ANALYSIS OF NOTCHES AND CRACKS: A NUMERICAL PROCEDURE FOR SOLVING THE EQUATIONS OF ELASTO-PLASTIC FLOW IN THREE INDEPENDENT VARIABLES

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FOREWORD

This report describes work performed in the Department of Mechanical Engineering at the Carnegie Institute of Technology of Carnegie-Mellon University for Langley Research Center, National Aeronautics and Space Administration, under NASA grant NGR-39-002-023, "Analysis of Notches and Cracks." The work was performed between October 1966 and December 1967, and is a continuation of earlier work by the author. Notes for this report are kept in File SM-7; it is one of several to be issued in conjunction with this effort, the others including:

Further Comment on the Association between Crack Opening and G_I, J. L. Swedlow, International Journal of Fracture Mechanics, 3, 1, March 1967, pp 75-79.

Character of the Equations of Elasto-Plastic Flow in Three Independent Variables, J. L. Swedlow, International Journal of Non-Linear Mechanics, to appear.

Analysis of Cracks and Notches, J. L. Swedlow, ASM Transactions Quarterly, 60, 3, September 1967, p 557.

Conversion of Uniaxial Stress-Strain Data for Use in Elasto-Plastic Analysis, J. L. Swedlow, Report SM-2, Department of Mechanical Engineering, Carnegie-Mellon University.

Invited Discussion of the Paper, Elastic-Plastic Stress and Strain Distributions Near Crack Tips Due to Antiplane Shear (ASME paper 67-WA/MET-19), J. L. Swedlow, Report SM-3, Department of Mechanical Engineering, Carnegie-Mellon University.

Equations of Elasto-Plastic Flow for Antiplane Shear, J. L. Swedlow, Report SM-4, Department of Mechanical Engineering, Carnegie-Mellon University.

Three-Dimensional Elastostatics -- A Direct Formulation, T. A. Cruse, Report SM-5, Department of Mechanical Engineering, Carnegie-Mellon University.

Analysis of Notches and Cracks: Progress in Pilot Comparisons between Experiment and Theory, J. L. Swedlow, Report SM-8, Department of Mechanical Engineering, Carnegie-Mellon University.

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ABSTRACT

Using a recently derived form of the equation for elasto-plastic flow, procedures are set up for the solution of initial- and boundary-value problems. Attention is directed specifically to the problem of a longitudinally stressed, axisymmetric notched rod. The problem is stated explicitly in terms of ellipsoidal coordinates, which seem most natural for the geometric character of the problem. The basic numerical procedures are outlined in some detail, and their application to the problem at hand is discussed. Means for managing and utilizing the output are considered although, at this writing, no actual data are in hand. The underlying feature of the whole procedure is to provide solutions to such problems in general, such that the solutions will be of consistently high accuracy.

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I. INTRODUCTION

It has become nearly truistic to observe that, in the context of fracture mechanics, our knowledge of the stress and strain fields in the vicinity of cracks suffers from the exclusion from analysis of the effects of plasticity, or yielding. The procedures now available derive mainly from linear elasticity and its results for highly idealized bodies.[1,2] That such procedures have been successful is hardly contradictory; much effort has been expended in defining the restrictions under which linear fracture mechanics, as it is called, is applicable.

The picture that seems to emerge is that linear fracture mechanics represents a limit case of sorts. That is, such procedures are meaningful for materials and/or geometries in which but little yielding occurs at the point of fracture initiation. Somewhat more yielding tends to invalidate the predictions of linear fracture mechanics, and the associated procedures appear incapable of being suitably modified to account fully for greater amounts of yielding.

Hence, a reexamination of the associated problems in continuum mechanics is indicated. The ultimate goal is, in a broad sense, to develop a more generalized understanding of the stress and strain fields accompanying the event of fracture (or rupture, fatigue, etc.), while retaining the useful simplicity now at hand. In particular, any further results should contain the present (elastic) results, both as a limit case and as a measure of the changes that occur as yielding proceeds.

In considering what sorts of features ought to be included in such work, one might define three main categories: material nonlinearities, geometric nonlinearities, and true geometric representation. The first of these involves mainly the event of plastic flow, and can include as well items such as creep, strain-weakening and -strengthening (in cyclic loading), and so on. The second incorporates the changing shape of a body as deformation proceeds, either locally or globally. The third category relaxes our historical limitations of treating problems in two spatial variables.

A fair amount of speculation and some very intense effort have been directed toward including these features in analysis. The present report is concerned with efforts in the first category, material nonlinearities. Moreover, a matter of continuing concern is to ensure that the procedures being developed do not necessarily preclude eventual consideration of the other features needed in analysis.

Earlier efforts[3,4,5] along these lines may be viewed in retrospect as pilot work. Those procedures involved formulation of the equations of elasto-plastic flow in a manner amenable to analysis, and development of efficient although crude numerical methods for problem solution. In effect, the earlier work has paved the way for the present effort by demonstrating that the procedures and methods are viable. Moreover, close examination of the results pointed up specific requirements to be met by more carefully prepared analyses.

As a result, the present program of research has been established with three main objectives. The first is development of very accurate numerical procedures for solving the exact equations of elasto-plastic flow, with emphasis on problems involving notches and cracks. Secondly, direct comparison between the results of such work and similarly precise experiments are to be made to establish the manner in which such analyses may be regarded as physically meaningful. Finally, the analytical tools thus developed are to be applied systematically to determine the interplay between material properties, geometry, loading conditions, and failure phenomena.

This report is concerned with the numerical procedures noted above. The first step in their establishment has been a reformulation of the equations of elasto-plastic flow.[6] This work has resulted in an explicit statement of Navier's equations, that is, the equilibrium equations written in terms of the displacement (rates). It has been found that these equations are quasi-linear, which greatly facilitates numerical solutions. Moreover, for materials whose stress-strain curve is monotonic, the equations are elliptic. This means that arbitrary work-hardening may, from a procedural standpoint, be treated merely as an extension of elasticity.

The next step has been the adoption of a generic view of notch and crack problems. Since numerical solution methods are unavoidable, computation of the infinite stresses associated with perfectly sharp cracks becomes problematic. On the other hand, actual cracks are not perfectly sharp but have root radii of the order of 10^{-4} in, or less. Such root radii engender stress concentration factors of the order of 10^2. It was thus concluded that geometries with similar stress concentrations are to be preferred, for two reasons. First, they model the physical situation more accurately than the mathematical abstraction of a perfectly sharp crack and, second, they preclude the necessity of finding an infinity on the computer.

Such an approach follows, of course, the pattern of using elliptic coordinates set by Inglis[7] and Griffith[8] long ago. As a test case, two numerical elastic analyses were performed and compared to exact analytical results.[9] The first, for torsion, was compared to the classical paper by Filon,[10] and the second for planar elasticity simulated Griffith's problem.[8] The comparison showed errors of the order of 1% or less for values of the ratio, root radius/semi-focal distance, as low as 10^{-4}. Thus the analysis of notches and cracks is to be performed in elliptic coordinates, with the stress concentrator being simply a preselected coordinate line. Distinction between a notch and a crack becomes arbitrary; for example, a crack might be defined as any stress raiser whose (elastic) stress concentration factor exceeds ten.

As a final step in establishing numerical procedures, detailed numerical methods must be defined. Based on earlier work,[3,9] it was found that the repeated application of Taylor's series is sufficient to meet this need. Details of these methods appear below.

In order to make this presentation explicit, attention herein is directed toward the problem of the tension of an axisymmetric notched rod. The procedures are equally applicable to planar elasto-plastic flow; indeed the latter cases are somewhat more simplified in that circumferential stress and strain components need not be calculated simultaneously with the others.

II. PROBLEM STATEMENT

The problem to be treated, as noted above, is that of the tension of an axisymmetric, notched rod. The rod has a radius, in its unnotched portion, of R; its length is 2H; and the notch is of hyperbolic shape* of depth D, root radius ρ. We presume loading to be axial only and insist that it, too, be axisymmetric. Loading may be generated either by displacements or stresses imposed on the rod ends; for simplicity we use here a constant axial stress. See Figure 2.1.

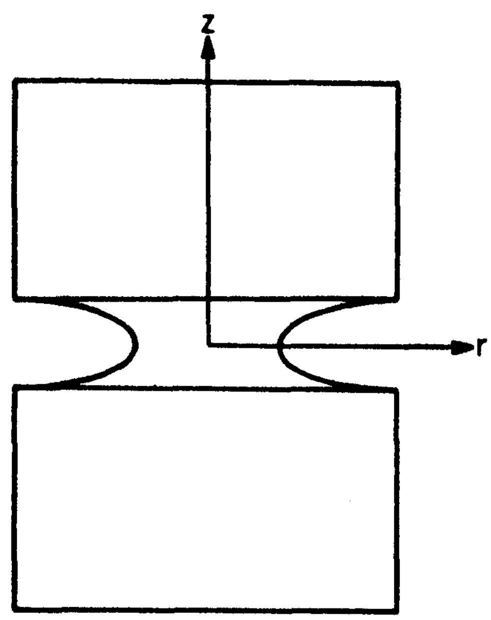


Figure 2.1 - Geometry of axisymmetric rod under tension, $\sigma_z = \bar{\sigma}$ at $z = \pm H$. Rod radius is R, notch depth D, and notch root radius ρ.

By symmetry, we need consider only the first quadrant of the domain; further detail appears below.

The governing equations for this case are derived elsewhere,[6] and that development need not be repeated here. For these equations

* More accurately, an hyperboloid of one sheet.

to become explicit, specification is to be made of two elastic constants, the inelastic portion of the stress-strain curve, and a loading function ϕ.

We may write the governing equations in an ellipsoidal system of coordinates (ξ, θ, η) derived from cylindrical coordinates (r, θ, z) by

$$\begin{aligned} r &= b \cosh\xi \cos\eta \\ \theta &= \theta \\ z &= b \sinh\xi \sin\eta \end{aligned} \tag{2.1}$$

where b is the semifocal distance of the family of confocal ellipsoids ξ = const and hyperboloids η = const. The following notation is then introduced:

$H^2 = \cosh 2\xi - \cos 2y$

u, w = displacement components in ξ, η directions

$\varepsilon_\xi, \varepsilon_\theta, \varepsilon_\eta, \varepsilon_{\xi\eta}$ = strain components

$\sigma_\xi, \sigma_\theta, \sigma_\eta, \tau_{\xi\eta}$ = stress components

and a dot over any dependent variable denotes its derivative with respect to time t. The three independent variables in the problem are seen to be ξ, η, t; all dependent variables are independent of the coordinate θ as a consequence of axisymmetry.

The strain rate - displacement rate relations are

$$\dot{\varepsilon}_\xi = \frac{\sqrt{2}}{bH} (\partial\dot{u}/\partial\xi + \dot{w} \sin 2\eta/H^2)$$

$$\dot{\varepsilon}_\theta = \frac{\sqrt{2}}{bH} (\dot{u} \tanh\xi - \dot{w} \tan\eta) \tag{2.2}$$

$$\dot{\varepsilon}_\eta = \frac{\sqrt{2}}{bH} (\partial\dot{w}/\partial\eta + \dot{u} \sinh 2\xi/H^2)$$

$$2\dot{\varepsilon}_{\xi\eta} = \frac{\sqrt{2}}{bH} [\partial\dot{u}/\partial\eta + \partial\dot{w}/\partial\xi - (\dot{u} \sin 2\eta + \dot{w} \sinh 2\xi)/H^2]$$

The equilibrium equations become

$$\partial\dot{\sigma}_\xi/\partial\xi + \partial\dot{\tau}_{\xi\eta}/\partial\eta + (\dot{\sigma}_\xi - \dot{\sigma}_\theta) \tanh\xi$$

$$+ (\dot{\sigma}_\xi - \dot{\sigma}_\eta) \sinh 2\xi/H^2 + \dot{\tau}_{\xi\eta}(2 \sin 2\eta/H^2 - \tan\eta) = 0$$

$$\tag{2.3}$$

$$\partial\dot{\tau}_{\xi\eta}/\partial\xi + \partial\dot{\sigma}_\eta/\partial\eta - (\dot{\sigma}_\eta - \dot{\sigma}_\theta) \tan\eta$$

$$- (\dot{\sigma}_\xi - \dot{\sigma}_\eta) \sin 2\eta/H^2 + \dot{\tau}_{\xi\eta}(2 \sin 2\xi/H^2 + \tanh\xi) = 0$$

It should be recalled that (2.2) and (2.3) require specification of initial conditions;[6] we presume all dependent variables to be null-valued at the beginning of loading.

The constitutive relations take the form

$$\dot{\sigma}_\xi = \frac{2\mu}{1-2\nu} (a_{11} \dot{\varepsilon}_\xi + a_{12} \dot{\varepsilon}_\theta + a_{13} \dot{\varepsilon}_\eta + 2a_{14} \dot{\varepsilon}_{\xi\eta})$$

$$\dot{\sigma}_\theta = \frac{2\mu}{1-2\nu} (a_{21} \dot{\varepsilon}_\xi + a_{22} \dot{\varepsilon}_\theta + a_{23} \dot{\varepsilon}_\eta + 2a_{24} \dot{\varepsilon}_{\xi\eta}) \tag{2.4}$$

$$\dot{\sigma}_\eta = \frac{2\mu}{1-2\nu} (a_{31} \dot{\varepsilon}_\xi + a_{32} \dot{\varepsilon}_\theta + a_{33} \dot{\varepsilon}_\eta + 2a_{34} \dot{\varepsilon}_{\xi\eta})$$

$$\dot{\tau}_{\xi\eta} = \frac{2\mu}{1-2\nu} (a_{41} \dot{\varepsilon}_\xi + a_{42} \dot{\varepsilon}_\theta + a_{43} \dot{\varepsilon}_\eta + 2a_{44} \dot{\varepsilon}_{\xi\eta})$$

where μ is the elastic shear modulus, ν is Poisson's ratio, $a_{ij} = a_{ji}$ (i, j = 1, 2, 3, 4), and

$$a_{11} = 1-\nu - (1-2\nu)\ \beta^2 a_\xi^2$$

$$a_{12} = \nu - (1-2\nu)\ \beta^2 a_\xi a_\theta$$

$$a_{13} = \nu - (1-2\nu)\ \beta^2 a_\xi a_\eta$$

$$a_{14} = -\frac{1}{2}(1-2\nu)\ \beta^2 a_\xi a_\tau$$

$$a_{22} = 1-\nu - (1-2\nu)\ \beta^2 a_\theta^2 \tag{2.5}$$

$$a_{23} = \nu - (1-2\nu)\ \beta^2 a_\theta a_\eta$$

$$a_{24} = -\frac{1}{2}(1-2\nu)\ \beta^2 a_\theta a_\tau$$

$$a_{33} = 1-\nu - (1-2\nu)\ \beta^2 a_\eta^2$$

$$a_{34} = -\frac{1}{2}(1-2\nu)\ \beta^2 a_\eta a_\tau$$

$$a_{44} = \frac{1}{2}-\nu-\frac{1}{4}(1-2\nu)\ \beta^2 a_\tau^2$$

To define the terms in (2.5) we note that the loading function ϕ is given by

$$\phi = \phi(J_2, J_3) \tag{2.6}$$

where J_2 and J_3 are the second and third invariants of the stress deviator tensor. Then

$$\begin{aligned}
\gamma a_\xi &= \partial\phi/\partial\sigma_\xi \\
&= (\partial\phi/\partial J_2)(\partial J_2/\partial\sigma_\xi) + (\partial\phi/\partial J_3)(\partial J_3/\partial\sigma_\xi) \\
\gamma a_\theta &= \partial\phi/\partial\sigma_\theta \\
\gamma a_\theta &= \partial\phi/\partial\sigma_\eta \\
\gamma a_\tau &= \partial\phi/\partial\tau_{\xi\eta}
\end{aligned} \tag{2.7}$$

and
$$\begin{aligned}
\gamma^2 &= 1 + \beta^2\,(a_\xi^{\,2} + a_\theta^{\,2} + a_\eta^{\,2} + \tfrac{1}{2}\,a_\tau^{\,2}) \\
\beta^2 &= \frac{\mu}{\mu_{eq}^{(p)}}\;\frac{\phi}{\gamma(\sigma_\xi a_\xi + \sigma_\theta a_\theta + \sigma_\eta a_\eta + 2\tau_{\xi\eta} a_\tau)}
\end{aligned} \tag{2.8}$$

The quantity $\mu_{eq}^{(p)}$ is the equivalent plastic shear (tangent) modulus.[6]

Inserting (2.2) into (2.4), and the result into (2.3) gives equilibrium equations in terms of the displacement rates. The result is written in terms of the following functions

$$A_\xi = \operatorname{sech}\xi\ \partial(a_{11}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{14}\cos\eta)/\partial\eta$$

$$B_\xi = \operatorname{sech}\xi\ \partial(a_{14}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{44}\cos\eta)/\partial\eta$$

$$C_\xi = \operatorname{sech}\xi\ \partial(a_{13}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{34}\cos\eta)/\partial\eta$$

$$D_\xi = \operatorname{sech}\xi\ \partial(a_{12}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{24}\cos\eta)/\partial\eta$$

(2.9)

$$A_\eta = \operatorname{sech}\xi\ \partial(a_{34}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{33}\cos\eta)/\partial\eta$$

$$B_\eta = \operatorname{sech}\xi\ \partial(a_{44}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{34}\cos\eta)/\partial\eta$$

$$C_\eta = \operatorname{sech}\xi\ \partial(a_{14}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{13}\cos\eta)/\partial\eta$$

$$D_\eta = \operatorname{sech}\xi\ \partial(a_{24}\cosh\xi)/\partial\xi + \sec\eta\ \partial(a_{23}\cos\eta)/\partial\eta$$

We thus have

$$a_{11}\partial^2\dot{u}/\partial\xi^2 + 2a_{14}\partial^2\dot{u}/\partial\xi\partial\eta + a_{44}\partial^2\dot{u}/\partial\eta^2$$

$$+ a_{14}\partial^2\dot{w}/\partial\xi^2 + (a_{13}+a_{44})\partial^2\dot{w}/\partial\xi\partial\eta + a_{34}\partial^2\dot{w}/\partial\eta^2$$

$$+ A_\xi\partial\dot{u}/\partial\xi + B_\xi\partial\dot{u}/\partial\eta + [B_\xi - a_{12}\tan\eta + a_{24}(\sinh 2\xi/H^2$$

$$- \tanh\xi) + (a_{11} + a_{44})\sin 2\eta/H^2]\ \partial\dot{w}/\partial\xi$$

$$+ [C_\xi - a_{23}\tanh\xi - a_{24}(\sin 2\eta/H^2 + \tan\eta) - (a_{33}$$

$$+ a_{44})\sinh 2\xi/H^2]\ \partial\dot{w}/\partial\eta + \{D_\xi\tanh\xi + a_{12}\operatorname{sech}^2\xi$$

$$- a_{22}\tanh^2\xi + (C_\xi\sinh 2\xi - B_\xi\sin 2\eta)/H^2 \qquad (2.10a)$$

$$+ 2(- a_{23}\sinh 2\xi\tanh\xi + a_{11}\cosh 2\xi + a_{24}\tanh\xi\sin 2\eta$$

$$- a_{44}\cos 2\eta)/H^2 + [- (2a_{13} + a_{33})\sinh^2 2\xi + 2a_{14}\sinh 2\xi\sin 2\eta$$

$$+ a_{44}\sin^2 2\eta]/H^4\}\ \dot{u} + \{- D_\xi\tan\eta - 2a_{14} + a_{22}\tanh\xi\tan\eta$$

$$- a_{24}\sec^2\eta - (B_\xi\sinh 2\xi - A_\xi\sin 2\eta)/H^2$$

$$+ [a_{24}(\sinh 2\xi\tanh\xi - \sin 2\eta\tan\eta) + a_{23}\sinh 2\xi\tan\eta$$

$$- a_{12}\tanh\xi\sin 2\eta]/H^2 + [a_{34}\sinh^2 2\xi + (a_{44} - a_{13}$$

$$- 2a_{11})\sinh 2\xi\sin 2\eta + a_{14}(2\sinh^2 2\xi - \sin^2 2\eta)]/H^4\}\ \dot{w} = 0$$

$$
\begin{aligned}
& a_{33}\partial^2\dot{w}/\partial\eta^2 + 2a_{34}\partial^2\dot{w}/\partial\eta\partial\xi + a_{44}\partial^2\dot{w}/\partial\xi^2 \\
& + a_{34}\partial^2\dot{u}/\partial\eta^2 + (a_{13}+a_{44})\partial^2\dot{u}/\partial\eta\partial\xi + a_{14}\partial^2\dot{u}/\partial\xi^2 \\
& + A_\eta\partial\dot{w}/\partial\eta + B_\eta\partial\dot{w}/\partial\xi + [B_\eta + a_{23}\tanh\xi + a_{24}(\sin2\eta/H^2 \\
& + \tan\eta) + (a_{33}+a_{44})\sinh2\xi/H^2]\ \partial\dot{u}/\partial\eta \\
& + [C_\eta + a_{12}\tan\eta - a_{24}(\sinh2\xi/H^2 - \tanh\xi) - (a_{11} \\
& + a_{44})\sin2\eta/H^2]\ \partial\dot{u}/\partial\xi + \{- D_\eta\tan\eta - a_{23}\sec^2\eta \\
& - a_{22}\tan^2\eta + (C_\eta\sin2\eta - B_\eta\sinh2\xi)/H^2 \\
& + 2(a_{12}\sin2\eta\tan\eta + a_{13}\cos2\eta - a_{24}\tan\eta\sinh2\xi \\
& - a_{44}\cosh2\xi)/H^2 + [-(2a_{13}+a_{11})\sin^2 2\eta + 2a_{34}\sin2\eta\sinh2\xi \\
& + a_{44}\sinh^2 2\xi]/H^4\}\ \dot{w} + \{D_\eta\tanh\xi + 2a_{34} + a_{22}\tan\eta\tanh\xi \\
& + a_{24}\operatorname{sech}^2\xi - (B_\eta\sin2\eta - A_\eta\sinh2\xi)/H^2 \\
& + [-a_{24}(\sin2\eta\tan\eta - \sinh2\xi\tanh\xi) - a_{12}\sin2\eta\tanh\xi \\
& + a_{23}\tan\eta\sinh2\xi]/H^2 + [a_{14}\sin^2 2\eta + (a_{44}-a_{13} \\
& - 2a_{33})\sin2\eta\sinh2\xi + a_{34}(2\sin^2 2\eta - \sinh^2 2\xi)]/H^4\}\ \dot{u} = 0
\end{aligned}
\qquad (2.10b)
$$

A certain duality may be observed between (2.10a) and (2.10b); indeed suitable permutations of the coefficients will allow one to be written directly from the other.

Boundary Equations: Conditions on various boundary surfaces may be specified in the usual manner, recalling only that time dependence must be taken into account for a problem to be properly posed. Should a boundary surface coincide with a coordinate surface, conditions may be obtained from the foregoing. Thus, displacements may be specified directly, and stresses from a suitable combination of (2.2) and (2.4).

Other surfaces, described more readily in the cylindrical coordinates (r, θ, z), require transformation formulae. These relations are, for stresses

$$\dot{\sigma}_r = [(\cosh 2\xi - 1)(1+\cos 2\eta)\dot{\sigma}_\xi + (\cosh 2\xi + 1)(1-\cos 2\eta)\dot{\sigma}_\eta - 2\sin 2\xi \sin 2\eta\ \dot{\tau}_{\xi\eta}]/(2H^2)$$

$$\dot{\sigma}_z = [(\cosh 2\xi + 1)(1-\cos 2\eta)\dot{\sigma}_\xi + (\cosh 2\xi - 1)(1+\cos 2\eta)\dot{\sigma}_\eta + 2\sinh 2\xi \sin 2\eta\ \dot{\tau}_{\xi\eta}]/(2H^2) \qquad (2.11)$$

$$\dot{\tau}_{rz} = [\sinh 2\xi \sin 2\eta(\dot{\sigma}_\xi - \dot{\sigma}_\eta) + 2(\cosh 2\xi \cos 2\eta - 1)\dot{\tau}_{\xi\eta}]/(2H^2)$$

and, for displacements

$$\dot{u}_r = \sqrt{2}(\dot{u}\ \sinh\xi\cos\eta - \dot{w}\ \cosh\xi\sin\eta)/H$$

$$\dot{w}_z = \sqrt{2}(\dot{u}\ \cosh\xi\sin\eta + \dot{w}\ \sinh\xi\cos\eta)/H \qquad (2.12)$$

where $\dot{u}_r$ and $\dot{w}_z$ are displacement components in cylindrical coordinates. The relations (2.11) and (2.12) may be integrated with respect to time merely by removing the dots above all dependent variables.

Specific Conditions: The problem at hand is to be solved in the domain

$$0 < \xi < \xi_u$$

$$\eta_o < \eta < \pi/2$$

where

$$\xi_u = \begin{cases} \sinh^{-1} [H/(b \sin\eta)] , & \eta > \eta_c \\ \cosh^{-1} [R/(b \cos\eta)] , & \eta > \eta_c \end{cases} \tag{2.13}$$

and

$$2(b \cos\eta_c)^2 = R^2+H^2+b^2 + [(R^2+H^2+b^2)^2 - 4R^2b^2]^{1/2} \tag{2.14}$$

Also, we have

$$b = (R-D) \sqrt{1 + \rho/(R-D)}$$

$$\eta_o = \cos^{-1} [(R-D)/b] \tag{2.15}$$

Hence the domain is specified once R, H, D, and ρ are known. A sketch of a typical domain appears in Figure 2.2, using R = H = 2 in, D = 1 in, ρ = 0.1 in. Both the "physical plane" and the "coordinate plane," i.e., diametrical cross-sections, are shown for later reference.

In the domain, we presume that $\phi = \phi(J_2, J_3)$ is given subject to the constraint[6] that ϕ has the dimensions of stress; that $\mu_{eq}^{(p)} = \mu_{eq}^{(p)}(\phi)$ is specified; and that μ and ν are known. Then (2.10a) and (2.10b) are the governing equations.

Boundary conditions are

$$\xi = 0,\ \eta_o \le \eta \le \pi/2:\quad \dot{u} = 0;\ \partial\dot{w}/\partial\xi = 0$$

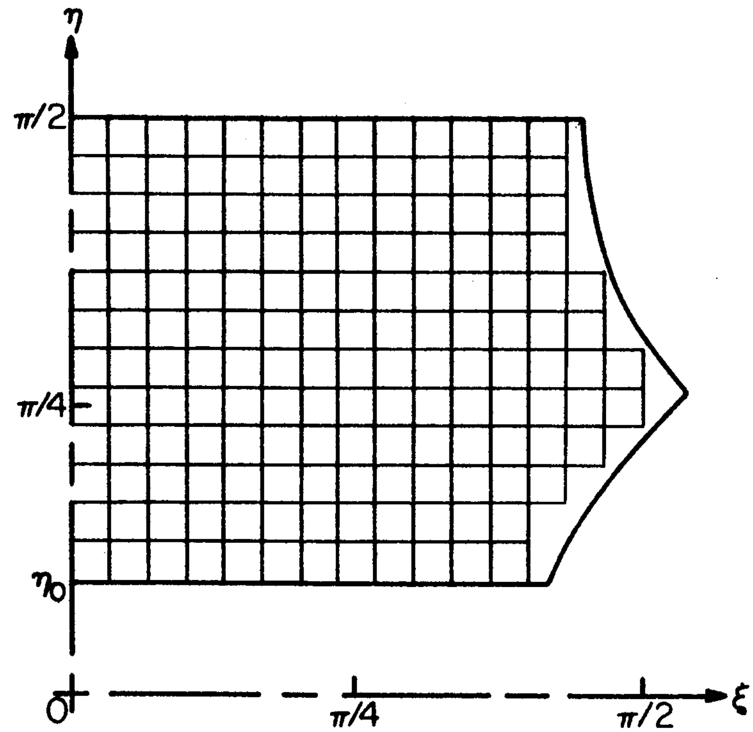
(equivalently, $\dot{u}$ is antisymmetric and $\dot{w}$ is symmetric with respect to ξ)

$$0 \le \xi \le \xi_u,\ \eta = \eta_o:\quad \dot{\sigma}_\eta = \dot{\tau}_{\xi\eta} = 0 \qquad (2.16)$$

$$\xi = \xi_u,\ \eta_o \le \eta \le \eta_c:\quad \dot{\sigma}_r = \dot{\tau}_{rz} = 0$$

$$\xi = \xi_u,\ \eta_c \le \eta \le \pi/2:\quad \dot{\sigma}_z = \dot{\bar{\sigma}};\ \dot{\tau}_{rz} = 0$$

$$0 \le \xi \le \xi_u,\ \eta = \pi/2:\quad \partial u/\partial\eta = 0;\ \dot{w} = 0$$



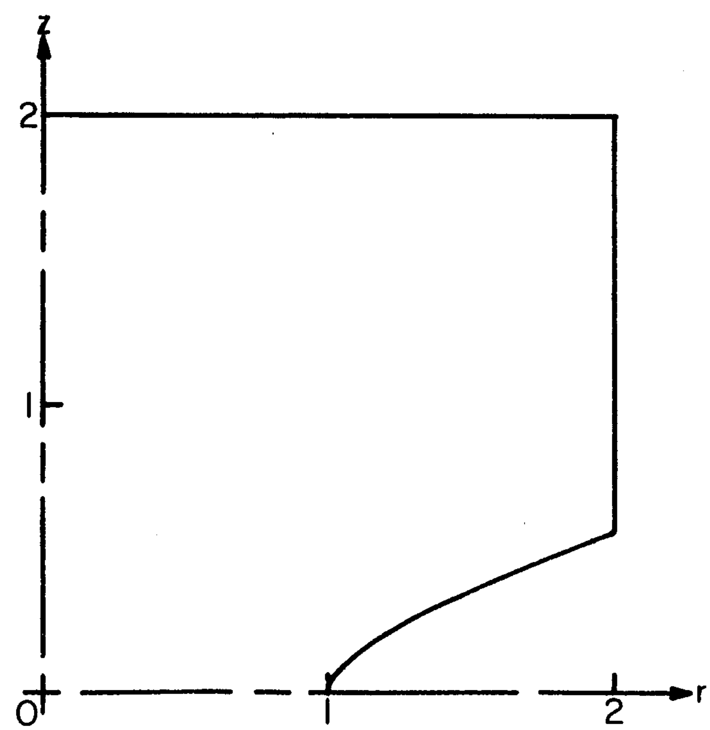


Figure 2.2 - Coordinate plane (above) and physical plane (below), for R = H = 2, D = 1, ρ = 0.1

and symmetry conditions on the last statement may be used in analogy to those specified for the first. The function $\overline{\sigma} = \overline{\sigma}(t)$ is the loading, required by the theory to accumulate sufficiently slowly that the deformation remains quasi-static.

III. PRIMARY NUMERICAL PROCEDURES

The overall numerical procedure rests on three primary techniques of numerical analysis. The first is repeated applications of Taylor's series, and is reviewed in some detail below. The second is a procedure for solving large sets of simultaneous (linear) algebraic equations, and is based on the work of C. W. McCormick.[11] It is a method applicable to sparse but banded matrices and offers the features of low storage in the computer, high speed, and controllable accuracy. The third is a standard predictor-corrector method, introduced below at a more appropriate point.

In addition, there are certain procedures for handling the mass of output anticipated. These are, at this writing, in initial developmental stages; further description appears at the end of the report.

TAYLOR SERIES

The theoretical aspects of Taylor's theorem are well documented and need not be repeated here. Its modes of application, however, are manifold and bear some discussion.

Consider a coordinate line as might be selected in a "coordinate plane," shown in Figure 3.1. At equi-spaced points along this line, the function ϕ - not to be confused with the loading function - assumes discrete values, as indicated by the subscripts i, j,,,q. Providing that ϕ is analytic along this line, it possesses

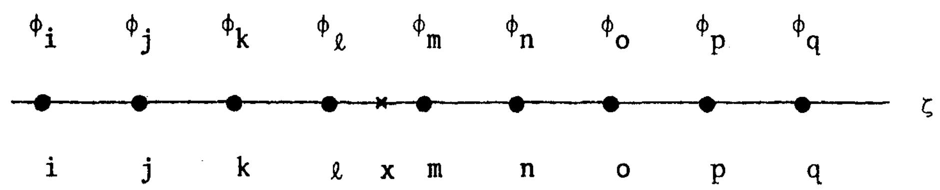


Figure 3.1 - Coordinate line in a coordinate plane and ϕ evaluated at equi-spaced points i, j,,,q.

Taylor's series expansions about any point on the coordinate line.

<u>Remark</u>: In the following development, equal spacing is presumed. Arbitrarily unequal spacing may be used, however, to derive analogous results; the formulae are considerably more involved.[12] The requirement of analyticity is not restrictive in the present problem for, as may be recalled, the domain is such as to exclude singularities in the solution.

A typical expansion takes the form

$$\phi = \phi_\ell + (\zeta-\zeta_\ell)\ \partial\phi_\ell/\partial\zeta + \frac{1}{2!}\ (\zeta-\zeta_\ell)^2\partial^2\phi_\ell/\partial\zeta^2 + \ldots$$

$$+ \frac{1}{n!}\ (\zeta-\zeta_\ell)^n\ \partial^n\partial_\ell/\partial\zeta^n + \ldots \tag{3.1}$$

In particular we have, with $\Delta\zeta = \zeta_m - \zeta_\ell = \zeta_\ell - \zeta_k$, etc.,

$$\phi_m = \phi_\ell + \Delta\zeta\ \partial\phi_\ell/\partial\zeta + \frac{1}{2}(\Delta\zeta)^2\partial^2\phi_\ell/\partial\zeta^2 + \ldots$$
$$+ \frac{1}{n!}\ (\Delta\zeta)^n\partial^n\phi_\ell/\partial\zeta^n + \ldots \tag{3.2}$$

If expressions similar to (3.2) are written for ϕ_j, ϕ_k, ϕ_ℓ, and ϕ_n, based on (3.1) and with attention paid to the sign and magnitude of $\zeta - \zeta_\ell$ in (3.1), we arrive at a set of algebraic relations between ϕ at points neighboring ζ_ℓ, and the value of ϕ and its derivatives at ζ_ℓ. For example, if we denote $\partial\phi_\ell/\partial\zeta$ by ϕ_ℓ', and so on, we have

$$\phi_m = \phi_\ell + \Delta\zeta\phi_\ell' + \frac{1}{2}\Delta\zeta^2\phi_\ell'' + \frac{1}{6}\Delta\zeta^3\phi_\ell''' + \frac{1}{24}\Delta\zeta^4\phi_\ell'''' + \ldots$$

$$\phi_\ell = \phi_\ell \tag{3.3}$$

$$\phi_k = \phi_\ell - \Delta\zeta\phi_\ell' + \frac{1}{2}\Delta\zeta^2\phi_\ell'' - \frac{1}{6}\Delta\zeta^3\phi_\ell''' + \frac{1}{24}\Delta\zeta^4\phi_\ell'''' + \ldots$$

It is well known that (3.3) may be solved to find an appropriate value of, say ϕ_ℓ''. The procedure is to multiply the first of (3.3) by a, the second by b, and the third by c, and equate the sum to ϕ_ℓ''.

Hence

$$a\phi_m + b\phi_\ell + c\phi_k = (a+b+c)\phi_\ell + (a-c)\Delta\zeta\phi_\ell' + \frac{1}{2}(a+c)\Delta\zeta^2\phi_\ell'' + \frac{1}{6}(a-c)\Delta\zeta^3\phi_\ell''' + \frac{1}{24}(a+c)\Delta\zeta^4\phi_\ell'''' + \ldots \quad (3.4)$$

The right-hand side of (3.4) is set equal to ϕ_ℓ'', whereby we require

$$\begin{aligned} a + b + c &= 0 \\ \Delta\zeta(a \quad - c) &= 0 \\ \frac{1}{2}\Delta\zeta^2(a \quad + c) &= 1 \end{aligned} \quad (3.5)$$

Note that we have thus placed conditions on the first three coefficients on the right of (3.4), because we had only three constants to evaluate. If more terms are desired, more points, e.g., ϕ_j and ϕ_n, must be included in (3.3).

The solution to (3.5) is obviously

$$a = -\frac{1}{2}b = c = 1/\Delta\zeta^2 \quad (3.6)$$

So that (3.4) becomes

$$(\phi_m - 2\phi_\ell + \phi_k)/\Delta\zeta^2 = \phi_\ell'' + \frac{1}{12}\Delta\zeta^2\phi_\ell'''' + \ldots$$

This is the well-known central difference formula for an ordinary second derivative. Its error is indicated by the second term on the right of (3.7) and has a magnitude largely determined by $\Delta\zeta^2$. Neglecting the error term gives the approximate relation desired.

In similar fashion other derivative formulae can be constructed. First derivatives, for example, may be written as

$$\begin{aligned}\phi_\ell' &= (-\phi_n + 4\phi_m - 3\phi_\ell)/2\Delta\zeta \\ \phi_\ell' &= (\phi_m - \phi_k)/2\Delta\zeta \\ \phi_\ell' &= (3\phi_\ell - 4\phi_k + \phi_j)/2\Delta\zeta\end{aligned} \qquad (3.8)$$

all with errors of the order of $\Delta\zeta^2$. The expressions in (3.8) are forward, central, and backward difference formulae, and are of use at boundaries of a domain and its interior.

Interpolation Formulae: The same procedure may be used to interpolate values of ϕ between specified points. Consider, for example, the problem of finding ϕ at ζ_x such that $\zeta_x - \zeta_\ell = x\Delta\zeta$. Expanding about ζ_x, we have

$$\phi_m = \phi_x + (1-x)\Delta\zeta\phi_x' + \frac{1}{2}(1-x)^2\Delta\zeta^2\phi_x'' + \frac{1}{6}(1-x)^3\Delta\zeta^3\phi_x''' + \ldots$$

$$\phi_\ell = \phi_x - x\Delta\zeta\phi_x' + \frac{1}{2}x^2\Delta\zeta^2\phi_x'' - \frac{1}{6}x^3\Delta\zeta^3\phi_x''' + \ldots \qquad (3.9)$$

$$\phi_j = \phi_x - (1+x)\Delta\zeta\phi_x' + \frac{1}{2}(1+x)^2\Delta\zeta^2\phi_x'' - \frac{1}{6}(1+x)^3\Delta\zeta^3\phi_x''' + \ldots$$

In this case, we have

$$\begin{aligned} a\phi_m + b\phi_\ell + c\phi_j = (a+b+c)\phi_x &+ [(1-x)a - xb \\ &- (1+x)c]\Delta\zeta\phi_x' + \frac{1}{2}[(1-x)^2a \\ &+ x^2b + (1+x)^2c]\Delta\zeta^2\phi_x'' \\ &+ \frac{1}{6}[(1-x)^3a - x^3b \\ &- (1+x)^3c]\Delta\zeta^3\phi_x''' + \ldots \end{aligned} \qquad (3.10)$$

and the right-hand side of (3.10) must equal ϕ_x. Hence

$$\begin{aligned} a + b + c &= 1 \\ (1-x)a - xb - (1+x)c &= 0 \qquad (3.11) \\ (1-x)^2a + x^2b + (1+x)^2c &= 0 \end{aligned}$$

giving

$$a = \frac{1}{2} x(1+x)$$

$$b = 1 - x^2 \qquad (3.12)$$

$$c = - \frac{1}{2} x(1-x)$$

and (3.10) becomes

$$\frac{1}{2} x(1+x)\phi_m + (1-x^2)\phi_\ell - \frac{1}{2} x(1-x)\phi_j = \phi_x + \frac{1}{6} x(1-x)^2 \Delta\zeta^3 \phi_x''' + \ldots \qquad (3.13)$$

which is seen to be a good approximation when the error term is neglected. In (3.13), the sign of x determines on which side of ζ_ℓ that ζ_x is to be found. There does not appear to be any reason that x be limited to the range $-1 \leq x \leq 1$, so that this relation may be reasonably used for extrapolation, i.e., $x > 1$ or $x < -1$, as well.

Further Results: Let us now apply the foregoing to a 3 x 3 grid of points, as sketched in Figure 3.2. We take

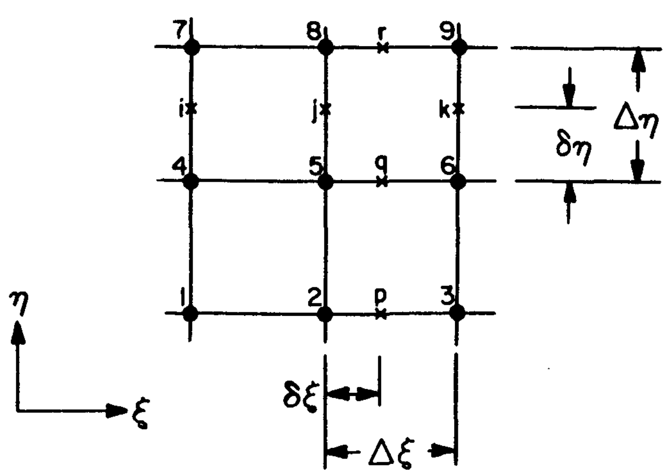


Figure 3.2 - Showing a 3 x 3 grid of points and intermediate points on one grid line only.

The function ϕ as known - or at least determinate - at the numbered points, and inquire as to the value of ϕ and its first derivatives at the lettered points. Following the procedure outlined above, we let $x = \delta\xi/\Delta\xi$, $y = \delta\eta/\Delta\eta$, taking care to observe the sign of x and y as set by the coordinate direction and the position of a given intermediate point relative to the one at center.

After some algebra we find

$$\phi_i = -\frac{1}{2}y(1-y)\phi_1 + (1-y^2)\phi_4 + \frac{1}{2}y(1+y)\phi_7$$

$$\phi_j = -\frac{1}{2}y(1-y)\phi_2 + (1-y^2)\phi_5 + \frac{1}{2}y(1+y)\phi_8 \qquad (3.14a)$$

$$\phi_k = -\frac{1}{2}y(1-y)\phi_3 + (1-y^2)\phi_6 + \frac{1}{2}y(1+y)\phi_9$$

$$\phi_p = -\frac{1}{2}x(1-x)\phi_1 + (1-x^2)\phi_2 + \frac{1}{2}x(1+x)\phi_3$$

$$\phi_q = -\frac{1}{2}x(1-x)\phi_4 + (1-x^2)\phi_5 + \frac{1}{2}x(1+x)\phi_6 \qquad (3.14b)$$

$$\phi_r = -\frac{1}{2}x(1-x)\phi_7 + (1-x^2)\phi_8 + \frac{1}{2}x(1+x)\phi_9$$

$$\partial\phi_i/\partial\xi = \{-\frac{1}{2}y(1-y)(-3\phi_1 + 4\phi_2 - \phi_3)$$

$$+ (1-y^2)(-3\phi_4 + 4\phi_5 - \phi_6)$$

$$+ \frac{1}{2}y(1+y)\ (-3\phi_7 + 4\phi_8 - \phi_9)\}/(2\Delta\xi)$$

$$\partial\phi_j/\partial\xi = \{-\frac{1}{2}y(1-y)(-\phi_1 + \phi_3) + (1-y^2)(-\phi_4 + \phi_6) \qquad (3.15a)$$

$$+ \frac{1}{2}y(1+y)(-\phi_7 + \phi_9)\}/(2\Delta\xi)$$

$$\partial\phi_k/\partial\xi = \{-\frac{1}{2}y(1-y)(\phi_1 - 4\phi_2 + 3\phi_3)$$

$$+ (1-y^2)(\phi_4 - 4\phi_5 + 3\phi_6)$$

$$+ \frac{1}{2}y(1+y)(\phi_7 - 4\phi_8 + 3\phi_9)\}/(2\Delta\xi)$$

$$\partial\phi_p/\partial\xi = \{-(1-2x)\phi_1 - 4x\phi_2 + (1+2x)\phi_3\}/(2\Delta\xi)$$

$$\partial\phi_q/\partial\xi = \{-(1-2x)\phi_4 - 4x\phi_5 + (1+2x)\phi_6\}/(2\Delta\xi) \qquad (3.15b)$$

$$\partial\phi_r/\partial\xi \quad \{-(1-2x)\phi_7 - 4x\phi_8 + (1+2x)\phi_9\}/(2\Delta\xi)$$

$$\partial\phi_i/\partial\eta = \{-\ (1-2y)\phi_1 - 4y\phi_4 + (1+2y)\phi_7\}/(2\Delta\eta)$$

$$\partial\phi_j/\partial\eta = \{-\ (1-2y)\phi_2 - 4y\phi_5 + (1+2y)\phi_8\}/(2\Delta\eta) \qquad (3.16\text{a})$$

$$\partial\phi_k/\partial\eta = \{-\ (1-2y)\phi_3 - 4y\phi_6 + (1+2y)\phi_9\}/(2\Delta\eta)$$

$$\partial\phi_p/\partial\eta = \{-\ \frac{1}{2}x(1-x)(-3\phi_1 + 4\phi_4 - \phi_7)$$

$$+ (1-x^2)(-3\phi_2 + 4\phi_5 - \phi_8)$$

$$+ \frac{1}{2}x(1+x)(-3\phi_3 + 4\phi_6 - \phi_9)\}/(2\Delta\eta)$$

$$\partial\phi_q/\partial\eta = \{-\ \frac{1}{2}x(1-x)(-\phi_1+\phi_7) + (1-x^2)(-\phi_2 + \phi_8)$$

$$+ \frac{1}{2}x(1+x)(-\phi_3+\phi_9)\}/(2\Delta\eta) \qquad (3.16\text{b})$$

$$\partial\phi_r/\partial\eta = \{-\ \frac{1}{2}x(1-x)(\phi_1 - 4\phi_4 + 3\phi_7)$$

$$+ (1-x^2)(\phi_2 - 4\phi_5 + 3\phi_8)$$

$$+ \frac{1}{2}x(1+x)(\phi_3 - 4\phi_6 + 3\phi_9)\}/(2\Delta\eta)$$

Examination of (3.15) and (3.16), in conjunction with (3.13) indicates that the errors of the first derivative formulae are comparable to those of (3.8).

APPLICATION TO THE DIFFERENTIAL EQUATIONS

The differential equations (2.10) may be converted directly to finite difference form by suitable use of formula such as (3.7) and the second of (3.8). Slightly different procedures are employed in finding the coefficients of (2.9), as noted below.

In order to make good use of McCormick's solution method,[11] some care needs to be exercised in writing the finite difference forms of (2.10) at each point. Accordingly, we write the equations at each grid point in the "coordinate plane" in a systematic manner. Starting at the lower left-hand corner in Figure 2.2, i.e., at $(\xi,\eta) = (0, \eta_o)$, we take the points in order from $\xi = 0$ to $\xi = \xi_u$ for $\eta = \eta_o$. Next we take the points along $\eta = \eta_o + \Delta\eta$, from left to right, and so on until reaching the upper right-hand corner in the top sketch of Figure 2.2.

Several features are presumed in this scheme. First we must have found the irregular right-hand boundary, i.e., the function $\xi_u(\eta)$. Second, for any point at or adjacent to a boundary, ficticious exterior points are implied. These points are temporary, and reference to them is removed when boundary conditions are imposed. Finally, since both equations are written at each point or, in other words, each point is considered but once, the order of appearance of the displacement rates is in pairs, one for each point.

Thus, our ultimate aim is to generate a system of algebraic equations whose matrix representation is

$$[K]\ \{\dot{\underline{u}}\} = \{\dot{\underline{T}}\} \tag{3.17}$$

where [ ] denotes a square matrix, { } denotes a column vector. In the present scheme, the entries in $\{\dot{u}\}$ are $\dot{u}(0, \eta_o)$, $\dot{w}(0, \eta_o)$; $\dot{u}(\Delta\xi, \eta_o)$, $\dot{w}(\Delta\xi, \eta_o)$; ...; $\dot{u}(n\Delta\xi, \pi/2)$, $\dot{w}(n\Delta\xi, \pi/2)$. Here we have used $n\Delta\xi$ to denote the grid point furthest right on the line $\eta = \pi/2$.

The structure of [K] is seen from the following observations. When writing the finite difference equations at any point (ξ, η), reference is made at most to the eight points surrounding it. For example, the points shown in Figure 3.2 are sufficient to write the finite difference equations at point 5 in the figure. This means that non-zero entries in [K] will be found only within some determinate distance from the main diagonal of the matrix. Such a matrix is referred to as banded.

An immediate consequence of this matrix structure is that only the band need be stored in the computer, thus allowing a much larger number of grid points to be used for a given storage capacity. Further, it becomes useful to refer to storage location relative to the main diagonal itself rather than absolute values.

The procedure begins by writing the finite difference equations at all points where the dependent variables are to be found, according to the foregoing scheme. The preliminary step of finding the coefficients is, for the most part, straightforward and requires no comment. Care is to be exercised, however, in finding the terms in (2.9) for points at or next to a boundary. Use of the forward and backward formulae of (3.8) is indicated in such cases.

Symmetry Conditions: The symmetry conditions in the first and last of (2.16) are inserted next. Along $\xi = 0$, for example, these take the form

$$\dot{u}(+\Delta\xi,\ \eta) = -\ \dot{u}(-\Delta\xi,\ \eta)$$
$$\dot{w}(+\Delta\xi,\ \eta) = +\ \dot{w}(-\Delta\xi,\ \eta) \qquad (3.18)$$

Hence, reference to $\dot{u}(-\Delta\xi, \eta)$, i.e., at a temporarily ficticious point, is replaced by suitable reference to $\dot{u}(+\Delta\xi, \eta)$. Thus the coefficient of the former is subtracted from the coefficient of the latter, and the entry in that location is set to zero. This step is straightforward; one need only take care that further entries are not made in that location subsequently.

Boundary Conditions on the Notch Surface: The second of (2.16) are imposed next. Performing the steps noted above, we arrive at the two simultaneous equations

$$a_{13}\partial\dot{u}/\partial\xi + a_{34}\partial\dot{u}/\partial\eta + a_{34}\partial\dot{w}/\partial\xi + a_{33}\partial\dot{w}/\partial\eta$$
$$+ c_1\dot{u} + c_2\dot{w} = 0$$
$$(3.19)$$
$$a_{14}\partial\dot{u}/\partial\xi + a_{44}\partial\dot{u}/\partial\eta + a_{44}\partial\dot{w}/\partial\xi + a_{34}\partial\dot{w}/\partial\eta$$
$$+ d_1\dot{u} + d_2\dot{w} = 0$$

where c_1, c_2, d_1, d_2 are abbreviations for more involved expressions readily derivable as noted. Solution of (3.19) for the normal derivatives gives

$$\partial\dot{u}/\partial\eta = -L_1(\partial\dot{u}/\partial\xi,\ \partial\dot{w}/\partial\xi,\ \dot{u},\ \dot{w})$$

$$\partial\dot{w}/\partial\eta = -L_2(\partial\dot{u}/\partial\xi,\ \partial\dot{w}/\partial\xi,\ \dot{u},\ \dot{w}) \qquad (3.20)$$

where L_1, L_2 are linear functions of their arguments. Putting, for example, the first of (3.20) in finite difference form, we have

$$\dot{u}(\xi,\ \eta_o-\Delta\eta) = \dot{u}(\xi,\ \eta_o+\Delta\eta) + 2\Delta\eta L_1 \qquad (3.21)$$

Hence reference in the matrix [K] to the ficticious $\dot{u}(\xi,\ \eta_o-\Delta\eta)$ may be replaced, via (3.21), to reference to points on the boundary or within the domain. Caution must be exercised in dealing with the derivative arguments of L_1 to use the appropriate formula of (3.8) so as to stay within the matrix band. Finally, the original coefficient of $u(\xi,\ \eta_o-\Delta\eta)$ is zeroed out, as above. The same procedure is used for the second of (3.20).

Remaining Boundary Conditions: While the entire motivation for this procedure is to provide great accuracy in the vicinity of the notch root, there is no reason to abandon accuracy near the free and loaded surfaces. Accordingly, methods have been devised to apply the remaining boundary conditions in a consistent manner even though the contour is irregular with respect to the coordinate system.

A more useful form of (2.11) is

$$N\dot{\sigma}_r = \alpha_1 \partial\dot{u}/\partial\xi + \alpha_4 \partial\dot{u}/\partial\eta + \alpha_4 \partial\dot{w}/\partial\xi + \alpha_3 \partial\dot{w}/\partial\eta$$

$$+ \gamma_1 \dot{u} + \gamma_2 \dot{w}$$

$$N\dot{\sigma}_z = \bar{\alpha}_1 \partial\dot{u}/\partial\xi + \bar{\alpha}_4 \partial\dot{u}/\partial\eta + \bar{\alpha}_4 \partial\dot{w}/\partial\xi + \bar{\alpha}_3 \partial\dot{w}/\partial\eta \qquad (3.22)$$

$$+ \bar{\gamma}_1 \dot{u} + \bar{\gamma}_2 \dot{w}$$

$$N\dot{\tau}_{rz} = \beta_1 \partial\dot{u}/\partial\xi + \beta_4 \partial\dot{u}/\partial\eta + \beta_4 \partial\dot{w}/\partial\xi + \beta_3 \partial\dot{w}/\partial\eta$$

$$+ \delta_1 \dot{u} + \delta_2 \dot{w}$$

where (See page 33 for (3.23)).

It is seen that (3.22) are in substantially the same form as (3.19), so that the remaining boundary conditions may be imposed in the same manner as those along the notch surface.

The one procedural difference is that (3.22) are not written at grid points, that is, the remaining boundary conditions pertain to points on grid lines η = const and at values of ξ greater than that of the grid point furthest to the right in the "coordinate plane" of Figure 2.2. As a result, the coefficients of (3.23) must be evaluated at the boundary itself; (3.14) are employed for the required extrapolation. Further, the derivatives in (3.22) are subjected to the same process using (3.15) and (3.16).

$$N = (1-2\nu)bH/(2\sqrt{2}\mu)$$

$$S = \sinh 2\xi \sin 2\eta / H^2$$

$$C = (\cosh 2\xi \cos 2\eta - 1)/H^2$$

$$\alpha_i = \frac{1}{2}(a_{1i}+a_{i3}) + \frac{1}{2}(a_{1i}-a_{i3})C - a_{i4}S$$

$$\bar{\alpha}_i = \frac{1}{2}(a_{1i}+a_{i3}) - \frac{1}{2}(a_{1i}-a_{i3})C + a_{i4}S$$

$$\beta_i = \frac{1}{2}(a_{1i}-a_{i3})S + a_{i4}C \tag{3.23}$$

$$\gamma_1 = \alpha_2 \tanh\xi + (\alpha_3 \sinh 2\xi - \alpha_4 \sin 2\eta)/H^2$$

$$\gamma_2 = -\alpha_2 \tan\eta + (\alpha_1 \sin 2\eta - \alpha_4 \sinh 2\xi)/H^2$$

$$\bar{\gamma}_1 = \bar{\alpha}_2 \tanh\xi + (\bar{\alpha}_3 \sinh\xi - \bar{\alpha}_4 \sin 2\eta)/H^2$$

$$\bar{\gamma}_2 = -\bar{\alpha}_2 \tan\eta + (\bar{\alpha}_1 \sin 2\eta - \bar{\alpha}_4 \sin 2\xi)/H^2$$

$$\delta_1 = \beta_2 \tanh\xi + (\beta_3 \sinh 2\xi - \beta_4 \sin 2\eta)/H^2$$

$$\delta_2 = -\beta_2 \tan\eta + (\beta_1 \sin 2\eta - \beta_4 \sinh 2\xi)/H^2$$

The main result is that the values of $\dot{u}$ and $\dot{w}$ at the temporary ficticious points external to the domain may be evaluated in terms of values internal to the domain. Thus the remaining boundary conditions are satisfied, and the problem is fully specified in a set of linear, algebraic, finite-difference equations.

IV. PROCEDURE FOR A TYPICAL LOAD INCREMENT

Strictly speaking, the problem as formulated is directed to solving for displacement rates at any instant of time, due to the excitation of a corresponding loading rate. In the present procedure, however, we make use of the fact that the displacement rates are nearly proportional to the loading rate, and thereby consider slightly different quantities. In effect, we look for the behavior over a "small" increment of time δt so that

$$\delta u = \int_t^{t+\delta t} \dot{u}\, dt \sim \dot{u}\, \delta t \tag{4.1}$$

and so on for all dependent variables. We thus speak interchangeably of rates and increments.

This procedure appears justified if the time increment is "small" enough. A rigorous definition of smallness does not present itself; certain features may nonetheless be distinguished. We would expect, for example, that $\delta\varepsilon/\varepsilon << 1$ for the various strain components. Looking at the types of stress-strain curve used in analysis (continously turning tangent; monotonic), we would anticipate further that the tangent modulus of the total strain curve, given by $\mu/(1+\mu/\mu_{eq}^{(p)})$, does not change rapidly from one load increment to the next.

In an operational sense, meeting such conditions becomes rather awkward. Allowance is thereby made for two modes of operation. The first, following earlier work,[4] requires load increments to be specified by the analyst as part of the input to the program. This mode requires some judgement but allows one to arrange for certain specific loads to be generated. Thus, for example, if comparison were to be made between theory and experiment at $\bar{\sigma} = 13{,}950$ lb/in^2, say, this load state could be achieved with no ambiguity.

The second mode sets the load increment internally. At first, a unit load increment is applied; the solution is then generated. Next, a load factor is determined such that the yielded zone grows to include one more grid point in the quadrant than before. Use of this mode is expected to provide information for better use of the first mode.

<u>Typical Load Increment</u>: The computer program embodying the procedures described in this report is written for a typical load increment. Implied in this phrase is a knowledge of the stress and strain fields at the start of the increment, which is sufficient to establish the position each grid point has reached on the stress-strain curve. Such information is adequate for finding the various coefficients required by the governing equations, both in the field and on the boundary.

A consequence of this view is that an elastic increment is typical. That is, if the stress-strain curve is initially linear, the entire procedure is applicable. Indeed we require this initial linearity; the first load increment is adjusted such that the most highly stressed (in terms of τ_{eq} or ϕ) grid point has just exceeded the proportional limit. The corresponding load increment is normally large relative to subsequent values, but this exception to the preceding comments in permissible because of the linearity of the problem at this point.

Writing the Matrix: As noted above, the matrix is banded, and we allocate storage only for the band of non-zero elements in [K] (cf. (3.17) et seq.). The number of columns in the matrix band is always odd because it contains the main diagonal of [K] and an equal number of diagonals to the right and left. The central column in the matrix band corresponds to the main diagonal of [K]. The number of rows is twice the number of grid points.

As a preliminary step, we evaluate the geometric data for the problem in such a fashion as to insure that the storage allocation of the matrix band is not exceeded, and that the ratio $\Delta\xi/\Delta\eta$ is close to unity.

Next, the differential equations (2.10), in finite difference form, are written at each grid point so generated. This step is preceded, of course, by evaluation of the pertinent coefficients in the equations using the data at the beginning of the load increment.

Finally, the various symmetry and boundary condition are inserted following the same order and procedure noted in the previous chapter. It is worth commenting that the conditions on $r = R$ and $z = H$ require particular care; in large measure, this is the price paid for increased accuracy and resolution in the vicinity of the notch or crack root.

Comment on Boundary Conditions: One aspect of the procedure for imposing boundary conditions on $r = R$ and $z = H$ should be discussed.

Starting at the axis $\xi = 0$ and moving along a line ξ = const for increasing ξ, we test to determine whether, for any grid point, any of the surrounding eight points falls outside the domain. If not, no action is required.

If any of the surrounding points are external, i.e., temporary and ficticious, several items of information must be established. First, the number of points that fall exterior must be determined. Second, we must find the intersections between the grid lines corresponding to the point in question, as in Figure 3.2, and the boundary lines; coordinates of these intersections and the associated values of x and y (cf. (3.14) - (3.16)) are to be found.

It happens that the different numbers and different combinations of points that can fall outside the domain number nearly forty cases. Most of these are such that general rules can be formulated for their disposition. A few exceptions occur and they require special attention.

Once the necessary information has been established, further steps are required. From (2.11) (or (2.12)) in the form of (3.22), it is seen that two boundary equations are to be written. These equations should be evaluated on the same point(s) of the boundary so that they may be put into a useable form.

As an example, consider the case that points 3 and 6 in Figure 3.2 are outside the domain, all others being inside. The boundary then cuts the grid lines at positions noted approximately by p, q, and k. The precise location of each point must, of course, be found. Let us state further that, for this case, $\dot{\sigma}_r$ and $\dot{\tau}_{rz}$ of (3.22) are each zero.

Then, using (3.14) - (3.16), (3.22) are to be put in a form such that each is a linear algebraic combination of $\dot{u}$ and $\dot{w}$ at points numbered (temporarily) 1 through 9. As a preliminary step, the coefficients α_1, α_2, ..., α_2 must be found. A problem arises, however, in that there are three points (p, q, k) where the boundary intersects the grid lines, but only two "grid" points exterior to the domain. The arbitrariness thus allowed permits selection of those points furthest apart, p and k for this case. Having made this choice, we write (3.22) at p and k in the linear form noted.

This set of four equations may then be solved to give $\dot{u}_3$, $\dot{w}_3$, $\dot{u}_6$, and $\dot{w}_6$ each as a linear combination of the values of $\dot{u}$ and $\dot{w}$ in the interior. The matrix band is then altered as before. Specifically, all reference to exterior points is replaced, via the rearranged boundary equations, by reference to interior points, and the former entries are zeroed.

Solution: Once the matrix band is in final form, the solution is obtained using Professor McCormick's elimination procedure. The result is the value of the displacement increments δu and δw at each grid point. Experience with this procedure indicates that the solution should be quite accurate although, with the reduced precision of the IBM System 360, this feature remains to be established.

Answer: Of all the information to be obtained, the displacements are perhaps the least important to the analyst. Accordingly we also compute strains, stresses, and several invariants.

The displacements are taken as the running sum of the displacement increments. By (2.2), the same procedure is employed for the strains.

The stresses, however, may not be found in the same manner, owing to the complexity of (2.4) - (2.8). This set of equations is regarded as differential equations in time, and they are integrated via standard predictor-corrector methods.

. Thus (2.4) are used to predict the stress increments by using the current strain increments and the beginning values of the coefficients a_{ij} (i, j = 1, 2, 3, 4). The stresses are tentatively evaluated as the beginning values plus the increments. The various coefficients are recomputed, and new values of the stress increments are determined. The stresses are then corrected. The cycle is repeated either for a prespecified number of times, or until convergence is achieved according to a pertinent criterion.

Final Data: Having thus obtained the primary dependent variables, we turn to computation of further data such as the plastic strains, principal stresses and strains, and energy densities, plastic, elastic, and total. These are found by standard formulae, and need not be detailed here.

Repetition of this procedure for a series of load increments produces the various stress and strain fields as functions of spatial position and time. The procedure is capable of handling both loading and unloading although, at present, the requisite coding is complete only for the former.

V. MANAGEMENT OF OUTPUT DATA

One of the problems attendant upon analyses of the sort described in this report concerns the manner in which output is handled. An estimate of the problem magnitude is obtained as follows. Let us suppose that the domain contains about 200 grid points, i.e., that the dependent variables are determined at each of these locations. Further, presume that there are 50 load increments. Since we compute some 25 variables* at each point in space and time, we are left with a quarter million pieces of data at the end of a single computation.

To confound matters further, the data in the form of numbers is hardly useful when interpreting the results. Characteristically, the analyst is more concerned with the functional dependence of the results along certain lines, e.g., the line of crack prolongation, or over a period of time.

As at least a partial solution to this difficulty, it is planned to have the output available in two forms. One is the usual form, i.e., printed on regular computer paper; the other is on tape. The latter form makes automatic review of the data relatively simple, particularly because of the arrangement in which the data are stored. If, for example, one were to decide to examine a particular feature of the data, a short computer program could be written to scan the tape and produce the information desired. Such features might include the indicated stress and strain intensities and singularities.

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* These include two displacements, four strains (total), four strains (plastic components only), four stresses, three each principal strains and stresses, the equivalent strain and stress, and three (elastic, plastic, and total) energy densities.

A second procedure, currently under development, is to be noted. This involves graphical display of the data in familiar form, on a cathode-ray tube face. Consider, for example, the matter of finding the size and shape of the yielded zone, or plastic enclave. According to the theory,[6] the plastic enclave is defined as the zone enclosed by points along which $\tau_{eq} = \tau_{\ell}$, i.e., where τ_{eq} is equal to its value at the proportional limit. Alternatively one might perfer[3] to examine contours along which $\tau_{eq} = c\tau_{\ell}$, where $c > 1$.

In any case, the problem becomes that of finding iso-contours of the data, either in its form as written on tape or in simple combinations. To meet this requirement, one need only use (3.14) in an inverted form. The vertical grid line corresponding to that through points 2, 5, and 8 in Figure 3.2 is scanned until values of, say, τ_{eq} are found which bracket the desired value τ_{ℓ}. Then we need only solve the second of (3.14a) to find the appropriate value of y. That is, starting with

$$\tau_{\ell} = -\frac{1}{2}\,y(1-y)\tau_{eq}^{(2)} + (1-y^2)\tau_{eq}^{(5)} + \frac{1}{2}\,y(1+y)\tau_{eq}^{(8)} \qquad (5.1)$$

we find

$$y = \frac{\tau_{eq}^{(2)}-\tau_{eq}^{(8)} - \left\{[\tau_{eq}^{(2)}-\tau_{eq}^{(8)}]^2 + 8[\tau_{\ell}-\tau_{eq}^{(5)}][\tau_{eq}^{(1)}-2\tau_{eq}^{(5)}+\tau_{eq}^{(8)}]\right\}^{\frac{1}{2}}}{2[\tau_{eq}^{(2)}-2\tau_{eq}^{(5)}+\tau_{eq}^{(8)}]} \qquad (5.2)$$

A similar formula may be computed for x from (3.14b).

Hence we would scan the data along each family of coordinate lines to establish a set of points along which $\tau_{eq} = \tau_{\ell}$. A smooth curve fitted to these points becomes, in this case, the boundary of the plastic enclave. Other cases might produce isochromatics, and so forth.

The procedure required to generate iso-contours is fairly simple. Its utility lies in repetitive application. Thus one might choose to examine the family of lines $\tau_{eq} = c\tau_{\ell}$, where now c assumes a sequence of values. This corresponds to a "snapshot" of behavior, i.e., a map in space at a selected instant of time.

Alternatively, one might look at the same relation with c fixed, but at a series of subsequent times, corresponding to a "movie" of the behavior. Combinations of the two types of usage are easily visualized.

Carrying this procedure one step further, one might conceive of using the graphical readout of the scope (CRT) as a means for seeing the results of a theoretical experiment. An enormous amount of numerical data may be scanned rapidly and efficiently, so that the analyst may retrieve that which is most pertinent to his purpose. Moreover, he is in a position to determine what constitutes pertinence, for each solution may be studied thoroughly. Finally, by having each solution permanently stored on tape, previous work may be re-examined as needed, as knowledge and understanding of this problem area grow.

VI. CONCLUDING REMARKS

It is one thing to describe a numerical procedure, as is done in this report, and another to reduce it to practice. The latter step is frequently made more difficult when a high speed digital computer is involved. Such is the case with the present effort. It is, at this writing, in the limbo between procedural formulation and routine operation, commonly termed "debugging." We are nonetheless confident of its feasibility, for a number of reasons. These include earlier work in plasticity,[3,4,5] careful problem formulation,[6] and appropriateness of the curvilinear coordinates.[9] What remains is the resolution of several, detailed procedural problems associated with the use of a digital computer, specifically, the IBM System 360.

There are other features of the present program for tension of an axisymmetric, elasto-plastic, notched rod not discussed here. One worth mentioning is a built-in check-point procedure. Should it be desired to interrupt the calculation after a certain number of load increments we may do so, and restart at some later time at the point of interruption. This means that a given analysis may be extended without duplicating its earlier portions, a feature of some advantage should a given problem need to be extended.

In addition to the computer program for the axisymmetric geometry, we have under concurrent development one for planar cases, i.e., plane stress or plane strain. The procedures are precisely the same as those discussed above, so that the remarks presented carry over to planar problems. Perhaps the major difference between axisymmetric and planar problems is that the differential equations are somewhat simpler for the latter case. The form is the same, however, and we are able to work with both programs together.

The procedures appear to be equally useful in solving other differential equations having the same mathematical character as (2.10). While we have yet to study this matter in depth, it would seem that the present method applies in general to elliptic, quasi-linear, coupled, partial differential equations subject to mixed boundary conditions. Although the present technique is presented in the context of two spatial variables, it has carefully been formulated so that extension to three spatial variables (and time) is permissible. Hence, the procedure appears to be one of considerable utility in mathematical physics.

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