BAFFLING OF FLUID SLOSHING IN CYLINDRICAL TANKS

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SECTION 1
INTRODUCTION

During the past few years a great deal of research has been conducted on the problem of sloshing of liquid propellants in missile and space vehicles. One of the primary objectives of these investigations was to study the effects of this liquid motion on the dynamic stability of the rocket.

It has been observed that, in any rocket flight, the vehicle body is subjected to translatory and oscillatory perturbations from external forces such as guidance and control inputs. Vehicle body motions of this type result in disturbances of the contained liquid. If the perturbations occur at a frequency near that of the control frequency of the vehicle, the liquid is forced to oscillate at amplitudes sufficiently large to cause severe de-stabilizing forces and moments on the vehicle. Bauer [1] remarks that with the increasing size of space vehicles and their larger tank diameters, which lower the natural frequencies of the propellants, the effects of propellant sloshing upon the stability of the vehicle may become extremely critical. This is especially true at launch since usually more than 90 percent of the total mass is in the form of liquid propellant. With increasing diameter, the oscillating propellant masses and the corresponding forces increase.

One of the methods of limiting the amplitude and duration of these liquid oscillations is the use of an annular baffle. The baffle will break up the flow pattern and thereby dampen the oscillations. Miles [2] gives approximate values for damping due to ring baffles in a right circular cylinder. He relies on a drag coefficient obtained from the experimental data of Keulegan and Carpenter [3]. Cole and Gambucci [4] and [5] conducted experimental tests for measuring the effectiveness of baffles in damping the liquid oscillations. There have been many other tests run to determine the effects of various baffles including [6] and [7].

The purpose of this study is to present a complete theoretical discussion of the effects of an annular ring baffle on the behavior of liquid within a right circular cylinder. The problem is solved by assuming two potential functions; one valid in the region above the baffle and the other valid in the region below the baffle. Matching the two functions in the plane of the baffle leads to a dual series. Having the solution of these series, the displacement of the free surface as well as the liquids' pressure, forces and moments are calculated.

SECTION 2

DEFINITION OF THE BOUNDARY VALUE PROBLEM

This report is concerned with the sloshing of an incompressible, inviscid liquid contained in a right circular cylinder that is mounted in a space vehicle moving along a prescribed path. Because of perturbations due to the deviation of the vehicle from its path, the system oscillates and produces waves on the surface of the liquid. It is proposed that the introduction of an annular baffle below the surface of the liquid will damp the induced oscillations.

Since the tank is in motion along some path, it appears to be reasonable to refer its motion to an inertial system, for example the earth. However, if any type of measuring device is attached to the tank, then it measures quantities in terms of a tank-fixed reference frame which is moving relative to the inertial system. Thus, it is necessary to be able to express the tank-fixed system in terms of the inertial system and vice versa.

Let Y_i, with coordinates y_i ($i = 1, 2, 3$) and origin O', be a fixed Cartesian reference frame and X_i with coordinates x_i and origin O be a Cartesian frame moving relative to Y_i. Then, instantaneously, it follows that

$$y_i = \overline{z}_i + a_{ji} x_j,$$

where the components of $\overline{z}_i$ are measured in Y_i, $\overline{z}_i$ (t) gives the instantaneous dispacement of O relative to O', and

$$a_{ij}(t) = \cos(x_i, y_j)$$

measures the instantaneous rotation of X_i with respect to Y_i.

In the following, a repeated index indicates summation over the range of values of the index. The coordinate transformation simultaneously gives the formulas of transformation for any free vector

$$\overline{B}_i(y) = a_{ji} B_j(x)$$

and

$$B_i(x) = a_{ij}\,\bar{B}_j(y),$$

where $\bar{B}_i$ and B_i are the same vector with components measured respectively in Y_i and X_i.

Since the a_{ij} are a set of direction cosines, they satisfy

$$a_{ik}\,a_{jk} = \delta_{ij}, \qquad a_{ki}\,a_{kj} = \delta_{ij}, \tag{2.1}$$

where δ_{ij} is the Kronecker delta, for any t. Letting $\dot{a}_{ij}$ denote da_{ij}/dt, a simple differentation of (2.1) yields

$$a_{ik}\,\dot{a}_{jk} + \dot{a}_{ik}\,a_{jk} = 0. \tag{2.2}$$

Define $\omega_{ij} = a_{ik}\,\dot{a}_{jk}$ so that according to (2.2)

$$\omega_{ji} = a_{jk}\,\dot{a}_{ik} = -a_{ik}\,\dot{a}_{jk} = -\omega_{ij}.$$

Thus ω_{ij} is a skew-symmetric second-order quantity, which can be shown to be a tensor. Hence there exists a dual vector ω_i defined by

$$\omega_{ij} = -\,\epsilon_{ijk}\,\omega_k,$$

where ϵ_{ijk} is the third-order alternating tensor. Consequently

$$a_{ik}\,\dot{a}_{jk} = -\,\epsilon_{ijk}\,\omega_k,$$

where ω_k may be identified as the angular velocity of X_i with respect to Y_i, measured along X_i.

The absolute velocity of a particle is given by

$$dy_i/dt = \dot{y}_i = \dot{\bar{z}}_i(y) + a_{ji}\,\dot{x}_j + \dot{a}_{ji}\,x_j = \bar{q}_i(y)\ .$$

However, the control instruments measure $q_i(x)$, where

$$q_i(x) = a_{ij}\,\bar{q}_j(y)\ .$$

But observing that q_i is a function of the coordinates x_i as well as t, it is to be noted that

$$\frac{d}{dt} q_i (x) = \frac{\partial q_i}{\partial t} (x) + \frac{\partial q_i}{\partial x_k} \frac{dx_k}{dt} . \tag{2.6}$$

It was shown above that

$$q_i (x) = \dot{z}_i + \dot{x}_i + \epsilon_{ijk} \omega_j x_k$$

or

$$\dot{x}_k = q_k (x) - \dot{z}_k - \epsilon_{kpg} \omega_p x_q ,$$

which is the same as

$$\underline{\dot{r}} = \underline{q} - \underline{\dot{z}} - \underline{\omega} \times \underline{r} . \tag{2.7}$$

Upon insertion of (2.7) into (2.6), the result is

$$\frac{d}{dt} q_i (x) = \frac{\partial q_i}{\partial t} (x) + \frac{\partial q_i}{\partial x_k} [q_k (x) - \dot{z}_k - \epsilon_{kpq} \omega_p x_q],$$

and finally the acceleration is found to be

$$a_i (x) = \frac{\partial q_i}{\partial t} (x) + \epsilon_{ijk} \omega_j q_k (x) + \frac{\partial q_i}{\partial x_k} [q_k (x) - \dot{z}_k (x) - \epsilon_{kpq} \omega_p x_q]$$

or, in vector symbolism,

$$\underline{a} = \frac{\partial \underline{q}}{\partial t} + \underline{\omega} \times \underline{q} + [(\underline{q} - \underline{\dot{z}} - \underline{\omega} \times \underline{r}) \cdot \nabla] \underline{q} .$$

For an incompressible fluid, the Eulerian equations of motion are

$$\bar{a}_i (y) = \bar{F}_i (y) - \frac{1}{\rho} \frac{\partial p}{\partial y_i} (y),$$

where $\bar{F}_i$ is the specific body force, ρ is the density, and p is the pressure.

Therefore

$$q_i(x) = a_{ij}\dot{\bar{z}}_j(y) + a_{ij}a_{kj}\dot{x}_k + a_{ij}\dot{a}_{kj}x_k$$

$$= \dot{z}_i(x) + \dot{x}_i - \epsilon_{jik}\omega_j x_k$$

$$= \dot{z}_i(x) + \dot{x}_i + \epsilon_{ijk}\omega_j x_k. \qquad (2.3)$$

In vector symbolism, (2.3) assumes the form

$$\underline{v} = \dot{\underline{z}} + \dot{\underline{r}} + \underline{\omega} \times \underline{r} ,$$

where $\underline{r}$ is the position vector in X_i. The absolute acceleration is obtained from

$$\bar{a}_i(y) = \frac{d}{dt}\bar{q}_i(y),$$

and the relation

$$\bar{q}_i(y) = a_{ji}q_j(x),$$

which yield

$$\bar{a}_i(y) = \frac{d}{dt}\bar{q}_i(y) = a_{ji}\frac{d}{dt}q_j(x) + \dot{a}_{ji}q_j(x). \qquad (2.4)$$

But since $a_i(x)$ is desired and

$$a_i(x) = a_{ij}\bar{a}_j(y), \qquad (2.5)$$

it follows from (2.4) and (2.5) that

$$a_i(x) = a_{ij}a_{kj}\frac{d}{dt}q_k(x) + a_{ij}\dot{a}_{kj}q_k(x)$$

$$= \frac{d}{dt}q_i(x) - \epsilon_{ikj}\omega_j q_k(x)$$

$$= \frac{d}{dt}q_i(x) + \epsilon_{ijk}\omega_j q_k(x).$$

an observer traveling with the tank, these are, of course, the only forcing motions he sees.

On the wetted surface of the tank, the boundary condition is that the velocity of liquid normal to the tank wall must equal the normal component of velocity of the tank itself (it is assumed here that the tank is rigid). Thus, if $\underline{\nu}$ denotes the unit exterior normal to the tank, then

$$\underline{q} \cdot \underline{\nu} = \underline{\nu} \cdot [\dot{\underline{z}} + \underline{\omega} \times \underline{r}] ,$$

since $\dot{\underline{r}} = 0$ for a rigid tank, or

$$-\underline{\nu} \cdot \nabla\varphi = \underline{\nu} \cdot [\dot{\underline{z}} + \underline{\omega} \times \underline{r}] .$$

There are two conditions at the free surface. If the disturbed free surface is denoted by $\eta(x_1, x_2, t)$ and the unit normal to the quiescent free surface is taken to be $\underline{n} = (0, 0, 1)$, then there exists a kinematic condition such that a particle of fluid that travels with the free surface as it moves must have the same velocity as the free surface itself, i.e., if x_3 is the displacement of a particle in the x_3- direction, then

$$\frac{d}{dt}(x_3 - \eta) \Big|_{x_3 = \eta} = 0. \tag{2.8}$$

Expanding the left-hand side of (2.8), it is found that

$$\left(\frac{dx_3}{dt} - \frac{\partial\eta}{\partial t} - \frac{\partial\eta}{\partial x_1}\dot{x}_1 - \frac{\partial\eta}{\partial x_2}\dot{x}_2\right)_{x_3 = \eta} = 0 ,$$

and since

$$\dot{\underline{r}} = \underline{q} - \dot{\underline{z}} - \underline{\omega} \times \underline{r},$$

it follows that

$$\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x_1}\dot{x}_1 + \frac{\partial\eta}{\partial x_2}\dot{x}_2 = (\underline{q} - \dot{\underline{z}} - \underline{\omega} \times \underline{r}) \cdot \underline{n}\Big|_{x_3 = \eta}.$$

Now

$$\frac{\partial p}{\partial y_i} = \frac{\partial p}{\partial x_k} \frac{\partial x_k}{\partial y_i} = a_{ki} \frac{\partial p}{\partial x_k} \cdot$$

Transforming to body-fixed axes, it is found that

$$a_i (x) = a_{ij} \bar{a}_j (y) = a_{ij} \bar{F}_j (y) - \frac{1}{\rho} a_{ij} a_{kj} \frac{\partial p}{\partial x_k} = F_i (x) - \frac{1}{\rho} \frac{\partial p}{\partial x_i}$$

If it is assumed that the fluid motion is irrotational, then there exists a potential function φ such that

$$\underline{q} = -\nabla \varphi \text{ or } \bar{q}_i (y) = - \frac{\partial \varphi}{\partial y_i} \cdot$$

It is easily verified that

$$q_i (x) = - \frac{\partial \varphi}{\partial x_i}$$

Then, since the fluid is incompressible ($\nabla \cdot \underline{q} = 0$), it follows that

$$\nabla^2 \varphi = 0.$$

Thus the fundamental differential equation to be solved is Laplace's equation for a velocity potential.

To describe the boundary conditions for the problem, consider a tank of arbitrary shape partially filled with liquid. Suppose that a constant acceleration is imposed along an axis of thrust — call it the x_3 axis. Then the liquid assumes a planar surface normal to the thrust axis — herein this surface is called the free surface. Choose the origin of X_i at the center of gravity of the fluid in this configuration. The motion of the tank-filled frame X_i relative to Y_i characterized by $\dot{z}_i$ (x) and ω_i are oscillatory motions superimposed on the constant-acceleration motion. For so that

$$\nabla[\underline{q} \cdot (\underline{\omega} \times \underline{r})] = [(\underline{\omega} \times \underline{r}) \cdot \nabla]\underline{q} + \underline{\omega} \times \underline{q} + 2\,\underline{q} \times \underline{\omega}$$

$$= [(\underline{\omega} \times \underline{r}) \cdot \nabla]\,\underline{q} + \underline{q} \times \underline{\omega} \;. \qquad (2.10)$$

When (2.10) is inserted into (2.9), the equation of motion, the result is

$$\nabla\left[\frac{\partial\varphi}{\partial t} - \frac{1}{2}(\underline{q} - \underline{\dot{z}})^2 - \Omega + \underline{q} \cdot (\underline{\omega} \times \underline{r}) - p/\rho\right] = 0,$$

or upon integration and replacement of Ω by αx_3, this becomes

$$\frac{p - p_o}{\rho} = -\alpha x_3 - \frac{1}{2}(\underline{q} - \underline{\dot{z}})^2 + \underline{q} \cdot (\underline{\omega} \times \underline{r}) + \frac{\partial\varphi}{\partial t}, \qquad (2.11)$$

where p_o is the constant ambient pressure, α is the magnitude of the acceleration of the liquid-tank system. Thus at

$$x_3 = \eta,$$

$$\frac{\partial\varphi}{\partial t} = \alpha\eta + \frac{1}{2}(\nabla\varphi + \underline{\dot{z}})^2 + (\underline{\omega} \times \underline{r}) \cdot \nabla\varphi.$$

For small free surface oscillations, the problem may be linearized. Thus second-order terms in velocities can be neglected and it is assumed that not only η is small, but also $\partial\eta/\partial x_i$, $i = 1, 2$.

Under these conditions, the boundary conditions on the free surface become

$$\frac{\partial\eta}{\partial t} = -\underline{n} \cdot \nabla\varphi - \underline{n} \cdot (\underline{\dot{z}} + \underline{\omega} \times \underline{r}),$$

$$(2.12)$$

$$\frac{\partial\varphi}{\partial t} = \alpha\eta,$$

The second condition is a dynamic one which states that the pressure at the free surface of the liquid must equal the ambient pressure. The form of this condition can be obtained from an integration of the equation of motion. Suppose that the only specific body force is that due to the gravitational field in which the liquid-tank system is operating. The equation of motion is

$$\frac{\partial \underline{q}}{\partial t} + \underline{\omega} \times \underline{q} + [(\underline{q} - \underline{\dot{z}} - \underline{\omega} \times \underline{r}) \cdot \nabla]\underline{q} = \underline{F} - \frac{1}{\rho} \nabla p.$$

Now

$$\underline{q} = - \nabla \varphi, \; \underline{F} = - \nabla \Omega, \frac{1}{\rho} \nabla p = \nabla (\frac{1}{\rho} p),$$

so that the equation of motion assumes the form

$$- \nabla \left(\frac{\partial \varphi}{\partial t} \right) + \underline{\omega} \times \underline{q} + [(\underline{q} - \underline{\dot{z}} - \underline{\omega} \times \underline{r}) \cdot \nabla]\underline{q} = - \nabla (\Omega + p/\rho).$$

Recall the vector identity

$$(\underline{q} - \underline{\dot{z}}) \cdot \nabla \underline{q} = (\underline{q} - \underline{\dot{z}}) \cdot \nabla (\underline{q} - \underline{\dot{z}}) = \frac{1}{2} \nabla (\underline{q} - \underline{\dot{z}})^2$$

and substitute into the differential equation of motion; the result is

$$\underline{\omega} \times \underline{q} - [(\underline{\omega} \times \underline{r}) \cdot \nabla]\underline{q} = \nabla \left(\frac{\partial \varphi}{\partial t} - \frac{1}{2} (\underline{q} - \underline{\dot{z}})^2 - \Omega - p/\rho \right) . \tag{2.9}$$

Consider next

$$\begin{aligned} \nabla [\underline{q} \cdot (\underline{\omega} \times \underline{r})] &= [(\underline{\omega} \times \underline{r}) \cdot \nabla]\underline{q} + (\underline{q} \cdot \nabla) (\underline{\omega} \times \underline{r}) \\ &\quad + (\underline{\omega} \times \underline{r}) \times (\nabla \times \underline{q}) + \underline{q} \times [\nabla \times (\underline{\omega} \times \underline{r})] \\ &= [(\underline{\omega} \times \underline{r}) \cdot \nabla]\underline{q} + (\underline{q} \cdot \nabla) (\underline{\omega} \times \underline{r}) + \underline{q} \times [\nabla \times (\underline{\omega} \times \underline{r})] \end{aligned}$$

since $\nabla \times \underline{q} = \underline{0}$. It is to be noted also that

$$\nabla \times (\underline{\omega} \times \underline{r}) = 2\underline{\omega} \text{ and } (\underline{q} \cdot \nabla) (\underline{\omega} \times \underline{r}) = \underline{\omega} \times \underline{q} ,$$

which are usually combined in the form

$$- \underline{n} \cdot \nabla \varphi - \frac{1}{\alpha} \frac{\partial^2 \varphi}{\partial t^2} = \underline{n} \cdot (\dot{\underline{z}} + \underline{\omega} \times \underline{r}) \Big|_{x_3 = \eta} \cdot$$

If the assumption is made that $\dot{\underline{z}}$ and $\underline{\omega}$ can be represented as harmonic oscillations,

$$\dot{\underline{z}} = \underline{u}\, e^{i\beta t} \text{ and } \underline{\omega} = \underline{\omega}_o\, e^{i\beta t},$$

it is usually assumed that

$$\varphi (x_1, x_2, x_3, t) = \psi (x_1, x_2, x_3)\, e^{i\beta t} .$$

The problem then reduces to solving

$$\nabla^2 \psi = 0$$

subject to

$$- \underline{\nu} \cdot \nabla \psi = \underline{\nu} \cdot (\underline{u} + \underline{\omega}_o \times \underline{r}) \tag{2.13}$$

on the wetted surface and

$$- \underline{n} \cdot \nabla \psi + \frac{\beta^2}{\alpha} \psi = \underline{n} \cdot (\underline{u} + \underline{\omega}_o \times \underline{r}) \tag{2.14}$$

on the free surface.

SECTION 3

PROPELLANT SLOSHING IN A CIRCULAR CYLINDRICAL TANK CONTAINING AN ANNULAR BAFFLE

In an attempt to minimize the effects of propellant sloshing, an annular baffle of infinitesimal thickness and width $(1 - \gamma)a$, $(0 < \gamma < 1)$, is mounted on the wall of a circular cylindrical tank of radius a. It is convenient to set up a coordinate system centered along the axis of symmetry of the cylinder with the origin located in the plane of the baffle. Accordingly, it is expedient to use cylindrical polar coordinates (r, θ, z), so that $z = -h$ denotes the bottom of the tank, $z = 0$ is the plane of the baffle, and $z = z_0$ is the quiescent free surface, whereas $r = a$ refers to the wall of the tank.

Therefore, in terms of cylindrical polar coordinates, the problem under consideration here is the solution of Laplace's equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \tag{3.1}$$

subject to the condition prescribed in (2.13) for the wetted surface of the tank and to the condition prescribed in (2.14) for the free surface.

In order to be more specific about these boundary conditions for the geometry of the tank involved, consider, first of all, the vertical wall of the tank $r = a$. The unit vector normal to the wall of the tank is

$$\underline{\nu} = \cos\theta\,\underline{i} + \sin\theta\,\underline{j} = \underline{e}_r . \tag{3.2}$$

The components of the vectors $\underline{u}$ and $\underline{\omega}_0$ that appear in (2.13) and (2.14) may be expressed as

$$\underline{u} = u_1\underline{i} + u_2\underline{j} + u_3\underline{k} \text{ and } \underline{\omega}_0 = \omega_1\underline{i} + \omega_2\underline{j} + \omega_3\underline{k},$$

where u_i and ω_i $(i = 1, 2, 3)$ are constants. It is evident that

$$-\nu\cdot\nabla\psi = -\frac{\partial\psi}{\partial r} \tag{3.3}$$

and

$$\underline{\nu}\cdot(\underline{u} + \underline{\omega}_o \times \underline{r}) = \underline{\nu}\cdot\underline{u} + \underline{\nu}\cdot(\underline{\omega}_o \times \underline{r})$$

$$= u_1 \cos\theta + u_2 \sin\theta + \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ \omega_1 & \omega_2 & \omega_3 \\ a\cos\theta & a\sin\theta & z \end{vmatrix}$$

$$= u_1 \cos\theta + u_2 \sin\theta + z(\omega_2 \cos\theta - \omega_1 \sin\theta). \tag{3.4}$$

Hence, in view of (2.13), (3.3) and (3.4), on the wall of the tank $r = a$, the boundary condition is

$$\left(\frac{\partial \psi}{\partial r}\right)_{r=a} = -u_1 \cos\theta - u_2 \sin\theta + z(\omega_1 \sin\theta - \omega_2 \cos\theta). \tag{3.5}$$

On the bottom of the tank, where $z=-h$ and $\nu = (0,0,-1)$, it follows again from (2.13) that

$$-\underline{\nu}\cdot\nabla\psi = \frac{\partial \psi}{\partial z} = -u_3 + \begin{vmatrix} 0 & 0 & -1 \\ \omega_1 & \omega_2 & \omega_3 \\ r\cos\theta & r\sin\theta & -h \end{vmatrix}$$

i.e.,

$$\left(\frac{\partial \psi}{\partial z}\right)_{z=-h} = -u_3 - r(\omega_1 \sin\theta - \omega_2 \cos\theta). \tag{3.6}$$

Now consider (2.14) and observe that $\underline{n} = (0,0,1)$ is the unit normal vector to the quiescent free surface $z = z_o$. Thus, this condition becomes

$$\left(\frac{\beta^2}{\alpha}\psi - \frac{\partial \psi}{\partial z}\right)_{z=z_o} = u_3 + \begin{vmatrix} 0 & 0 & 1 \\ \omega_1 & \omega_2 & \omega_3 \\ r\cos\theta & r\sin\theta & z_o \end{vmatrix}$$

or

$$\left(\frac{\beta^2}{\alpha}\psi - \frac{\partial \psi}{\partial z}\right)_{z=z_o} = u_3 + r(\omega_1 \sin\theta - \omega_2 \cos\theta). \tag{3.7}$$

Finally, because of (2.13), the condition

$$\left(\frac{\partial \psi}{\partial z}\right)_{z=0} = -u_3 + r(\omega_2 \cos\theta - \omega_1 \sin\theta) \tag{3.8}$$

holds on the baffle, i.e., for $\gamma a < r < a$, where ν has been taken to be $(0,0,\pm 1)$.

Furthermore, it seems to be reasonable that the z-component of the fluid particle velocity vector should be continuous in the region $z = 0$, $0 \le r < \gamma a$ which corresponds to the opening in the baffle; and ψ itself should be continuous there. In other words, for $z = 0$ and $0 \le r < \gamma a$, the conditions

$$\frac{\partial \psi}{\partial z}(r, \theta, 0+) = \frac{\partial \psi}{\partial z}(r, \theta, 0-)$$

$$\psi(r, \theta, 0+) = \psi(r, \theta, 0-) \tag{3.9}$$

must hold.

Therefore, the linearized sloshing problem for a cylindrical tank containing an annular baffle is now completely specified in (3.1) to (3.9), inclusive.

The following discussion treats only the case of pure translational oscillations ($\underline{\omega} = \underline{0}$). The details of the solution of the problem for the case of pure rotational oscillations ($\underline{u} = \underline{0}$) are essentially the same, and by the superposition principle the solutions for the two cases may be added to obtain the complete solution of the sloshing problem involving both translational and rotational oscillations of the tank. Therefore the problem is reduced to the solution of Laplace's equation

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} = 0,$$

subject to the prescribed conditions

$$\left(\frac{\partial \psi}{\partial r}\right)_{r=a} = -u_1 \cos\theta - u_2 \sin\theta, \; 0 \le \theta \le 2\pi, \; -h \le z < 0, \; 0 < z < z_o$$

$$\left(\frac{\partial \psi}{\partial z}\right)_{z=0} = -u_3, \; \gamma a < r < a, \; 0 \le \theta \le 2\pi,$$

$$\left(\frac{\partial \psi}{\partial z}\right)_{z=-h} = -u_3, \; 0 \le r < a, \; 0 \le \theta \le 2\pi,$$

$$\left(\frac{\beta^2}{a}\psi - \frac{\partial \psi}{\partial z}\right)_{z=z_o} = u_3, \; 0 \le r < a, \; 0 \le \theta \le 2\pi,$$

$$\psi(r, \theta, 0-) = \psi(r, \theta, 0+), \; 0 \le r < \gamma a, \; 0 \le \theta \le 2\pi,$$

$$\frac{\partial \psi}{\partial z}(r, \theta, 0-) = \frac{\partial \psi}{\partial z}(r, \theta, 0+), \; 0 \le r < \gamma a, \; 0 \le \theta \le 2\pi.$$

From a mathematical point of view, it is necessary to treat the tank as though it were composed of two regions, one above the baffle labeled I and the second below the baffle labeled II, which leads to two "potential functions" $\psi_1(r, \theta, z)$ and $\psi_2(r, \theta, z)$ for regions I and II, respectively, defined such that

$$\psi(r, \theta, z) = \begin{cases} \psi_1(r, \theta, z) & \text{for } 0 \le z \le z_o, \\ \psi_2(r, \theta, z) & \text{for } -h \le z \le 0, \end{cases}$$

where ψ_1 satisfies Laplace's equation

$$\frac{\partial^2 \psi_1}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_1}{\partial \theta^2} + \frac{\partial^2 \psi_1}{\partial z^2} = 0$$

and the boundary conditions

$$\left(\frac{\partial \psi_1}{\partial r}\right)_{r=a} = -u_1 \cos\theta - u_2 \sin\theta$$

$$\left(\frac{\partial \psi_1}{\partial z}\right)_{z=0} = -u_3, \; \gamma a < r < a, \qquad (3.10)$$

$$\left(\frac{\beta^2}{\alpha}\psi_1 - \frac{\partial \psi_1}{\partial z}\right)_{z=z_o} = u_3,$$

and ψ_2 also satisfies Laplace's equation

$$\frac{\partial^2 \psi_2}{\partial r^2} + \frac{1}{r}\frac{\partial \psi_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi_2}{\partial \theta^2} + \frac{\partial^2 \psi_2}{\partial z^2} = 0$$

plus the boundary conditions

$$\left(\frac{\partial \psi_2}{\partial r}\right)_{r=a} = -u_1 \cos \theta - u_2 \sin \theta$$

$$\left(\frac{\partial \psi_2}{\partial z}\right)_{z=-h} = -u_3$$

$$\left(\frac{\partial \psi_2}{\partial z}\right)_{z=0} = -u_3, \gamma a < r < a. \tag{3.11}$$

In addition to these conditions, the functions ψ_1 and ψ_2 must also satisfy the "continuity conditions"

$$\left.\begin{aligned} &\psi_1(r, \theta, 0) = \psi_2(r, \theta, 0) \\ &\frac{\partial \psi_1}{\partial z}(r, \theta, 0) = \frac{\partial \psi_2}{\partial z}(r, \theta, 0) \end{aligned}\right\} \text{ for } 0 \le r < \gamma a.$$

Since none of the boundary conditions embodied in (3.10) and (3.11) are homogeneous, it is convenient to make a change of dependent variable such that certain transformed boundary conditions are homogeneous. This is accomplished by defining a function $\bar{\varphi}_1$ (r, θ , z), so that

$$\bar{\varphi}_1 (r, \theta, z) = r(u_1 \cos\theta + u_2 \sin \theta) + u_3 z + \psi_1(r, \theta, z) \tag{3.12}$$

A few simple calculations lead to the new boundary value problem

$$\frac{\partial^2 \bar{\varphi}_1}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\varphi}_1}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\varphi}_1}{\partial \theta^2} + \frac{\partial^2 \bar{\varphi}_1}{\partial z^2} = 0, \tag{3.13}$$

$$\left(\frac{\partial \bar{\varphi}_1}{\partial r}\right)_{r=a} = 0,$$

$$\left(\frac{\partial \bar{\varphi}_1}{\partial z}\right)_{z=0} = 0, \qquad \text{for } \gamma a < r < a \tag{3-14}$$

$$\left(\frac{\beta^2}{a}\bar{\varphi}_1 - \frac{\partial \bar{\varphi}_1}{\partial z}\right)_{z=z_o} = \frac{\beta^2}{a}\left[u_3 z_o + r(u_1 \cos\theta + u_2 \sin\theta) \right] \tag{3-15}$$

In a similar fashion, defining for region II the function $\bar{\varphi}_2 (r, \theta, z)$ by

$$\bar{\varphi}_2(r, \theta, z) = \psi_2(r, \theta, z) + r(u_1 \cos\theta + u_2 \sin\theta) + u_3 z, \tag{3.16}$$

a new problem is obtained:

$$\frac{\partial^2 \bar{\varphi}_2}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\varphi}_2}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \bar{\varphi}_2}{\partial \theta^2} + \frac{\partial^2 \bar{\varphi}_2}{\partial z^2} = 0, \tag{3.17}$$

$$\left(\frac{\partial \bar{\varphi}_2}{\partial r}\right)_{r=a} = 0,$$

$$\left(\frac{\partial \bar{\varphi}_2}{\partial z}\right)_{z=-h} = 0, \tag{3.18}$$

$$\left(\frac{\partial \bar{\varphi}_2}{\partial z}\right)_{z=0} = 0, \qquad \text{for } \gamma a < r < a. \tag{3.19}$$

From the conditions of continuity, it follows that

$$\left.\begin{aligned} \bar{\varphi}_1 (r, \theta, 0) &= \bar{\varphi}_2 (r, \theta, 0) \\ \frac{\partial \bar{\varphi}_1}{\partial z}(r, \theta, 0) &= \frac{\partial \bar{\varphi}_2}{\partial z}(r, \theta, 0) \end{aligned}\right\} \text{ for } 0 \le r < \gamma a. \tag{3.20}$$

Separating the variables in (3.13) in the classical fashion, it is possible to show, after some labor, that the solution of Laplace's equation assumes the form

$$\bar{\varphi}_1(r, \theta, z) = A_o + \sum_{m=1}^{\infty} \left[A_m \cosh(\xi_m z/a) + B_m \sinh(\xi_m z/a) \right][$$

$$\cos\theta\, J_1(\xi_m r/a)] + \sum_{m=1}^{\infty} [C_m \cosh(\xi_m z/a)$$

$$+ D_m \sinh(\xi_m z/a)] \sin\theta\, J_1(\xi_m r/a) \tag{3.21}$$

where A_o, A_m, B_m, C_m, D_m are constants, $J_1(\xi_m r/a)$ is the Bessel function of the first kind of order one, and the ξ_m $(m = 1,2,3...)$ are the positive roots of the transcendental equation

$$J_1'(x) = 0.$$

In view of the boundary condition in (3.14) the function $\bar{\varphi}_1$ must satisfy the condition

$$\left(\frac{\partial\bar{\varphi}_1}{\partial z}\right)_{z=0} = \sum_{m=1}^{\infty} \frac{1}{a}\xi_m B_m \cos\theta\, J_1(\xi_m r/a)$$

$$+\sum_{m=1}^{\infty} \frac{1}{a}\xi_m D_m \sin\theta\, J_1(\xi_m r/a) = 0, \tag{3.22}$$

for $\gamma a < r < a$. Multiply (3.22) by $\cos\theta$ and integrate with respect to θ from 0 to 2π to obtain

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = 0, \text{ for } \gamma a < r < a. \tag{3.23}$$

A similar statement results for the term containing $\sin\theta$, i.e.,

$$\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = 0, \text{ for } \gamma a < r < a. \tag{3.24}$$

From the free surface condition given by (3.15), it follows that

$$\left(\frac{\beta^2}{\alpha}\bar{\varphi}_1 - \frac{\partial\bar{\varphi}_1}{\partial z}\right)_{z=z_o} = \frac{\beta^2}{\alpha} A_o + \sum_{m=1}^{\infty} [A_m \text{chsh}(m;z_o) + B_m \text{shch}(m;z_o)][$$

$$\cos\theta \; J_1(\xi_m r/a)] \; + \sum_{m=1}^{\infty} [C_m \, \text{chsh} \, (m;z_o) + C_m \, \text{shch} \, (m;z_o)][$$

$$\sin\theta \; J_1(\xi_m r/a)] \; = \frac{\beta^2}{\alpha}[u_3 z_o + r(u_1 \cos\theta + u_2 \sin\theta)], \tag{3.25}$$

where

$$\text{chsh}(m;z_o) = \frac{\beta^2}{\alpha}\cosh(\xi_m z_o/a) - \frac{1}{a}\,\xi_m \sinh(\xi_m z_o/a)$$

and

$$\text{shch}(m;z_o) = \frac{\beta^2}{\alpha}\sinh(\xi_m z_o/a) - \frac{1}{a}\,\xi_m \cosh(\xi_m z_o/a). \tag{3.26}$$

Integrating (3.25) with respect to θ from 0 to 2π, it is easily found that

$$A_o = u_3 z_o. \tag{3.27}$$

If (3.25) is multiplied by $\cos\theta$ and integrated with respect to θ from 0 to 2π, it becomes

$$\sum_{m=1}^{\infty} [A_m \, \text{chsh}(m;z_o) + B_m \, \text{shch}(m;z_o)] \, J_1(\xi_m r/a) = \frac{\beta^2}{\alpha} u_1 r; \tag{3.28}$$

and, in turn, if (3.28) is multiplied by $rJ_1(\xi_k r/a)$ and integrated with respect to r from 0 to a, then

$$[A_m \, \text{chsh}(m;z_o) + B_m \, \text{shch}(m;z_o)] \int_0^a rJ_1^2(\xi_m r/a)dr$$

$$= \frac{\beta^2}{\alpha} u_1 \int_0^a r^2 J_1(\xi_m r/a)dr,$$

making use of the fact that the set $\{J_1(\xi_m r/a)\}$ is a complete orthogonal set on the interval (0,a). Using the well-known integrals

$$\int_0^a rJ_1^2(\xi_m r/a)dr = \frac{a^2}{2}\,\frac{(\xi_m^2 - 1)}{\xi_m^2}\, J_1^2(\xi_m) \tag{3.29}$$

and

$$\int_0^a r^2 J_1(\xi_m r/a)dr = \frac{a^3}{\xi_m^2} J_1(\xi_m), \tag{3.30}$$

it follows immediately that

$$A_m \operatorname{chsh}(m;z_o) + B_m \operatorname{shch}(m;z_o) = \frac{2a\beta^2}{\alpha(\xi_m^2 - 1)\, J_1(\xi_m)} u_1. \tag{3.31}$$

Proceeding in an analogous fashion with the term in (3.25) containing sin θ, a similar relation is obtained:

$$C_m \operatorname{chsh}(m;z_o) + D_m \operatorname{shch}(m;z_o) = \frac{2a\beta^2}{\alpha(\xi_m^2 - 1)\, J_1(\xi_m)} u_2, \tag{3.32}$$

where again (3.29) and (3.30) have been employed. Inserting (3.27), (3.31) and (3.32) into (3.21) and rearranging terms, the expression for the function $\bar{\varphi}_1(r, \theta, z)$ now assumes the form

$$\bar{\varphi}_1(r,\theta,z) = u_3 z_o + \frac{2a\beta^2}{\alpha}(u_1 \cos\theta + u_2 \sin\theta)\sum_{m=1}^{\infty}$$

$$\frac{\cosh(\xi_m z/a)\; J_1(\xi_m r/a)}{J_1(\xi_m)(\xi^2 - 1)\operatorname{chsh}(m;z_o)} - \sum_{m=1}^{\infty} \frac{(B_m \cos\theta + D_m \sin\theta)}{\operatorname{chsh}(m;z_o)}$$

$$\operatorname{shch}(m;z_o - z)\; J_1(\xi_m r/a). \tag{3.33}$$

At this point, no specific statements can yet be made regarding the coefficients B_m and D_m which appear in (3.33), despite the fact that these coefficients satisfy (3.23) and (3.24) respectively. It is necessary now to investigate the function $\bar{\varphi}_2(r, \theta, z)$ as well as the continuity conditions given in (3.20) in order to determine the coefficients B_m and D_m.

In analogy to (3.21), the solution of (3.17) may be expressed in the form

$$\bar{\varphi}_2(r,\theta,z) = E_o + \sum_{m=1}^{\infty} [F_m \cosh(\xi_m z/a) + G_m \sinh(\xi_m z/a)][\cos\theta\, J_1(\xi_m r/a)] + \sum_{m=1}^{\infty} [H_m \cosh(\xi_m z/a) + K_m \sinh(\xi_m z/a)][\sin\theta\, J_1(\xi_m r/a)]. \quad (3.34)$$

According to (3.18), it follows that

$$\sum_{m=1}^{\infty} \xi_m[-F_m \sinh(\xi_m h/a) + G_m \cosh(\xi_m h/a)]\cos\theta\, J_1(\xi_m r/a) + \sum_{m=1}^{\infty} \xi_m[-H_m \sinh(\xi_m h/a) + K_m \cosh(\xi_m h/a)]\sin\theta\, J_1(\xi_m r/a) = 0,$$

which leads immediately to

$$F_m \sinh(\xi_m h/a) = G_m \cosh(\xi_m h/a)$$

$$H_m \sinh(\xi_m h/a) = K_m \cosh(\xi_m h/a)$$

because of the orthogonality property of the trigonometric and Bessel functions. Consequently (3.34) may be put in the somewhat more compact form

$$\bar{\varphi}_2(r,\theta,z) = E_0 + \sum_{m=1}^{\infty} \frac{(G_m \cos\theta + K_m \sin\theta)\cosh[\xi_m (h+z)/a]\, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)}$$

For $z = 0$ and $\gamma a < r < a$, (3.19) comes into play and yields

$$\left(\frac{\partial\bar{\varphi}_2}{\partial z}\right)_{z=0} = \sum_{m=1}^{\infty} \frac{1}{a}\xi_m(G_m \cos\theta + K_m \sin\theta)\, J_1(\xi_m r/a) = 0,$$

and therefore

$$\sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a) = 0, \quad (3.35)$$

for $\gamma a < r < a$. The so-called continuity conditions given in (3.20) may now be applied. Setting $\bar{\varphi}_1 = \bar{\varphi}_2$ at $z = 0$ for $0 \leq r < \gamma a$, it follows that

$$u_3 z_o + \frac{2a\beta^2}{\alpha}(u_1 \cos\theta + u_2 \sin\theta) \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\,\mathrm{chsh}(m;z_o)}$$

$$- \sum_{m=1}^{\infty} \frac{(B_m \cos\theta + D_m \sin\theta)}{\mathrm{chsh}(m;z_0)} \mathrm{shch}(m;z_o)\, J_1(\xi_m r/a) =$$

$$E_o + \sum_{m=1}^{\infty} \frac{(G_m \cos\theta + K_m \sin\theta)}{\sinh(\xi_m h/a)} \cosh(\xi_m h/a)\, J_1(\xi_m r/a).$$

The orthogonality properties of the trigonometric functions may next be exploited to obtain the result

$$E_o = u_3 z_o$$

as well as

$$\sum_{m=1}^{\infty} B_m \frac{\mathrm{shch}(m;z_o)}{\mathrm{chsh}(m;z_o)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} G_m \coth(\xi_m h/a)\, J_1(\xi_m r/a)$$

$$= \frac{2a\beta^2}{\alpha} u_1 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\,\mathrm{chsh}(m;z_o)}, \qquad (3.36)$$

$$\sum_{m=1}^{\infty} D_m \frac{\mathrm{shch}(m;z_o)}{\mathrm{chsh}(m;z_0)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} K_m \coth(\xi_m h/a)\, J_1(\xi_m r/a)$$

$$= \frac{2a\beta^2}{\alpha} u_2 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\,\mathrm{chsh}(m;z_o)} \qquad (3.37)$$

which are valid for $0 \le r < \gamma a$. Matching the derivatives according to (3.20) and again observing the orthogonality of the trigonometric functions on the interval $(0, 2\pi)$, it can be shown that

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a), \quad 0 \le r < \gamma a,$$

and

$$\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a), \quad 0 \le r < \gamma a.$$

In summary, then, the pertinent results are

$$\sum_{m=1}^{\infty} B_m \frac{\mathrm{shch}(m;z_o)}{\mathrm{chsh}(m;z_o)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} G_m \coth(\xi_m h/a) J_1(\xi_m r/a)$$

$$= \frac{2a\beta^2}{\alpha} u_1 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)}, \quad 0 \le r < \gamma a \qquad (3.38)$$

$$\sum_{m=1}^{\infty} D_m \frac{\mathrm{shch}(m;z_o)}{\mathrm{chsh}(m;z_o)} J_1(\xi_m r/a) + \sum_{m=1}^{\infty} K_m \coth(\xi_m h/a) J_1(\xi_m r/a)$$

$$= \frac{2a\beta^2}{\alpha} u_2 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)}, \quad 0 \le r < \gamma a \qquad (3.39)$$

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a), \quad 0 \le r < \gamma a \qquad (3.40)$$

$$\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a), \quad 0 \le r < \gamma a$$

$$\sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a) = 0, \quad \gamma a < r < a, \qquad (3.35)$$

$$\sum_{m=1}^{\infty} \xi_m K_m J_1(\xi_m r/a) = 0, \quad \gamma a < r < a,$$

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = 0, \; \gamma a < r < a, \tag{3.23}$$

$$\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = 0, \; \gamma a, < r < a, \tag{3.24}$$

Suppose now that the infinite series in (3.40) converge to a function F (r), i.e.,

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = \sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a) = F(r), \; 0 \le r < \gamma a. \tag{3.41}$$

But according to (3.23), (3.35) and (3.41), it is observed that

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = \begin{cases} F(r), & 0 \le r < \gamma a, \\ 0, & \gamma a < r < a, \end{cases} \tag{3.42}$$

and

$$\sum_{m=1}^{\infty} \xi_m G_m J_1(\xi_m r/a) = \begin{cases} F(r), & 0 \le r < \gamma a \\ 0, & \gamma a < r < a. \end{cases} \tag{3.43}$$

From (3.42) and (3.43), it is apparent that $G_m = B_m$ since the two infinite series converge to the same function in the fundamental interval (0, a). In an analogous fashion, it can also be shown that $D_m = K_m$ (m = 1,2,3,...). Because of these last two statements (3.23), (3.24), (3.38) and (3.39) can be collected in the forms

$$\sum_{m=1}^{\infty} \frac{B_m \operatorname{chsh}(m;z_o + h)\, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)\operatorname{chsh}(m;z_o)}$$

$$= \frac{2a\beta^2}{\alpha} u_1 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\operatorname{chsh}(m;z_o)} \quad , \; 0 \le r < \gamma a,$$

$$\sum_{m=1}^{\infty} \xi_m B_m J_1(\xi_m r/a) = 0, \; \gamma a < r < a \tag{3.23}$$

and

$$\sum_{m=1}^{\infty} \frac{D_m \operatorname{chsh}(m;z_o + h)\, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)\operatorname{chsh}(m;z_o)}$$

$$= \frac{2a\beta^2}{\alpha} u_2 \sum_{m=1}^{\infty} \frac{J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)}, \quad 0 \le r < \gamma a,$$

$$\sum_{m=1}^{\infty} \xi_m D_m J_1(\xi_m r/a) = 0, \quad \gamma a < r < a.$$

Therefore, the coefficients B_m and D_m must be determined from these last four equations, which reduce to the single pair

$$\sum_{m=1}^{\infty} \frac{B_m^* \,\mathrm{chsh}(m;z_o + h)\, J_1(\xi_m x)}{\sinh(\xi_m h/a)\mathrm{chsh}(m;z_o)}$$

$$= \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)}, \quad 0 \le x < \gamma, \tag{3.44}$$

$$\sum_{m=1}^{\infty} \xi_m B_m^* J_1(\xi_m x) = 0, \quad \gamma < x < 1, \tag{3.45}$$

upon setting $x = r/a$ and defining $B_m = 2a\beta^2 u_1 B_m^*/\alpha$ and $D_m = 2a\beta^2 u_2 B_m^*/\alpha$. (3.44) and (3.45) may be called a dual series pair, and, since the ξ_m are the zeros of $J_1'(x) = 0$, the pair may be termed a dual Dini series. The solution of the dual series for the coefficients B_m^* presents a real problem in itself and will be considered in some detail in Section 5. Thus, once the values of B_m^* $(m = 1,2,3,\ldots)$ have been determined, all the required coefficients in the expressions for the functions $\bar{\varphi}_1(r,\theta,z)$ and $\bar{\varphi}_2(r,\theta,z)$ will be known.

SECTION 4

FREE SURFACE DISPLACEMENTS, PRESSURE, FORCES, AND MOMENTS

Supposing for the moment that the dual Dini series embodied in (3.44) and (3.45) have been solved for the coefficients B_m^*, the functions $\bar{\varphi}_1(r, \theta, z)$ and $\bar{\varphi}_2(r, \theta, z)$ may now be expressed in the form

$$\bar{\varphi}_1(r,\theta,z) = u_3 z_o + \frac{2a\beta^2}{\alpha}(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a)\, J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)} - \sum_{m=1}^{\infty} \frac{B_m^* \mathrm{shch}(m;z_o - z)\, J_1(\xi_m r/a)}{\mathrm{chsh}(m;z_o)}\right],$$

and

$$\bar{\varphi}_2(r,\theta,z) = u_3 z_o + \frac{2a\beta^2}{\alpha}(u_1 \cos\theta + u_2 \sin\theta) \sum_{m=1}^{\infty} \frac{B_m^* \cosh[\xi_m (h+z)/a]\, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)},$$

and, by (3.12) and (3.16), the functions $\psi_1(r,\theta,z)$ and $\psi_2(r,\theta,z)$ are defined by

$$\psi_i(r,\theta,z) = \bar{\varphi}_i(r,\theta,z) - r(u_1 \cos\theta + u_2 \sin\theta) - u_3 z, \tag{4.1}$$

for $i = 1,2$. Thus

$$\psi_1(r,\theta,z) = u_3(z_o - z) + \frac{2a\beta^2}{\alpha}(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a)\, J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)} - \sum_{m=1}^{\infty} \frac{B_m^* \mathrm{shch}(m;z_o - z)\, J_1(\xi_m r/a)}{\mathrm{chsh}(m;z_o)} - \frac{\alpha r}{2a\beta^2}\right] \tag{4.2}$$

and

$$\psi_2(r,\theta,z) = u_3(z_o - z) + \frac{2a\beta^2}{\alpha}(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{B_m^* \cosh(\xi_m(h+z)/a)\, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)} - \frac{\alpha r}{2a\beta^2}\right]. \tag{4.3}$$

Next define

$$\varphi(r,\theta,z,t) = \begin{cases} \varphi_1(r,\theta,z,t), & 0 < z \le z_o, \\ \varphi_2(r,\theta,z,t), & -h \le z < 0, \end{cases}$$

so that $\varphi_j(r,\theta,z,t) = e^{i\beta t}\psi_j(r,\theta,z)$, ($j = 1,2$ and $i = \sqrt{-1}$). Then from (2.12) the free surface displacement of the fluid in the baffled tank may be computed as

$$\eta(r,\theta,t) = \frac{1}{\alpha}\left(\frac{\partial\varphi_1}{\partial t}\right)_{z=z_o} = \frac{i\beta}{\alpha} e^{i\beta t} \psi_1(r,\theta,z_o)$$

$$= \frac{2ia\beta^3}{\alpha^2}(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{\cosh(\xi_m z_o/a)\, J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)} + \frac{1}{a}\sum_{m=1}^{\infty} \frac{B_m^* \xi_m J_1(\xi_m r/a)}{\mathrm{chsh}(m;z_o)} - \frac{\alpha r}{2a\beta^2}\right],$$

where ψ_1 as given in (4.2) has been utilized.

The linearized form of (2.11) is

$$p(r,\theta,z,t) = p_o - \rho\left(\alpha z - \frac{\partial\varphi}{\partial t}\right), \tag{4.4}$$

and hence the expression for the pressure may more conveniently be written in the form

$$p(r,\theta,z,t) = p_o - \rho\,[\alpha z - i\beta\, e^{i\beta t}\, \psi_j(r,\theta,z)],\ (j = 1,2), \tag{4.5}$$

in virtue of (4.1) and (4.4). Thus, for the region above the baffle, the appropriate expression for the pressure in the fluid is

$$p_1(r,\theta,z,t) = p_o - \rho\,[\alpha z - i\beta e^{i\beta t}\, u_3(z_o - z)]$$

$$+ \frac{2i\rho a\beta^3}{\alpha}\, e^{i\beta t}\,(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a)\, J_1(\xi_m r/a)}{J_1(\xi_m)(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)}\right.$$

$$\left. - \sum_{m=1}^{\infty} \frac{B_m^*\mathrm{shch}(m;z_o - z)\, J_1(\xi_m r/a)}{\mathrm{chsh}(m;z_o)} - \frac{\alpha r}{2a\beta^2}\right].$$

For $r = a$, the pressure on the wall of the tank above the baffle is

$$p_1(a,\theta,z,t) = p_o - \rho\,[\alpha z - i\beta\, e^{i\beta t}\, u_3(z_o - z)] + \frac{2i\rho a\beta^3}{\alpha}\, e^{i\beta t}$$

$$(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{\cosh(\xi_m z/a)}{(\xi_m^2 - 1)\mathrm{chsh}(m;z_o)}\right.$$

$$\left. - \sum_{m=1}^{\infty} \frac{B_m^*\,\mathrm{shch}(m;z_o - z)\, J_1(\xi_m)}{\mathrm{chsh}(m;z_o)} - \frac{\alpha}{2\beta^2}\right]. \tag{4.6}$$

From (4.3) and (4.5), the expression for the pressure in the fluid below the baffle is found to be

$$p_2(r,\theta,z,t) = p_o - \rho[\alpha z - i\beta e^{i\beta t} u_3\,(z_o - z)] + \frac{2i\rho a\beta^3}{\alpha}\, e^{i\beta t}(u_1 \cos\theta$$

$$+ u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{B_m^*\, \cosh[\xi_m(z+h)/a]\, J_1(\xi_m r/a)}{\sinh(\xi_m h/a)} - \frac{\alpha r}{2a\beta^2}\right]; \tag{4.7}$$

and consequently, on the wall r = a, the pressure distribution is

$$p_2(a,\theta,z,t) = p_o - \rho[\alpha z - i\beta\, e^{i\beta t}\, u_3(z_o - z)]$$

$$+ \frac{2i\rho a\beta^3}{\alpha} e^{i\beta t}(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{B^*_m \cosh[(\xi_m(z+h)/a]\, J_1(\xi_m)}{\sinh(\xi_m h/a)} - \frac{\alpha}{2\beta^2}\right]. \quad (4.8)$$

Setting z = -h in (4.7), the pressure on the bottom of the tank is

$$p(r,\theta,-h,t) = p_o - \rho[-\alpha h - i\beta\, e^{i\beta t}\, u_3(h + z_o)]$$

$$+ \frac{2i\rho a\beta^3}{\alpha} e^{i\beta t}(u_1 \cos\theta + u_2 \sin\theta)\left[\sum_{m=1}^{\infty} \frac{B^*_m J_1(\xi_m r/a)}{\sinh(\xi_m h/a)} - \frac{\alpha r}{2a\beta^2}\right]. \quad (4.9)$$

The net force $\underline{F}$ acting on an area S can be computed from the surface integral

$$\underline{F} = \int_S (p - p_o)\, \underline{\nu}\, dS, \quad (4.10)$$

where $p - p_o$ denotes the net pressure at a point and $\underline{\nu}$ is the unit exterior normal to the surface S.

The unit normal to the vertical wall (r = a) of the tank is

$$\underline{\nu} = \cos\theta\, \underline{i} + \sin\theta\, \underline{j}, \quad (4.11)$$

so that the force acting on the wall above the baffle can be found from (4.10) upon replacement of $p - p_o$ from (4.6); this leads to

$$\underline{F} = \frac{2i\rho a\beta^3}{\alpha} e^{i\beta t}\int_0^{2\pi} (u_1 \cos\theta + u_2 \sin\theta)(\cos\theta\, \underline{i} + \sin\theta\, \underline{j})\, d\theta$$

$$\int_0^{z_o}\left[\sum_{m=1}^{\infty}\frac{\cosh(\xi_m z/a)}{(\xi_m^2-1)\mathrm{chsh}(m;z_o)}-\sum_{m=1}^{\infty}\frac{B_m^*\,\mathrm{shch}(m;z_o-z)\,J_1(\xi_m)}{\mathrm{chsh}(m;z_o)}\right.$$

$$\left.-\frac{\alpha}{2\beta^2}\right]adz,$$

where $dS = ad\theta dz$, $0 \le \theta \le 2\pi$, and $0 \le z \le z_o$ have been used. If $\underline{F} = F_1\underline{i} + F_2\underline{j}$ and $\underline{u} = u_1\underline{i} + u_2\underline{j}$, then for $j = 1,2$,

$$F_j = \frac{2i\pi a^2\rho\beta^3}{\alpha}u_j\;e^{i\beta t}\left[a\sum_{m=1}^{\infty}\frac{\sinh(\xi_m z_o/a)}{\xi_m(\xi_m^2-1)\mathrm{chsh}(m;z_o)}\right.$$

$$\left.-\sum_{m=1}^{\infty}\frac{B_m^*\left[\frac{a\beta^2}{\alpha\xi_m}\left(\cosh(\xi_m z_o/a)-1\right)-\sinh(\xi_m z_o/a)\right]J_1(\xi_m)}{\mathrm{chsh}(m;z_o)}-\frac{\alpha a z_o}{2\beta^2}\right],$$

where in view of (3.26)

$$\int_0^{z_o}\mathrm{shch}(m;z_o-z)dz = \frac{a\beta^2}{\alpha\xi_m}\left[\cosh(\xi_m z_o/a)-1\right]-\sinh(\xi_m z_o/a)$$

In the region $-h \le z < 0$ on the wall $r = a$, the net force is found to be

$$F_j = \frac{2i\pi a^2\rho\beta^3}{\alpha}u_j\;e^{i\beta t}\left[a\sum_{m=1}^{\infty}\frac{B_m^*\;J_1(\xi_m)}{\xi_m}-\frac{\alpha h}{2\beta^2}\right],\; j = 1,2,$$

using (4.8), (4.10) and (4.11).

Finally, on the bottom of the tank, the unit exterior normal is $-\underline{k}$ and the pressure difference is given by (4.9), so that the net force acting on that portion of the tank is, since $dS = rdrd\theta$,

$$\underline{F} = \rho\underline{k}\int_0^{2\pi}d\theta\int_0^a\left[-\alpha h - i\beta e^{i\beta t}u_3(h+z_0)\right]rdr$$

$$= \pi a^2\rho\left[-i\beta\,e^{i\beta t}\;u_3(h+z_o)-\alpha h\right]\underline{k}.$$

Letting $\underline{\nu}$ denote the unit normal vector to a surface S, $\underline{r}$ the position vector of a point, and $p - p_o$ the pressure difference at a point on the surface, the moment, $\underline{M}$, of force taken with respect to the origin of coordinates may be computed from the surface integral

$$\underline{M} = + \int_S (p - p_o)\, \underline{r} \times \underline{\nu}\, dS. \tag{4.12}$$

For the portion of the wall of the tank above the baffle, the appropriate pressure distribution is given by (4.6) and by (4.11), so that

$$\underline{M} = \frac{2ia^2 \rho \beta^3 \pi}{\alpha} e^{i\beta t} (-u_2 \underline{i} + u_1 \underline{j}) \left[\sum_{m=1}^{\infty} \frac{\int_0^{z_o} z \cosh(\xi_m z/a) dz}{(\xi_m^2 - 1) \mathrm{chsh}(m;z_o)} \right.$$

$$\left. - \sum_{m=1}^{\infty} \frac{B_m^* J_1(\xi_m) \int_0^{z_o} z\, \mathrm{shch}(m;z_o - z) dz}{\mathrm{chsh}(m;z_o)} - \frac{\alpha z_o^2}{4\beta^2} \right]$$

$$= \frac{2\pi a^2 i \beta^3 \rho}{\alpha} e^{i\beta t} (-u_2 \underline{i} + u_1 \underline{j}) \left[\vphantom{\sum} \right.$$

$$a \sum_{m=1}^{\infty} \frac{[\xi_m z_o \sinh \xi_m z_o/a - a(\cosh \xi_m z_o/a - 1)]}{\xi_m^2 (\xi_m^2 - 1) \mathrm{chsh}(m;z_o)}$$

$$-a \sum_{m=1}^{\infty} \frac{B_m^* J_1(\xi_m)[\xi_m(1 - \cosh \xi_m z_o/a) - \frac{\beta^2}{\alpha}(\xi_m z_o - a \sinh \xi_m z_o/a)]}{\xi_m^2\, \mathrm{chsh}(m;z_o)}$$

$$\left. - \frac{\alpha z_o^2}{4\beta^2} \right],$$

where the integrals

$$\int_0^{z_o} z \cosh \xi_m (z_o - z)/a \, dz = -\frac{a^2}{\xi_m^2}\left[1 - \cosh(\xi_m z_o/a)\right]$$

$$\int_0^{z_o} z \cosh(\xi_m z/a) dz = \frac{a z_o}{\xi_m} \sinh(\xi_m z_o/a) - \frac{a^2}{\xi_m^2}\left[\cosh(\xi_m z_o/a) - 1\right]$$

$$\int_0^{z_o} z \sinh \xi_m (z_o - z)/a \, dz = -\frac{a z_o}{\xi_m} + \frac{a^2}{\xi_m^2} \sinh \xi_m z_o/a$$

have been used. Using (4.8), $\underline{r} \times \underline{\nu} = -z(\sin\theta\, \underline{i} - \cos\theta\, \underline{j})$, and (4.12), it is found that

$$\underline{M} = \frac{2 i \pi a^2 \rho \beta^3}{\alpha} e^{i\beta t}(-u_2 \underline{i} + u_1 \underline{j})\left[a^2 \sum_{m=1}^{\infty} \frac{B^*_m \left[1 - \cosh(\xi_m h/a)\right] J_1(\xi_m)}{\xi_m^2 \sinh(\xi_m h/a)} + \frac{\alpha h^2}{4\beta^2}\right]$$

gives the moment of force acting on the portion of the wall of the tank below the baffle. Lastly, the moment of force on the bottom of the tank is evaluated from the integral

$$\underline{M} = -\int_0^{2\pi} d\theta \int_0^a (p - p_o)\Big|_{z=-h} r(\sin\theta\, \underline{i} - \cos\theta\, \underline{j})\, r dr$$

which, in view of (4.9), leads to the expression

$$\underline{M} = -\frac{2 i \pi a^4 \rho \beta^3}{\alpha} e^{i\beta t}(u_2 \underline{i} - u_1 \underline{j})\left[\sum_{m=1}^{\infty} \frac{B^*_m J_1(\xi_m)}{\xi_m^2 \sinh(\xi_m h/a)} - \frac{\alpha}{8\beta^2}\right].$$

SECTION 5

SOLUTION OF THE DUAL DINI SERIES

The solution of the dual Dini Series

$$\sum_{m=1}^{\infty} \frac{B_m^* \operatorname{chsh}(m;z_o+h) J_1(\xi_m x)}{\sinh(\xi_m h/a) \operatorname{chsh}(m;z_o)}$$

$$= \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2 - 1) \operatorname{chsh}(m;z_o)}, \quad 0 \le x < \gamma \tag{3.44}$$

$$\sum_{m=1}^{\infty} \xi_m B_m^* J_1(\xi_m x) = 0, \; \gamma < x < 1, \tag{3.45}$$

may be expressed in a variety of ways, and in the following pages several methods of determining the coefficients B_m^* will be given.

Being somewhat more general than above, consider the dual series

$$\sum_{m=1}^{\infty} \xi_m^p G(m) a_m J_v(\xi_m x) = F(x), \; 0 \le x < \gamma \tag{5.1}$$

$$\sum_{m=1}^{\infty} a_m J_v(\xi_m x) = 0, \; \gamma < x < 1, \tag{5.2}$$

where $J_v(\xi_m x)$ is the Bessel function of the first kind of order v, $\xi_m (m = 1,2,3,\cdots)$ are the positive roots of $J_v'(x) = 0$, F(x) is a known function, $-1 \le p \le 1$, G(m) is known, and the coefficients a_m are to be determined. The technique of determining the a_m is based on a method due to Tranter and Cooke [8].

The following theorem, which is analogous to a theorem stated in [8], is necessary for the ensuing discussions:

Theorem. If n is zero or a positive integer, $v > -1$, $k > 0$, and ξ_m are the positive roots of the transcendental equation $J_v'(x) = 0$, then

$$\sum_{m=1}^{\infty} \frac{J_{v+2n+k}(\gamma\xi_m)J_v(\xi_m x)}{\xi_m^{k-2}(\xi_m^2-v^2)J_v^2(\xi_m)} = \frac{1}{2}\int_0^{\infty} r^{1-k} J_{v+2n+k}(\gamma r)J_v(xr)\, dr$$

$$= \begin{cases} \dfrac{\Gamma(v+n+1)}{2^k \gamma^{v-k+2}\Gamma(v+1)\Gamma(n+k)} x^v(1-x^2/\gamma^2)^{k-1} \mathcal{F}_n(k+v, v+1, x^2/\gamma^2), & 0 \le x < \gamma \\ 0, \ \gamma < x < 1 \end{cases}$$

where

$$\mathcal{F}_n(k+v, v+1, x^2/\gamma^2) = {}_2F_1(-n,\ k+v+n, v+1, x^2/\gamma^2)$$

is Jacobi's polynomial, as given by Magnus and Oberhettinger [9].

Proof. Consider the Weber-Schafheitlin integral

$$\int_0^{\infty} r^{1-k} J_{v+2n+k}(\gamma r)J_v(xr)\, dr$$

$$= \begin{cases} \dfrac{\Gamma(v+n+1)}{2^{k-1} \gamma^{v-k+2}\Gamma(v+1)\Gamma(n+k)} x^v(1-x^2/\gamma^2)^{k-1} \mathcal{F}_n(k+v, v+1, x^2/\gamma^2), & 0 \le x < \gamma \\ 0, \ \gamma < x \end{cases}$$

which is given by Watson [10], and then apply the Hankel inversion formula to obtain

$$r^{-k} J_{v+2n+k}(\gamma r) = \frac{\Gamma(v+n+1)}{2^{k-1}\gamma^{v-k+2}\Gamma(v+1)\Gamma(n+k)} \int_0^{\gamma} x^{v+1}(1$$

$$-x^2/\gamma^2)^{k-1} \mathcal{F}_n(k+v, v+1, x^2/\gamma^2) J_v(rx)\, dx \tag{5.3}$$

Now the Dini expansion of the function f(x) defined by

$$f(x) = \begin{cases} \dfrac{\Gamma(v+n+1)}{2^{k-1}\gamma^{v-k+2}\Gamma(v+1)\Gamma(n+k)} x^{v}(1-x^2/\gamma^2)^{k-1} \mathscr{F}_n(k+v, v+1, x^2/\gamma^2), & 0 \le x < \gamma \\ 0, \ \gamma < x < 1 \end{cases}$$

is

$$f(x) = \sum_{m=1}^{\infty} A_m J_v(\xi_m x), \tag{5.4}$$

The coefficients A_m being determined from

$$A_m = \frac{2\xi_m^2}{(\xi_m^2 - v^2)J_v^2(\xi_m)} \int_0^1 xJ_v(\xi_m x) f(x)\, dx. \tag{5.5}$$

Therefore, using (5.5), the A_m are found to be

$$A_m = \frac{2\xi_m^2}{(\xi_m^2 - v^2)J_v^2(\xi_m)} \frac{\Gamma(v+n+1)}{2^{k-1}\gamma^{v-k+2}\Gamma(v+1)\Gamma(n+k)}$$

$$\int_0^{\gamma} x^{v+1}(1-x^2/\gamma^2)^{k-1} \mathscr{F}_n(k+v, v+1, x^2/\gamma^2) J_v(\xi_m x)\, dx$$

$$= \frac{2J_{v+2n+k}(\gamma\xi_m)}{\xi_m^{k-2}(\xi_m^2 - v^2)J_v^2(\xi_m)}\ , \tag{5.6}$$

and hence, upon substitution of the result of (5.6) into (5.4), it is evident that

$$\sum_{m=1}^{\infty} \frac{J_v(\xi_m x) J_{v+2n+k}(\gamma\xi_m)}{\xi_m^{k-2}(\xi_m^2 - v^2)J_v^2(\xi_m)}$$

$$= \begin{cases} \dfrac{\Gamma(v+n+1)}{2^{k}\gamma^{v-k+2}\Gamma(v+1)\Gamma(n+k)} x^{v}(1-x^2/\gamma^2)^{k-1} \mathscr{F}_n(k+v, v+1, x^2/\gamma^2), & 0 \le x < \gamma, \\ 0, \ \gamma < x < 1, \end{cases}$$

which is the desired result.

With the results of the above theorem now available, it is observed that

$$\sum_{m=1}^{\infty} \frac{J_{v+2n+p/2+1}(\gamma \xi_m) J_v(\xi_m x)}{\xi_m^{p/2-1}(\xi_m^2 - v^2) J_v^2(\xi_m)} = 0, \; \gamma < x < 1, \tag{5.7}$$

where k has been replaced by 1 + p/2. Define

$$a_m = \frac{1}{\xi_m^{p/2-1}(\xi_m^2 - v^2) J_v^2(\xi_m)} \sum_{n=0}^{\infty} b_n J_{v+2n+p/2+1}(\gamma \xi_m) \tag{5.8}$$

so that (5.2), i.e., the second equation of the pair forming the dual series, is satisfied identically. Thus

$$\sum_{m=1}^{\infty} a_m J_v(\xi_m x) = \sum_{m=1}^{\infty} \frac{J_v(\xi_m x)}{\xi^{p/2-1}(\xi_m^2 - v^2) J_v^2(\xi_m)} \sum_{n=0}^{\infty} b_n J_{v+2n+p/2+1}(\gamma \xi_m)$$

$$= \sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} \frac{J_{v+2n+p/2+1}(\gamma \xi_m) J_v(\xi_m x)}{\xi_m^{p/2-1}(\xi_m^2 - v^2) J_v^2(\xi_m)} = 0, \; \gamma < x < 1,$$

by (5.7), assuming that the order of the summations may be interchanged.

Next substitute (5.8) into (5.1), the first equation in the dual series, to obtain

$$\sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m^{p/2+1} G(m) J_v(\xi_m x) J_{v+2n+p/2+1}(\gamma \xi_m)}{(\xi_m^2 - v^2) J_v^2(\xi_m)}$$

$$= F(x), \; 0 \le x < \gamma, \tag{5.9}$$

after interchanging the order of summation. But (5.3) can be put in the form

$$\frac{J_{v+2k+p/2+1}(\gamma \xi_m)}{\xi_m^{1+p/2}} = \frac{\Gamma(v+k+1)}{2^{p/2} \gamma^{v-p/2+1} \Gamma(v+1) \Gamma(k+1+p/2)}$$

$$\int_0^{\gamma} x^{v+1}(1-x^2/\gamma^2)^{p/2}\ \mathscr{F}_k(1+p/2+v, v+1, x^2/\gamma^2) J_v(\xi_m x)\, dx.$$

Then if (5.9) is multiplied by

$$x^{v+1}(1-x^2/\gamma^2)^{p/2}\ \mathscr{F}_k(1+p/2+v, v+1, x^2/\gamma^2)$$

and integrated with respect to x from 0 to γ, the result is

$$\sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} \frac{G(m) J_{v+2n+p/2+1}(\gamma\xi_m) J_{v+2k+p/2+1}(\gamma\xi_m)}{(\xi_m^2 - v^2) J_v^2(\xi_m)}$$

$$= \frac{\Gamma(v+k+1)}{2^{p/2}\gamma^{v-p/2+1}\Gamma(v+1)\Gamma(k+1+p/2)} \int_0^{\gamma} x^{v+1}(1$$

$$-x^2/\gamma^2)^{p/2}\ \mathscr{F}_k(1+p/2+v, v+1, x^2/\gamma^2) F(x)\, dx. \tag{5.10}$$

For the sake of notional brevity, define

$$S(k,n;v,p) = \sum_{m=1}^{\infty} \frac{G(m) J_{v+2k+p/2+1}(\gamma\xi_m) J_{v+2n+p/2+1}(\gamma\xi_m)}{(\xi_m^2 - v^2) J_v^2(\xi_m)}, \tag{5.11}$$

and

$$E(k;v,p) = \frac{\Gamma(v+k+1)}{2^{p/2}\gamma^{v-p/2+1}\Gamma(v+1)\Gamma(k+p/2+1)} \int_0^{\gamma} x^{v+1}(1$$

$$-x^2/\gamma^2)^{p/2}\ \mathscr{F}_k(1+p/2+v, v+1, x^2/\gamma^2) F(x)\, dx, \tag{5.12}$$

so that the form of (5.10) becomes

$$\sum_{n=0}^{\infty} b_n S(k,n;v,p) = E(k;v,p),$$

from which the coefficients b_n may be calculated by considering the result as an infinite system of linear equations in b_n, for each choice of v and p.

For the dual series given by (3.44) and (3.45), it is convenient to define

$$C^*_m = \xi_m B^*_m, \tag{5.13}$$

so that the dual series

$$\sum_{m=1}^{\infty} \frac{\xi_m^{-1} C^*_m \,\mathrm{chsh}(m;z_o+h) J_1(\xi_m x)}{\sinh(\xi_m h/a)\,\mathrm{chsh}(m;z_o)}$$

$$= \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)\,\mathrm{chsh}(m;z_o)}, \quad 0 \le x < \gamma,$$

$$\sum_{m=1}^{\infty} C^*_m J_1(\xi_m x) = 0, \quad \gamma < x < 1,$$

is of the form expressed in (5.1) and (5.2), where p=-1, v=1,

$$G(m) = \frac{\mathrm{chsh}(m;z_o+h)}{\sinh(\xi_m h/a)\,\mathrm{chsh}(m;z_o)}, \tag{5.14}$$

and C^*_m has replaced a_m. Therefore (5.8) is now written

$$C^*_m = \frac{\xi_m^{3/2}}{(\xi_m^2-1)J_1^2(\xi_m)} \sum_{n=0}^{\infty} b_n J_{2n+3/2}(\gamma\xi_m),$$

so that, in view of (5.13),

$$B^*_m = \frac{\xi_m^{1/2}}{(\xi_m^2-1)J_1^2(\xi_m)} \sum_{n=0}^{\infty} b_n J_{2n+3/2}(\gamma\xi_m). \tag{5.15}$$

According to (5.11), it follows that S(k,n;1,-1) = S(k,n)

$$= \sum_{m=1}^{\infty} \frac{\mathrm{chsh}(m;z_o+h) J_{2n+3/2}(\gamma\xi_m) J_{2k+3/2}(\gamma\xi_m)}{\sinh(\xi_m h/a) J_1^2(\xi_m)(\xi_m^2-1)\,\mathrm{chsh}(m;z_o)}, \tag{5.16}$$

and from (5.12)

$$E(k;1,-1) = E(k) = \sum_{m=1}^{\infty} \frac{1}{J_1(\xi_m)(\xi_m^2-1)\mathrm{chsh}(m;z_o)} \Bigg[$$

$$\frac{\Gamma(k+2)}{2^{-1/2}\gamma^{3/2}\Gamma(2)\Gamma(k+1/2)} \int_0^{\gamma} x^2(1-x^2/\gamma^2)^{-1/2} \mathscr{F}_k(3/2,2,x^2/\gamma^2) J_1(\xi_m x)\, dx \Bigg]$$

$$= \sum_{m=1}^{\infty} \frac{J_{2k+3/2}(\gamma\xi_m)}{J_1(\xi_m)\xi_m^{1/2}(\xi_m^2-1)\mathrm{chsh}(m;z_o)} \tag{5.17}$$

by (5.3).

To summarize, it is desired to obtain the coefficients b_n from the following equation

$$\sum_{n=0}^{\infty} b_n S(k,n) = E(k), \tag{5.18}$$

where S(k,n) and E(k) are given by (5.16) and (5.17) respectively. Then once the values of $b_n (n=0,1,2,\cdots)$ are known, the values of the coefficients B_m^* $(m=1,2,3,\cdots)$ can be computed from (5.15).

Making use of a certain contour integral in the complex plane, Tranter and Cooke [8] were able to sum the infinite series that corresponds to S(k,n,1,-1) in this paper, and this sum was expressible in the form of an improper integral of the first kind involving the modified Bessel functions of both the first and second kinds. Because the form of S(k,n,1,-1), as it appears here, is much more complicated than the corresponding expression appearing in [8], the contour integral approach yielded for the sum of S(k,n,1,-1) an improper integral, as expected, and, in addition, another infinite series. In other words, it was found that the infinite series in (5.14) could be replaced by another infinite series plus an improper integral.

At this point it appeared futile to pursue the Tranter-Cooke [8] method further, and it was decided to attempt to determine the b_n's numerically using the IBM 7074 computer, given values of the various parameters appearing in the problem.

Now (5.18) represents an infinite system of linear algebraic equations in the b_n's, and a set of numerical values of the quantities $b_0, b_1, b_2, \cdots$ is called a solution of the system if on substituting these values in the left member of (5.18) the infinite series converges and all the equations are satisfied for $k=0,1,2\cdots$. According to Kantorovich and Krylov [11], approximate solutions of (5.16) may, under certain conditions, be obtained by terminating the infinite series at, say, n=N and by then assigning to k the values, 0, 1, 2, ⋯N, in such a way that an N+1 by N+1 system of linear algebraic equations is obtained, i.e.,

$$S(0,0)b_o+S(0,1)b_1+\cdots+S(0,N)b_N = E(0)$$

$$S(1,0)b_o+S(1,1)b_1+\cdots+S(1,N)b_N = E(1)$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$S(N,0)b_o+S(N,1)b_1+\cdots+S(N,N)b_N = E(N).$$

A somewhat different system of equations can be obtained by making use of a special case of the integral [12]

$$\int_0^a x^{v+1}(a^2-x^2)^u J_v(yx)\,dx = \frac{2^u\Gamma(u+1)a^{v+u+1}}{y^{u+1}} J_{v+u+1}(ay), \tag{5.19}$$

provided that $y > 0$, $a > 0$, $\mathrm{Re}(u) > -1$, and $\mathrm{Re}(v) > -1$. Set $v=1$, $a=\gamma$, $u=k-1/2$, and $y=\xi_m$; the form of (5.19) now becomes

$$\int_0^\gamma x^2(\gamma^2-x^2)^{k-1/2} J_1(\xi_m x)\,dx = \frac{2^{k-1/2}\Gamma(k+1/2)\gamma^{k+3/2}}{\xi_m^{k+1/2}} J_{k+3/2}(\gamma\xi_m). \tag{5.20}$$

Multiply (5.9) by $x^2(\gamma^2-x^2)^{k-1/2}$, and integrate with respect to x from 0 to γ to obtain (using the same conditions as for (3.44)

$$\sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\mathrm{chsh}(m;z_o+h)J_{2n+3/2}(\gamma\xi_m)J_{k+3/2}(\gamma\xi_m)}{\sinh(\xi_m h/a)J_1^2(\xi_m)\xi_m^k(\xi_m^2-1)\mathrm{chsh}(m;z_o)}$$

$$=\sum_{m=1}^{\infty} \frac{J_{k+3/2}(\gamma\xi_m)}{J_1(\xi_m)\xi_m^{k+1/2}(\xi_m^2-1)\mathrm{chsh}(m;z_o)},$$

where the integral of (5.20) has been used, assuming that the interchange of the order of integration and summation is permissible.

Again define

$$S(k,n) = \sum_{m=1}^{\infty} \frac{\text{chsh}(m;z_o+h)J_{2n+3/2}(\gamma\xi_m)J_{k+3/2}(\gamma\xi_m)}{\sinh(\xi_m h/a)J_1^2(\xi_m)\xi_m^k(\xi_m^2-1)\text{chsh}(m;z_o)}, \tag{5.21}$$

and

$$E(k) = \sum_{m=1}^{\infty} \frac{J_{k+3/2}(\gamma\xi_m)}{J_1(\xi_m)\xi_m^{k+1/2}(\xi_m^2-1)\text{chsh}(m;z_o)}, \tag{5.22}$$

where $k=1,2,3,\cdots$, so that an infinite system of equations

$$\sum_{n=0}^{\infty} b_n S(k,n) = E(k) \tag{5.23}$$

is obtained, and the values of B_m^* can be determined from (5.15) as soon as the values of the b_n are known.

As a matter of fact, it is possible to obtain the formal solution of the dual Dini series given by (3.44) and (3.45) in a variety of forms depending upon one's ability to find a Dini expansion of a function that converges to zero from $\gamma < x < 1$. This ability seems to be dictated by the availability of integrals involving the Bessel function $J_1(\xi_m x)$ and two free parameters, e.g., consider (5.3) and (5.19). As might be expected, one method of solving the dual series may be more amenable to numerical computation than another method. To be more specific, it has been observed that the solutions represented by (5.16), (5.17), and (5.18) and by (5.21, (5.22), and (5.23) are not well suited to machine computation since the matrices represented here by S(k,n) become ill-conditioned as the dimensions of the matrices exceed 15 × 15 in one case and 20 × 20 in the other case. Therefore, in the subsequent pages, other formulas for the solution of the dual series are given. Basically the method of solution is the same as that given above; however different integrals involving $J_1(\xi_m x)$ are employed to give various forms for S(k,n) and E(k). From a numerical standpoint, it is desirable to obtain infinite series for S(k,n) that converge fairly rapidly and that are such that the elements of the S(k,n) matrix do not become too small or lead to an ill-conditioned system.

Consider a function f(x) defined as follows:

$$f(x) = \begin{cases} x^{-1}(\gamma^2-x^2)^{1/2} \cos(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma \\ 0, & \gamma < x < 1. \end{cases}$$

If the $\xi_m (m=1,2,3,\cdots)$ are the positive roots of the transcendental equation $J_1'(x) = 0$, then f(x) may be expanded in a Dini series of the form

$$f(x) = \sum_{m=1}^{\infty} A_m J_1(\xi_m x) \tag{5.24}$$

where

$$A_m = \frac{2\xi_m^2}{(\xi_m^2-1)J_1^2(\xi_m)} \int_0^1 xf(x)J_1(\xi_m x)\, dx \tag{5.25}$$

$$= \frac{2\xi_m^2}{(\xi_m^2-1)J_1^2(\xi_m)} \int_0^{\gamma} (\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2})J_1(\xi_m x)\, dx. \tag{5.26}$$

According to Erdelyi [12], Page 39, Formula 47, it is known that

$$\int_0^{\gamma} (\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2})J_1(\xi_m x)\, dx$$

$$= \frac{\pi}{2} J_{1/2}\left(\frac{1}{2}\gamma\left[\sqrt{n^2+\xi_m^2}-n\right]\right) J_{1/2}\left(\frac{1}{2}\gamma\left[\sqrt{n^2+\xi_m^2}+n\right]\right)$$

$$= \frac{1}{\gamma\xi_m}\left[\cos(\gamma n) - \cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right],$$

and therefore (5.26) assumes the form

$$A_m = \frac{2\xi_m}{\gamma J_1^2(\xi_m)(\xi_m^2-1)} \left[\cos(\gamma n) - \cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right].$$

Hence it has been established that

$$\sum_{m=1}^{\infty} \frac{\xi_m\left[\cos(\gamma n) - \cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right] J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)}$$

$$= \begin{cases} \frac{1}{2}\gamma x^{-1}(\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma, \\ 0, & \gamma < x < 1. \end{cases} \tag{5.27}$$

Now define

$$C_m^* = \frac{\xi_m}{J_1^2(\xi_m)(\xi_m^2-1)} \sum_{n=1}^{\infty} b_n \left[\cos(\gamma n) - \cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right], \tag{5.28}$$

where the $C_m^*(m=1,2,3,\cdots)$ are the unknown coefficients in the dual series

$$\sum_{m=1}^{\infty} \xi_m^{-1} C_m^* G(m) J_1(\xi_m x) = F(x), \quad 0 \le x < \gamma, \tag{5.29}$$

$$\sum_{m=1}^{\infty} C_m^* J_1(\xi_m x) = 0, \quad \gamma < x < 1, \tag{5.30}$$

where G(m) is given by (5.14) and

$$F(x) = \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)\operatorname{chsh}(m;z_0)}. \tag{5.31}$$

(5.30) is formally satisfied identically for the choice of C_m^* as made in (5.28) as is seen by direct substitution

$$\sum_{m=1}^{\infty}\frac{\xi_m J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)}\sum_{n=1}^{\infty} b_n\left[\cos(\gamma n)-\cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right]$$

$$=\sum_{n=1}^{\infty} b_n\sum_{m=1}^{\infty}\frac{\xi_m\left[\cos(\gamma n)-\cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right]J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)} = 0,\ \gamma < x < 1,$$

because of (5.27), assuming that the interchange of the order of summation is permissible.

Returning to (5.29) and again replacing the C_m^* by (5.28),

$$\sum_{m=1}^{\infty}\frac{G(m)J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)}\sum_{n=1}^{\infty} b_n\left[\cos(\gamma n)-\cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right]$$

$$= F(x),\ 0 \le x < \gamma,$$

or upon interchanging the order of summation,

$$\sum_{n=1}^{\infty} b_n\sum_{m=1}^{\infty}\frac{G(m)\left[\cos(\gamma n)-\cos\left(\gamma\sqrt{n^2+\xi_m^2}\right)\right]J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)} = F(x). \qquad (5.32)$$

Multiply (5.32) by $(\gamma^2-x^2)^{-1/2}\cos(k\sqrt{\gamma^2-x^2})$ and integrate with respect to x from 0 to γ to obtain

$$\sum_{n=1}^{\infty} b_n\sum_{m=1}^{\infty}\frac{G(m)\left[\cos(\gamma n)-\cos(\gamma\sqrt{n^2+\xi_m^2})\right]\left[\cos(\gamma k)-\cos(\gamma\sqrt{k^2+\xi_m^2})\right]}{J_1^2(\xi_m)\xi_m(\xi_m^2-1)}$$

$$=\sum_{m=1}^{\infty}\frac{\left[\cos(\gamma k)-\cos(\gamma\sqrt{k^2+\xi_m^2})\right]}{J_1(\xi_m)\xi_m(\xi_m^2-1)\mathrm{chsh}(m;z_o)}, \qquad (5.33)$$

where (5.26) and (5.31) have been employed.

Therefore, the infinite system

$$\sum_{n=1}^{\infty} b_n S(k,n) = E(k),$$

where

$$S(k,n) = \sum_{m=1}^{\infty} \frac{G(m)\left[\cos(\gamma n)-\cos(\gamma\sqrt{n^2+\xi_m^2})\right]\left[\cos(\gamma k)-\cos(\gamma\sqrt{k^2+\xi_m^2})\right]}{J_1^2(\xi_m)\xi_m(\xi_m^2-1)},$$

and

$$E(k) = \sum \frac{\left[\cos(\gamma k)-\cos(\gamma\sqrt{k^2+\xi_m^2})\right]}{J_1(\xi_m)\xi_m(\xi_m^2-1)\mathrm{chsh}(m;z_o)}, \qquad n,k = 1,2,3,\cdots$$

must now be solved for the coefficients b_n.

Alternatively, one might make the change of variable $x = \gamma \sin y$ in (5.32) and then multiply by $\sin^2 y \cos(k \cos y)$ and integrate with respect to y from 0 to $\pi/2$ to obtain

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m G(m)\left[\cos(\gamma n)-\cos(\gamma\sqrt{n^2+\xi_m^2})\right]J_{3/2}(\sqrt{k^2+\gamma^2\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(k^2+\gamma^2\xi_m^2)^{3/4}}$$

$$= \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\sqrt{k^2+\gamma^2\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\gamma^2\xi_m^2)^{3/4}\mathrm{chsh}(m;z_o)} \tag{5.34}$$

using the integral

$$\int_0^{\pi/2} \sin^2 y \cos(k\cos y) J_1(\gamma\xi_m \sin y)\, dy$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \frac{\gamma\xi_m}{(k^2+\gamma^2\xi_m^2)^{3/4}} J_{3/2}(\sqrt{k^2+\gamma^2\xi_m^2}),$$

which is a special case of a more general integral given by Erdelyi [12], Page 361, Formula 19. Now define

$$S(k,n) = \sum_{m=1}^{\infty} \frac{\xi_m G(m)\left[\cos(\gamma n) - \cos(\gamma\sqrt{n^2+\xi_m^2})\right] J_{3/2}(\sqrt{k^2+\gamma^2\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(k^2+\gamma^2\xi_m^2)^{3/4}}$$

and

$$E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\sqrt{k^2+\gamma^2\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\gamma^2\xi_m^2)^{3/4}\operatorname{chsh}(m;z_0)}, \quad n,\ k = 1,2,3\cdots$$

In practice, it was observed that the values of the b_n obtained by solving the system in (5.33) and the systems in (5.34) were roughly of the same order of magnitude, and were such as to cast serious doubt on the convergence of the series for C_m^*, (5.28). Consequently it would appear that the infinite series in (5.28) converges too slowly, if at all, to be of any practical value.

Consider next a function f(x) defined as follows:

$$f(x) = \begin{cases} x \sin(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma, \\ 0, \gamma < x < 1. \end{cases}$$

With reference to (5.24) and (5.25), the coefficients A_m in the Dini expansion of f(x) are obtained from the integral

$$A_m = \frac{2\xi_m^2}{J_1^2(\xi_m)(\xi_m^2-1)} \int_0^{\gamma} x^2 \sin(n\sqrt{\gamma^2-x^2}) J_1(\xi_m x)\, dx$$

$$= \frac{n\gamma^2\sqrt{2\pi\gamma}\,\xi_m^3}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}} J_{5/2}(\gamma\sqrt{n^2+\xi_m^2}),$$

where the integral

$$\int_0^{\gamma} x^2 \sin(n\sqrt{\gamma^2-x^2}) J_1(\xi_m x)\, dx$$

$$= \left(\frac{\pi}{2}\right)^{1/2} \frac{n\gamma^{5/2}\xi_m}{(n^2+\xi_m^2)^{5/4}} J_{5/2}(\gamma\sqrt{n^2+\xi_m^2}), \qquad (5.35)$$

as given by Erdelyi [12], Page 335, Formula 19, has been used.

Therefore, it has been established that

$$\sum_{m=1}^{\infty} \frac{\xi_m^3 J_{5/2}(\gamma\sqrt{n^2+\xi_m^2}) J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}}$$

$$= \begin{cases} \dfrac{x}{n\gamma^2\sqrt{2\pi\gamma}} \sin(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma, \\ 0, & \gamma < x \le 1. \end{cases}$$

Define

$$C^*_m = \frac{\xi_m^3}{J_1^2(\xi_m)(\xi_m^2-1)} \sum_{n=1}^{\infty} \frac{b_n J_{5/2}(\gamma\sqrt{n^2+\xi_m^2})}{(n^2+\xi_m^2)^{5/4}},$$

so that (5.30) is satisfied identically. The details of verifying this last statement are essentially the same as in the previous cases outlined a few pages earlier, and therefore they are omitted here. However, (5.29) assumes the form

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m^2 G(m) J_{5/2}(\gamma\sqrt{n^2+\xi_m^2}) J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}}$$

$$= \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)\operatorname{chsh}(m;z_o)}, \quad 0 \le x < \gamma.$$

Proceeding as before and using (5.35), it is found that

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_{5/2}(\gamma\sqrt{n^2+\xi_m^2}) J_{5/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}(k^2+\xi_m^2)^{5/4}}$$

$$= \sum_{m=1}^{\infty} \frac{\xi_m J_{5/2}(\gamma\sqrt{\xi_m^2+k^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\xi_m^2)^{5/4}\operatorname{chsh}(m;z_o)},$$

which leads to the infinite system of equations

$$\sum_{n=1}^{\infty} b_n S(k,n) = E(k), \quad k = 1,2,3,\cdots$$

where

$$S(k,n) = \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_{5/2}(\gamma\sqrt{n^2+\xi_m^2}) J_{5/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{5/4}(k^2+\xi_m^2)^{5/4}}, \tag{5.36}$$

and

$$E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_{5/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\xi_m^2)^{5/4} \operatorname{chsh}(m;z_o)}.$$

While the infinite series in (5.36) converges fairly rapidly, it is found that the matrix S(k,n) was ill-conditioned for k=n=15.

The integral

$$\int_0^{\pi/2} J_u(z \sin t) J_v(s \cos t) \sin^{u+1} t \cos^{v+1} t \, dt$$

$$= \frac{1}{y}\left(\frac{z}{y}\right)^u \left(\frac{s}{y}\right)^v J_{u+v+1}(y), \tag{5.37}$$

where $y^2=z^2+s^2$, Re(u) > -1, and Re(v) > -1, is given in Luke [13], Page 299, Formula 26. If the substitutions $z=\gamma\xi_m$ and $s=\gamma w$ and the change of variable $x=\gamma \sin t$ are made, then the form of the integral in (5.37) becomes

$$\int_0^{\gamma} J_u(\xi_m x) J_v(w\sqrt{\gamma^2-x^2}) x^{u+1} (\gamma^2-x^2)^{v/2} \, dx$$

$$= \frac{\gamma^{u+v+1} w^v \xi_m^u}{(\xi_m^2+w^2)^{(1/2)(u+v+1)}} J_{u+v+1}(\gamma\sqrt{w^2+\xi_m^2}).$$

Consider the special case of this last integral for which u=1, v=0, and w=n, a positive integer:

$$\int_0^{\gamma} x^2 J_0(n\sqrt{\gamma^2-x^2}) J_1(\xi_m x)\, dx = \frac{\gamma^2 \xi_m}{(\xi_m^2+n^2)} J_2(\gamma\sqrt{n^2+\xi_m^2}). \tag{5.38}$$

Recalling the integrand of (5.38), define a function f(x) such that

$$f(x) = \begin{cases} x\, J_0(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma \\ 0, & \gamma < x < 1. \end{cases}$$

Now the coefficients A_m which appear in the Dini expansion of f(x) can be found by carrying out the integration

$$A_m = \frac{2\xi_m^2}{J_1^2(\xi_m)(\xi_m^2-1)} \int_0^1 x\, f(x) J_1(\xi_m x)\, dx$$

$$= \frac{2\xi_m^2}{J_1^2(\xi_m)(\xi_m^2-1)} \int_0^{\gamma} x^2 J_0(n\sqrt{\gamma^2-x^2}) J_1(\xi_m x)\, dx$$

$$= \frac{2\gamma^2 \xi_m^3 J_2(\gamma\sqrt{n^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(\xi_m^2+n^2)},$$

according to (5.38). Therefore, it has been shown that

$$\sum_{m=1}^{\infty} \frac{\xi_m^3 J_2(\gamma\sqrt{n^2+\xi_m^2}) J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)(\xi_m^2+n^2)}$$

$$= \begin{cases} \dfrac{x}{2\gamma^2} J_0(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma \\ 0, & \gamma < x < 1. \end{cases}$$

Defining

$$C^*_m = \frac{\xi_m^3}{J_1^2(\xi_m)(\xi_m^2-1)} \sum_{n=1}^{\infty} \frac{b_n J_2(\gamma\sqrt{n^2+\xi_m^2})}{(\xi_m^2+n^2)}, \qquad (5.39)$$

it is evident that the second equation which appears in the dual series is satisfied identically. If (5.39) is inserted into (5.29), it is found that the following equation arises:

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m^2 G(m) J_2(\gamma\sqrt{n^2+\xi_m^2}) J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)(\xi_n^2+n^2)}$$

$$= \sum_{m=1}^{\infty} \frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)\operatorname{chsh}(m;z_o)}, \quad 0 \le x < \gamma. \qquad (5.40)$$

Multiply (5.40) by $x^2 J_0(k\sqrt{\gamma^2-x^2})$ and integrate with respect to x from 0 to γ in to obtain

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_2(\gamma\sqrt{n^2+\xi_m^2}) J_2(\gamma\sqrt{k^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(\xi_m^2+n^2)(\xi_m^2+k^2)}$$

$$= \sum_{m=1}^{\infty} \frac{\xi_m J_2(\gamma\sqrt{\xi_m^2+k^2})}{J_1(\xi_m)(\xi_m^2-1)\operatorname{chsh}(m;z_o)(\xi_m^2+k^2)},$$

which may be abbreviated as

$$\sum_{n=1}^{\infty} b_n S(k,n) = E(k),$$

where

$$S(k,n) = \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_2(\gamma\sqrt{n^2+\xi_m^2}) J_2(\gamma\sqrt{k^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(\xi_m^2+n^2)(\xi_m^2+k^2)} \qquad (5.41)$$

$$E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_2(\gamma \sqrt{\xi_m^2+k^2})}{J_1(\xi_m)(\xi_m^2-1)\text{chsh}(m;z_o)(\xi_m^2+k^2)}, \quad n,\ k = 1,2,3,\cdots$$

It is not difficult to show that the infinite series of (5.41) converges absolutely, and, as a matter of fact, the asymptotic form of the terms in the series for large m is

$$\frac{1}{\xi_m^3 \sin^2(\xi_m - \pi/4)},$$

neglecting constant multiplicative factors. Nonetheless, the matrix S(k,n) proved to be too ill-conditioned for k=n=15 to be handled even by a double precision matrix inversion routine on the computer.

Finally, define

$$f(x) = \begin{cases} x(\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2}), & 0 \le x < \gamma \\ 0, & \gamma < x < 1, \end{cases}$$

and determine its Dini expansion for the interval (0,1). The coefficients A_m which appear in the expansion are to be computed from the integral

$$A_m = \frac{2\xi_m^2}{J_1^2(\xi_m)(\xi_m^2-1)} \int_0^1 x\, f(x) J_1(\xi_m x)dx.$$

Thus

$$A_m = \frac{2\xi_m^2}{J_1^2(\xi_m)(\xi_m^2-1)} \int_0^{\gamma} x^2(\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2})J_1(\xi_m x)\, dx$$

$$= \frac{\gamma(2\gamma\pi)^{1/2}\xi_m^3}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}} J_{3/2}(\gamma\sqrt{n^2+\xi_m^2}),$$

where the integral

$$\int_0^{\gamma} x^2(\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2})J_1(\xi_m x)\,dx$$

$$=\left(\frac{\gamma\pi}{2}\right)^{1/2}\frac{\gamma\xi_m}{(n^2+\xi_m^2)^{3/4}}J_{3/2}(\gamma\sqrt{n^2+\xi_m^2}),$$

given by Erdelyi [12], Page 361, Formula 19, has been employed. Hence it has been established that

$$\sum_{m=1}^{\infty}\frac{\xi_m^3 J_{3/2}(\gamma\sqrt{n^2+\xi_m^2})J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}}$$

$$=\begin{cases}\dfrac{x}{\gamma(2\pi\gamma)^{1/2}}(\gamma^2-x^2)^{-1/2}\cos(n\sqrt{\gamma^2-x^2}), & 0\le x<\gamma\\[2ex] 0,\ \gamma<x<1.\end{cases}$$

Continuing in the usual fashion, define now

$$C_m^* = \frac{\xi_m^3}{J_1^2(\xi_m)(\xi_m^2-1)}\sum_{n=1}^{\infty}\frac{b_n J_{3/2}(\gamma\sqrt{n^2+\xi_m^2})}{(n^2+\xi_m^2)^{3/4}}. \tag{5.42}$$

This choice of the form of C_m^* is appropriate because (5.30) is satisfied identically as is easily shown by direct substitution.

Inserting the expression for C_m^* given in (5.42) into (5.29), it follows that

$$\sum_{n=1}^{\infty}b_n\sum_{m=1}^{\infty}\frac{\xi_m^2 G(m)J_{3/2}(\gamma\sqrt{n^2+\xi_m^2})J_1(\xi_m x)}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}}$$

$$=\sum_{m=1}^{\infty}\frac{J_1(\xi_m x)}{J_1(\xi_m)(\xi_m^2-1)\mathrm{chsh}(m;z_0)},\quad 0\le x<\gamma.$$

Multiply this last equation by $x^2(\gamma^2-x^2)^{-1/2}\cos(k\sqrt{\gamma^2-x^2})$, and integrate with respect to x from 0 to γ to obtain

$$\sum_{n=1}^{\infty} b_n \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_{3/2}(\gamma\sqrt{n^2+\xi_m^2}) J_{3/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}(k^2+\xi_m^2)^{3/4}}$$

$$= \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\xi_m^2)^{3/4}\operatorname{chsh}(m;z_o)},$$

which leads to the infinite system of linear equations in b_n

$$\sum_{n=1}^{\infty} b_n S(k,n) = E(k),$$

where

$$S(k,n) = \sum_{m=1}^{\infty} \frac{\xi_m^3 G(m) J_{3/2}(\gamma\sqrt{n^2+\xi_m^2}) J_{3/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1^2(\xi_m)(\xi_m^2-1)(n^2+\xi_m^2)^{3/4}(k^2+\xi_m^2)^{3/4}} \tag{5.43}$$

and

$$E(k) = \sum_{m=1}^{\infty} \frac{\xi_m J_{3/2}(\gamma\sqrt{k^2+\xi_m^2})}{J_1(\xi_m)(\xi_m^2-1)(k^2+\xi_m^2)^{3/4}\operatorname{chsh}(m;z_o)}.$$

For large m, the terms in the infinite series in (5.43) behave as

$$\frac{1}{\xi_m^2 \sin^2(\xi_m-\pi/4)},$$

so that it would appear that the series should converge. It should be pointed out that $\sin(\xi_m-\pi/4) \neq 0$ for any $m=1,2,3,\cdots$.

SECTION 6
SOME REMARKS ON THE NUMERICAL ASPECTS OF THE PROBLEM

Because the infinite series in (5.43) converges slowly, methods of speeding convergence have been investigated. In particular, Lubkin [14] has discussed the transformation of a given infinite series

$$S = a_o + a_1 + a_2 + a_3 + \ldots + a_n + \ldots \tag{6.1}$$

into a new series

$$T = b_o + b_1 + b_2 + b_3 + \ldots + b_n + \ldots$$

Define the partial sums of the series by

$$S_n = a_o + a_1 + a_2 + \ldots + a_n \tag{6.2}$$

and

$$T_n = b_o + b_1 + b_2 + \ldots + b_n,$$

and define the T series by the relation

$$T_n = S_n + \frac{a_{n+1}}{1 - (a_{n+1}/a_n)}, \quad n \geq 0,$$

which, after minor manipulations, can be expressed as

$$T_n = \frac{S_n^2 - S_{n-1}S_{n+1}}{2S_n - S_{n-1} - S_{n+1}}, \quad n > 0, \quad T_o = \frac{S_o^2}{2S_o - S_1}$$

Going one step further, it can be shown that the individual terms of the T series may be computed from

$$b_o = \frac{a_o R_o}{R_o - 1}, \quad \text{and } b_n = a_n \left[\frac{R_{n-1} - R_n}{(R_n - 1)(R_{n-1} - 1)} \right], \quad n \geq 1,$$

where $R_n = a_n/a_{n+1}$.

In accord with usual terminology, a series $C = \sum_{n=0}^{\infty} c_n$ with partial sums $C_n = \sum_{m=0}^{n} c_m$ is: 1) said to be more rapidly convergent than S, see (6.1), if both S and and C converge, and $(C - C_n)/(S - S_n)$, the ratio of corresponding remainders, tends to zero as n tends to infinity; 2) of the same order of rapidity of convergence as S if both series converge, and $\left|(C - C_n)/(S - S_n)\right|$ remains in value between two finite positive constants for all sufficiently large n; and 3) no less rapidly convergent that S if both series converge and the ratio of corresponding remainders is bounded as n tends to infinity.

In particular, the following theorem has proved useful:

Theorem. Let

$$Q_n = n\,(R_n - 1)$$

and

$$Q = \lim_{n \to \infty} Q_n.$$

If S converges, Q exists, $\neq 1$, and $n(Q_n - Q_{n-1}) \to 0$ as $n \to \infty$, then the series $U = \sum_{n=0}^{\infty} u_n$ with

$$u_n = \frac{(Qb_n - a_n)}{Q - 1}$$

converges more rapidly than S and has the same sum.

However, since T converges and T = S, Lubkin [14] shows that

$$U_n = \sum_{m=0}^{n} u_m = \frac{QT_n - S_n}{Q - 1} \to \frac{QS - S}{Q - 1} = S,$$

and thus the approximation

$$U_n = \frac{QT_n - S_n}{Q - 1} \tag{6.3}$$

to the value of S is more accurate than is S_n itself.

Another nonlinear transformation, called the W transformation, is useful, especially so since the value of Q is not required. Lubkin [14] points out that there are also peculiar cases where the T transformation is usable but not the W.

Define

$$P_o = \frac{b_o}{a_o} = \frac{R_o}{R_o - 1}$$

and

$$P_n = \frac{b_n}{a_n} = \frac{R_{n-1} - R_n}{(R_n - 1)(R_{n-1} - 1)}, \quad n \geq 1.$$

The W transformation is then defined by the relations

$$W = w_o + w_1 + w_2 + \ldots + w_n + \ldots ,$$

$$W_n = w_o + w_1 + w_2 + \ldots + w_n,$$

$$= \frac{T_n - P_n S_n}{1 - P_n} . \tag{6.4}$$

The conditions under which the W transformation is applicable are given in the following theorem as given by Lubkin [14]: Theorem. If S converges, Q exists, $\neq 1$, $n(Q_n - Q_{n-1}) \rightarrow 0$, and $n[(n+1)(Q_{n+1} - Q_n) - n(Q_n - Q_{n-1})] \rightarrow 0$ as n tends to infinity, then W converges more rapidly than S and has the same sum.

Examination of the numerical values obtained for the series given in (5.43) has revealed that the transformations appearing in (6.3) and (6.4) lead to results that are not consistent with the partial sums of the series itself. The complexity of the general term in the series makes it rather difficult to show that the conditions of the appropriate theorems are actually fulfilled.

Salzer [15] discusses a method of summing certain slowly convergent series which is well-suited for machine computation. The application of Salzer's technique may be widespread since it involves a purely numerical device which is employed without any specific analytic work upon the series. The basic idea of this approach is to consider S_n see (6.2) as the tabulated value of a certain function of x, say S(x), at $x = n$, from which one would like to calculate $S(\infty)$ by the m-point Lagrangian interpolation polynomial. Then to calculate the limit S of a sequence $S_1, S_2, \ldots, S_n$, using the m-point extrapolation formula, one multiplies each of the last m terms, S_{n-i}, $i = 1, 2, \ldots, m-1$, by the corresponding extrapolation coefficient $B_{n,n-i}^{(m)}/D_n^{(m)}$, and sums; thus

$$S \sim (1/D_n^{(m)}) \sum_{i=0}^{m-1} B_{n,n-i}^{(m)} S_{n-i}. \tag{6.5}$$

The coefficients required in (6.5) are listed in Table I in Salzer's paper.

For machine purposes, numerical values of the Bessel functions involved in various phases of the problem can be obtained by writing a machine language subroutine following a recent paper by Gautschi [16]. This procedure evaluates to d significant digits the Bessel functions $J_{a+n}(x)$ for $n = 0, 1, 2, \ldots$, $0 \le a < 1$, and $x > 0$. The method of computation is a variant of the backward recurrence algorithm of J.C.P. Miller as discussed by Gautschi [17]. The algorithm is most efficient when x is small or moderately large, although near a zero of one of the Bessel functions generated, the accuracy of that particular Bessel function may deteriorate to less than d significant digits.

Abramowitz and Stegun [18] have given polynomial approximations for $J_o(x)$ and $J_1(x)$ for small argument x as well as large. In particular, for $|x| \le 3$,

$$J_o(x) = 1 - 2.24999\,97(x/3)^2 + 1.26562\,08(x/3)^4$$
$$-0.31638\,66(x/3)^6$$

$$+ 0.04444\ 79(x/3)^8 - 0.00394\ 44(x/3)^{10} + 0.00021\ 00(x/3)^{12}$$

$$+ e, \quad |e| < 5\times10^{-8}$$

whereas for $3 \leq x < \infty$,

$$J_o(x) = x^{-\frac{1}{2}} f_o(x) \cos \theta_o(x),$$

where

$$f_o(x) = 0.79788\ 456 - 0.00000\ 077(3/x) - 0.00552\ 740(3/x)^2$$

$$-0.00009\ 512(3/x)^3 + 0.00137\ 237(3/x)^4 - 0.00072\ 805(3/x)^5$$

$$+0.00014\ 476(3/x)^6 + e, \quad |e| < 1.6\times10^{-8},$$

and

$$\theta_o(x) = x - 0.78539\ 816 - 0.04166\ 397(3/x) - 0.00003\ 954(3/x)^2$$

$$+ 0.00262\ 573(3/x)^3 - 0.00054\ 125(3/x)^4 - 0.00029\ 333(3/x)^5$$

$$+ 0.00013\ 558(3/x)^6 + e, \quad |e| < 7\times10^{-8}.$$

Similar expressions are given for $J_1(x)$. They are, for

$|x| \leq 3$,

$$x^{-1} J_1(x) = \frac{1}{2} - 0.56249\ 985(x/3)^2 + 0.21093\ 573(x/3)^4$$

$$- 0.03954\ 289(x/3)^6 + 0.00443\ 319(x/3)^8 - 0.00031\ 761(x/3)^{10}$$

$$+ 0.00001\ 109(x/3)^{12} + e, \quad |e| < 1.3\times10^{-8}$$

and for $3 \leq x < \infty$

$$J_1(x) = x^{-\frac{1}{2}} f_1(x) \cos \theta_1(x),$$

where

$$f_1(x) = 0.79788\ 456 + 0.00000\ 156(3/x) + 0.01659\ 667(3/x)^2$$

$$+ 0.00017\ 105(3/x)^3 - 0.00249\ 511(3/x)^4 + 0.00113\ 653(3/x)^5$$

$$- 0.00020\ 033(3/x)^6 + e, \quad |e| < 4\times10^{-8}$$

and

$$\theta_1(x) = x - 2.35619\ 449 + 0.12499\ 612(3/x) + 0.00005\ 650(3/x)^2$$
$$-0.00637\ 879(3/x)^3 + 0.00074\ 348(3/x)^4 + 0.00079\ 824(3/x)^5$$
$$-0.00029\ 166(3/x)^6 + e, \quad |e| < 9\times10^{-8}.$$

When v is real, the function $J_v'(x)$ has an infinite number of real zeros, all of which are simple with the possible exception of x = 0. For non-negative v the mth positive zero of this function is denoted by $j_{v,m}'$. Large zeros may be obtained from McMahon's expansion as given by Abramowitz and Stegun [18]; i.e., when v is fixed, m >> v, and $u=4v^4$, then

$$j_{v,m}' = b - \frac{u+3}{8b} - \frac{4(7u^2 + 82u - 9)}{3(8b)^3} - \frac{32(83u^3 + 2075u^2 - 3039u + 3537)}{15(8b)^5}$$
$$- \frac{64(6949u^4 + 296,492u^3 - 1,248,002u^2 + 7,414,380u - 5,853,627)}{105(8b)^7} - \ldots,$$

where $b=(m+v/2-3/4)\pi$. On the other hand, the well-known Newton-Raphson technique may be employed to obtain the small zeros as well as the large. This method involves an iterative process in which an initial approximation x_o to a desired real root is obtained, by rough graphical methods or otherwise, and the recurrence relation

$$x_{n+1} = x_n - f(x_n)/f'(x_n)$$

is used to generate a sequence of successive approximations $x_1, x_2, \ldots, x_n, \ldots$ which converges to the desired root. Starting with

$$f(x) = xJ_1'(x) = xJ_o(x) - J_1(x),$$

it is easy to show that the zeros of $J_1'(x) = 0$ can be obtained from

$$x_{n+1} = \frac{x_n[(x_n^2-2)J_1(x_n) + x_nJ_o(x_n)]}{(x_n^2-1)J_1(x_n)}$$

Thus, once the first zero $j'_{1,1}$ has been determined to the desired accuarcy, the next, $j'_{1,2}$, can be obtained by repeating the above process starting with the initial approximation $j'_{1,2} \doteq j'_{1,1} + \pi$, since the zeros of $J'_1(x)$ are separated roughly by π. This process has been utilized to obtain the first 40 zeros $j'_{1,m} = \xi_m$, (m=1, 2, 3, . . . , 40), which are tabulated along with the values of $J_1(\xi_m)$ in Table 1.

Table 1. Some Zeros of $J_1'(x)$ and Values of $J_1(\xi_m)$

m	ξ_m	$J_1(\xi_m)$
1	1.84118	0.58186 512
2	5.33144	-0.34612 619
3	8.53632	0.27329 993
4	11.70600	-0.23330 440
5	14.86359	0.20701 284
6	18.01553	-0.18801 748
7	21.16437	0.17345 904
8	24.31133	-0.16183 821
9	27.45705	0.15228 206
10	30.60192	-0.14424 289
11	33.74618	0.13735 718
12	36.88999	-0.13137 284
13	40.03344	0.12610 881
14	43.17663	-0.12143 116
15	46.31960	0.11723 850
16	49.46239	-0.11345 236
17	52.60504	0.11001 101
18	55.74757	-0.10686 507
19	58.89000	0.10397 455
20	62.03235	-0.10130 665
21	65.17416	0.09883 418
22	68.31683	-0.09653 438
23	71.45898	0.09438 803
24	74.60109	-0.09237 880
25	77.74315	0.09049 268
26	80.88519	-0.08871 756
27	84.02718	0.08704 301
28	87.16916	-0.08545 986
29	90.31110	0.08396 007
30	93.45301	-0.08253 658
31	96.59491	0.08118 313
32	99.73679	-0.07989 417
33	102.87870	0.07866 471
34	106.02054	-0.07749 031
35	109.16236	0.07636 701
36	112.30417	-0.07529 119
37	115.44597	0.07425 961
38	118.58775	-0.07326 029
39	121.72954	0.07231 757
40	124.87131	-0.07140 201

SECTION 7

CONCLUSION

This report considers the irrotational motion of an incompressible, inviscid liquid contained in a partially filled cylindrical tank on the vertical wall of which is mounted a thin ring for the purpose of damping the free surface oscillations of the liquid. The tank is subjected to both transverse and rotational harmonic vibrations. In the solution of Laplace's equation for the velocity potential a dual Dini series arises because the boundary conditions in the plane of the annular baffle are of the mixed type. Following a method due to Tranter and Cooke [8], several forms of the formal solution of the dual series are given which, in every case, lead to an infinite system of linear algebraic equations. From a numerical point of view, many of these systems are plagued with an ill-conditioned coefficient matrix, and, also, it should be pointed out that in certain cases great care must be taken to obtain accurately the elements of these matrices since they are obtained by summing rather slowly convergent infinite series. Some methods of speeding the convergence of these series are discussed. For the numerical solution of the dual series, since the zeros of the transcendental equation $J_1'(x) = 0$ and numerical values of Bessel functions of the first kind of various orders and arguments are needed, special formulas are tabulated and machine language algorithms are referenced.

SECTION 8

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