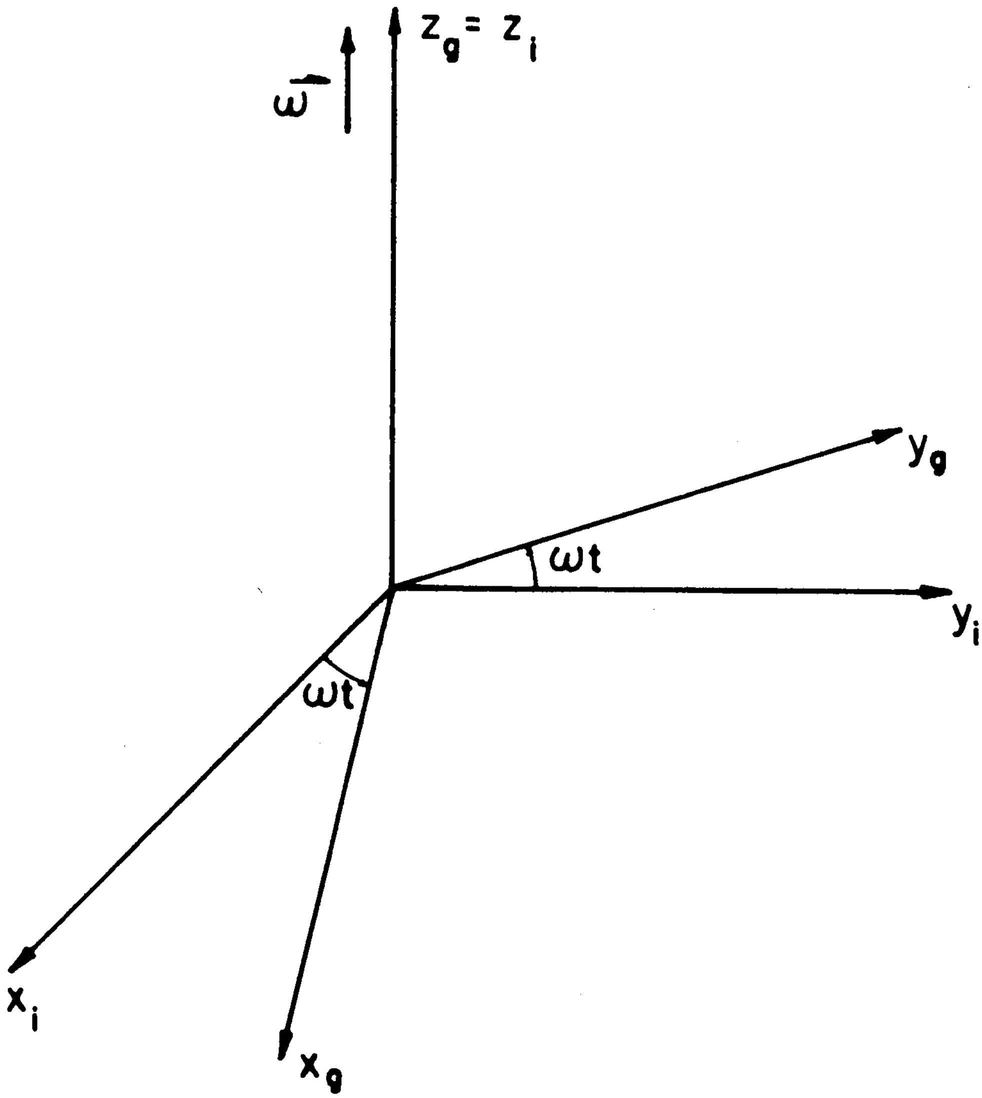

FIGURE 3. Transformation from geocentric system to inertial geocentric system.

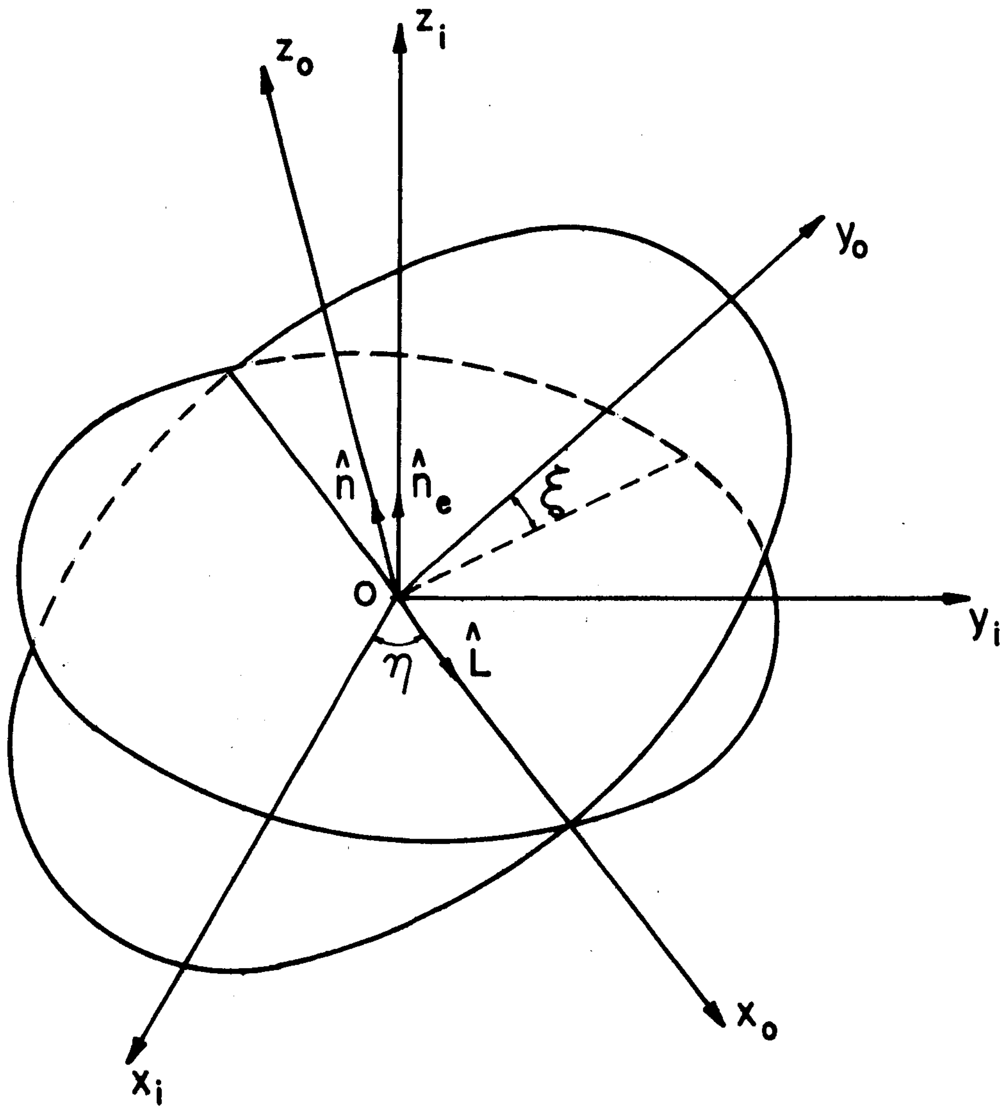

FIGURE 4. Orbital plane coordinate system and the geocentric inertial coordinate system.

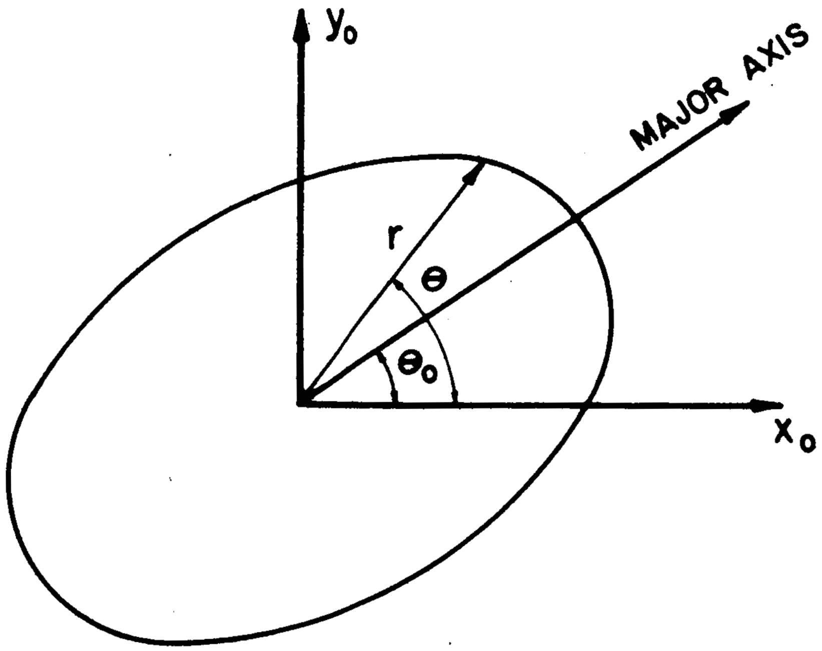

FIGURE 5. Polar coordinates in the orbital plane.

PART III

DATA SMOOTHING IN THE DETERMINATION OF THE ORBITAL PARAMETERS.

A. INTRODUCTION

A-1 Statement of the Problem.

The orbit of a satellite is determined from the data obtained with a radio tracking system. Various techniques are available which use range, velocity and angle measurements. In this study, the position of the satellite is determined by three simultaneous range measurements, one from each of a group of three earth stations called station-group. The distance between stations is small compared to the distance of the stations to the satellite; most probably, the stations will form an equilateral triangle but this is not necessary.

Only free flight orbits (no powered flight maneuvers) have been considered. Assuming a two-body problem, the orbit is an ellipse, an hyperbola or a parabola. Only the case of elliptic orbits with small excentricity is investigated. One focus of the ellipse is the center of the earth and the orbital parameters can be computed knowing three positions of the satellite. When the excentricity tends to zero, the elliptic orbit tends to a circular orbit with center 0, the center of the earth; the radius can be computed knowing one position of the satellite. The errors in the location of the stations and in the range measurements cause an uncertainty on the satellite position which produces errors in the computation of the orbit and of the orbital parameters. Three position measurements are necessary to determine an elliptic orbit and only one to determine a circular orbit, the additional position measurements can be used to reduce the uncertainty of the orbital parameters. This technique which uses sampling theory is known as data smoothing.

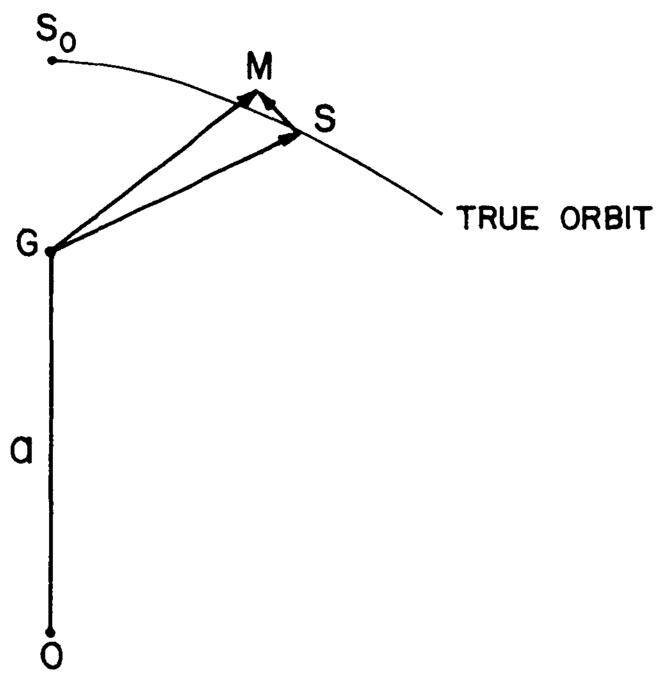

Fig. A-1 All the errors in the determination of the satellite position are lumped into an error vector $\vec{SM}$. S true satellite position, M measured satellite position.

A-2 The Method of Approach.

The position of the stations is known. The position of the satellite is determined by a vector with origin at the center of gravity of the station group and with tip on the satellite. The location of the stations is assumed to be known exactly, all the various errors being lumped together as an error on the satellite position.

Let S and M, be respectively the true and the measured position of the satellite and denote by G the center of the station group. The measured vector $\vec{GM}$ is the vector sum of the true vector $\vec{GS}$ and of an error vector $\vec{SM}$, as shown in Fig. A-1. Denote by O the center of the earth. It is advantageous to express the position of S in polar coordinates: angle $\vec{OG}$, $\vec{OS} = \alpha$ and modulus $|OS| = r$. The error vector $\vec{SM}$ is defined by its modulus ρ and the angle $\vec{OS}$, $\vec{SM} = \varphi$. The vector $\vec{OS}$ and $\vec{OM}$ will be denoted respectively as true position vector and measured position vector. The time when the satellite is at the closest approach to the station is taken for origin of time. The position of the satellite is measured every Δt sec.; the sampling time is denoted by t_{-k} t_{-k+1} ... t_0 ... t_{k-1} t_k where $t_k = k\Delta t$ and $t_{-k} = -k\Delta t$. At time t_k the true and the measured satellite position are denoted by S_k and M_k, respectively; the phase and the modulus of the vector error are denoted by φ_k and ρ_k, respectively. It is assumed that φ_k is uniformly distributed and that ρ_k is Rayleigh[†] distributed with a variance σ_k^2 increasing with $|k|$. The definition of the various parameters is illustrated by Fig. A-2.

In Section B, a formula for the estimation of the radius R in terms of m_k, the modulus of the kth measured position vector, is derived. A smoothing technique is developed where the optimum number of samples and the optimum accuracy are determined.

The case of an elliptic orbit is considered in Section C. The three orbital parameters are defined implicitly as a function of m_{k-1}, m_k and m_{k+1}. A smoothing technique is proposed.

Section D contains the conclusions and a discussion of the assumptions.

[†] This standard assumption corresponds to measured satellite coordinates, independent and normally distributed about the true satellite coordinates.

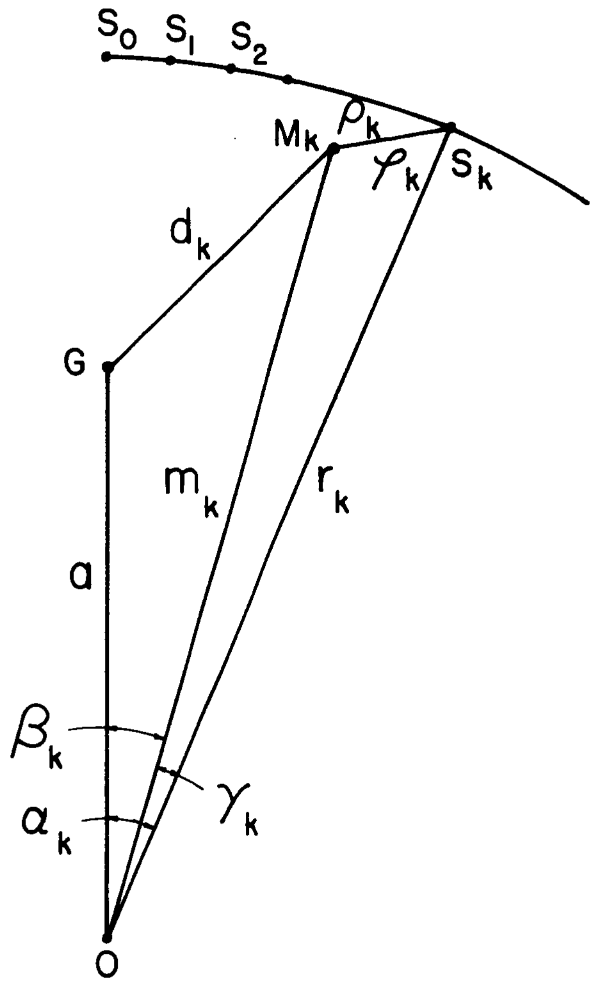

Fig. A-2 Polar coordinate representation of the true and measured satellite position at time t_k.

B. DATA SMOOTHING FOR CIRCULAR ORBIT

B-1 The Measured Radius.

Assuming that the orbit is circular, the center of the orbit is the center of the earth and the only unknown orbital parameter is the radius R. Consider again Fig. A-2. At time $t = t_k$, $m_k = |OS_k|$ is the measured value of the radius. In the triangle $OM_k S_k$, m_k can be evaluated in terms of $r_k = R$ and the modulus and phase of the vector error, i.e. ρ_k and φ_k.

$$m_k^2 = r_k^2 + \rho_k^2 - 2 r_k \rho_k \cos \varphi_k$$
$$m_k = \sqrt{r_k^2 + \rho_k^2 - 2 r_k \rho_k \cos \varphi_k} \tag{B-1}$$

The vector error is unknown by definition, but its statistics is assumed to be known. The random phase φ_k is uniformly distributed and the modulus ρ_k is Rayleigh distributed. The vector error $\overrightarrow{S_k M_k}$ is represented by a joint probability density

$$p(\rho_k, \varphi_k) = \frac{1}{2\pi} \frac{2\rho_k}{\sigma_k^2} e^{-\rho_k^2 / \sigma_k^2} \tag{B-2}$$

where σ_k^2 is the mean square modulus, i. e. $\sigma_k^2 = \overline{\rho_k^2}$. †

It is assumed that σ^2 is a known function of α. The rotation of the earth is neglected as a first approximation since the satellite rotates much faster than the earth.

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† A bar above a random variable means average.

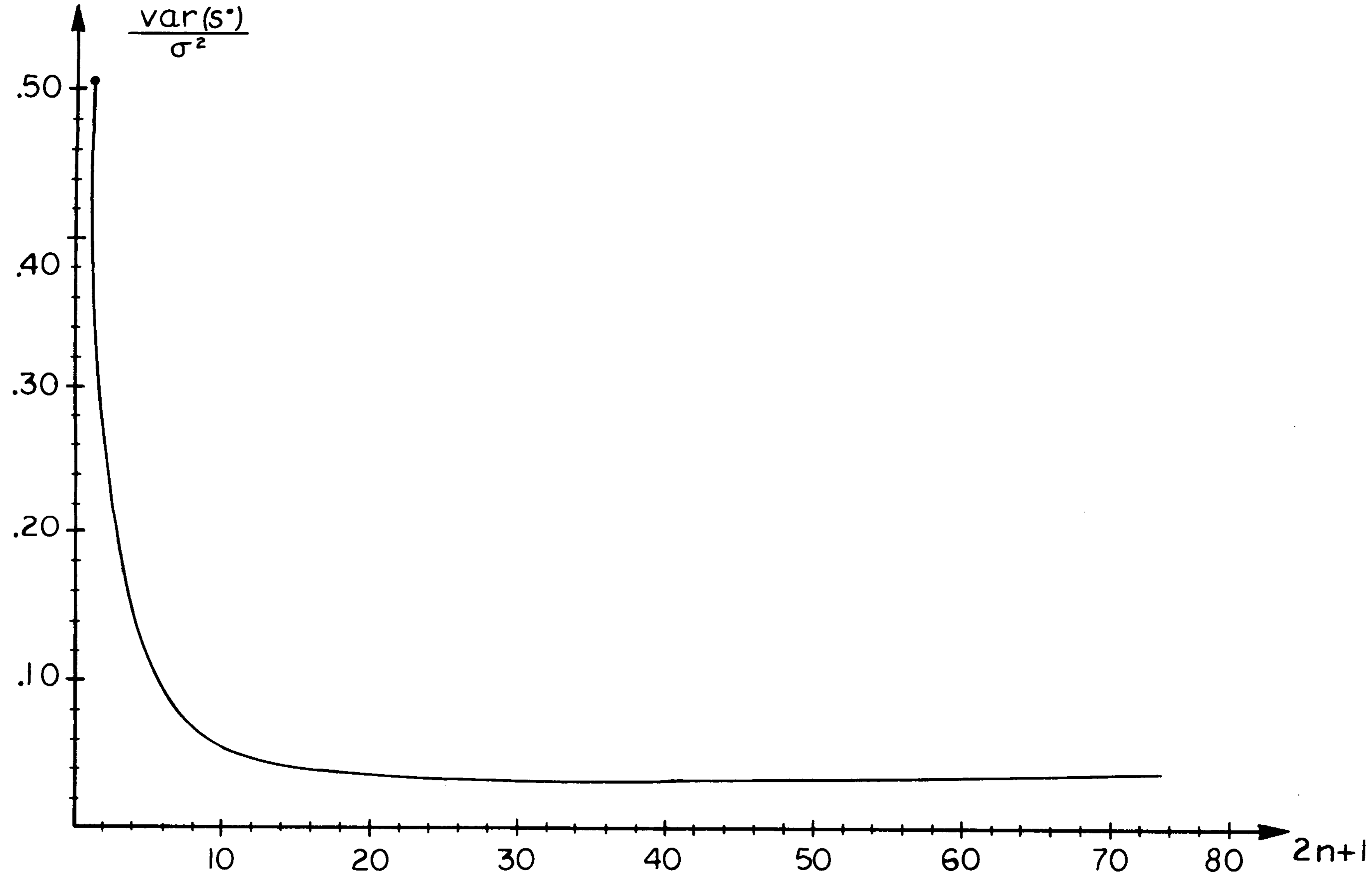

Figure B-1: The variance of the sample mean (var $(s^{\cdot})$) versus the number of samples $(2n + 1)$ for the example of Par. B-4.

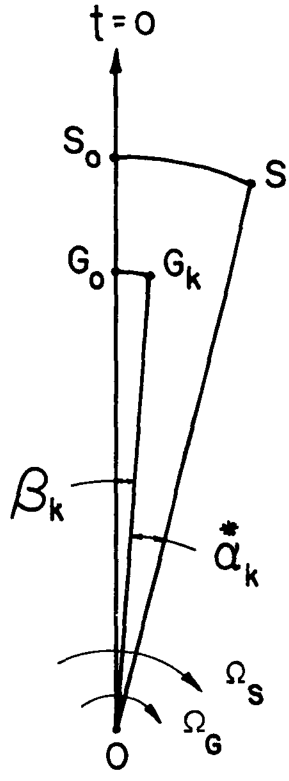

Fig. B-2 Effect of earth rotation for circular orbit. Both, the satellite S and the station group G rotate about the center of the earth. The subscript k refers to time t_k.

B-2 Estimation of the Radius.

The radius R can be estimated from one measured value m_k. An unbiased estimator s_k is obtained by comparing $\overline{m_k}$ to R.

By definition

$$\overline{m_k} = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} m_k \frac{2\rho_k}{\sigma_k^2} e^{-\rho_k^2/\sigma_k^2} \, d\varphi_k \, d\rho_k \tag{B-3}$$

and $\overline{m_k}$ is the average of m_k with respect to both φ_k and ρ_k.

After integration with respect to φ_k [1],

$$\overline{m_k} = \frac{1}{\pi} \int_0^\infty 2(R + \rho_k) \, E\left(\sqrt{\frac{4R\rho_k}{(R+\rho_k)^2}}\right) \frac{2\rho_k e^{-\rho_k^2/\sigma_k^2}}{\sigma_k^2} \, d\rho_k \tag{B-4}$$

where $E(\ )$ is the complete elliptic integral of the second kind.

Since ρ_k is much smaller than R, it is convenient to develop $E(\ )$ in series and keep only the first three terms. Then

$$\begin{aligned} \frac{2}{\pi}(R+\rho_k)\, E\left(\sqrt{\frac{4R\rho_k}{(R+\rho_k)^2}}\right) &\approx R + \rho_k - \frac{R\rho_k}{R+\rho_k} - \frac{3}{4}\frac{R^2\rho_k^2}{(R+\rho_k)^2} \\ &\approx R + \frac{\rho_k^2}{4R} \end{aligned} \tag{B-5}$$

Integrating with respect to ρ_k yields,

$$\begin{aligned} \overline{m_k} &\approx \int_0^\infty \left(R + \frac{\rho_k^2}{4R}\right) \frac{2\rho_k}{\sigma_k^2} e^{-\rho_k^2/\sigma_k^2} \, d\rho_k \\ &\approx R + \frac{\sigma_k^2}{4R} \end{aligned} \tag{B-6}$$

The mean of $\overline{m_k}$ is not equal to R, i.e. m_k is not an unbiased estimator. Solving (B-6) yields

$$R = \frac{\overline{m_k} + \sqrt{\overline{m_k}^2 - \sigma_k^2}}{2} \approx \overline{m_k}\,(1 - \frac{\sigma_k^2}{4\overline{m_k}^2}) \tag{B-7}$$

From (B-7), it follows that

$$s_k = m_k - \frac{\sigma_k^2}{4\, m_k} \tag{B-8}$$

is an unbiased estimator for R, i. e. $\overline{s_k} = R$. The variance of m_k is easily obtained

$$\mathrm{var}\,(m_k) = \overline{m_k^2} - \overline{m_k}^2 \tag{B-9}$$

Using (B-1) and (B-2) and integrating,

$$\begin{aligned} \overline{m_k^2} &= \int_0^\infty \int_0^{2\pi} m_k^2\, p(\rho_k\,, \varphi_k)\, d\varphi_k \; d\rho_k \\ &= R^2 + \sigma_k^2 \end{aligned} \tag{B-10}$$

Combining (B-9), (B-10) and (B-6) yields

$$\begin{aligned} \mathrm{var}\,(m_k) &= R^2 + \sigma_k^2 - (R + \frac{\sigma_k^2}{4R})^2 \\ &\approx \frac{\sigma_k^2}{2} \end{aligned} \tag{B-11}$$

The variance of s_k is

$$\mathrm{var}\,(s_k) = \overline{s_k^2} - \overline{s_k}^2 \tag{B-12}$$

$$\overline{s_k} = \overline{m_k} - \left(\frac{\overline{\sigma_k^2}}{4m_k}\right) \tag{B-13}$$

$$\overline{\left(\frac{1}{m_k}\right)} = \int_0^{\infty}\int_0^{2\pi} \frac{1}{2\pi}\,\frac{2\rho_k}{\sigma_k^2}\,\frac{e^{-\rho_k^2/\sigma_k^2}}{\sqrt{R^2 + \rho_k^2 - R\rho_k \cos\varphi_k}}\, d\varphi_k\, d\rho_k \tag{B-14}$$

$$= \int_0^{\infty} \frac{2}{\pi(R + \rho_k)}\, K\left(\sqrt{\frac{4R\rho_k}{(R + \rho_k)^2}}\right) \frac{2\rho_k}{\sigma_k^2}\, e^{-\rho_k^2/\sigma_k^2}\, d\rho_k \tag{B-15}$$

where $K(\ )$ is the complete elliptic integral of the first kind. Since $R \gg \rho_k$,

$$\frac{2}{\pi(R + \rho_k)}\, K\left(\sqrt{\frac{4R\rho_k}{(R + \rho_k)^2}}\right) \approx \frac{1}{R}\left(1 + \frac{\rho_k}{R}\right) \tag{B-16}$$

Combining (B-13) and (B-14) and integrating yields

$$\overline{\left(\frac{1}{m_k}\right)} \approx \frac{1}{R}\left(1 + \frac{\sigma\sqrt{\pi}}{2R}\right) \tag{B-17}$$

Combining (B-6), (B-17) and (B-13) and neglecting the smaller terms yields, as expected,

$$\overline{s_k} = R \tag{B-18}$$

From (B-8)

$$s_k^2 = m_k^2 - \frac{\sigma_k^2}{2} + \frac{\sigma_k^4}{16\, m_k^2} \tag{B-19}$$

From (B-1)

$$\frac{1}{m_k^2} = \frac{1}{(R^2 + \rho_k^2) - 2\, R\, \rho_k \cos \varphi_k} \tag{B-20}$$

$$\frac{1}{\pi} \int_0^{\pi} \frac{1}{m_k^2}\, d\varphi_k = \frac{1}{R^2 - \rho_k^2}$$

$$\approx \frac{1}{R^2} \left(1 + \frac{\rho_k^2}{R^2}\right) \tag{B-21}$$

Integrating (B-21) with respect to ρ_k yields

$$\left(\overline{\frac{1}{m_k^2}}\right) \approx \frac{1}{R^2} \left(1 + \frac{\sigma_k^2}{R^2}\right) \approx \frac{1}{R^2} \tag{B-22}$$

From (B-19)

$$\overline{s_k^2} = \overline{m_k^2} - \frac{\sigma_k^2}{2} + \left(\frac{\overline{\sigma_k^4}}{16 m_k^2}\right) \tag{B-23}$$

Combining, (B-10), (B-22) and (B-23)

$$\overline{s_k^2} = R^2 + \sigma_k^2 - \frac{\sigma_k^2}{2} + \frac{\sigma_k^4}{16\, R^2} \tag{B-24}$$

Combining (B-24), (B-18) and (B-12) yields

$$\operatorname{var}(s_k) = \frac{\sigma_k^2}{2} + \frac{\sigma_k^4}{16R^2}$$

$$\approx \frac{\sigma_k^2}{2} \quad . \tag{B-25}$$

That is, the unbiased estimator s_k has the same variance $\frac{\sigma_k^2}{2} = \frac{\overline{\rho_k^2}}{2}$ as the measured radius m_k.

B-3 Data Smoothing for Radius Estimation.

In a practical situation, many measured m_k and, therefore, many samples s_k of the unbiased estimator are available. It is shown next that the best estimate of R is the sample mean $s^\bullet$ of size $2n+1$.

$$s^\bullet = \frac{1}{2n+1} \sum_{k=-n}^{n} s_k \quad . \tag{B-26}$$

Clearly, the sample mean is an unbiased estimator

$$\overline{s^\bullet} = \overline{s_k} = R \quad . \tag{B-27}$$

The variance of the sample mean of $s^\bullet$ is

$$\overline{(s^\bullet - \overline{s^\bullet})^2} = \overline{\left(\frac{1}{2n+1} \sum_{i=-n}^{n} (s_i - R)\right)\left(\frac{1}{2n+1} \sum_{k=-n}^{n} (s_k - R)\right)} . \tag{B-28}$$

Since the successive samples are independent and unbiased, all the cross terms are zero,

$$\overline{(s^\bullet - \overline{s^\bullet})^2} = \frac{1}{(2n+1)^2} \sum_{k=-n}^{n} \overline{(s_k - R)^2} \, , \tag{B-29}$$

$$\operatorname{var}\ (s^{\cdot}) = \frac{1}{(2n+1)^2} \sum_{k=-n}^{n} \frac{\sigma_k^2}{2} \ . \tag{B-30}$$

If all the measurements had the same accuracy, σ_k^2 = constant, the variance of $s^{\cdot}$ would decrease continuously when n increases. In the actual situation, σ_k^2 increases rapidly with the distance to the station, i. e. with k. The variance of $s^{\cdot}$ decreases first and then increases when n increases.

The optimum number of samples N is the choice of n which makes the variance of $s^{\cdot}$ minimum.

In other words N is the maximum value of n which satisfies the inequality

$$\operatorname{var}\ (s^{\cdot}_{(2N+3)}) - \operatorname{var}\ (s^{\cdot}_{(2N+1)}) \leq 0 \ , \tag{B-31}$$

and N can be obtained if the variance σ_k^2 of each measurement m_k is known as a function of the satellite position α_k . This technique is illustrated best by an example.

B-4 Example of Data Smoothing.

Assume that the variance of the modulus of the error vector is a parabolic function of the true position angle α_k and that the satellite position is measured at intervals of one degree.

For example,

$$\sigma_k^2 = \sigma^2 (a + b\ k^2). \tag{B-32}$$

From (B-30), it follows

$$\operatorname{var}\ (s^{\cdot}_{(2n+1)}) = \frac{\sigma^2}{2} \left(\frac{a}{2n+1} + \frac{2b}{(2n+1)^2} \sum_{k=1}^{n} k^2 \right) . \tag{B-33}$$

From standard tables [2],

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \tag{B-34}$$

so that (B-33) reduces to

$$\text{var}\,(s^{\bullet}_{(2n+1)}) = \frac{\sigma^2}{2(2n+1)}\,(a + \frac{b\,n(n+1)}{3}). \tag{B-35}$$

Similarly,

$$\text{var}\,(s^{\bullet}_{(2n+3)}) = \frac{\sigma^2}{2(2n+3)}\,(a + \frac{b(n+1)(n+2)}{3}). \tag{B-36}$$

The optimum value of n is determined by (B-31), i.e.

$$\frac{\sigma^2}{2(2n+3)}\,(a + \frac{b(n+1)(n+2)}{3}) - \frac{\sigma^2}{2(2n+1)}\,(a + \frac{b\,n(n+1)}{3}) \leq 0 \tag{B-37}$$

which reduces to

$$\frac{\sigma^2}{(2n+3)(2n+1)}\,[-a + \frac{b(n+1)^2}{3}] \geq 0\,. \tag{B-38}$$

The solution is,

$$n \leq \sqrt{\frac{3a}{b}} - 1\,. \tag{B-39}$$

The optimum value of N is obtained by truncation, †

$$N = \text{Trunc}\,(\sqrt{\frac{3a}{b}} - 1) \tag{B-40}$$

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† The truncation of a positive number x is the largest integer less or equal to x.

and the optimum number of samples is $2N + 1$.

Assume $a = 1$ and $b = .01$ to continue the example. From (B-40),

$$N = 16 \tag{B-41}$$

The optimum number of samples is 33, the variance of the sample mean is

$$\mathrm{var}\,(s^{\bullet}_{33}) = .0346\,\sigma^2. \tag{B-42}$$

If only the best sample ($k = 0$) is used, the variance is,

$$\mathrm{var}\,(s_0) = .5\;\sigma^2. \tag{B-43}$$

Therefore, the variance of the estimator is about fourteen times smaller after data smoothing. To illustrate the behavior of the variance of the sample mean as a function of the number of samples, the relation (B-36) is plotted in Fig. B-1 for $a = 1$ and $b = 0.01$.

B-5 Earth Rotation and Data Smoothing.

The effect of earth rotation is investigated assuming that the trajectories of the satellite and of the station group are circular with center O and coplanar. When the rotation of the earth is neglected G is fixed and the points O, G and S_0 are continuously aligned. The rotation of the earth produces an angular displacement, $\beta = \vec{O}S_0, \vec{O}G$ of $\vec{O}G$ with respect to $\vec{O}S_0$ as shown in Fig. B-2. Denote the angle, $\vec{O}G, \vec{O}S$ by α^*.† Let Ω_G and Ω_S be respectively the constant angular velocity of the earth and of the satellite in circular orbit. The position of S_0 corresponds to the origin of time. Assume again that the positions of the satellite and of the earth are sampled at regular time intervals, i.e. $t_k = k\Delta\tau$ where k is an integer. At time t_k, the angular position of the satellite with respect to the fixed vector $\vec{O}S_0$ is

† α and α^* denote the angle $\vec{O}G$, $\vec{O}S$ without and with earth rotation.

$$\vec{OS}_0 , \vec{OS}_k = \vec{OS}_0 , \vec{OG}_k + \vec{OG}, \vec{OS}_k$$

$$= \beta_k + \alpha_k^*$$

$$= \Omega_S t_k \qquad \text{(B-44)}$$

Since, $\beta_k = \Omega_G t_k$, it follows

$$\alpha_k^* = \Omega_S - \Omega_G$$

$$= k(\Omega_S - \Omega_G)\Delta\tau \qquad \text{(B-45)}$$

which can be written in the form,

$$\alpha_k^* = k^* \Omega_S \Delta\tau , \qquad \text{(B-46)}$$

where $k^* = k(1 - \frac{\Omega_G}{\Omega_S})$.

It follows that the effect of the rotation of the earth in the determination of the radius of the orbit can be accounted for by substitution of k^* for k in Par. B-3. To illustrate the procedure, the example of Par. B-3 is continued by taking into account earth rotation.

Assume that satellite and earth rotates in the same direction, the satellite being twenty times faster, i. e. $\frac{\Omega_G}{\Omega_S} = 0.05$ and $k^* = .95\, k$.

Let $(2n^* + 1)$ be the number of samples when taking the earth rotation into account. Then (B-33) becomes,

---

† Assuming that earth and satellite rotates in the same direction.

$$\operatorname{var}(s^{\cdot}_{(2n^*+1)}) = \frac{\sigma^2}{2}\left(\frac{a}{2n^*+1} + \frac{2b}{(2n^*+1)^2}\sum_{k=1}^{n^*}(.95\,K)^2\right) \tag{B-47}$$

Using (B-34), (B-47) reduces to,

$$\operatorname{var}(s^{\cdot}_{(2n^*+1)}) = \frac{\sigma^2}{2(2n^*+1)}\left(a + \frac{(.95)^2\, b\, n^*(n^*+1)}{3}\right) \tag{B-48}$$

Comparing (B-35) and (B-48), it is clear that the effect of earth rotation is to replace b by

$$b^* = b\left(1 - \frac{\Omega_G}{\Omega_S}\right).$$

Therefore, the optimum value of n^*, N^*, is obtained by substitution of b by b^* in (B-40).

$$N^* = \operatorname{Trunc}\left(\frac{1}{.95}\sqrt{\frac{3a}{b}} - 1\right) \tag{B-49}$$

when $a = 1$ and $b = .01$, $N^* = 17$.

The optimum number of samples becomes 35, the variance of the estimator becomes $.0274\,\sigma^2$.

In conclusion, when the relative velocity earth satellite decreases (earth rotates in same direction as satellite), the optimum number of samples to evaluate the radius increases and the variance of the sample mean decreases. If the earth and the satellite had the same angular speed, the satellite will look stationary, the variance of the sample mean will then decrease linearly with the number of samples. The conclusions are just opposite if the earth and satellite rotate in opposite directions.

C. DATA SMOOTHING FOR ELLIPTICAL ORBIT

C-1 Determination of the Orbital Parameters.

The orbit is determined in polar coordinates with respect to the vector $\vec{OS_0}$. The shape of the orbit depends on two numbers: p the semi-latus rectum or parameter, and e, the excentricity. The position of the foci line with respect to $\vec{OS_0}$ is defined by the angle θ. The three orbital parameters p, e and θ can be determined by three positions of the satellite.

Assuming that three exact positions of the satellite, S_1, S_2 and S_3 are known, the orbital parameters could be determined exactly. Let (r_1, α_1), (r_2, α_2) and (r_3, α_3) be the polar coordinates of S_1, S_2, S_3. Then p, e and θ are the solutions of a set of equations:

$$r_1 = \frac{p}{1 + e \cos(\theta - \alpha_1)}$$

$$r_2 = \frac{p}{1 + e \cos(\theta - \alpha_2)} \qquad \text{(C-1)}$$

$$r_3 = \frac{p}{1 + e \cos(\theta - \alpha_3)}$$

Expanding $\cos(\theta - \alpha)$ and making the change of variables $x = e \cos \theta$ and $y = e \sin \theta$, yields a set of linear equations:

$$1\, p - (r_1 \cos \alpha_1)\, x - (r_1 \sin \alpha_1)\, y = r_1$$

$$1\, p - (r_2 \cos \alpha_2)\, x - (r_2 \sin \alpha_2)\, y = r_2 \qquad \text{(C-2)}$$

$$1\, p - (r_3 \cos \alpha_3)\, x - (r_3 \sin \alpha_3)\, y = r_3$$

The solution is easily obtained using matrix algebra:

$$p = \frac{r_1 r_2 r_3 \left(\sin(\alpha_3 - \alpha_2) + \sin(\alpha_1 - \alpha_3) + \sin(\alpha_2 - \alpha_1)\right)}{r_3 r_2 \sin(\alpha_3 - \alpha_2) + r_1 r_3 \sin(\alpha_1 - \alpha_3) + r_2 r_1 \sin(\alpha_2 - \alpha_1)} \tag{C-3}$$

$$x = \frac{r_3 r_2(\sin\alpha_2 - \sin\alpha_3) + r_1 r_3 (\sin\alpha_3 - \sin\alpha_1) + r_2 r_1 (\sin\alpha_1 - \sin\alpha_2)}{r_3 r_2 \sin(\alpha_3 - \alpha_2) + r_1 r_3 \sin(\alpha_1 - \alpha_3) + r_2 r_1 \sin(\alpha_2 - \alpha_1)} \tag{C-4}$$

$$y = \frac{r_3 r_2(\cos\alpha_3 - \cos\alpha_2) + r_1 r_3(\cos\alpha_1 - \cos\alpha_3) + r_2 r_1 (\cos\alpha_2 - \cos\alpha_1)}{r_3 r_2 \sin(\alpha_3 - \alpha_2) + r_1 r_3 \sin(\alpha_1 - \alpha_3) + r_2 r_1 \sin(\alpha_2 - \alpha_1)} \tag{C-5}$$

Then $$e = \sqrt{x^2 + y^2} \tag{C-6}$$

and $$\theta = \tan^{-1} (y/x) . \tag{C-7}$$

C-2 Estimation of p, e and θ.

Refer again to Fig. A-2. In a practical situation, the exact polar coordinates (r_1, α_1), (r_2, α_2) and (r_3, α_3) are not available. It is necessary to estimate p, e and θ, using the measured polar coordinates (m_1, β_1), (m_2, β_2) and (m_3, β_3).

Clearly, $\alpha_k = \beta_k + \gamma_k$. In the triangle $O M_k S_k$

$$\frac{\sin\gamma_k}{\rho_k} = \frac{\sin\varphi_k}{m_k}; \tag{C-8}$$

therefore, $$\alpha_k = \beta_k + \sin^{-1} \left(\frac{\rho_k \sin\varphi_k}{m_k}\right) \tag{C-9}$$

where $$m_k = \sqrt{r_k^2 + \rho_k^2 - 2 r_k \rho_k \cos \varphi_k} \tag{C-10}$$

As a first approximation, the estimate E_p, E_x and E_y or p, x and y are obtained by replacing α_k by $\hat{\beta}_k$ and r_k by m_k in (C-3), (C-4) and (C-5):

$$E_p = \frac{m_1 m_2 m_3 \left(\sin(\hat{\beta}_3 - \hat{\beta}_2) + \sin(\hat{\beta}_1 - \hat{\beta}_3) + \sin(\hat{\beta}_2 - \hat{\beta}_1)\right)}{m_3 m_2 \sin(\beta_3 - \beta_2) + m_1 m_3 \sin(\beta_1 - \beta_3) + m_2 m_1 \sin(\beta_2 - \beta_1)} \tag{C-11}$$

$$E_x = \frac{m_3 m_2 (\sin\beta_2 - \sin\beta_3) + m_1 m_3 (\sin\beta_3 - \sin\beta_1) + m_2 m_1 (\sin\beta_1 - \sin\beta_2)}{m_3 m_2 \sin(\beta_3 - \beta_2) + m_1 m_3 \sin(\beta_1 - \beta_3) + m_2 m_1 \sin(\beta_2 - \beta_1)} \tag{C-12}$$

$$E_y = \frac{m_3 m_2 (\cos\beta_3 - \cos\beta_2) + m_1 m_3 (\cos\beta_1 - \cos\beta_3) + m_2 m_1 (\cos\beta_2 - \cos\beta_1)}{m_3 m_2 \sin(\beta_3 - \hat{\beta}_2) + m_1 m_3 \sin(\hat{\beta}_1 - \beta_3) + m_2 m_1 \sin(\beta_2 - \beta_1)} \tag{C-13}$$

To find the mean and the variance of E_p, E_x and E_y requires six integrations too difficult to be practical. The solution is to replace E_p, E_x and E_y by approximate power series which can be integrated. For this purpose, it is convenient to transform the random variables from Polar to Cartesian coordinates. Let $u_k = \rho_k \cos\varphi_k$ and $v_k = \rho_k \sin\varphi_k$. The polar density distribution

$$p_1(\rho_k, \varphi_k) = \frac{1}{2\pi} \frac{2\rho_k}{\sigma_k^2} e^{-\frac{\rho_k^2}{\sigma_k^2}} \tag{C-14}$$

becomes

$$p_2(u_k, v_k) = \frac{1}{\pi \sigma_k^2} e^{-\left(\frac{u_k^2 + v_k^2}{\sigma_k^2}\right)} . \tag{C-15}$$

The measured samples m_k and β_k become:

$$m_k = \sqrt{(r_k - u_k)^2 + v_k^2} \tag{C-16}$$

$$\beta_k = \alpha_k - \sin^{-1}\left(\frac{v_k}{\sqrt{(r_k - u_k)^2 + v_k^2}}\right). \tag{C-17}$$

The estimated values of p, x and y are functions of the six independent variables $u_1, v_1, u_2, v_2, u_3, v_3$ which have zero mean. As a first approximation,

$$E_p = (E_p)_0 + \sum_{i=1}^{3} \left(\frac{\partial E_p}{\partial u_i}\right)_0 u_i + \sum_{i=1}^{3} \left(\frac{\partial E_p}{\partial v_i}\right)_0 v_i , \tag{C-18}$$

where the subscript 0 means evaluated at $u_1 = u_2 = u_3 = v_1 = v_2 = v_3 = 0$. Clearly, $(E_p)_0 = p$, the true value of p.

With the approximation (C- 18), the mean and the variance of the sampled value E_p are easily obtained:

$$\bar{E}_p = p , \tag{C-19}$$

$$\mathrm{Var}\,(E_p) = \sum_{i=1}^{3} \left(\frac{\partial E_p}{\partial u_i}\right)_0^2 \frac{\sigma_i^2}{2} + \sum_{i=1}^{3} \left(\frac{\partial E_p}{\partial v_i}\right)_0^2 \frac{\sigma_i^2}{2} . \tag{C-20}$$

The cross terms disappear because all the u_i and v_i are independent. If the excentricity e is very small, the variance σ_i^2 of the modulus of the vector error, which corresponds to a position α_i of the satellite, can be read on the curve made for circular orbit.

C-3 Data Smoothing for Elliptic Orbit.

It was shown in C-2 that an estimate of the orbital parameters can be computed after measuring three positions of the satellite, say $i = 1, 2$ and 3. If in a practical situation n measured values of m_k and β_k are available, $\frac{n}{3}$ estimates of p, e and θ can be computed: $(E_p)_1$, $(E_p)_2$ .... .

The sample mean of size $2N + 1$, $(E_p)^{\cdot}_{2N+1}$, is a better estimate of p than any single estimate $(E_p)_1$ or $(E_p)_2$ ... , for a proper choice of N. (The same is true for e and θ.)

With the approximation (C-18),

$$(E_p)^{\cdot}_{2N+1} = \frac{1}{2N+1} \sum_{k=-N}^{N} (E_p)_k \; . \qquad \text{(C-21)}$$

Assuming that three consecutive measurements are combined to form an estimate of p, the kth sample is

$$(E_p)_k = p + \sum_{i=1}^{3} \left(\frac{\partial E_{pk}}{\partial u_{i+k}}\right)_0 u_{i+k} + \sum_{j=1}^{3} \left(\frac{\partial E_{pk}}{\partial v_{j+k}}\right)_0 v_{j+k} \; . \qquad \text{(C-22)}$$

Combination of (C-21) and (C-22)yields,

$$(E_p)^{\cdot}_{2N+1} = p + \frac{1}{2N+1} \left\{ \sum_{k=-N}^{N} \sum_{i=1}^{3} \left(\frac{\partial E_{pk}}{\partial u_{i+k}}\right)_0 u_{i+k} + \sum_{k=-N}^{N} \sum_{j=1}^{3} \left(\frac{\partial E_{pk}}{\partial v_{j+k}}\right)_0 v_{j+k} \right\}. \qquad \text{(C-23)}$$

The mean of the sample mean is p. The variance of the sample mean of size $2N + 1$ is

$$\text{Var. } (E_p)^{\bullet}_{2N+1} =$$

$$\frac{1}{(2N+1)^2} \overline{\left\{ \sum_{k=-N}^{N} \sum_{i=1}^{3} \left(\frac{\partial E_{pk}}{\partial u_{i+k}}\right)_0 u_{i+k} + \sum_{k=-N}^{N} \sum_{i=1}^{3} \left(\frac{\partial E_{pk}}{\partial v_{i+k}}\right)_0 v_{i+k} \right\}^2} \qquad \text{(C-24)}$$

Since all the random variables are independent with zero mean, all the cross products disappear:

$$\text{Var. } (E_p)^{\bullet}_{2N+1} = \frac{1}{(2N+1)^2} \left\{ \sum_{k=-N}^{N} \sum_{i=1}^{3} \left(\frac{\partial E_{pk}}{\partial u_{i+k}}\right)_0^2 \overline{u^2_{i+k}} + \sum_{k=-N}^{N} \sum_{i=1}^{3} \left(\frac{\partial E_{pk}}{\partial v_{i+k}}\right)^2 \overline{v^2_{i+k}} \right\} \qquad \text{(C-25)}$$

Since, $\overline{u^2_{i+k}} = \overline{v^2_{i+k}} = \frac{\sigma^2_{ik}}{2}$, it follows

$$\text{Var } (E_p)^{\bullet}_{2N+1} = \frac{1}{2N+1} \sum_{k=-N}^{N} \sum_{i=1}^{3} \left[\left(\frac{\partial E_{pk}}{\partial u_{i+k}}\right)_0^2 + \left(\frac{\partial E_{pk}}{\partial v_{i+k}}\right)_0^2 \right] \frac{\sigma^2_{i+k}}{2} . \qquad \text{(C-26)}$$

$\left(\frac{\partial E_{pk}}{\partial u_{i+k}}\right)_0$ and $\left(\frac{\partial E_{pk}}{\partial v_{i+k}}\right)_0$ can be obtained by taking the partial derivatives of (C-11) with respect to u and v and then substituting 0 for u and v. If the curve modulus variance versus satellite location is available, everything is known in (C-26) except N; that is, $\text{Var } (E_p)^{\bullet}_{2N+1}$ is a function of N. The optimum value N^* is the value of N which minimizes $\text{Var } (E_p)^{\bullet}_{2N+1}$. The best estimate of the parameter p of the elliptical orbit is then $(E_p)^{\bullet}_{2N^*+1}$. Similar results could be obtained for e and p.

D. CONCLUSION.

The purpose of this report was to develop a data smoothing technique for an optimum determination of the orbital parameters of a free flight orbit. A position vector for the satellite is determined from three range measurements, one from each of the three stations. The measured position differs from the true position by an error vector which combines all the various errors. The data smoothing technique developed here differs from conventional sampling theory because the variance of a sample (measured position-vector) is not a constant but a function of the relative position of the satellite and of the station.

If the orbit is circular, the only orbital parameter is the radius. A formula to estimate the radius with one vector position measurement is derived. A better estimate is obtained by using a sample mean of size N^* as an estimate. The optimum number of samples N^*, i.e. the optimum orbital arc length to measure, is determined. An example is completely solved showing that the accuracy increases considerably. The effect of earth rotation is easily accounted for.

If the orbit is elliptical, the three orbital parameters p, e and θ can be determined by three position vector measurements. Formulas for the estimates E_p, E_e and E_θ of p, e and θ in terms of the modulus and angles of the measured position vectors are derived. Since the formulas are very complex, the mean and the variance of the estimates cannot be obtained in closed form. The method proposed is to expand each estimate in powers of the components u_i and $v_i (i = 1, 2, 3)$ of the three corresponding error vectors. A technique of data smoothing is proposed to increase the accuracy. This technique will determine the optimum number of samples for best accuracy and the variance to expect.

The problem of data smoothing of the orbital parameters was completely solved for a circular orbit and a method was developed for an elliptical orbit. Much work remains to be done for the elliptical orbit: more terms in the series development, formulas for all the partial derivatives, and application to specific examples. Only the information contained in the range measurements was used. Obviously, for free flight orbit and equal sampling intervals, the same area is covered between two samples; this information should be used to improve the accuracy. Only free flight orbits have been considered while an increase in the accuracy of the orbit determination during a satellite transit is very desirable.

The data smoothing technique developed in this report is quite powerful and general. It can be used to determine constant parameters, which are explicitly or implicitly functions of sampled values and of random variables. The random variables may be stationary or time-varying. A minimum number of samples is necessary; for example, three position samples (three measured position-vectors) are necessary to determine the orbital parameters. If more samples are available, the accuracy can be increased. It is quite probable that this data smoothing technique could be modified for the determination of time-varying parameters in presence of time varying random variables.

LIST OF SYMBOLS IN PART III

a, d	Modulus of $\vec{OG}$ and $\vec{GM}$
E	Elliptic integral; E_p E_x E_y estimates of p, x, and y
e	Excentricity
G	Center of the station group
k	Subscript denotes successive satellite positions
M, S	Measured and true satellite positions
m, r	Modulus of $\vec{OM}$ and $\vec{OS}$
O	Center of the Earth
p	Elliptical parameter; p( ) Probability density
R	Radius of circular orbit
s	Unbiased estimator for the radius
$\vec{SM}$	Error vector
u, v	Cartesian components of vector error
x, y	Define e and θ $(x = e \cos \theta$, $y = e \sin \theta)$
α, β, γ	Angles $\vec{OG}$, $\vec{OS}$; $\vec{OG}$, $\vec{OM}$; $\vec{OM}$, $\vec{OS}$
θ	Angular position of foci-line with respect to $\vec{OS_0}$
ρ, φ	Modulus and phase of vector error
Ω_G, Ω_S	Angular rotation of station group and satellite
$\cdot$	A dot as a superscript means sample mean.
$-$	A bar as a superscript means average.

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