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RADIATION DIFFUSION IN A MEDIUM WITH A STRONGLY ELONGATED SCATTERING INDICATRIX

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SUMMARY

It is usually postulated in the theory of diffusion that the scattering indicatrix is spherical, or little differing from it. With such a postulate the working out of theory achieved to date a considerable success. However, the problem of radiation diffusion in a medium with a strongly elongated scattering indicatrix has been studied to a considerably lesser degree. Meantime such media are quite often encountered in practice. Related to them in particular are galactic dust nebulae, planetary atmospheres and water basins.

An approximate method is proposed in the present note for the solution of the indicated problem, which consists in the reduction of the integral-differential equation for radiation transfer to a differential equation of second order in partial derivatives. The results obtained are applied to the case of Henyey-Greenstein's scattering indicatrix, and it is anticipated that similar results will be obtained in the case of other problems.

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Let us consider for the sake of simplicity that radiation diffusion takes place in a medium consisting of plane-parallel layers. We shall denote by $I(\tau, \vartheta, \varphi)$ the intensity of radiation proceeding at the optical depth τ at the angle ϑ to the normal for the azimuth φ. As is well known (see, for example [1], the radiation transfer equation has the form

$$\cos\vartheta \frac{\partial I(\tau, \vartheta, \varphi)}{\partial \tau} = -I(\tau, \vartheta, \varphi) + \frac{\lambda}{4\pi}\int_0^{2\pi} d\varphi' \int_0^{\pi} x(\gamma)\, I(\tau, \vartheta', \varphi') \sin\vartheta' \, d\vartheta', \tag{1}$$

where $x(\gamma)$ is the scattering indicatrix (γ being the angle between the directions of the scattered and incident rays), λ is the probability of photon survival in the course of an elementary scattering event.

(*) DIFFUZIYA IZLUCHENIYA PRI SIL'NO VYTYANUTOY INDIKATRISE RASSEYANIYA

We consider that the scattering indicatrix is strongly stretched forward, i. e. function $(x(\gamma)$ has a sharp maximum in the direction characterized by the angles ϑ and ϕ. This is why the quantity $I(\tau, \vartheta', \varphi')$ may be substituted approximately by its expansion in Taylor series by powers $\vartheta' - \vartheta$ and $\varphi' - \varphi$. Limiting ourselves to quadratic terms of this equation, we find instead of Eq.(1)

$$\cos\vartheta \frac{\partial I}{\partial\tau} + (1-\lambda) I = \lambda u \left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial I}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 I}{\partial\varphi^2}\right], \tag{2}$$

where

$$u = \frac{1}{8}\int_0^\pi x(\gamma)\sin^3\gamma\, d\gamma. \tag{3}$$

Eq.(2) may be also obtained in a somewhat different fashion. Let us substitute the assigned scattering indicatrix approximately by the indicatrix

$$x(\gamma) = \begin{cases} c, & \gamma \leqslant \gamma_0, \\ 0, & \gamma > \gamma_0, \end{cases} \tag{4}$$

where the quantities γ_0 and $\underline{c}$ are interrelated by the normalization condition

$$c(1 - \cos\gamma_0) = 2. \tag{5}$$

Substituting (4) into (1), expanding $I(\tau, \vartheta', \varphi')$ in Taylor series by and taking advantage of the smallness of γ_0, we again arrive at Eq.(2) in place of Eq.(1), in which

$$u = 1/2c \tag{6}$$

The more elongated the scattering indicatrix is, the smaller the angle γ_0 and the greater the value of $\underline{c}$. Usually the indicatrix's prolation is characterized by the parameter $\underline{x_1}$, constituting the first term of its expansion by Legendre polynomials, i. e.

$$x_1 = \frac{3}{2}\int_0^\pi x(\gamma)\cos\gamma\sin\gamma\, d\gamma. \tag{7}$$

Finding the value of $\underline{x}$ for the indicatrix (4) and equating it to the value of $\underline{x}$ for the real indicatrix, we obtain

$$c = 3 / (3 - x_1) \tag{8}$$

Substituting (8) into (6), we have

$$u = (3 - x_1) / 6 \tag{9}$$

The determination of parameter u by formula (9) may be found to be more fortunate than its determination by formula (3).

Thus, for an approximate finding of radiation intensity $I(\tau, \vartheta, \varphi)$ we obtained Eq.(2), in which the value of u is determined by either formula (3) or (9) (or by any other convenient formula). Obviously, the results of utilization of Eq.(2) will be so much the more precise as the photon undergoes more scatterings in the given medium (that is, the greater is the optical thickness of the medium and the closer to unity is the quantity λ). This is explained by the fact that after numerous scatterings, even in the case of indicatrix (4), the photon may sharply change its direction, as a consequence of which there appear backscattered photons*.

For the solution of any concrete problem boundary conditions must be added to Eq.(2). If the radiation sources are located outside the medium, these conditions must determine the intensity of the radiation incident upon the boundaries from without. If they are inside the medium, the boundary conditions must be expressing the absence of outer radiation. In the latter case a term should be introduced in Eq,(2), characterizing the radiating capacity or the emissivity of the medium.

As an example, let us resolve with the aid of Eq.(2) the problem of light regime in deep layers of a semi-infinite medium in case of outer emission sources. This problem may also be resolved with precision (see [1]), and the results of approximate and precise solutions can be mutually compared.

In the given case the radiation intensity is independent of the azimuth and may be represented in the form

$$I(\tau, \vartheta) = y(\vartheta)e^{-k\tau}, \tag{10}$$

where k is a constant depending only on the scattering indicatrix and the quantity $\bar{.}$

Substituting (10) into (2), we obtain the following equation for the determination of function $y(\vartheta)$:

$$\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial y}{\partial\vartheta}\right) = (a - b\cos\vartheta)\,y, \tag{11}$$

where

$$a = (1-\lambda)/\lambda u, \quad b = k/\lambda u. \tag{12}$$

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* An equation similar to (2) was utilized in the theory of particle diffusion (with substitution of $\sin\vartheta$ by ϑ and without derivative with respect to ϕ), for the study of small particle deflections from the initial direction.

We shall seek the solution of Eq.(11) in the form of expansion by Legendre polynomials, i. e. we shall postulate

$$y(\vartheta) = \sum_{0}^{\infty} y_n P_n(\vartheta). \tag{13}$$

The substitution of (13) into (11) leads to the following recurrent formula for the determination of the factors y_n:

$$[a + n(n+1)]\, y_n = b\left(\frac{n}{2n-1} y_{n-1} + \frac{n+1}{2n+3} y_{n+1}\right). \tag{14}$$

From the condition of resolvability of the system of homogenous equations (14) consisting in that its determinant be zero, it is possible to find the dependence between the quantities a and b. Hence, with the aid of formulas (12) a dependence is obtained between the quantities k, λ, u.

TABLE 1

b	a	$u = 0,05$		$u = 0,10$		$u = 0,15$	
		λ	k	λ	k	λ	k
0	0	1	0	1	0	1	0
1	0,16	0,992	0,050	0,984	0,098	0,977	0,146
2	0,56	0,973	0,097	0,947	0,189	0,923	0,277
3	1,09	0,948	0,142	0,902	0,271	0,859	0,387
4	1,70	0,922	0,184	0,854	0,342	0,796	0,478
5	2,37	0,894	0,224	0,809	0,404	0,738	0,554
6	3,06	0,867	0,260	0,766	0,460	0,685	0,647
7	3,78	0,841	0,294	0,726	0,503	0,638	0,670
8	4,52	0,816	0,326	0,689	0,551	0,596	0,715
9	5,28	0,791	0,356	0,655	0,590	0,558	0,753
10	6,05	0,768	0,384	0,623	0,623	0,524	0,787

The determinant of system (14) is

$$\Delta = \begin{vmatrix} a & -\frac{1}{3}b & 0 & 0 & \cdots \\ -b & a+2 & -\frac{2}{5}b & 0 & \cdots \\ 0 & -\frac{2}{3}b & a+6 & -\frac{3}{7}b & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \tag{15}$$

Taking advantage of the well known approach (see, for example, [2]), from the condition $\Delta = 0$ we may express a as a function of b in the form of continued fraction. Let us denote by Δ_n the determinant obtainable from (15) by striking out the n first lines and columns. Then we have

$$\Delta = a\Delta_1 - \frac{1}{3}b^2\Delta_2, \tag{16}$$

$$\Delta_1 = (a+2)\Delta_2 - \frac{4}{15}b^2\Delta_3 \tag{17}$$

and so forth. Since $\Delta = 0$, it follows from (16) that

$$a = {}^1/_3 b^2 \Delta_2 / \Delta_1. \qquad (18)$$

Substituing into (18) the expression (17) and the subsequent expressions for Δ_2, Δ_3,..., we obtain

$$a = \frac{1}{3} \cfrac{b^2}{a + 2 - \cfrac{4}{15} \cfrac{b^2}{a + 6 - \cfrac{9}{35} \cfrac{b^2}{a + 12 - \dots}}} \qquad (19)$$

For small b formula (19) gives $a = {}^1/_6 b^2$. Hence, on the basis of (12) and (9) we find

$$\lambda = 1 - k^2 / (3 - x_1). \quad x\ ) \qquad (20)$$

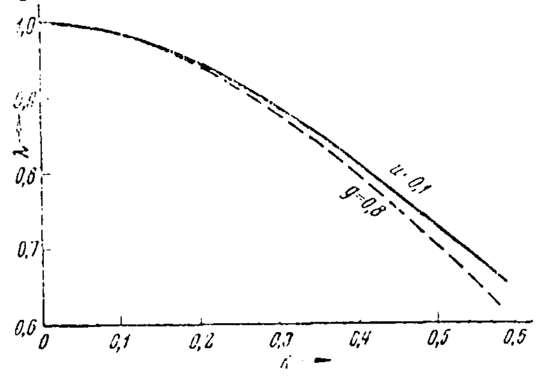
Utilizing (20), from (13) and (14) we obtain

$$y(\vartheta) = y_0 \left(1 + \frac{3k}{3 - x_1} \cos\vartheta\right). \qquad (21)$$

The rigorous theory for small k also leads to formulas (20) and (21). The determination of parameter u by formula (9) is also so some extent justified by it.

Compiled in Table 1 are the values of a for the values of b from 0 to 10 computed by formula (19). Given there also is the dependence in the numerical form between the quantities λ and k for three values of parameter u.

Upon finding the dependence between a and b, the radiation intensity may be determined by formulas (13) and (14) with a precision to the constant multiplier y_0).



Let us now apply the results obtained to the case of Heyney-Greenstein scattering indicatrix [3]:

$$x(\gamma) = (1 - g^2) / (1 + g^2 - 2g\cos\gamma)^{3/2}, \qquad (22)$$

often used by astrophysicists. Since for it $x_1 = 3g$, we have on the basis of formulas (5) and (7)

$$u = {}^1/_2 (1 - g). \qquad (23)$$

This means that the approximate dependence between the quantities λ and k brought out in Table 1, corresponds to indicatrix (22) at g = 0.7, 0.8, 0.9.

Two curves are given in Fig.1 for the indicatrix (22), of which one conveys the precise dependence between λ and k (at g = 0.8), and the other represents the approximate dependence (at g = 0.1). We may see that for values close to the unity both curves are near one another. Consequently, the utilization of Eq.(2) leads in the given case to satisfactory results.

It may be expected that similar results will be obtained also in the case of solutions of other problems with the aid of Eq.(2).

****** T H E E N D ******

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