A STATISTICAL RAY ANALYSIS OF THE SCATTERING OF RADIO WAVES BY AN ANISOTROPICALLY TURBULENT, NON-HOMOGENEOUS SOLAR CORONA

by

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ABSTRACT

The scattering of radio waves by an anisotropically turbulent solar corona exhibiting large-scale refraction effects (due to a radial gradient in average electron density) is discussed in terms of a statistical ray analysis similar to that of Chandrasekhar (1952). The corona is assumed to be spherically symmetric throughout. The ray equations of geometrical optics are written in terms of the spherical coordinate system natural to the solar corona, and discussed for both the case of a spherically symmetric average corona for which the electron density is an exactly specifiable function of position, and the case of an anisotropically turbulent corona for which the electron density may be known in only a statistical sense.

For the case where the corona is turbulent, and therefore known in only a statistical sense, a linear perturbation analysis is employed to obtain, for the first time, quadrature solutions for the statistical fluctuations in the ray position, signal phase, and pulse propagation time for a corona exhibiting large scale refraction. The general expressions thus obtained are discussed in particular for the special case of nearly linear rays. It is shown that at appropriate frequencies even very slight ray curvatures can have a significant effect on the fluctuations in the times of propagation of pulse signals across the corona, a conclusion discussed in terms of the Sunblazer experiment (Harrington (1965)) and the test of general relativity proposed by Shapiro (1964). Throughout the work we seek to provide a proper analytical framework in which to interpret observed fluctuations in the signal angle of arrival (related to the redistribution of signal energy in the sky), fluctuations in the arrival times of pulse signals, and variations in the signal bandwidth. Our attention is drawn specifically to deducing, as functions of distance from the sun, the mean square fluctuations in electron density, the statistical correlation lengths, and the degree of anisotropy. We also consider what effects might be observed to arise from non-radial coronal outflow and curvature of the general solar magnetic field lines. The work concludes with a discussion of scattering data available at present; this is shown to be consistent, beyond some ten solar radii, with a coronal density behaving as r^{-2}, a degree of anisotropy nearly constant with distance from the sun, and a statistical correlation length which during solar minimum does not vary with (r), but which tends to increase linearly with (r) near solar maximum indicating that the interplanetary plasma develops a radial filamentary structure as solar maximum is approached.

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CHAPTER I

Introduction to the Problem

Background

Machin and Smith (1951) and Vitkevich (1951) suggested that the observation of astronomical radio sources passing close to the sun could lead to the determination of a coronal electron density profile if the effective occultation radius could be found as a function of frequency. Early observations seeking to utilize this predicted effect (Hewish (1955)), Machin and Smith (1952), Vitkevich (1955)) observed, however, only what proved to be an apparent broadening of the source due to scattering of the radio waves by irregularities in the electron density of the corona. The observed scattering was shown by Hewish (1955) to be consistent with the multiple scattering wave theory of Fejer (1953) and the multiple scattering ray theory of Chandrasekhar (1952). The existence of multiple scattering of radio waves from many coronal irregularities has since been verified by numerous observations. The promise of utilizing this observed radio scattering as a means of investigating coronal irregularities and solar wind turbulence has led during the past ten years to a great number of observations by a number of workers of coronal scattering of radio sources passing close to the sun (Hewish, et al (1955, 1958, 1963, 1964, 1965, 1966), Slee (1959, 1961, 1966), Hogbom (1960), Gorgolewski and Hewish (1960), Erickson (1964), Vitkevich (1960, 1966), Pisareva (1959), Little, et al (1966a, b), Briggs (1961), Dennison and Hewish (1967), Yakovlev, et al (1966), Douglas and Smith (1967), Cohen, et al (1966, 1967)).

In spite of the proliferation of observational work, however, little has been accomplished toward a proper theoretical foundation upon which the interpretation of the data must rest. As a result even the most basic observations have been at times incorrectly interpreted, and the groundwork for the proper understanding of effects due to coronal anisotropy and large scale coronal density gradients has not been laid. The ray theory of Chandrasekhar (1952) is applicable only to media which are isotropically turbulent and, on the average, uniform. The ray analysis of Vitkevich (1966) is somewhat more general, but does not include effects of overall refraction by the corona and is limited only to an isotropically turbulent or to an extremely anisotropically turbulent corona the density of which exhibits a simple power law behavior with distance from the solar center. Similarly, although recent progress has been made in wave theories of scattering, only the "thin screen" approximation has been worked out for isotropically turbulent media exhibiting no large scale refraction (Fejer (1953), Ratcliffe (1956), Chernov (1960), Tatarski (1961), Wagner (1962), Mercier (1962), Briggs and Parkin (1963), Little and Hewish (1966), Salpeter (1967)). Furthermore, the existing theories are oriented primarily toward the scattering of incoherent sources, and have for the most part not been extended to coherent artificial sources.

It is therefore the intent of this dissertation to provide an analytical framework in which to interpret the scattering of coherent radio sources when statistical anisotropy and large scale refraction effects of the medium may be important. The use of coherent sources as a means of investigating the average coronal electron density has been suggested by Harrington (1965) and ways will be suggested in this work to utilize the same coherent source as a means of investigating the (smaller scale) statistical structure of the corona. We shall seek to include large scale refraction effects and turbulent anisotropy because the coronal electron density is a strong function of distance from the sun (refraction) and anisotropic scattering has been observed.

As stated above the problem of this work is a broad one indeed, and some restriction of the problem is necessary. We shall restrict our attention to a ray analysis of small angle scattering in a spherically symmetric corona. The assumption of spherical symmetry should be quite valid near the solar maximum , but somewhat less so near solar minimum . The departures from sphericity in the latter case seem, however, to be of sufficiently small magnitude that ray trajectories calculated on the basis of sphericity will not be much in error, particularly for those rays lying near the solar equatorial plane. The assumption of small angle scattering would seem to be justified on the basis of the present observations which indicate that for frequencies above some tens of megacycles the scattering does not exceed a few degrees; we may thus employ a linear theory of perturbation. The choice of a ray analysis, rather than a diffraction theory, has been made for a number of reasons. Foremost among these is simplicity. Calculations based on a wave theory are exceedingly difficult, and are at present limited to the thin screen approximation for isotropically turbulent media exhibiting no large scale refraction. Since our purpose is to examine the effects of large scale refraction and statistical anisotropy it appears that a ray theory will be most suitable. Use of a ray theory is difficult to formally justify, however. One necessary condition is certainly that the wavelength be small compared to the smallest scale size of the coronal irregularities. As we will be dealing with wavelengths on the order of meters and inhomogeneities on the order of hundreds of kilometers in size this condition will certainly be met; no effects due to very small scale turbulence have been seen, and we do not include this possibility in this work. However, in order to neglect diffraction we require

$$\lambda (s-s_o) << \tau_o^2 \qquad (1.1)$$

This condition says that in order for diffraction to be neglected the scale of the diffraction pattern produced by an irregularity of scale size τ_o at a distance $(s-s_o)$ must be much less than the scale of the irregularities themselves. It may be also interpreted as saying that diffraction may be neglected if many Fresnel zones lie within the scale size of the coronal inhomogeneities. This condition is not met for $\lambda = 1$ meter, $(s-s_o) = 1$ AU, and $\tau_o = 200$ kilometers (Hewish and Dennison (1966)). Thus we might expect to see diffraction effects. Furthermore, it occurs to us that if the variation in phase upon traversing the coronal scattering region is greater than about 1 radian it will then make little sense to talk about the scale of a diffraction pattern or the dimensions of the Fresnel zones in the usual way. Thus for large variations in phase condition (1.1) would seem to be incorrect. In that case the proper condition for neglect of diffraction is (Salpeter (1967), Cohen, et al (1967))

$$\lambda (s-s_o) << \frac{2\pi \tau_o^2}{\delta\Phi_{r.m.s.}} \tag{1.2}$$

where $\delta\Phi_{r.m.s.}$ is the root-mean-square fluctuation in phase upon traversing the medium. This condition says that diffraction may be neglected if the fluctuations in angle of arrival of a signal are such that the (point) source location in the sky never appears to fluctuate by more than the average angular dimensions of an inhomogeneity, i.e. $\delta\psi_{r.m.s.} << \tau_o/(s-s_o)$, where $\delta\psi_{r.m.s.}$ is the root-mean-square fluctuation in signal angle of arrival. For λ = 1 meter, $(s-s_o)$ = 1 AU, and τ_o = 200 km condition (1.2) is not satisfied, suggesting once again that we may expect to see diffraction effects. But let us see what the effects of diffraction might be. We may consider the coronal inhomogeneities as redistributing the energy from a point source over some non-zero solid angle in the sky. Diffraction due to the coronal irregularities will result in this redistribution of energy taking the form of a diffraction pattern with an angular scale λ/τ_o radians. If λ = 3 meters and τ_o = 200 km this scale is on the order of 10^{-3} degrees. But for rays passing within about 100 solar radii from the sun and frequencies on the order of 100 MHz the redistribution of energy is observed to occur over at least several minutes of arc, indicating that the scattering due to local fluctuations in refractive index (and the consequent ray bending) can preponderate that due to interference effects. In fact, it may be shown (Chandrasekhar (1952)) that the angular redistribution of energy due to diffraction will be much less than that calculated on the basis of a ray theory as long as

$$\delta\Phi_{r.m.s.} >> 1 \tag{1.3}$$

The fact that the wave theory of Fejer (1953) reduces to the ray theory of Chandrasekhar in the limit of large phase variations validates this conclusion. Thus inequality (1.3) represents a fundamental condition on the validity of our results. Furthermore, if multiple random scattering occurs we expect, according to the central limit theorem, the energy distribution in the sky to follow a normal distribution, indicating that the fine structure of an interference pattern will be masked by the dominant small scale refractive effects. Since these refractive effects are adequately described by a ray theory, we proceed in that direction. Discussion of the validity of a ray analysis in terms of the propagation of a wave through a plasma awaits Chapter II.

Having thus introduced our theoretical model, we shall now further describe the specific problems we will (and will not) treat. We shall not be concerned explicitly with the behavior in time of the scattering phenomena, but only with temporal averages of the statistical quantities. This neglect of the temporal behavior of the fluctuating quantities would at first inspection appear to require that the time of flight of a signal across the scattering region be much less than the time required for a density fluctuation to drift across the line of sight, if the fluctuations are "stable" for this length of time (the observations of Hewish and Okoye (1965) support this latter notion). However, if the individual fluctuations (blobs) are statistically independent we need then only require that the signal time of flight across a fluctuation be much less than the time in which a single "blob" drifts across a line of sight; this condition is equivalent to the requirement that the drift velocity component normal to the line of sight be much less than the speed of light, a condition certainly fulfilled for the solar corona. We shall furthermore confine our attention only to what might be learned from observations at a single observing station (this includes interferometers with baselines much less than r_0), neglecting correlation phenomena between widely separated stations. The latter type of observation requires a number of stations appropriately connected, and is therefore more difficult than the use of one station only; as we shall see, the use of coherent sources allows much to be learned with just one observing station. Next, although we shall initially consider any degree of large scale ray bending we shall discuss in detail only the case of slight ray curvature, utilizing only the zeroth and first degree terms in an expansion in ω_p^2/ω^2. Finally, we shall not consider the effects of fluctuations in the general solar magnetic field on the scattering, although this might be expected to result in observed fluctuations in signal polarization.

We thus see that a number of problems have been omitted. The temporal behavior of the fluctuations, correlations between widely separated observation stations, pronounced ray curvature (closely related to this arc scattering phenomena associated with radar observations of the sun), and magnetic effects are all problems of importance, and subjects for future investigation. It is felt, however, that the proper definition of this work must exclude these problems.

We are now ready to proceed. In Chapter II we present the basic equations we shall use, and, for orientation, we discuss them for a spherically symmetric corona exhibiting no small scale irregularities. In Chapter III are solved, for the first time, the first order perturbation equations for rays propagating through a spherically symmetric average corona upon which are "superposed" small scale turbulent fluctuations. The solutions thus obtained are used to derive proper expressions for fluctuations in the ray position, signal phase, and pulse signal propagation times. Finally, in Chapter IV these expressions for the scattering quantities are discussed for the special case of nearly linear rays. It is shown that even very slight ray curvature can have a significant effect on the fluctuations in the times of propagation of pulse signals across the corona, a conclusion discussed in terms of the Sunblazer experiment and the test of general relativity proposed by Shapiro. We also provide there a proper analytical framework in which to interpret fluctuations in the angle of arrival of a signal (related to the redistribution of signal energy in the sky), fluctuations in the arrival times of pulse signals, and fluctuations in the signal bandwidth. Our attention is drawn specifically to deducing, as functions of distance from the sun, the mean square fluctuations in electron density, the correlation lengths, and the degree of anisotropy. Our attention is also drawn to examining the possible effects of non-radial coronal outflow and curvature of the general solar magnetic field lines. The chapter concludes with a discussion of present data in the context of our analytical findings.

Having thus anticipated some of the principal contributions of this work, we are now ready to proceed.

CHAPTER II

The Average Corona

Introduction

It is the purpose of this chapter to obtain a mathematical formalism with which we may discuss the behavior of electromagnetic rays in a refractive medium, such as the solar corona, and to examine in some detail the radio ray trajectories in a specific model for a spherically symmetric average corona for which we may specify exactly the spatial dependence of the index of refraction.

The Basic Ray Equations

As was discussed in Chapter I we shall confine our attention throughout this work to a description of the radio propagation through the solar corona based on the behavior of radio rays (in the same sense as is usually employed in geometrical optics) in a refractive medium. We begin therefore with the well known equation for ray trajectories in a medium of arbitrary refractive index (see, for example, Rossi, "Optics", Addison-Wesley (1957), §2-3):

$$\frac{d}{ds}\left(\mu \frac{d\vec{r}}{ds}\right) = \nabla \mu \qquad (2.1)$$

where $\vec{r}$ = position vector to ray
μ = refractive index (a function of space)
s = arc length along the ray

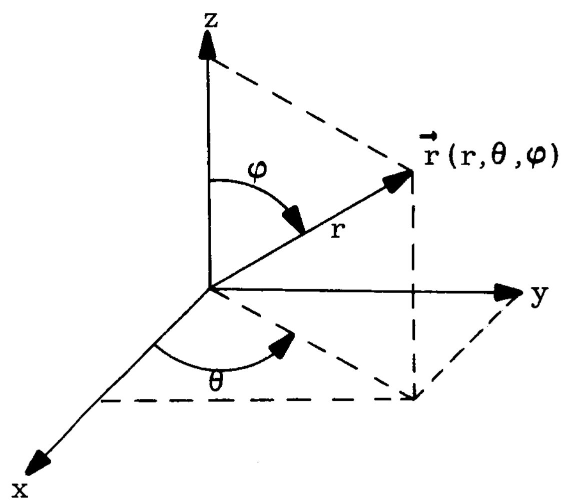
In the usual notation of a right-handed Cartesian coordinate system (2.1) becomes

$$\frac{d}{ds}\left\{\mu \frac{dx}{ds}\right\} = \frac{\partial\mu}{\partial x}$$

$$\frac{d}{ds}\left\{\mu \frac{dy}{ds}\right\} = \frac{\partial\mu}{\partial y} \qquad (2.2)$$

$$\frac{d}{ds}\left\{\mu \frac{dz}{ds}\right\} = \frac{\partial\mu}{\partial z}$$

while in the spherical coordinate system defined by the following figure



equations (2.1) take the forms

$$\frac{d}{ds}\left\{\mu\left[\sin\varphi\cos\theta\frac{dr}{ds} - r\sin\varphi\sin\theta\frac{d\theta}{ds}\right.\right.$$

$$\left.\left. + r\cos\varphi\cos\theta\frac{d\varphi}{ds}\right]\right\} = \frac{\partial\mu}{\partial r}(\sin\varphi\cos\theta) \qquad (2.3)$$

$$- \frac{\partial\mu}{\partial\theta}\left(\frac{\sin\theta}{r\sin\varphi}\right) + \frac{\partial\mu}{\partial\varphi}\left(\frac{\cos\varphi\cos\theta}{r}\right)$$

$$\frac{d}{ds}\left\{\mu\left[\sin\varphi\sin\theta\frac{dr}{ds}+r\sin\varphi\cos\theta\frac{d\theta}{ds}\right.\right.$$

$$\left.\left.+r\cos\varphi\sin\theta\frac{d\varphi}{ds}\right]\right\}=\frac{\partial\mu}{\partial r}(\sin\varphi\sin\theta) \tag{2.4}$$

$$+\frac{\partial\mu}{\partial\theta}\left(\frac{\cos\theta}{r\sin\varphi}\right)+\frac{\partial\mu}{\partial\varphi}\left(\frac{\cos\varphi\sin\theta}{r}\right)$$

$$\frac{d}{ds}\left\{\mu\left[\cos\varphi\frac{dr}{ds}-r\sin\varphi\frac{d\varphi}{ds}\right]\right\}=\frac{\partial\mu}{\partial r}(\cos\varphi)$$

$$-\frac{\partial\mu}{\partial\varphi}\left(\frac{\sin\varphi}{r}\right). \tag{2.5}$$

The (r,θ,φ) representation of equations (2.3)-(2.5) will be most useful here as the solar corona, which closely exhibits spherical symmetry, is most conveniently described in a spherical coordinate system.

Equations (2.3)-(2.5) are the fundamental equations that shall be used; they incorporate no assumptions other than the appropriateness of the ray description. The independent variable in these equations is the arc length, s, along a ray, and the ray coordinates (r,θ,φ) may be regarded as functions of s. However, the right-hand sides of equations (2.3)-(2.5) are noted to contain the factors $\frac{\partial\mu}{\partial r}$, $\frac{\partial\mu}{\partial\theta}$, and $\frac{\partial\mu}{\partial\varphi}$ which are explicitly functions of (r,θ,φ). A formal connection can be made, however, between s and the position coordinates (r,θ,φ) via the relation

$$(ds)^2=(dr)^2+r^2\sin^2\varphi\,(d\theta)^2+r^2\,(d\varphi)^2. \tag{2.6}$$

Thus, the set of equations (2.3)-(2.6) can completely determine the ray trajectories in a medium of specified refractive index $\mu(r,\theta,\varphi)$. In what follows in this chapter we shall confine our attention to an interesting special case.

The Spherically Symmetric Corona

In this paragraph our attention will be confined to the useful special case of a spherically symmetric solar corona, with refractive index a function of the radial coordinate r only:

$$\mu_o = \mu_o(r) \tag{2.7}$$

where the subscript $(\ )_o$ designates that we are dealing with a specifiable, rather than a random, configuration. The usefulness of this notation will become apparent in Chapter III, where the ray trajectories here described become the basis of a perturbation description of scattering. The assumption of spherical symmetry appears to be quite valid near the maximum of the solar activity cycle, but less so near solar minimum . The departures from sphericity in the latter case seem, on purely qualitative grounds, to be of sufficiently small magnitude, however, that ray trajectories calculated on the basis of equation (2.7) will not be much in error. But it should be pointed out that occasional departures from spherical symmetry occur which are of such magnitude as to blatantly violate equation (2.7) . These cases will not be considered.

We begin by inserting equation (2.7) into (2.5) to obtain

$$\frac{d}{ds}\left\{\mu_o\left[\cos\varphi_o\frac{dr_o}{ds} - r_o\sin\varphi_o\frac{d\varphi_o}{ds}\right]\right\} = \frac{d\mu_o}{dr}\cos\varphi_o . \tag{2.8}$$

This equation is identically satisfied by

$$\varphi_o(s) = \pi/2 . \tag{2.9}$$

That is, a solution exists in which the ray trajectories lie wholly in the plane $\varphi = \pi/2$. Since, in a spherically symmetric corona, there is nothing special about the orientation of the $\varphi = 0$ axis we can conclude that in the case of sphericity the ray trajectories are planar. (This is analagous to the central force problem of classical mechanics.) We shall therefore confine our attention only to those rays lying in the plane specified by equation (2.9); by virtue of the assumed spherical symmetry the complete set of ray trajectories in that plane represents the total set of ray geometries for the entire corona, and therefore we lose no information by confining our attention only to the plane $\varphi = \pi/2$.

If equations (2.7) and (2.9) are inserted into equations (2.3) and (2.4) we obtain the two relations

$$\frac{d}{ds}\left\{\mu_o\left[\cos\theta_o \frac{dr_o}{ds} - r_o \sin\theta_o \frac{d\theta_o}{ds}\right]\right\} = \cos\theta_o \frac{d\mu_o}{dr} \tag{2.10}$$

$$\frac{d}{ds}\left\{\mu_o\left[\sin\theta_o \frac{dr_o}{ds} + r_o \cos\theta_o \frac{d\theta_o}{ds}\right]\right\} = \sin\theta_o \frac{d\mu_o}{dr} . \tag{2.11}$$

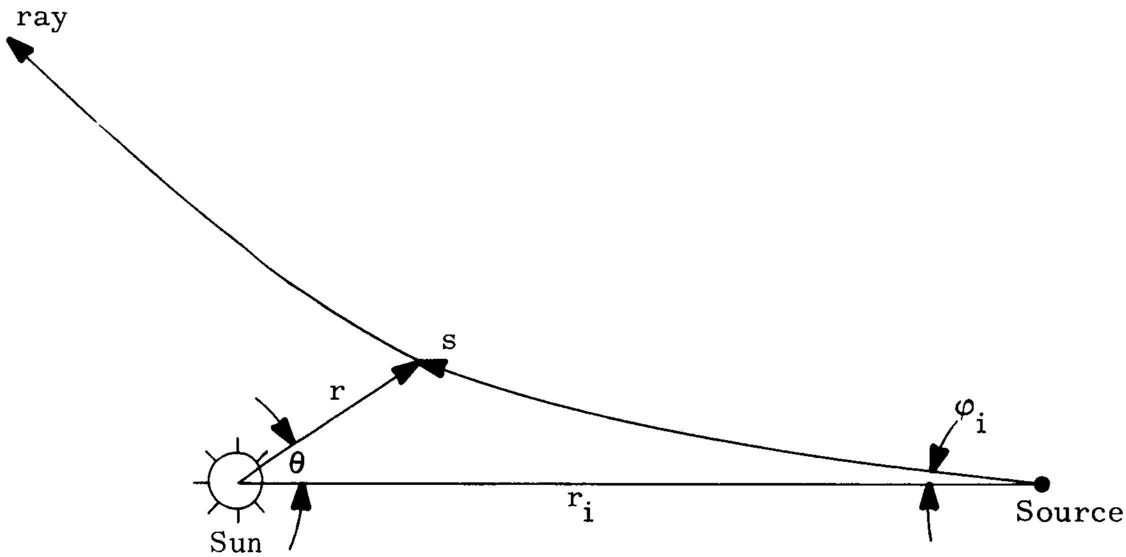
Multiplying equation (2.10) by $\sin\theta_o$ and (2.11) by $\cos\theta_o$ and subtracting the resultant equations yields, after some simplification,

$$\frac{d}{ds}\left\{\mu_o\, r_o^2 \frac{d\theta_o}{ds}\right\} = 0 . \tag{2.12}$$

This important relation (analagous to the conservation of angular momentum) states that the bracketed quantity in equation (2.12) is constant along the ray; denoting the constant by C we have

$$\mu_o r_o^2 \frac{d\theta_o}{ds} = C . \qquad (2.13)$$

Now, constant C may be readily evaluated by considering the geometry of the rays in the plane $\varphi = \pi/2$:



As the figure implies, we shall consider a ray to originate at position ($r = r_i$, $\theta = 0$) at an angle φ_i with respect to the source-sun line. Letting the subscript $(\ )_i$ represent the initial conditions of the ray, it is easy to show that

$$\left(\frac{d\theta_o}{ds}\right)_i = \frac{\sin\varphi_i}{r_i} \qquad (2.14)$$

which, when inserted into equation (2.13), gives

$$C = \mu_i r_i \sin\varphi_i . \qquad (2.15)$$

Combining equations (2.13) and (2.15) yields the equation determining the rays in a spherically symmetric corona:

$$\mu_o r_o^2 \frac{d\theta_o}{ds} = \mu_i r_i \sin\varphi_i = C . \tag{2.16}$$

Equation (2.16) is most useful when the variable s is eliminated. If we set $\varphi_o = \pi/2$ in equation (2.6) we obtain

$$(ds)^2 = (dr_o)^2 + r_o^2 (d\theta_o)^2 \tag{2.17}$$

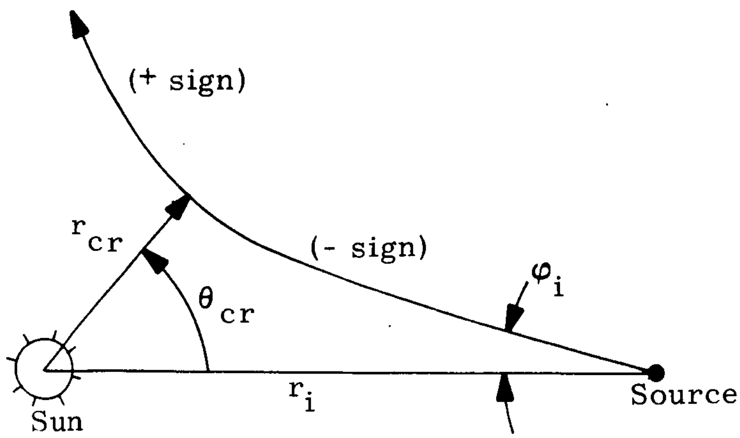
which may be immediately put in the form

$$\left(\frac{d\theta_o}{ds}\right)^{-2} = r_o^2 + \left(\frac{dr_o}{d\theta_o}\right)^2 . \tag{2.18}$$

Equation (2.18) may be used to eliminate the variable s from (2.16). We obtain

$$d\theta_o = \frac{\pm\, dr_o}{r_o\left\{\dfrac{\mu_o^2 r_o^2}{\mu_i^2 r_i^2 \sin^2\varphi_i} - 1\right\}^{1/2}} . \tag{2.19}$$

If the rays are concave away from the sun, as will be the case in a corona exhibiting a decreasing electron density with increasing distance from the sun, the minus sign will be appropriate when $\theta_o \leq \theta_{cr}$, while the plus sign will be appropriate for $\theta_o \geq \theta_{cr}$, where θ_{cr} is defined to be that value of θ_o for which $dr_o = 0$, i.e. the turning point of the ray. We shall similarly define r_{cr} to be the value of r_o at the turning point. These relationships are illustrated in the figure below.



To proceed, we define the following quantities:

$R_{\odot}$ = solar radius

ρ_o = $r_o/R_{\odot}$

$\rho_o^{(-)}$ = ρ_o for $\theta_o \leq \theta_{cr}$

$\rho_o^{(+)}$ = ρ_o for $\theta_o \geq \theta_{cr}$

ρ_{cr} = $r_{cr}/R_{\odot}$

ρ_i = $r_i/R_{\odot}$

$\theta_o^{(-)}$ = θ_o for $\theta_o \leq \theta_{cr}$

$\theta_o^{(+)}$ = θ_o for $\theta_o \geq \theta_{cr}$.

With these definitions we may integrate equation (2.19) to obtain an explicit quadrature representation of the ray trajectories:

$$\theta_o^{(-)}(\rho_o) = \int_{\rho_o}^{\rho_i} \frac{d\rho_o}{\rho_o\left\{\frac{\mu_o^2\rho_o^2}{\mu_i^2\rho_i^2\sin^2\varphi_i} - 1\right\}^{1/2}} \qquad (2.20)$$

$$\theta_o^{(+)}(\rho_o) = \theta_{cr} + \int_{\rho_{cr}}^{\rho_o} \frac{d\rho_o}{\rho_o \left\{ \frac{\mu_o^2 \rho_o^2}{\mu_i^2 \rho_i^2 \sin^2 \varphi_i} - 1 \right\}^{1/2}} \tag{2.21}$$

where we have implicitly used the initial condition that $\theta_o^{(-)} = 0$ when $\rho_o^{(-)} = \rho_i$. The value of θ_{cr} may be similarly found to be

$$\theta_{cr} = \int_{\rho_{cr}}^{\rho_i} \frac{d\rho_o}{\rho_o \left\{ \frac{\mu_o^2 \rho_o^2}{\mu_i^2 \rho_i^2 \sin^2 \varphi_i} - 1 \right\}^{1/2}} \tag{2.22}$$

where ρ_{cr} is determined by noting that at the ray turning point $\frac{dr_o}{d\theta_o}$ is zero which implies, by equation (2.19), that

$$\mu_o^2 (r_{cr}) \rho_{cr}^2 = \mu_i^2 \rho_i^2 \sin^2 \varphi_i \tag{2.23}$$

which may be construed as an equation determining ρ_{cr} (or r_{cr}). Thus equations (2.20)-(2.23) determine completely the set of ray trajectories in the plane $\varphi = \pi/2$ once $\mu_o(r)$ is specified. In general it will be impossible to carry out the necessary integrations explicitly, and one must resort to numerical techniques; it is this kind of calculation that has been discussed by Jaeger and Westfold (1950) and by Bracewell and Preston (1956). We shall pursue some numerical calculations later in this chapter.

It is easy to demonstrate that the ray trajectories are symmetrical about the line $\theta = \theta_{cr}$. Equations (2.20) and (2.22) may be combined to give

$$\theta_{cr} - \theta_o^{(-)} = \int_{\rho_{cr}}^{\rho_o^{(-)}} (\quad) \tag{2.24}$$

while equation (2.21) is

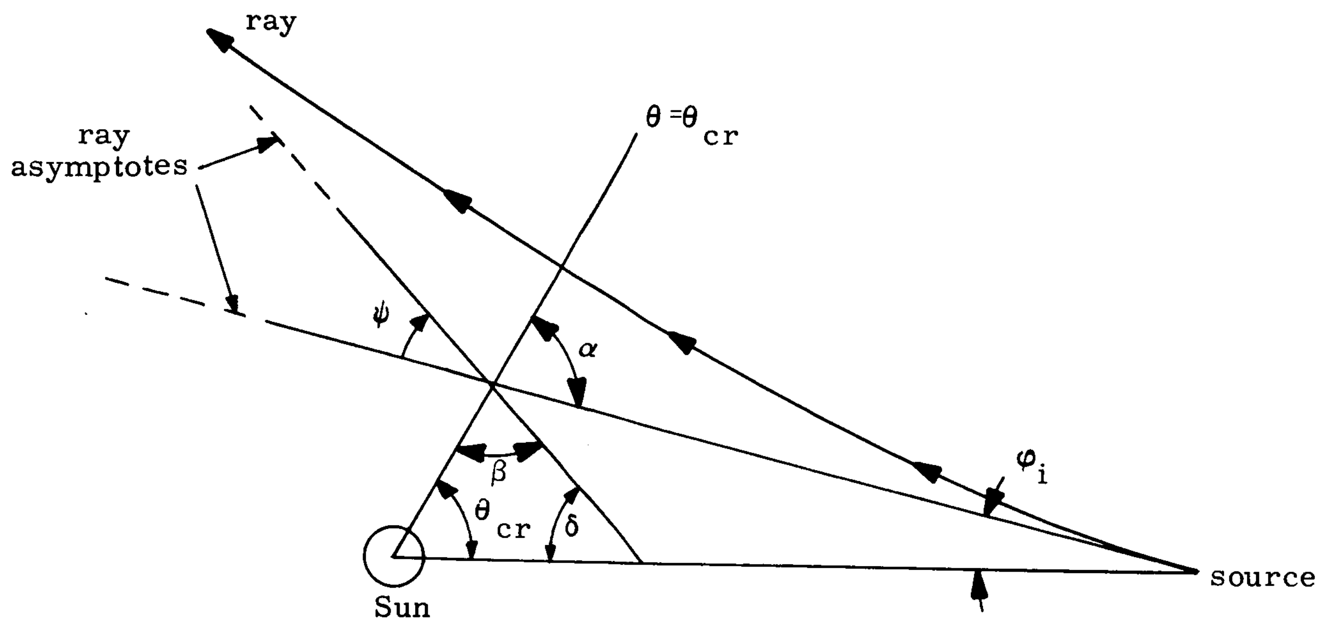
$$\theta_o^{(+)} - \theta_{cr} = \int_{\rho_{cr}}^{\rho_o^{(+)}} (\quad) . \tag{2.25}$$

It is clear that if $\rho_o^{(-)} = \rho_o^{(+)}$, then

$$\theta_o^{(+)} - \theta_{cr} = \theta_{cr} - \theta_o^{(-)} , \tag{2.26}$$

proving the desired symmetry.

This symmetry about the line $\theta = \theta_{cr}$ will facilitate calculation of the net angle, ψ, through which a ray turns upon traversing the corona. For simplicity we shall consider the source of radiation to lie sufficiently far from the sun that no ray curvature occurs beyond that point; this is a useful assumption for sources of interest to us, but it must be pointed out that it is violate for calculations of radiation originating within the corona itself. Consider the geometry below



In drawing the figure the demonstrated symmetry about the $\theta = \theta_{cr}$ line has been utilized in constructing the two ray asymptotes to intersect the $\theta = \theta_{cr}$ line at the same point. Now, from the figure it is clear that

$$\begin{aligned} \psi + \varphi_i &= \delta \\ \beta + \delta + \theta_{cr} &= \pi \\ \psi + \alpha + \beta &= \pi \ . \end{aligned} \tag{2.27}$$

But symmetry about $\theta = \theta_{cr}$ implies that

$$\alpha = \beta . \tag{2.28}$$

Equations (2.27a) and (2.27b) yield for the ray turning angle, ψ :

$$\psi = \pi - 2(\varphi_i + \theta_{cr}) . \tag{2.29}$$

The actual evaluation of the effects due to ray curvature must generally be executed by a machine calculation based on equations (2.22), (2.23), and (2.29). However, it has been shown by Bracewell and Preston (1956) that in the limit of a nearly linear ray a simple formula for ψ may be obtained anayltically. First, let us specify what we mean by a "nearly linear ray." Since the ray geometry depends on the properties of the coronal medium at the ray itself it is clear that a useful approximation to a ray trajectory necessitates that the coronal properties at the approximate ray differ only slightly from those at the actual ray. We thus require that the maximum offset distance of the ray from its linear approximation (this maximum offset will occur near the point of closest approach of the ray to the sun) be much less than the distance over which some coronal property in which we are interested (to be denoted by X) changes significantly. This requirement may be written

$$(s-s_o) \tan \psi \left[\frac{1}{X} \frac{dX}{dr} \right] \ll 1$$

for a distant source, and

$$(s-s_o) \tan \frac{\psi}{2} \left[\frac{1}{X} \frac{dX}{dr} \right] \ll 1$$

for source and observer equidistant from the sun; here $(s-s_o)$ is the distance from the observer to the point on the ray closest to the sun. Since the radial gradient of the inner corona is large this condition is quite restrictive for most rays.

Now, following Bracewell and Preston (1956), if the above conditions are fulfilled and the ray under consideration is nearly linear we may approximate equation (2. 22) for θ_{cr} as

$$\theta_{cr} \simeq \int_{\rho_{cr}}^{\rho_i} \frac{d\rho_o}{\rho_o \left\{ \frac{\rho_o^2}{\rho_i^2 \sin^2 \varphi_i} - 1 \right\}^{1/2}}$$

which becomes, with the help of equation (2. 23) with $\mu_i = 1$,

$$\theta_{cr} \simeq \sin^{-1} \left[\mu_o(\rho_{cr}) \right] - \varphi_i \ .$$

This last expression gives the bulk of the actual value of θ_{cr} for nearly linear rays. We can do better however by writing equation (2.22) without approximation as

$$\theta_{cr} = \int_{\rho_{cr}}^{\rho_i} \frac{1}{\rho_o} \left\{ \frac{1}{\left[\dfrac{\mu_o^2 \rho_o^2}{\mu_i^2 \rho_i^2 \sin^2 \varphi_i} - 1\right]^{1/2}} - \frac{1}{\left[\dfrac{\rho_o^2}{\rho_i^2 \sin^2 \varphi_i} - 1\right]^{1/2}} \right\} d\rho_o$$

$$+ \sin^{-1}\left[\mu_o(\rho_{cr})\right] - \varphi_i \quad .$$

The integral above is small and has been evaluated approximately; the result is

$$\theta_{cr} \simeq \frac{\pi}{2} - \phi_i - \frac{\pi}{4} \rho_{cr} \left(\frac{d\mu_o}{d\rho}\right)_{\rho_{cr}}$$

Inserting this into (2.29) we obtain for the ray turning angle

$$\psi \simeq \frac{\pi}{2} \rho_{cr} \left(\frac{d\mu_o}{d\rho}\right)_{\rho_{cr}}$$

If we take the refractive index to be specified by

$$\mu^2 = 1 - \frac{\omega_\rho^2}{\omega^2} \quad ; \quad \omega_p^2 \sim 1/\rho^{2n}$$

where ω_ρ is the plasma frequency and ω the (angular) frequency of the radiation, and note that for the nearly linear rays considered $\mu_o(\rho_{cr})$ will be close to unity, we may then obtain

$$\psi \simeq n\frac{\pi}{4}\left(\frac{\omega_p^2}{\omega^2}\right)_{\rho_{cr}} \tag{2.30}$$

This essentially completes the analytical description of the determination of the ray trajectories lying in the plane $\varphi = \pi/2$ of a spherically symmetric solar corona. In Chapter III we will use these rays as the basis of a perturbation solution of scattering, and it will therefore be of value to consider them numerically. Before doing so, however, we first wish to present a number of equations, which we shall find useful later, pertaining to the rays in a spherically symmetric corona. We begin with equations (2.10) and (2.11), which were obtained under only the assumption of sphericity, for rays in the plane $\varphi = \pi/2$. If we multiply (2.10) by $\cos\theta_o$, and (2.11) by $\sin\theta_o$, and add the resultant equations, we obtain, after some simplification,

$$\frac{d\mu_o}{dr} = \frac{d}{ds}\left\{\mu_o\frac{dr_o}{ds}\right\} - \mu_o r_o\left(\frac{d\theta_o}{ds}\right)^2. \tag{2.31}$$

Expanding the right-hand side, and noting that for the spherically symmetric corona under consideration

$$\frac{d\mu_o}{ds} = \frac{d\mu_o}{dr}\frac{dr_o}{ds}$$

we obtain

$$\frac{d^2r_o}{ds^2} = \left[\frac{d\ln\mu_o}{dr} + \frac{1}{r_o}\right] r_o^2\left(\frac{d\theta_o}{ds}\right)^2 \tag{2.32}$$

where equation (2.17) has been used. Having thus obtained an explicit expression for $d^2 r_o / ds^2$, it is also of value to explicitly determine $d^2\theta_o / ds^2$; formally expanding equation (2.12) gives immediately

$$\frac{d^2\theta_o}{ds^2} = - \frac{d\theta_o}{ds} \frac{d}{ds} \left[\ln (\mu_o r_o^2) \right] . \tag{2.33}$$

The single remaining quantity we wish to calculate is the curvature, to be denoted by K_o , of the rays. If we let x_o and y_o denote the x and y coordinates of a ray for the spherically symmetric case, the curvature is given by (see, for example, Thomas, "Calculus and Analytic Geometry," Addison-Wesley (1960), §12-6)

$$K_o = \frac{dx_o}{ds} \frac{d^2 y_o}{ds^2} - \frac{dy_o}{ds} \frac{d^2 x_o}{ds^2} . \tag{2.34}$$

Letting

$$\begin{aligned} x_o &= r_o \cos \theta_o \\ y_o &= r_o \sin \theta_o \end{aligned} \tag{2.35}$$

and inserting these into equation (2.34) gives, after simplification involving use of equations (2.31)-(2.33),

$$K_o = \frac{r_o}{\mu_o} \frac{d\theta_o}{ds} \frac{d\mu_o}{dr} . \tag{2.36}$$

We shall pursue these relationships no further. Let it simply be said that the results of equations (2.31)-(2.36) will prove to be of value in subsequent chapters.

Propagation in a Plasma: The Refractive Index

It is our purpose in this section to first discuss the validity of the ray treatment for the propagation of an electromagnetic signal through the interplanetary plasma, and to then derive, for an appropriate model of the coronal plasma, a suitable expression for the refractive index of the (high frequency) waves of interest to us here.

The proper description of the propagation of a wave through a non-homogeneous medium must in general rest upon solution of a wave equation. If, however, the wave number $k = 2\pi/\lambda$ (λ is the wavelength) varies slowly over distances on the order of a wavelength one may then employ a WKB approximation for the solution to the wave equation; it is the WKB solution that yields the ray trajectories. The validity criteria for such a description are:

$$\left|\frac{d^2k}{ds^2}\right| << \left|k \frac{dk}{ds}\right|$$

$$\left|\frac{dk}{ds}\right| << \left|k^2\right| \qquad (2.37)$$

In terms of the refractive index equations (2.37) become:

$$\frac{1}{\mu^2}\left|\frac{d\mu}{ds}\right| \lambda_o << 2\pi$$

$$\frac{1}{\mu \frac{d\mu}{ds}}\left|\frac{d^2\mu}{ds^2}\right| \lambda_o << 2\pi \qquad (2.38)$$

where here λ_o is the wavelength in free space. Now we shall show below that for the coronal plasma the refractive index may be expected to become zero when $\omega^2 = \omega_p^2$; a refractive index of zero will in general lead to violation of conditions (2.38), thus requiring a proper wave description in the vicinity of $\mu^2 = 0$. This restriction on the validity of the ray analysis is a weak one, however, for if we evaluate equations (2.38) near $\mu = 0$ for a refractive index given by

$$\mu^2 = 1 - \frac{\omega_p^2}{\omega^2}$$

we obtain for the worst case of $ds = dr$

$$\mu^3 >> \frac{n}{4\pi} \frac{\lambda_o}{r} \tag{2.39}$$

where here (n) refers to the exponent in an assumed power law dependence for the electron density:

$$n(r) \sim \frac{1}{r^n}$$

Since we expect (r) in (2.39) to be at least R_o, and λ_o to be on the order of meters, we see that equations (2.38) represent very weak restrictions indeed. Only problems dealing with the reflection of radar pulses from coronal layers where $\mu^2 = 0$ will encounter difficulty in this regard; this problem will not concern us here, however.

We turn now to discuss whence the expression above for the refractive index arises, and the conditions for its validity. We begin with Maxwell's equation, using Gaussian units:

$$\nabla \cdot \vec{E} = 4\pi\rho_c$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \qquad (2.40)$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

where $\vec{E}$ and $\vec{B}$ are the electric and magnetic field vectors, respectively, (ρ_c) is the charge density, $\vec{J}$ is the current vector and (c) the speed of light. Equations (2.40) automatically satisfy charge conservation:

$$\frac{\partial\rho_c}{\partial t} + \nabla \cdot \vec{J} = 0 \qquad (2.41)$$

If all quantities are now considered to vary as $\exp[i(\vec{k}\cdot\vec{r} - \omega t)]$ we obtain

$$i\vec{k} \cdot \vec{E} = \frac{4\pi}{\omega}\vec{k} \cdot \bar{\bar{\sigma}} \cdot \vec{E}$$

$$\vec{k} \cdot \vec{B} = 0$$

$$\vec{k} \times \vec{E} = \frac{\omega}{c}\vec{B} \qquad (2.42)$$

$$\vec{k} \times \vec{B} = \frac{\omega}{c}\left[\frac{4\pi}{i\omega}\bar{\bar{\sigma}} \cdot \vec{E} - \vec{E}\right]$$

$$\omega\rho_c = \vec{k} \cdot \bar{\bar{\sigma}} \cdot \vec{E}$$

where $\bar{\bar{\sigma}}$ is the conductivity tensor

$$\vec{J} = \bar{\bar{\sigma}} \cdot \vec{E} \qquad (2.43)$$

Eliminating $\vec{B}$ from equations (2.42) yields

$$\vec{k} \times (\vec{k} \times \vec{E}) + \left(\frac{\omega}{c}\right)^2 \overline{\overline{K}} \cdot \vec{E} = 0 \qquad (2.44)$$

where we have defined:

$$\overline{\overline{K}} = \overline{\overline{I}} - \frac{4\pi}{i\omega}\overline{\overline{\sigma}} \qquad (2.45)$$

The requirement that the set of equations (2.44) be soluble leads to a dispersion relation. But we must first specify $\overline{\overline{\sigma}}$.

The conductivity tensor $\overline{\overline{\sigma}}$ represents the motional response to an electric field of charged particles in the medium. To obtain this response we examine the equation of motion for an electron, the species which we assume to carry the bulk of the current:

$$m\frac{d\vec{v}}{dt} = -e\left[\vec{E} + \frac{\vec{v}}{c} \times (\vec{B}_0 + \vec{B})\right] - \nu m\vec{v} \qquad (2.46)$$

where (m) is the electron mass, (e) the electron charge (magnitude), (ν) the electron collision frequency, $\vec{B}_0$ the externally applied magnetic field, and $\vec{v}$ the (induced) electron velocity. Fourier analysing (2.46) yields

$$\vec{v} = \frac{-e\vec{E}}{m(\nu - i\omega)}$$

or

$$\overline{\overline{\sigma}} = \frac{e^2 n}{m(\nu - i\omega)}\,\overline{\overline{I}} \qquad (2.47)$$

If we insert (2.47) into (2.45) we obtain:

$$\overline{\overline{K}} = \left\{1 - \frac{\omega_p^2}{\omega^2(1 + i\frac{\nu}{\omega})}\right\}\overline{\overline{I}} \qquad (2.48)$$

With this expression the solubility condition on Equation (2.44) yields:

$$\mu^2 = 1 - \frac{\omega_p^2}{\omega^2(1 + i\frac{\nu}{\omega})} \tag{2.49}$$

The collision frequency (ν) has been given by Smerd and Westfold (1949) to be

$$\nu = 42 \frac{n}{T^{3/2}} \tag{2.50}$$

where (T) is the electron temperature ($^{\circ}$K) and (n) is here the number density (cm^{-3}). If we take $T = 10^{6\circ}K$ and $n = 10^8$ cm^{-3} we obtain $\nu = 4$ sec^{-1} demonstrating that for the frequencies of interest we may write (2.49) as:

$$\mu^2 = 1 - \frac{\omega_p^2}{\omega^2} \tag{2.51}$$

(The collision frequency will, however, be important in producing collisional absorption of a wave.) Equation (2.51) represents the refractive index we shall use in this work. Its validity is subject to a number of assumptions, which we discuss now.

We note first that we have neglected the effects of the magnetic fields on the induced velocity ($\vec{v}$). Neglect of the wave magnetic field ($\vec{B}$) may be shown to imply

$$|\vec{v}| \ll v_{ph} \tag{2.52}$$

where v_{ph} is the signal phase velocity.

Since equation (2.51) yields $v_{ph} \geq c$ we expect inequality (2.52) to be satisfied for sufficiently weak signals. If (S) is the signal power flow (watts/m^2) we may show that condition (2.52) is, for $v_{ph} = c$,

$$S \ll 2.5 \times 10^{-7} f^2 \qquad (2.53)$$

where (f) is the signal frequency. For (f) on the order of megacycles we expect satisfaction of (2.53). Neglect of the external magnetic field B_0 represents a more serious restriction, however. It may be shown, though, (Allis et al (1963)) that neglect of $\vec{B}_0$ is justified as long as

$$\frac{eB_0}{m\omega} \ll \mu^2 \qquad (2.54)$$

If we are in a region of small refractive effects ($\mu = 1$) condition (2.54) implies, for a one Gauss field,

$$f \gg 2.8 \text{ MHz} \qquad (2.55)$$

Condition (2.54) is more difficult to satisfy, however, in regions where $\mu < 1$.

We note next that in the derivation of (2.51) we have considered only electron motions, neglecting those of the much heavier ions. This assumption will be justified as long as:

$$\omega \gg \frac{eB_0}{m_i} \qquad (2.56)$$

where m_i is the ion mass; satisfaction of (2.54) guarantees satisfaction of (2.56).

In the derivation of equation (2.51) we have also neglected the thermal motions of the electrons. In equation (2.46) for the electron motion we neglected the variation of $\vec{E}$ over the path of an electron, the validity of which requires

$$v_{Th} << v_{ph}, \qquad (2.57)$$

where $v_{Th}^2 = \frac{2kT}{m}$ is the thermal velocity. For $T = 10^6\ ^{o}K$ we expect satisfaction of the above.

A non-zero electron temperature can result in other effects, however. (For a more complete discussion of these points see Allis et al (1963) and Stix (1962)). An electromagnetic wave may be Landau damped if there is a sufficient number of electrons present at the phase speed of the wave to interact appreciably with it. Satisfaction of (2.57) implies, however, that there should be no appreciable Landau damping; this conclusion is reinforced when we realize that $v_{ph} \geq c$. Perhaps a more important effect of a non-zero electron temperature is the introduction of a new wave mode (the electron plasma wave) propagating with a phase speed on the order of v_{Th} (Bohm and Gross (1949 a, b; 1950)). This wave will not couple with the electromagnetic wave unless their phase velocities become equal, a situation which does not occur as long as condition (2.54) is satisfied (Allis et al (1963)). The final effect of a non-zero electron temperature which we shall mention is the introduction of resonance at not only the electron cyclotron frequency, but also at the harmonics of the electron cyclotron frequency. These resonances result in an absorption of the signal. The resonant absorption at the harmonics of the electron cyclotron frequency is generally small, and it may be shown (Stix (1962)) that although absorption at the cyclotron frequency can be strong, the width of the resonance line for a 10^6 °K plasma will be only some 50 KHz if the magnetic field strength is 1 Gauss. Thus if (2.54) is satisfied we should not expect to see cyclotron absorption.

Having thus described in some detail the conditions upon which rests the validity of the ray analysis, and having presented an appropriate expression for the coronal refractive index, we turn now to a brief consideration of coronal collisional absorption.

Absorption

In order to adequately discuss coronal absorption we need only specify the radial distribution of electron density and temperature for the corona. Following many other authors we choose the Allen-Baumbach formula for the electron density

$$n(\rho) = 10^8\left[\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right] \text{cm}^{-3}$$

(Baumbach (1937), Allen (1947)) but we shall add to this an additional term corresponding to the theoretical results of Parker (1958; 1960 a,b) and the results of solar probe measurements near 1 AU. Thus, as suggested by Harrington (1965) we shall use for the corona ($\rho > 1.03$)

$$n_o(\rho) = 10^8\left[\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right] + \frac{10^6}{\rho^2} \qquad (2.58)$$

This expression is valid only for the coronal regions, and is significantly in error for the chromospheric and photospheric regions where the densities are considerably higher than those given by equation (2.58). As the top of the chromosphere is generally considered to be at $\rho = 1.03$ (see, for example, Brandt and Hodge, "Solar System Astrophysics," McGraw-Hill (1964), page 99) equation (2.58) will be considered correct only for $\rho > 1.03$. However, as the radio propagation phenomena to be described in this work will be limited to only those rays lying wholly in the corona, we shall use equation (2.58) without corrections for the chromosphere.

For the electron temperature we shall assume a constant representative value of 10^6 °K. Actually, the coronal temperature is probably constant out to only 3-4 $R_\odot$ (Brandt and Hodge, "Solar System Astrophysics," McGraw-Hill (1964), page 110) but since the temperature will be used only in the evaluation of absorption, and since by far most of the absorption occurs within 3 $R_\odot$, we will regard this neglect of the spatial variation of temperature as justified.

The absorption may be calculated as follows. If we define an optical depth τ as the integral along a ray of the absorption coefficient κ, that is

$$\tau = \int_{\text{path}} \kappa \, ds$$

then it is easily shown (see, for example, Shklovsky, "Cosmic Radio Waves," Harvard University Press (1960), pages 141-142) that the intensity of a signal after having traversed the absorbing medium is

$e^{-\tau}$ times its initial intensity. The absorption coefficient κ may be shown to be (ibid., pages 144-145)

$$\kappa = \frac{\omega_p^2 \nu}{\omega^2 \mu c}$$

as long as

$$\frac{\nu^2}{\omega^2} << \mu^4$$

a condition fulfilled in the present calculation. We have numerically evaluated $e^{-\tau}$ for rays in the model corona specified by equation (2.58); for this specific model we denote τ as τ_o. The results of the calculation are displayed in Graphs 1 and 2. We conclude that above 50 MHz absorption is not important for rays with $\varphi_i > .4^o$, or $\rho_i \sin \varphi_i > 1.4$. Thus rays <u>appearing</u> to an observer to originate outside of a region about the sun with radius 1.4 $R_\odot$ may be expected to be not appreciably absorbed, whereas rays appearing to originate within that region should suffer appreciable absorption.

Collisionless absorption has not been considered, for, as we have argued in the previous section, Landau damping should not be important for $T = 10^6$ °K and $v_{ph} > c$, while if $e B_o/m\omega << \mu^2$ we do not expect to see resonant cyclotron absorption. Furthermore, we point out that anomalously high collisional absorption near resonances ($\mu = \infty$) will not occur in this model, in virtue of the form of the refractive index given by equation (2.51).

Summary

In this chapter it has been our task to establish a basic set of equations allowing description of radio rays in a medium of arbitrary refractive index. We proceeded then to restrict discussion of these equations to the specific case of spherical symmetry. We then discussed in some detail the validity of the ray analysis and the refractive index we shall use in this work. We shall see in the next chapter that the rays discussed in this chapter will form the basis of perturbation solution of scattering.

CHAPTER III

Quadrature Solutions for Scattering

Introduction

In the present chapter we seek to obtain, via a first order perturbation analysis, general quadrature solutions for rays scattered by a random medium.

Basic Equations

Equations (2.3)-(2.5) are general equations for the behavior of rays in a medium of arbitrary refractive index, $\mu(\vec{r})$. They may be solved in principle to give the ray coordinates r, θ, and φ as functions of the arc length, s, along the ray, if one is given some appropriate initial conditions (tantamount to r_i and φ_i of Chapter II) and if the refractive index $\mu(\vec{r})$ is known. For a turbulent medium, however, it will be impossible to specify $\mu(\vec{r})$ exactly; only its statistical properties may be regarded as known. Direct utilization of equations (2.3)-(2.5) is thus impossible; as we are unable to specify $\mu(\vec{r})$ exactly we cannot solve them numerically, while the very form of the set (non-linear, coupled) precludes obtaining quadrature solutions for r, θ, and φ as functions of s and μ.

However, it will be usually possible to specify a <u>local</u> average of the refractive index, to be denoted by $\mu_o(\vec{r})$, about which occur turbulent fluctuations. If we denote this latter component by $\delta\mu(\vec{r})$ we may write

$$\mu(\vec{r}) = \mu_o(\vec{r}) + \delta\mu(\vec{r}) \tag{3.1}$$

in which the first term on the right hand side represents the known average component and the second term represents the fluctuating component, known only in a statistical sense, of the refractive index field. If we let angular brackets, < >, represent an ensemble average (or a time average, if coronal processes are ergodic) we clearly have

$$\begin{aligned} \langle \mu(\vec{r}) \rangle &= \mu_o(\vec{r}) \\ \langle \delta\mu(\vec{r}) \rangle &= 0 \ . \end{aligned} \tag{3.2}$$

We have thus far said nothing about the form of $\mu_o(\vec{r})$. However, it behooves us for the sake of simplicity to consider, as in Chapter II, a spherically symmetric average corona in which μ_o is a function only of distance from the sun's center, i.e.

$$\mu_o(\vec{r}) = \mu_o(r) \ . \tag{3.3}$$

Having thus split the refractive index field into an average component and a fluctuating component, it seems natural to similarly divide the ray coordinates into average and fluctuating components. We therefore write

$$\begin{aligned} r(s) &= r_o(s) + \delta r(s) \\ \theta(s) &= \theta_o(s) + \delta\theta(s) \\ \varphi(s) &= \varphi_o(s) + \delta\varphi(s) \end{aligned} \tag{3.4}$$

where s again refers to the arc length along the ray. Here $r_o(s)$, $\theta_o(s)$, and $\varphi_o(s)$ are those ray coordinates which would be obtained if the fluctuating component, $\delta\mu$, of the refractive index were identically zero; they are thus those values of r, θ, and φ which would be obtained from equations (2.3)-(2.5) were we to use $\mu(\vec{r}) = \mu_o(\vec{r})$. On the other hand, $\delta r(s)$, $\delta\theta(s)$, and $\delta\varphi(s)$ are the portions of the ray coordinates which are induced by the statistical component of the refractive index, and are themselves therefore fluctuating statistical quantities.

Following Chapter II, if we assume spherical symmetry (equation (3.3)) we may for convenience in equation (3.4) let

$$\varphi_o(s) = \pi/2 . \tag{3.5}$$

The set of all rays lying in this plane, it will be remembered, represents the set of all possible ray geometries, and we suffer no loss in generality by considering only rays in the single plane $\varphi_o = \pi/2$.

Equations (3.4) may be interpreted as follows. We may regard $\delta\mu(\vec{r})$ as a statistical perturbation about some average refractive index field, $\mu_o(\vec{r})$, and then $\delta r(s)$, $\delta\theta(s)$, and $\delta\varphi(s)$ are the corresponding perturbations of the ray components about some average ray given by $r_o(s)$ and $\theta_o(s)$; these components have been discussed in Chapter II. Thus we may regard the average rays, about which we are perturbing, as well defined. There is, however, a subtle point, which may have already occurred to the reader. In equations (3.4) the argument of the left hand side is the arc length along the perturbed ray; this is also the argument of r_o and θ_o on the right hand side. However, in our calculations in Chapter II of $r_o(s)$ and $\theta_o(s)$ the argument there referred to the arc length along the average, unperturbed ray. Thus the arguments of $r_o(s)$ and $\theta_o(s)$ used in Chapter II and here are different, and the relevancy of the calculations of Chapter II to the present analysis becomes obscured. If, however, the perturbation quantities $\delta r(s)$, $\delta\theta(s)$, and $\delta\varphi(s)$ are small, so that a perturbed ray at all times lies close to its corresponding average unperturbed ray, the two arc lengths in question will be approximately equal, and the calculations of Chapter II and the present chapter regain their compatibility. In what follows we shall indeed treat the perturbation quantities $\delta\mu$, δr, $\delta\theta$, and $\delta\varphi$ as small, and we shall limit ourselves to a first order analysis. In that case, as shall be shown below, the arc lengths of both the average and perturbed rays are, to first order, the same, thus insuring compatibility of the work of Chapter II with the scattering computations to follow.

Let us begin. Inserting equations (3.1), (3.3), and (3.4) into equations (2.3)-(2.5) for the ray trajectories, and dropping all terms of second order in $\delta\mu$, δr, $\delta\theta$, $\delta\varphi$, and their derivatives, we obtain the following three equations for the perturbation ray coordinates:

$$\frac{d}{ds}\left\{\left(\mu_o \cos\theta_o\right)\frac{d\delta r}{ds} - \left(\mu_o \sin\theta_o \frac{dr_o}{ds} + \mu_o r_o \cos\theta_o \frac{d\theta_o}{ds}\right)\delta\theta\right.$$

$$+\left(\cos\theta_o \frac{dr_o}{ds} - r_o \sin\theta_o \frac{d\theta_o}{ds}\right)\delta\mu - \left(\mu_o r_o \sin\theta_o\right)\frac{d\delta\theta}{ds}$$

$$\left.-\left(\mu_o \sin\theta_o \frac{d\theta_o}{ds}\right)\delta r\right\} + \left(\frac{d\mu_o}{dr}\sin\theta_o\right)\delta\theta =$$

$$\cos\theta_o \frac{\partial\delta\mu}{\partial r} - \frac{\sin\theta_o}{r_o}\frac{\partial\delta\mu}{\partial\theta} \qquad (3.6)$$

$$\frac{d}{ds}\left\{\left(\mu_o \sin\theta_o\right)\frac{d\delta r}{ds} + \left(\mu_o \cos\theta_o \frac{dr_o}{ds} - \mu_o r_o \sin\theta_o \frac{d\theta_o}{ds}\right)\delta\theta\right.$$

$$+ \left(\sin\theta_o \frac{dr_o}{ds} + r_o \cos\theta_o \frac{d\theta_o}{ds}\right)\delta\mu + \left(\mu_o r_o \cos\theta_o\right)\frac{d\delta\theta}{ds}$$

$$\left. + \left(\mu_o \cos\theta_o \frac{d\theta_o}{ds}\right)\delta r\right\} - \left(\frac{d\mu_o}{dr}\cos\theta_o\right)\delta\theta =$$

$$\sin\theta_o \frac{\partial\delta\mu}{\partial r} + \frac{\cos\theta_o}{r_o}\frac{\partial\delta\mu}{\partial\theta} \qquad (3.7)$$

$$\frac{d}{ds}\left\{\mu_o \frac{d}{ds}\left(r_o\,\delta\varphi\right)\right\} - \frac{d\mu_o}{dr}\delta\varphi = \frac{1}{r_o}\frac{\partial\delta\mu}{\partial\varphi}\,. \qquad (3.8)$$

In these equations terms of zeroth order in the perturbation quantities have been eliminated in virtue of equations (2.10) and (2.11). The first and second of the above three equations are linear, second order, coupled differential equations for $\delta r(s)$ and $\delta\theta(s)$, driven by the perturbation refractive index field. The third is a linear second order differential equation for $\delta\varphi(s)$, driven by the perturbation refractive index field.

As written in equations (3.6)-(3.8) the driving terms appear as functions of $\vec{r}$, whereas all other variables are to be construed as functions of s alone; the question arises as to where we are to evaluate the terms in $\delta\mu$. Strictly speaking, the evaluation is to be along the perturbed ray, but as we do not yet know where this is, we must find another procedure. What we do is to divide $\delta\mu(\vec{r})$ into two parts; introducing a change in notation:

$$\delta\mu(\vec{r}) \rightarrow \delta\mu + \frac{d\mu_o}{dr}\,\delta r\,. \tag{3.9}$$

The first term on the right hand side represents fluctuations in the refractive index of the medium which may to first order be evaluated along the basic ray. The second term represents the change in refractive index seen by an observer on the perturbed ray due to displacements of that ray in the non-uniform average corona; the value of $\frac{d\mu_o}{dr}$ may, again to first order, be evaluated along the basic ray. Equation (3.9) is correct to first order and represents the proper evaluation of $\delta\mu(\vec{r})$ in (3.6)-(3.8). The contribution of the $\frac{d\mu_o}{dr}\delta r$ term will be zero in the equation for $\delta\varphi$, as there only $\frac{\partial\delta\mu}{\partial\varphi}$ appears, and

$$\frac{\partial\delta\mu}{\partial\varphi} \rightarrow \frac{\partial\delta\mu}{\partial\varphi} + \delta r\,\cancel{\frac{d^2\mu_o}{dr\,d\varphi}}\,.$$

We shall see, however, that the $\frac{d\mu_o}{dr}\delta r$ term is important in the equations for $\delta r(s)$ and $\delta\theta(s)$.

In summary, we restate the assumptions that have been used in deriving equations (3.6)-(3.8). We have been throughout this discussion working with radio rays, the adequacy of which has been discussed in Chapter I. In addition, two other assumptions have thus far been made. We have taken the local average corona, about which occur statistical fluctuations, to be spherically symmetric and consequently the basic rays are taken to be those determined by this spherically symmetric corona. This assumption seems quite good indeed around sunspot maximum, but becomes less so near sunspot minimum. In the latter event the departure from spherical symmetry seems small enough that basic rays calculated on the basis of spherical symmetry are probably quite valid indeed; however, the departure from spherical symmetry will manifest itself as an asphericity in the strength of scattering in the corona. It appears, therefore, that calculations of scattering about basic rays which have been derived on the basis of spherical symmetry will not be far in error if the asphericity is considered to manifest itself only in the terms in $\delta\mu(\vec{r})$ (the driving terms) in equations (3.6)-(3.8). In any event, it is the purpose of this discussion to probe the effects of overall refraction and anisotropic turbulence in the corona, and a detailed discussion of asphericity will only serve to cloud the issue. Finally, we have assumed the statistical departures from the average configuration to be sufficiently small to permit use of a first-order theory; as the scattering of radio astronomical sources has been found not to exceed one or two degrees this may be regarded as a valid assumption.

The Equation for $\delta\varphi(s)$

Equation (3.8) for $\delta\varphi(s)$ is the simplest of our three basic equations and will be treated first. We had

$$\frac{d}{ds}\left\{\mu_o \frac{d}{ds}\left(r_o\delta\varphi\right)\right\} - \frac{d\mu_o}{dr}\,\delta\varphi = \frac{1}{r_o}\,\frac{\partial\delta\mu}{\partial\varphi}\ . \qquad (3.10)$$

This is an inhomogeneous linear, second order ordinary differential equation for $\delta\varphi(s)$; its coefficients as well as $\frac{\partial\delta\mu}{\partial\varphi}$ refer to the basic ray and are thus implicitly known functions of s.

The general solution to such an equation will be of the form

$$\delta\varphi = \delta\varphi_h + \delta\varphi_p \qquad (3.11)$$

where $\delta\varphi_h$ is the general solution of the associated homogeneous equation and $\delta\varphi_p$ is some particular solution of the complete equation. The homogeneous solution itself will be of the form

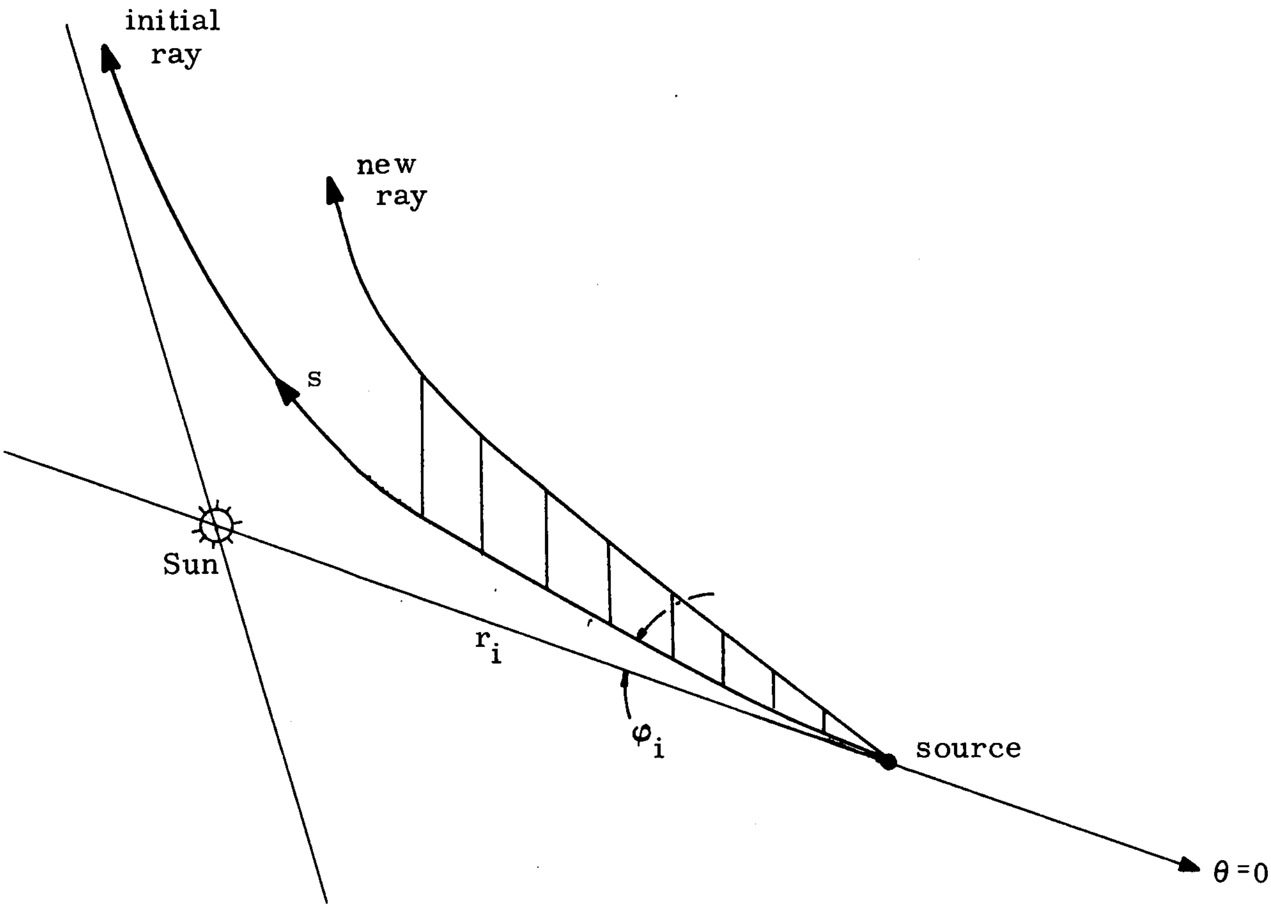
$$\delta\varphi_h = c_1\,\delta\varphi_{h1} + c_2\,\delta\varphi_{h2} \tag{3.12}$$

where $\delta\varphi_{h1}$ and $\delta\varphi_{h2}$ are the two independent solutions to the homogeneous equation associated with equation (3.10), and c_1 and c_2 are constants chosen to satisfy the initial conditions. Now, it is a property of such an equation that once one of the homogeneous solutions, say $\delta\varphi_{h1}$, is found both the second homogeneous solution, $\delta\varphi_{h2}$, and the particular solution, $\delta\varphi_p$, may be found in direct fashion by the method of variation of Parameters (see, for example, Morse and Feshbach, "Methods of Theoretical Physics," McGraw-Hill (1953), §5.2). Thus equation (3.10) may be considered solved once one of the homogeneous solutions has been found.

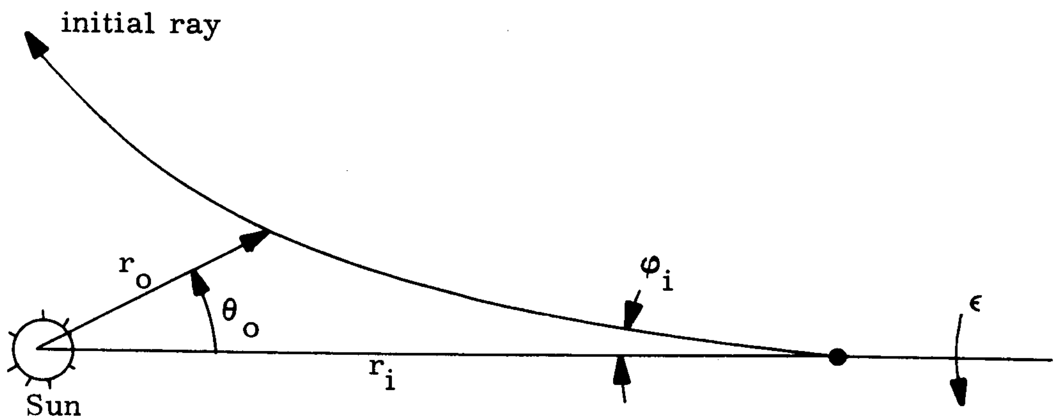
The homogeneous equation associated with (3.10) is

$$\frac{d}{ds}\left\{\mu_o \frac{d}{ds}\left(r_o\,\delta\varphi_h\right)\right\} - \frac{d\mu_o}{dr}\,\delta\varphi_h = 0\,. \tag{3.13}$$

Its solution may be readily obtained if we note that by having let the driving term, $\frac{\partial\delta\mu}{\partial\varphi}$, of equation (3.10) go to zero we have obtained nothing more than an equation for rays in an unperturbed average corona specified by $\mu_o(r)$. Thus $\delta\varphi_h$ is simply the angular displacement from the initial ray of some other ray which is also determined by $\mu_o(r)$ and which lies close to the initial ray in question. As a simple specific case consider a ray which originates at the same point (r_i) as the initial ray and with the same orientation (φ_i) in the plane of the initial ray but with a component out of that plane, i.e.



Now, if we can for this simple geometry obtain an expression for $\delta\varphi_h(s)$ we will have obtained $\delta\varphi_{h1}$ and thus essentially solved equation (3.10). But this is easy, for it is clear from the above figure that if the new ray lies very close to the initial ray, as it must for a first order analysis, it can be generated simply by rotating the initial ray by some small angle ϵ about the source-sun line. Consider the geometry of the initial ray:



It is clear that the displacement of the rotated initial ray perpendicular to the plane is, if ϵ is small,

$$r_o \delta\varphi_{h1} = r_o \sin\theta_o \, \epsilon \, .$$

If we ignore the ϵ in the above equation we get

$$\delta\varphi_{h1}(s) = \sin\theta_o \, . \tag{3.14}$$

This can be verified by insertion into the homogeneous equation, (3.13):

$$\frac{d}{ds}\left\{\mu_o \frac{d}{ds}\left(r_o \sin\theta_o\right)\right\} - \frac{d\mu_o}{dr} \sin\theta_o \overset{?}{=} 0 \, . \tag{3.15}$$

This is, however, exactly equation (2.11), thus verifying (3.14).

Equation (3.14) may be immediately used to find, via the method of variation of parameters, $\delta\varphi_{h2}$ to be

$$\delta\varphi_{h2}(s) = \cos\theta_o \,. \tag{3.16}$$

This may be checked by insertion into equation (3.13):

$$\frac{d}{ds}\left\{\mu_o \frac{d}{ds}\left(r_o \cos\theta_o\right)\right\} - \frac{d\mu_o}{dr}\cos\theta_o \overset{?}{=} 0 \tag{3.17}$$

which is exactly equation (2.10), thus verifying (3.16).

In virtue of equation (3.12) the general homogeneous solution to the equation for $\delta\varphi(s)$ may be written as

$$\delta\varphi_h(s) = c_1 \sin\theta_o + c_2 \cos\theta_o \,. \tag{3.18}$$

The Wronskian of the two homogeneous solutions is

$$\Delta = \begin{vmatrix} \delta\varphi_{h1}(s) & \delta\varphi_{h2}(s) \\ \dfrac{d\delta\varphi_{h1}(s)}{ds} & \dfrac{d\delta\varphi_{h2}(s)}{ds} \end{vmatrix} = -\frac{d\theta_o(s)}{ds} \tag{3.19}$$

thus verifying that the two solutions are independent, as $\frac{d\theta_o}{ds}$ is never zero (except in the degenerate case $\theta_o = 0$, which is of no concern to us here).

To obtain the particular solution, $\delta\varphi_p$, it is convenient to put equation (3.10) into standard form. This may be done in straightforward fashion, and we obtain

$$\frac{d^2\delta\varphi}{ds^2} + \frac{d}{ds}\left[\ln\left(\mu_o r_o^{\,2}\right)\right]\frac{d\delta\varphi}{ds} + \left(\frac{d\theta_o}{ds}\right)^2 \delta\varphi$$

$$= \frac{1}{\mu_o r_o^{\,2}} \quad \frac{\partial\delta\mu}{\partial\varphi} \tag{3.20}$$

where equation (2.30) has been used to eliminate $\frac{d\mu_o}{dr}$. The method of variation of parameters then gives immediately

$$\delta\varphi_p = \frac{1}{C}\left\{\sin\theta_o \int \frac{\partial\delta\mu}{\partial\varphi} \cos\theta_o \, ds \right.$$

$$\left. - \cos\theta_o \int \frac{\partial\delta\mu}{\partial\varphi} \sin\theta_o \, ds\right\} \tag{3.21}$$

where C is a constant specifying about which of the basic rays the scattering occurs:

$$C = \mu_o r_o^{\,2} \frac{d\theta_o}{ds} = \mu_i r_i \sin\varphi_i \tag{3.22}$$

in virtue of equation (2.16). $\delta\varphi_p$ as given in (3.21) may be checked by insertion into equation (3.10). The procedure is somewhat tedious and will not be presented here; suffice it to say that equation (3.21) has indeed been verified as correct in this manner.

The complete general solution for $\delta\varphi(s)$ may be found therefore to be (equations (3.11), (3.12), (3.14), (3.16) and (3.21)):

$$\delta\varphi(s) = c_1 \sin\theta_o(s) + c_2 \cos\theta_o(s)$$

$$+ \frac{\sin\theta_o(s)}{C}\int_o^s \left(\frac{\partial\delta\mu}{\partial\varphi}\right)' \cos\theta_o\,(s')\, ds' \tag{3.23}$$

$$- \frac{\cos\theta_o(s)}{C}\int_o^s \left(\frac{\partial\delta\mu}{\partial\varphi}\right)' \sin\theta_o\,(s')\, ds'$$

where the prime merely denotes a dummy variable of integration. This may be put in somewhat more compact form as

$$\begin{aligned}\delta\varphi(s) &= c_1 \sin\theta_o(s) + c_2 \cos\theta_o(s) \\ &+ \frac{1}{C}\int_o^s \left(\frac{\partial\delta\mu}{\partial\varphi}\right)' \sin(\theta_o - \theta_o')\, ds' .\end{aligned} \tag{3.24}$$

The lower limit of integration, $s = 0$, refers to the source.

The values of c_1 and c_2 in equation (3.24) may be ascertained by considering boundary conditions on $\delta\varphi(s)$. In all cases we shall consider the ray to "leave" the source in unperturbed fashion so that

$$\begin{aligned}\delta\varphi(s = 0) &= 0 \\ \left(\frac{d\delta\varphi}{ds}\right)_{s=0} &= 0 .\end{aligned} \tag{3.25}$$

Taking $\theta_o(s = 0) = 0$ we then obtain from (3.24)

$$c_1 = c_2 = 0 . \tag{3.26}$$

Thus we obtain finally

$$\boxed{\delta\varphi(s) = \frac{1}{C}\int_o^s \left(\frac{\partial\delta\mu}{\partial\varphi}\right)' \sin(\theta_o - \theta_o')\, ds'.} \tag{3.27}$$

To summarize, equation (3.27) is the general solution for the scattering parameter $\delta\varphi(s)$ of a ray traversing a spherically symmetric average corona about which occur (small) statistical fluctuations. Its validity is limited only by the validity of the following assumptions:

i) applicability of a ray analysis

ii) spherical symmetry: $\mu_o = \mu_o(r)$
iii) applicability of a first order treatment
iv) the initial conditions of equation (3.25).

We turn now to a consideration of equations (3.6) and (3.7), for $\delta r(s)$ and $\delta\theta(s)$.

The Equations for $\delta r(s)$ and $\delta\theta(s)$

Equations (3.6) and (3.7) are linear, second order, ordinary differential equations for the scattering parameters $\delta r(s)$ and $\delta\theta(s)$. They are coupled and must be treated together. We had:

$$\begin{aligned} \frac{d}{ds}\Big\{ &\left(\mu_o \cos\theta_o\right)\frac{d\delta r}{ds} - \left(\mu_o r_o \sin\theta_o\right)\frac{d\delta\theta}{ds} \\ &- \left(\mu_o \sin\theta_o \frac{dr_o}{ds} + \mu_o r_o \cos\theta_o \frac{d\theta_o}{ds}\right)\delta\theta \\ &- \left(\mu_o \sin\theta_o \frac{d\theta_o}{ds}\right)\delta r \\ &+ \left(\cos\theta_o \frac{dr_o}{ds} - r_o \sin\theta_o \frac{d\theta_o}{ds}\right)\delta\mu\Big\} \\ &+ \left(\frac{d\mu_o}{dr}\sin\theta_o\right)\delta\theta = \cos\theta_o \frac{\partial\delta\mu}{\partial r} - \frac{\sin\theta_o}{r_o}\frac{\partial\delta\mu}{\partial\theta} \end{aligned} \qquad (3.28)$$

$$\frac{d}{ds}\left\{ \left(\mu_o \sin\theta_o\right) \frac{\partial\delta r}{\partial s} + \left(\mu_o r_o \cos\theta_o\right) \frac{d\delta\theta}{ds} \right.$$

$$+ \left(\mu_o \cos\theta_o \frac{dr_o}{ds} - \mu_o r_o \sin\theta_o \frac{d\theta_o}{ds}\right) \delta\theta$$

$$+ \left(\mu_o \cos\theta_o \frac{d\theta_o}{ds}\right) \delta r \qquad (3.29)$$

$$\left. + \left(\sin\theta_o \frac{dr_o}{ds} + r_o \cos\theta_o \frac{d\theta_o}{ds}\right) \delta\mu \right\}$$

$$- \left(\frac{d\mu_o}{dr} \cos\theta_o\right) \delta\theta = \sin\theta_o \frac{\partial\delta\mu}{\partial r} + \frac{\cos\theta_o}{r_o} \frac{\partial\delta\mu}{\partial\theta} .$$

It is to be remembered (see equation (3. 9) and corresponding discussion) that here $\delta\mu$ is to be evaluated along the perturbed ray; it has not yet been divided into the two parts of equation (3. 9).

We begin by simplifying equations (3.28) and (3.29) as follows: If we multiply (3.28) by $\cos\theta_o$ and (3.29) by $\sin\theta_o$, and add the resultant equations we obtain

$$\frac{d}{ds}\left\{\mu_o\left[\frac{d\delta r}{ds} - r_o \frac{d\theta_o}{ds} \delta\theta\right] + \frac{dr_o}{ds}\delta\mu\right\}$$

$$- \frac{d\theta_o}{ds}\left\{ \left(\mu_o r_o\right) \frac{\partial\delta\theta}{ds} + \left(\mu_o \frac{dr_o}{ds}\right)\delta\theta + \left(\mu_o \frac{d\theta_o}{ds}\right)\delta r \right.$$

$$\left. + \left(r_o \frac{d\theta_o}{ds}\right)\delta\mu \right\} = \frac{\partial\delta\mu}{\partial r} . \qquad (3.30)$$

Similarly, if we multiply equation (3.28) by $(-\sin\theta_o)$ and (3.29) by $\cos\theta_o$, and add the resultant equations we obtain

$$\frac{d}{ds}\left\{\mu_o\left[r_o\frac{d\delta\theta}{ds}+\frac{dr_o}{ds}\delta\theta+\frac{d\theta_o}{ds}\delta r\right]+\left(r_o\frac{d\theta_o}{ds}\right)\delta\mu\right\}$$

$$+\frac{d\theta_o}{ds}\left\{\mu_o\frac{d\delta r}{ds}-\left(\mu_o r_o\frac{d\theta_o}{ds}\right)\delta\theta+\frac{dr_o}{ds}\delta u\right\} \qquad (3.31)$$

$$-\frac{d\mu_o}{dr}\delta\theta=\frac{1}{r_o}\frac{\partial\delta\mu}{\partial\theta}\ .$$

We now make use of the duopartite division of $\delta\mu$ effected by equation (3. 9):

$$\delta\mu \rightarrow \delta\mu + \frac{d\mu_o}{dr}\delta r \qquad (3.32)$$

where the left hand side implies evaluation along the perturbed ray, and the right hand side is to be evaluated along the basic ray. This division of the "complete" $\delta\mu$ (left hand side) into the two parts of the right hand side is correct to first order. From expression (3. 32) may immediately be derived the following useful expressions, also valid to first order:

$$\frac{\partial\delta\mu}{\partial\theta} \rightarrow \frac{\partial\delta\mu}{\partial\theta} \qquad (3.33)$$

$$\frac{\partial\delta\mu}{\partial r} \rightarrow \frac{\partial\delta\mu}{\partial r}+\frac{d^2\mu_o}{dr^2}\delta r \qquad (3.34)$$

$$\frac{d\delta\mu}{ds} \rightarrow \frac{\partial\delta\mu}{\partial r}\frac{dr_o}{ds}+\frac{\partial\delta\mu}{\partial\theta}\frac{d\theta_o}{ds}$$

$$+\frac{d\mu_o}{dr}\frac{d\delta r}{ds}+\frac{d^2\mu_o}{dr^2}\frac{dr_o}{ds}\delta r \qquad (3.35)$$

where we have used the fact that to first order along the basic ray we have

$$\frac{d}{ds} = \frac{dr_o}{ds}\frac{\partial}{\partial r} + \frac{d\theta_o}{ds}\frac{\partial}{\partial \theta} . \tag{3.36}$$

Once again, the right hand sides of equations (3.33)-(3.35) are to be evaluated along the basic ray.

Now, if relations (3.32)-(3.35) are inserted into equations (3.30) and (3.31) we obtain the following:

$$\begin{aligned} &\frac{d}{ds}\left\{\mu_o \frac{d\delta r}{ds}\right\} + \left(\frac{d\mu_o}{dr}\frac{dr_o}{ds}\right)\frac{d\delta r}{ds} - \mu_o\left(\frac{d\theta_o}{ds}\right)^2 \delta r \\ &- \left(2\mu_o r_o \frac{d\theta_o}{ds}\right)\frac{d\delta\theta}{ds} = -\left(\frac{dr_o}{ds}\frac{d\theta_o}{ds}\right)\frac{\partial\delta\mu}{\partial\theta} \\ &+ \mu_o r_o^2\left(\frac{d\theta_o}{ds}\right)^2 \frac{\partial}{\partial r}\left[\frac{1}{\mu_o}\left(\delta\mu + \frac{d\mu_o}{dr}\delta r\right)\right] \end{aligned} \tag{3.37}$$

$$\begin{aligned} &\frac{d}{ds}\left\{\mu_o r_o \frac{d\delta\theta}{ds}\right\} + \left(\mu_o \frac{dr_o}{ds}\right)\frac{d\delta\theta}{ds} \\ &+ \frac{d\theta_o}{ds}\left[2\mu_o + r_o\frac{d\mu_o}{dr}\right]\frac{d\delta r}{ds} + \frac{d}{ds}\left(\mu_o\frac{d\theta_o}{ds}\right)\delta r \\ &= \frac{1}{r_o}\left(\frac{dr_o}{ds}\right)^2\frac{\partial\delta\mu}{\partial\theta} - \mu_o r_o \frac{d\theta_o}{ds}\frac{dr_o}{ds}\frac{\partial}{\partial r}\left[\frac{1}{\mu_o}\left(\delta\mu + \frac{d\mu_o}{dr}\delta r\right)\right] \end{aligned} \tag{3.38}$$

where we have in carrying out some simplifying expansions used such relations as (2.16), (2.17), (2.30), (2.32) and (2.33). In equations (3.37) and (3.38), and in what follows, $\delta\mu$ is to be evaluated along the appropriate basic ray.

We now observe a very useful fact. If we multiply equation (3.37) by $\left(\frac{dr_o}{ds}\right)$ and equation (3.38) by $\left(r_o \frac{d\theta_o}{ds}\right)$, and add the resultant equations we obtain

$$\left(\mu_o \frac{dr_o}{ds}\right) \frac{d^2\delta r}{ds^2} + \left\{ \frac{d\mu_o}{dr}\left[1 + \left(\frac{dr_o}{ds}\right)^2\right] + 2\mu_o r_o \left(\frac{d\theta_o}{ds}\right)^2 \right\} \frac{d\delta r}{ds}$$

$$+ \left[\frac{d\theta_o}{ds} r_o^2 \frac{d}{ds}\left(\frac{\mu_o}{r_o} \frac{d\theta_o}{ds}\right)\right] \delta r + \left(\mu_o r_o^2 \frac{d\theta_o}{ds}\right) \frac{d^2\delta\theta}{ds^2}$$

$$+ \left(r_o^2 \frac{d\theta_o}{ds} \frac{d\mu_o}{ds}\right) \frac{d\delta\theta}{ds} = 0 \qquad (3.39)$$

The driving terms have identically vanished! This implies that the system posses a "constant of motion" which it is our task to find. We begin by recalling the discussion in the second section of this chapter, where it was noted that the present analysis required the arc lengths of both the perturbed and basic rays to be very nearly the same; in fact, in a sense our analysis has required them to be identical. Thus it occurs to us that the ray arc length might be our constant of motion. Now, for rays lying near the plane $\varphi_o = \pi/2$ we write for the element of ray arc length, ds:

$$(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2(d\varphi)^2. \qquad (3.40)$$

Inserting equations (3.4) we obtain

$$(ds)^2 = (dr_o + d\delta r)^2 + (r_o + \delta r)^2 (d\theta_o + d\delta\theta)^2$$

$$+ (r_o + \delta r)^2 (d\delta\varphi)^2 \qquad (3.41)$$

which to first order is

$$(ds)^2 = (dr_o)^2 + 2\, dr_o d\delta r + r_o^{\,2}\,(d\theta_o)^2$$

$$+ 2\, r_o^{\,2}\, d\theta_o d\delta\theta + 2\, r_o (d\theta_o)^2\, \delta r\, . \qquad (3.42)$$

As the element of arc length along the basic ray is given by

$$(ds_o)^2 = (dr_o)^2 + r_o^{\,2}\,(d\theta_o)^2 \qquad (3.43)$$

The difference between the arc length element on the perturbed ray and that on the basic ray is clearly

$$\Delta\,(ds)^2 = (ds)^2 - (ds_o)^2 = 2\, dr_o d\delta r$$

$$+ 2\, r_o^{\,2}\, d\theta_o d\delta\theta + 2\, r_o (d\theta_o)^2\, \delta r \qquad (3.44)$$

But, for the reasons stated above, we expect $\Delta(ds)^2$ to be zero, and we therefore guess that our "constant" of motion is

$$E(s) = \frac{dr_o}{ds}\frac{d\delta r}{ds} + r_o^{\,2}\,\frac{d\theta_o}{ds}\frac{d\delta\theta}{ds} + r_o\left(\frac{d\theta_o}{ds}\right)^2 \delta r \overset{?}{=} 0\, . \qquad (3.45)$$

This must yet be verified.

Now, it is easy to show that equation (3.39) may be written as

$$\frac{d}{ds}\left(\mu_o E\right) + \frac{d\mu_o}{ds}\, E = 0 \qquad (3.46)$$

or

$$\frac{dE(s)}{ds} + \frac{2}{\mu_o}\,\frac{d\mu_o}{ds}\, E = 0\, . \qquad (3.47)$$

The latter equation is immediately solved to give:

$$E(s) = \frac{c_o}{\mu_o^{\,2}(s)} \qquad (3.48)$$

where c_o is a constant of integration. It may be evaluated if we impose the boundary conditions (as in the discussion of $\delta\varphi(s)$) that a ray "leaves" the source in unperturbed fashion, so that

$$\begin{aligned} \delta r(s=0) &= 0 \\ \left(\frac{d\delta r}{ds}\right)_{s=0} &= 0 \\ \delta\theta(s=0) &= 0 \\ \left(\frac{d\delta\theta}{ds}\right)_{s=0} &= 0 \end{aligned} \tag{3.49}$$

If equations (3.49) are inserted into (3.45) and (3.48) we obtain immediately $c_o = 0$. Thus $E(s) = 0$ is the proper solution to equation (3.39) and we have as our constant of motion

$$\frac{dr_o}{ds}\frac{d\delta r}{ds} + r_o^2 \frac{d\theta_o}{ds}\frac{d\delta\theta}{ds} + r_o\left(\frac{d\theta_o}{ds}\right)^2 \delta r = 0 \, . \tag{3.50}$$

Thus we have proven that

$$\Delta (ds)^2 = 0$$

i.e., that to first order the ray arc length remains constant under perturbation. This fact is useful for several reasons. Besides helping us to solve our equations it also tells us what we mean by a point on a perturbed ray "corresponding to" a point on its basic ray. Since the arc length is a conserved quantity we mean simply that corresponding points are points of equal (s). This fact will become useful later in our discussion of phase shifts.

Now, in equation (3.37) the only term involving $\delta\theta$ is its first derivative, $d\delta\theta/ds$. But equation (3.50) allows us to write $d\delta\theta/ds$ only in terms of δr. Thus if we use equation (3.50) to eliminate $d\delta\theta/ds$ from equation (3.37) we are left with an equation for δr alone:

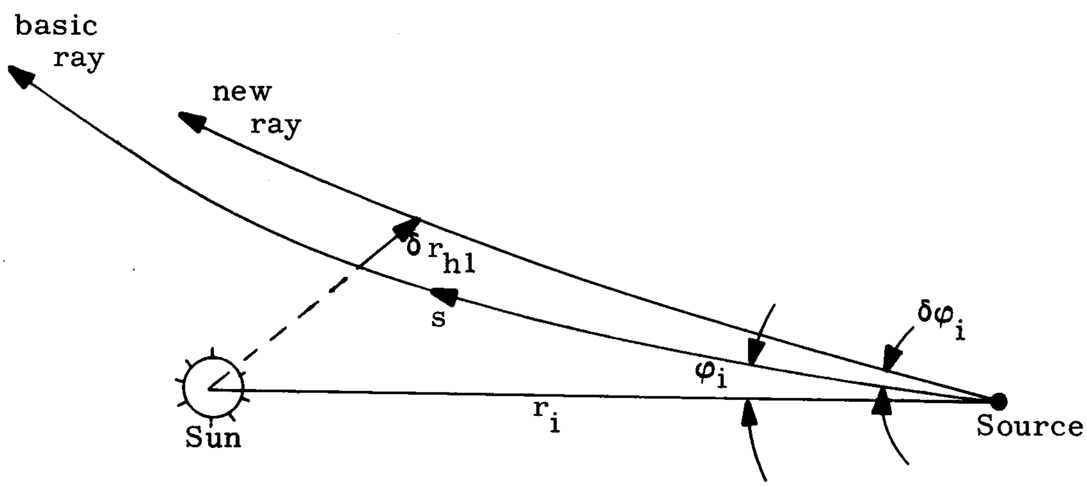
$$\frac{d^2\delta r}{ds^2} + 2\frac{dr_o}{ds}\left(\frac{1}{\mu_o}\frac{d\mu_o}{dr} + \frac{1}{r_o}\right)\frac{d\delta r}{ds}$$
$$+ \left(\frac{d\theta_o}{ds}\right)^2 \left[1 - r_o^2 \frac{d}{dr}\left(\frac{1}{\mu_o}\frac{d\mu_o}{dr}\right)\right]\delta r \tag{3.51}$$
$$= r_o^2 \left(\frac{d\theta_o}{ds}\right)^2 \frac{\partial}{\partial r}\left(\frac{\delta\mu}{\mu_o}\right) - \frac{dr_o}{ds}\frac{d\theta_o}{ds}\frac{\partial(\delta\mu/\mu_o)}{\partial\theta} .$$

This is an inhomogeneous, linear, second order ordinary differential equation for $\delta r(s)$; its coefficients, as well as the terms in $\delta\mu$, refer to the basic ray and are thus implicitly known functions of s. We may write its solution in the form

$$\delta r(s) = c_3\,\delta r_{h1} + c_4\,\delta r_{h2} + \delta r_p \tag{3.52}$$

where δr_{h1} and δr_{h2} are the two independent solutions to the homogeneous equations associated with (3.51), δr_p is a particular solution to equation (3.51), and c_3 and c_4 are constants chosen to satisfy initial conditions. Once δr_{h1} is found, δr_{h2} and δr_p may be directly determined by the method of variation of parameters.

δr_{h1} may be readily found by noting that the homogeneous version of (3.51) is obtained by letting the terms in $\delta\mu$ be zero; the homogeneous solutions thus refer to rays in an unperturbed average corona specified by $\mu_o(r)$. Thus δr_h is simply the radial displacement from the initial ray of some other ray which is also determined by $\mu_o(r)$ and which lies close to the initial ray in question. As a simple specific case we will consider a ray which originates at the same point and lies in the same plane as the initial ray, but which is specified by a value of φ_i slightly different than that of the basic ray, i.e.



We may find δr_h for this case, which we shall take as our δr_{h1}, by recalling equation (2.16) for the basic rays:

$$\mu_o r_o^2 \frac{d\theta_o}{ds} = \mu_i r_i \sin\varphi_i \; . \tag{3.53}$$

If $\delta\varphi_i$ is small a perturbation approach is appropriate and we write

$$(\mu_o + \delta\mu_h)(r_o + \delta r_h)^2 \frac{d(\theta_o + \delta\theta_h)}{ds} = \mu_i r_i \sin(\varphi_i + \delta\varphi_i)$$

which becomes to first order

$$\frac{\frac{d\delta\theta_h}{ds}}{\frac{d\theta_o}{ds}} + 2\,\frac{\delta r_h}{r_o} + \frac{\delta\mu_h}{\mu_o} = \cot\varphi_i\,\delta\varphi_i\,. \tag{3.54}$$

As there are no medium changes we shall take in equation (3.54)

$$\delta\mu_h = \frac{d\mu_o}{dr}\,\delta r_h\,. \tag{3.55}$$

If now we use the constant of motion of equation (3.50) to eliminate $d\delta\theta_h/ds$ from (3.54), we obtain

$$\frac{d\delta r_h}{ds} - \frac{\frac{d^2 r_o}{ds^2}}{\frac{dr_o}{ds}}\,\delta r_h = -\,\frac{r_o^{\,2}\left(\frac{d\theta_o}{ds}\right)^2}{\frac{dr_o}{ds}}\,\cot\varphi_i\,\delta\varphi_i \tag{3.56}$$

where d^2r_o/ds^2 has been introduced via equation (2.32). This is an inhomogeneous, linear, first order ordinary differential equation for δr_h which may be solved to give

$$\delta r_h = \cot\varphi_i\,\delta\varphi_i\,\frac{dr_o}{ds}\left\{ s - \int_o^s \frac{ds}{\left(\frac{dr_o}{ds}\right)^2} \right\} \tag{3.57}$$

Dropping the irrelevant multiplicative constant we will take the first homogeneous solution to equation (3.51) to be

$$\delta r_{h1} = \frac{dr_o}{ds}\left\{ s - \int_o^s \frac{ds}{\left(\frac{dr_o}{ds}\right)^2} \right\}$$

$$= \frac{dr_o}{ds}\left\{ \int_o^s \left[1 - \frac{1}{\left(\frac{dr_o}{ds}\right)^2} \right] ds \right\} \qquad (3.58)$$

$$= - \frac{dr_o}{ds} \int_o^s \frac{r_o^{\,2}\left(\frac{d\theta_o}{ds}\right)^2}{\left(\frac{dr_o}{ds}\right)^2} ds \; .$$

Equation (3.58) may be verified by direct substitution into the homogeneous form of (3.51). The procedure is too tedious to be presented here; suffice it to say that equation (3.58) has been verified as correct in this manner.

Having thus found $\delta r_{h1}(s)$, it is a simple matter to find $\delta r_{h2}(s)$ by the method of variation of parameters; we find thereby:

$$\delta r_{h2}(s) = \frac{dr_o}{ds} \qquad (3.59)$$

which may be verified directly by insertion into the homogeneous representation of equation (3.51).

Thus equations (3.52), (3.58) and (3.59) give the general solution to the homogeneous equation associated with (3.51) to be

$$\delta r_h(s) = c_3 \frac{dr_o}{ds} \int_o^s \left[1 - \frac{1}{\left(\frac{dr_o}{ds}\right)^2} \right] ds + c_4 \frac{dr_o}{ds} \qquad (3.60)$$

with a Wronskian

$$\Delta = \begin{vmatrix} \delta r_{h1}(s) & \delta r_{h2}(s) \\ \frac{d\delta r_{h1}(s)}{ds} & \frac{d\delta r_{h2}(s)}{ds} \end{vmatrix} = r_o^{\,2} \left(\frac{d\theta_o}{ds} \right)^2 . \qquad (3.61)$$

Except in the degenerate case of a basic ray directly approaching the solar center Δ is never zero, thus verifying that the two homogeneous solutions are independent.

The particular solution to equation (3.51) may be found from δr_{h1} and δr_{h2} by the method of variation of parameters, and is

$$\delta r_p(s) = \frac{1}{C} \frac{dr_o}{ds} \int_o^s \left[\int_o^{s'} \left(\frac{\partial \delta \mu}{\partial \theta} \right)'' ds'' \right] \left(1 - \frac{1}{\left(\frac{dr_o}{ds} \right)'^2} \right) ds'$$

$$- \frac{dr_o}{ds} \int_o^s \left(\frac{\delta \mu}{\mu_o} \right)_o^{s'} \left(1 - \frac{1}{\left(\frac{dr_o}{ds} \right)'^2} \right) ds' \qquad (3.62)$$

where the primes and double primes merely indicate dummy variables of integration. It has been verified by direct substitution into equation (3.51); the details will not be presented here.

Equations (3.52), (3.60), and (3.62) give the general solution of equation (3.51) to be

$$\frac{\delta r(s)}{\left(\frac{dr_o}{ds}\right)} = c_4 + c_3 \int_o^s \left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds' \tag{3.63}$$

$$+ \int_o^s \left[\frac{1}{C} \int_o^{s'} \left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'} \right] \left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds' .$$

Having thus found $\delta r(s)$ we now proceed to obtain $\delta\theta(s)$. The system constant of motion (equation (3.50)) is again useful in this regard, for if we use equation (3.63) above to eliminate $\delta r(s)$ from (3.50) we obtain the following elementary equation for $\delta\theta(s)$:

$$\frac{d\delta\theta}{ds} = \frac{d\theta_o}{ds} \left\{ c_3 + \frac{1}{C} \int_o^s \left(\frac{\partial\delta\mu}{\partial\theta}\right)' ds' - \left(\frac{\delta\mu}{\mu_o}\right)_o^s \right\}$$

$$+ \frac{d^2\theta_o}{ds^2} \left\{ c_4 + c_3 \int_o^s \left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds' \right. \tag{3.64}$$

$$+ \frac{1}{C} \int_o^s \left[\int_o^{s'} \left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' \right] \left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds'$$

$$\left. - \int_o^s \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'} \left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds' \right\}$$

here $d^2\theta_o/ds^2$ has been introduced in virtue of equation (2.33). This equation may be integrated directly to give, after much simplification,

$$\delta\theta(s) = c_5 + c_4\left(\frac{d\theta_o}{ds}\right)_o^s + c_3\left(\theta_o\right)_o^s + c_3\int_o^s\left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right)\left(\frac{d\theta_o}{ds}\right)_{s'}^s ds'$$

$$+ \frac{d\theta_o}{ds}\left\{\int_o^s\left[\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'}\right]\left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds'\right\}$$

$$+ \int_o^s\left[\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'}\right]\frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\left(\frac{d\theta_o}{ds}\right)' ds' . \qquad (3.65)$$

This has been checked by direct insertion into equation (3.50), but the calculation is too tedious to present here.

The constants of integration, c_3, c_4 and c_5, may be evaluated by imposing on equations (3.63) and (3.65) the first three boundary conditions of (3.49) (the fourth is redundant in virtue of equation (3.50)); they are

$$c_3 = c_4 = c_5 = 0 \qquad (3.66)$$

Thus we obtain finally

$$\delta r(s) = \frac{dr_o}{ds}\int_o^s\left[\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'}\right]\left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds' \qquad (3.67)$$

$$\delta\theta(s) = \frac{d\theta_o}{ds}\int_o^s\left[\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'}\right]\left(1 - \frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\right) ds'$$

$$+ \int_o^s\left[\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'}\right]\frac{1}{\left(\frac{dr_o}{ds}\right)'^2}\left(\frac{d\theta_o}{ds}\right)' ds' \qquad (3.68)$$

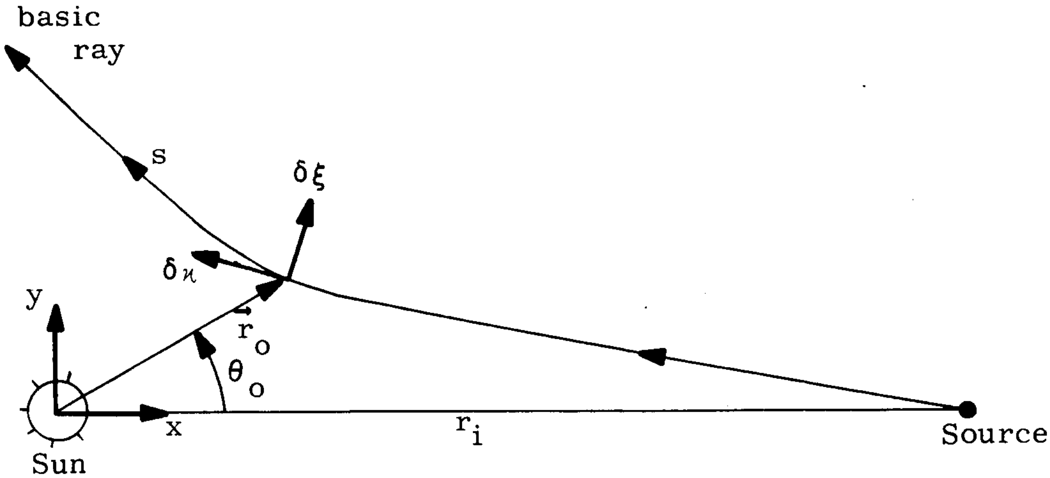
To summarize, equations (3.67) and (3.68) are general solutions, in quadrature form, for the scattering parameters $\delta r(s)$ and $\delta\theta(s)$ for a ray traversing a spherically symmetric average corona about which occur (small) statistical fluctuations. Their validity is subject only to the following assumptions:

i) applicability of a ray analysis
ii) spherical symmetry: $\mu_o = \mu_o(r)$
iii) applicability of a first order treatment
iv) the initial conditions of equations (3.49) .

The solutions obtained for $\delta r(s)$, $\delta\theta(s)$ and $\delta\varphi(s)$ constitute a complete description of the behavior of a ray as the medium departs from spherical symmetry. They comprise therefore the very basis of this work.

The Ray Coordinate System

Our attention has thus far been confined to the spherical coordinate system (r, θ, φ) introduced in Chapter II. For the purposes of scattering calculations, however, a much more usable and natural coordinate system is that formed by the basic rays themselves, in which the displacement of a perturbed ray from its corresponding initial ray may be specified by two components normal to that ray and by one component parallel to it. One of the normal displacement components shall be taken in the plane of the basic ray, and will be called $\delta\xi(s)$; the other will be taken to be perpendicular to that plane, and is therefore simply $r_o\delta\varphi(s)$. The displacement component parallel to the basic ray shall be denoted $\delta\varkappa(s)$. Thus



We proceed as follows. The position vector of the basic ray is written as

$$\vec{r}_o = \vec{i}\, r_o \cos\theta_o + \vec{j}\, r_o \sin\theta_o \tag{3.69}$$

where $\vec{i}$ and $\vec{j}$ are unit vectors along the x and y axes, respectively. The unit tangent vector to the basic ray, denoted by $\vec{T}_o$, is given by

$$\begin{aligned}\vec{T}_o = \frac{d\vec{r}_o}{ds} &= \vec{i}\left(\cos\theta_o \frac{dr_o}{ds} - r_o \sin\theta_o \frac{d\theta_o}{ds}\right) \\ &+ \vec{j}\left(\sin\theta_o \frac{dr_o}{ds} + r_o \cos\theta_o \frac{d\theta_o}{ds}\right)\end{aligned} \tag{3.70}$$

and the unit normal vector, $\vec{N}_o$, to the basic ray is given by (see for example, Thomas, "Calculus and Analytic Geometry," Addison-Wesley (1960), §12.5 - §12.7)

$$\vec{N}_o K_o = \frac{d\vec{T}_o}{ds} = \vec{i}\left(\cos\theta_o \frac{d^2 r_o}{ds^2} - 2\sin\theta_o \frac{dr_o}{ds}\frac{d\theta_o}{ds}\right.$$

$$- r_o \cos\theta_o \left(\frac{d\theta_o}{ds}\right)^2 - r_o \sin\theta_o \left.\frac{d^2\theta_o}{ds^2}\right)$$

$$+ \vec{j}\left(\sin\theta_o \frac{d^2 r_o}{ds^2} + 2\cos\theta_o \frac{dr_o}{ds}\frac{d\theta_o}{ds}\right. \tag{3.71}$$

$$- r_o \sin\theta_o \left(\frac{d\theta_o}{ds}\right)^2 + r_o \cos\theta_o \left.\frac{d^2\theta_o}{ds^2}\right)$$

where K_o is the basic ray curvature, and is given by equation (2.36).

Now, let us denote by $\delta\vec{r}$ the vector ray displacement, which is clearly given by

$$\delta\vec{r} = \delta r\left(\vec{i}\cos\theta_o + \vec{j}\sin\theta_o\right)$$

$$+ r_o\,\delta\theta\left(-\vec{i}\sin\theta_o + \vec{j}\cos\theta_o\right) \tag{3.72}$$

$$+ r_o\,\delta\varphi\,\vec{k}$$

where $\vec{k}$ is a unit vector along the z axis. The component of $\delta\vec{r}$ along $\vec{N}_o$ is $\delta\xi(s)$; thus

$$K_o \delta\xi(s) = \delta\vec{r}\cdot(\vec{N}_o K_o)\,. \tag{3.73}$$

If equations (3.71), (3.72) and (2.36) are inserted into equation (3.73) we obtain

$$\delta\xi(s) = \left(r_o \frac{d\theta_o}{ds}\right)\delta r(s) - \left(r_o \frac{dr_o}{ds}\right)\delta\theta(s)\,. \tag{3.74}$$

Similarly, the component of $\delta\vec{r}$ along $\vec{T}_o$ is $\delta\varkappa(s)$; thus

$$\delta\varkappa(s) = \delta\vec{r} \cdot \vec{T}_o \ . \qquad (3.75)$$

If equations (3.70) and (3.72) are inserted into equation (3.75) we obtain

$$\delta\varkappa(s) = \left(\frac{dr_o}{ds}\right) \delta r(s) + \left(r_o^{\ 2} \frac{d\theta_o}{ds}\right) \delta\theta(s) \ . \qquad (3.76)$$

Equations (3.74) and (3.76) thus represent the desired set of coordinate transformations from the spherical coordinate system (r, θ, φ) to the natural ray coordinate system $(\xi, \varkappa, \varphi)$.

Of interest to us also is the determination of the spatial derivatives of $\delta\mu$ in the $(\xi, \varkappa, \varphi)$ representation; only the normal derivative, $\frac{\partial\delta\mu}{\partial\xi}$, will be needed here. If we let $\vec{u}_\xi$ be a unit vector in the ξ direction we have

$$\frac{\partial(\delta\mu/\mu_o)}{\partial\xi} = \left(\vec{u}_\xi \cdot \nabla\right)\left(\frac{\delta\mu}{\mu_o}\right). \qquad (3.77)$$

But it is clear from equation (3.74) that

$$\vec{u}_\xi = r_o \frac{d\theta_o}{ds} \vec{u}_r - \frac{dr_o}{ds} \vec{u}_\theta \qquad (3.78)$$

where $\vec{u}_r$ and $\vec{u}_\theta$ are unit vectors in the r and θ directions, respectively. The gradient operation, ∇, may be written in spherical coordinates as

$$\nabla = \vec{u}_r \frac{\partial}{\partial r} + \frac{\vec{u}_\theta}{r_o \sin\varphi_o} \frac{\partial}{\partial\theta} + \frac{\vec{u}_\varphi}{r_o} \frac{\partial}{\partial\varphi} \qquad (3.79)$$

where we have implicitly noted that all derivatives are to be evaluated along the basic ray. When equations (3.78) and (3.79) are inserted into (3.77) we obtain

$$\frac{\partial}{\partial \xi}\left(\frac{\delta\mu}{\mu_o}\right) = r_o \frac{d\theta_o}{ds}\frac{\partial}{\partial r}\left(\frac{\delta\mu}{\mu_o}\right) - \frac{1}{r_o}\frac{dr_o}{ds}\frac{\partial}{\partial\theta}\left(\frac{\delta\mu}{\mu_o}\right). \qquad (3.80)$$

The Equations for $\delta\xi(s)$ and $\delta\varkappa(s)$

We begin with an important simplification. The term in the square brackets in equations (3.67) and (3.68) for $\delta r(s)$ and $\delta\theta(s)$ may be written

$$\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'}$$

$$= \int_o^{s'} \frac{1}{C}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \int_o^{s'}\frac{d}{ds''}\left(\frac{\delta\mu}{\mu_o}\right)'' ds'' \qquad (3.81)$$

$$= \int_o^{s'}\left[\left(\frac{\partial\delta\mu}{\partial\theta}\right)''\left(\frac{1}{C} - \frac{1}{\mu_o(s'')}\frac{d\theta_o(s'')}{ds''}\right) - \frac{dr_o(s'')}{ds''}\frac{\partial}{\partial r}\left(\frac{\delta\mu}{\mu_o}\right)''\right] ds''$$

$$= \int_o^{s'}\left(\frac{dr_o}{ds}\right)'' \frac{1}{\left(r_o\frac{d\theta_o}{ds}\right)''}\left[\left(\frac{1}{r_o}\frac{dr_o}{ds}\right)''\frac{\partial\left(\frac{\delta\mu}{\mu_o}\right)''}{\partial\theta} - \left(r_o\frac{d\theta_o}{ds}\right)''\frac{\partial}{\partial r}\left(\frac{\delta\mu}{\mu_o}\right)''\right] ds''$$

where equations (2.16) and (2.17) have been utilized. But by equation (3.80) the term in square brackets in the final integral of (3.81) is simply $-\frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)$, and

$$\frac{1}{C}\int_o^{s'}\left(\frac{\partial\delta\mu}{\partial\theta}\right)'' ds'' - \left(\frac{\delta\mu}{\mu_o}\right)_o^{s'} = -\frac{1}{C}\int_o^{s'}\left(\mu_o r_o\frac{dr_o}{ds}\right)''\frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)'' ds''. \qquad (3.82)$$

If, now, equations (3.67), (3.68), and (3.82) are inserted into equation (3.74) for $\delta\xi(s)$ we get

$$\delta\xi(s) = \frac{d(r_o^2)}{ds}\int_0^s\left[\int_0^{s'}\left(\mu_o\,\frac{d(r_o^2)}{ds}\right)''\frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)''ds''\right]\frac{ds'}{\left\{\mu_o\left[\frac{d(r_o^2)}{ds}\right]^2\right\}}, \tag{3.83}$$

Similarly, inserting equations (3.67), (3.68) and (3.82) into equation (3.76) for $\delta\varkappa(s)$ we obtain

$$\delta\varkappa(s) = 2C\int_0^s\left[\int_0^{s'}\left(\mu_o\,\frac{d(r_o^2)}{ds}\right)''\frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)''ds''\right]\left(\frac{1}{\mu_o(s')}-\frac{1}{\mu_o(s)}\right)\frac{ds'}{\left\{\mu_o\left[\frac{d(r_o^2)}{ds}\right]^2\right\}}, \tag{3.84}$$

Thus, to recapitulate, equations (3.27), (3.83), and (3.84) for $\delta\varphi(s)$, $\delta\xi(s)$, and $\delta\varkappa(s)$ represent the most natural set of parameters for our discussion of the scattering of radio rays by a turbulent medium.

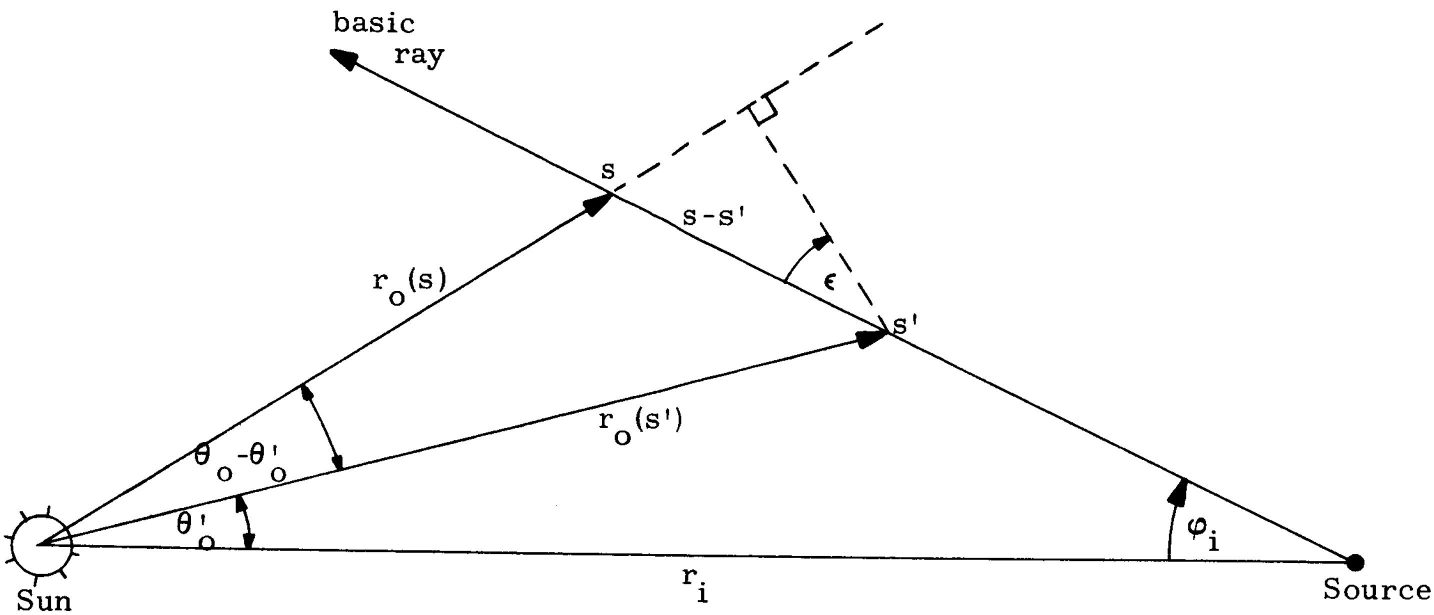
The Case μ_o = Constant

Before proceeding to a discussion of scattering phenomena we will find it instructive to examine the behavior of the equations for $\delta\varphi(s)$, $\delta\xi(s)$, and $\delta\varkappa(s)$ for the case of a non-refractive average corona, i.e., for $\mu_o(r)$ = constant.

We begin with the equation for $\delta\varphi(s)$, (3.27):

$$\delta\varphi(s) = \frac{1}{C}\int_0^s\left(\frac{\partial\delta\mu}{\partial\varphi}\right)'\sin\left(\theta_o-\theta_o'\right)ds' \,. \tag{3.85}$$

In the case μ_o = constant the rays will be linear, and we have the following geometry:



It is clear from the figure that

$$\sin\left(\theta_o - \theta_o'\right) = \frac{(s-s')\cos\epsilon}{r_o'} \tag{3.86}$$

where it may be easily shown that

$$\epsilon \text{ (radians)} = \frac{\pi}{2} - \left(\theta_o + \varphi_i\right). \tag{3.87}$$

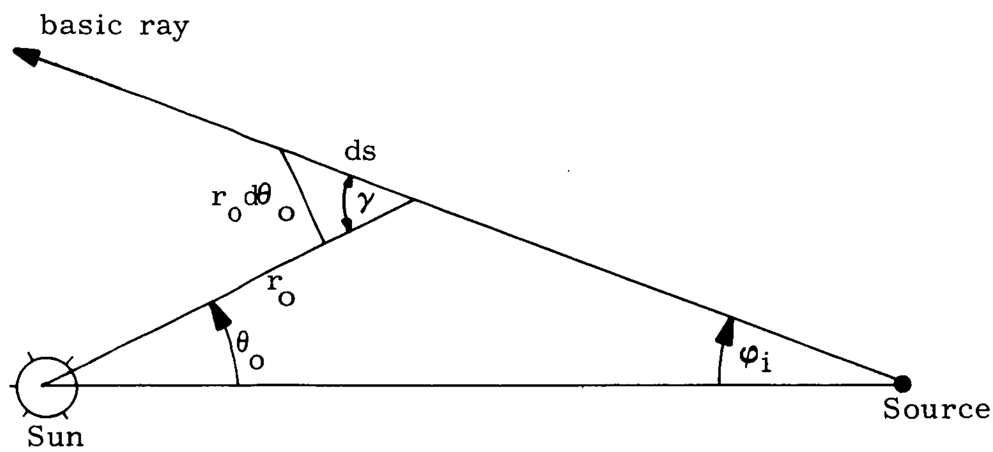
Inserting equation (3.87) into (3.86) gives

$$\sin\left(\theta_o - \theta_o'\right) = \frac{s-s'}{r_o'}\sin\left(\theta_o + \varphi_i\right) \tag{3.88}$$

which, when inserted into equation (3.85), gives

$$r_o\delta\varphi(s) = \int_0^s \frac{1}{r_o(s')}\frac{\partial}{\partial\varphi}\left(\frac{\delta\mu}{\mu_o}\right)' \frac{(s-s')}{r_o\frac{d\theta_o}{ds}}\sin\left(\theta_o + \varphi_i\right) ds'. \tag{3.89}$$

But it is clear from the following figure:



that

$$\frac{r_o d\theta_o}{ds} = \sin(\gamma) = \sin\left(\theta_o + \varphi_i\right) . \tag{3.90}$$

Thus when equation (3.90) is inserted into (3.89) we obtain for the displacement of a scattered ray normal to the plane of its corresponding basic ray

$$\delta z(s) = r_o \, \delta\varphi(s) = \int_o^s \frac{\partial}{\partial z} \left(\frac{\delta\mu}{\mu_o}\right)' (s-s') \, ds' \tag{3.91}$$

where we have let

$$\frac{\partial}{\partial z} = \frac{1}{r_o} \frac{\partial}{\partial\varphi} . \tag{3.92}$$

This may be put in the form

$$\delta z(s) = \int_o^s \left[\int_{s'}^s ds'' \right] \frac{\partial}{\partial z} \left(\frac{\delta\mu}{\mu_o}\right)' ds' \tag{3.93}$$

which upon a simple interchange of the order of integration becomes

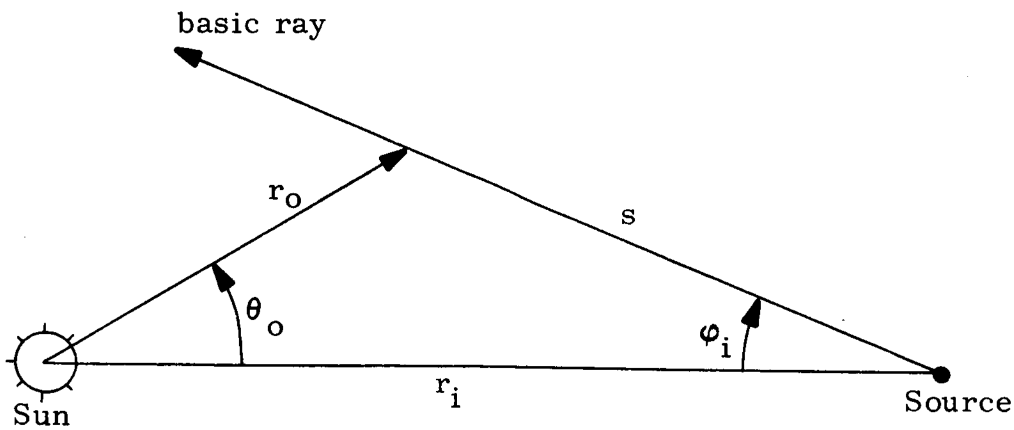
$$\boxed{\delta z(s) = \int_0^{s} \left[\int_0^{s'} \frac{\partial}{\partial z}\left(\frac{\delta\mu}{\mu_0}\right)'' ds'' \right] ds'} \qquad \mu_0 = \text{const.} \tag{3.94}$$

The form of equation (3.94) is exactly that obtained by Chandrasekhar (1952), as it ought to be, for in the present limit of μ_0 = constant our model is identical to his.

We discuss next the form taken by the equation for $\delta\xi(s)$, (3.83), in the limit of μ_0 = constant. In that case equation (3.83) is

$$\delta\xi(s) = \frac{d(r_0^{\,2})}{ds}\int_0^{s}\left[\int_0^{s'}\left(\frac{d(r_0^{\,2})}{ds}\right)'' \frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_0}\right)'' ds''\right]\frac{ds'}{\left[\frac{d(r_0^{\,2})}{ds}\right]'^{\,2}} \,. \tag{3.95}$$

Now consider the following geometry:



By the law of cosines we write

$$r_0^{\,2} = r_i^{\,2} + s^2 - 2sr_i \cos\varphi_i$$

$$\frac{d(r_0^{\,2})}{ds} = 2\,(s - r_i \cos\varphi_i)\,. \tag{3.96}$$

Inserting equation (3.96) into (3.95) yields

$$\delta\xi(s) = (s-s_o)\int_o^s \left[\int_o^{s'} (s''-s_o)\, \frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)'' ds'' \right] \frac{1}{(s'-s_o)^2}\, ds' \qquad (3.97)$$

where we have for convenience let

$$s_o = r_i \cos\varphi_i \;.$$

Interchanging the order of integration and performing the integral over ds' gives, after a simple change of notation

$$\delta\xi(s) = \int_o^s \frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)' (s-s')\, ds' \qquad (3.98)$$

which may (c.f. equations (3.91-3.94)) be put in the form

$$\boxed{\delta\xi(s) = \int_o^s \left[\int_o^{s'} \frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)'' ds'' \right] ds'} \qquad \mu_o = \text{const.} \qquad (3.99)$$

This, too, agrees with the results of Chandrasekhar (1952)), as it indeed should.

Finally, it is clear from equation (3.84) that in the limit of μ_o = constant we have $\delta\varkappa(s) = 0$, a convenient result.

Recapitulating, equations (3.94) and (3.99) for $r_o\delta\varphi(s)$ and $\delta\xi(s)$ are valid if μ_o is sufficiently constant that the basic rays, about which we are perturbing, are closely approximated by straight lines. They are identical to the results of Chandrasekhar (1952).

<u>The Equations for $\delta\Phi(s)$, $\delta t_{ph}(s)$, $\delta t_{gr}(s)$ and $\delta f(s)$</u>

To further our preparation for the subsequent discussion of scattering by a non-homogeneous, non-isotropically turbulent solar corona, we introduce now an additional scattering parameter, the variation in phase, $\delta\Phi(s)$, of a wave as it traverses the scattering medium. This variation may be conveniently written in terms of the source-observer optical length difference between the perturbed ray and its corresponding basic ray:

$$-\frac{\delta\Phi(s)}{\left(\frac{2\pi}{\lambda}\right)} = \int_{\text{source}}^{\text{observer}} \mu \, ds - \int_{(\text{source})_o}^{(\text{observer})_o} \mu_o \, ds_o \quad . \tag{3.100}$$

The first integral is to be taken along a perturbed ray and the second is to be taken along the corresponding unperturbed ray. Let us consider the first integral, along the perturbed ray. Since the perturbed ray at all points lies close to its corresponding average ray we seek to relate the refractive index at a point on the perturbed ray to the refractive index at the corresponding point on the basic ray. Since the corresponding points are points of equal (s) we may set $s = s_o$ and write, to first order,

$$\mu(s=s_o) = \mu_o(s_o) + \left(\frac{d\mu_o}{dr}\right)_{s_o} \delta r(s_o) + \delta\mu(s_o) \, .$$

Equation (3.100) becomes then

$$-\frac{\delta\Phi(s)}{\left(\frac{2\pi}{\lambda}\right)} = \int_{\text{source}}^{\text{observer}} \left(\mu_o + \delta\mu + \frac{d\mu_o}{dr} \delta r\right) ds_o - \int_{(\text{source})_o}^{(\text{observer})_o} \mu_o ds_o \tag{3.101}$$

where both integrands are to be evaluated along the basic ray. Now only the limits of these integrals remain to be considered. If the arc length between source and observer were the same for both the perturbed ray and the unperturbed ray the limits would be the same and we could eliminate $\int \mu_o ds_o$ in the equation above. However, the arc length between source and observer is not necessarily the same for the perturbed and unperturbed rays, since upon perturbation the ray may be displaced parallel to itself, i.e. $\delta\varkappa = 0$. Since by parallel displacement we mean that corresponding points, i.e. points of equal (s), have a displacement component parallel to the basic ray it is easy to see that the arc length between source and observer will not be the same in the perturbed and unperturbed cases, but is in fact different by $\delta\varkappa(s)$. Then equation (3.101) takes the form, to first order,

$$-\frac{\delta\Phi(s)}{\left(\frac{2\pi}{\lambda}\right)} = \int_o^{s-\delta\varkappa(s)} \left(\mu_o + \delta\mu + \frac{d\mu_o}{dr}\,\delta r\right) ds_o - \int_o^{s} \mu_o ds_o$$

where s is the source-observer distance measured along the basic ray. This may be put in the form

$$-\frac{\delta\Phi(s)}{\left(\frac{2\pi}{\lambda}\right)} = \int_o^{s} \left(\cancel{\mu_o} + \delta\mu + \frac{d\mu_o}{dr}\,\delta r\right) ds_o - \int_o^{s} \cancel{\mu_o} ds_o$$

$$- \int_{s-\delta\varkappa(s)}^{s} \left(\mu_o + \delta\mu + \frac{d\mu_o}{dr}\,\delta r\right) ds_o \quad .$$

Remembering that $\delta\varkappa(s)$ is a small quantity we obtain finally for the net variation in phase

$$\delta\Phi(s) = \frac{2\pi}{\lambda}\left\{\mu_o(s)\,\delta\varkappa(s) - \int_o^s\left[\delta\mu(s') + \left(\frac{d\mu_o}{dr}\right)'\delta r(s')\right]ds'\right\} \tag{3.102}$$

where the integration is to be carried out along the basic ray. Thus $\delta\Phi(s)$ is seen to have as its origin two causes, one appearing as an integration along the basic ray of the variations in optical length, and the other appearing simply as a shift at the observer of the ray parallel to itself. We may have intuited these terms at the start, but the derivation presented here is more convincing.

In the special case that μ_o = constant equation 3.102 becomes

$$\delta\Phi(s) = -\frac{2\pi}{\lambda}\int_o^s \delta\mu(s')\,ds' \qquad \mu_o = \text{const.} \tag{3.103}$$

which is the expression used by Chandrasekhar (1952).

Now, with regard to observability the most convenient interpretation of a phase fluctuation is as a fluctuation in the time necessary for a "phase front" to traverse, from source to observer, the ray path. If we denote this fluctuation in arrival time as $\delta t_{ph}(s)$, we may write

$$\delta t_{ph}(s) = -\frac{\lambda}{2\pi c}\,\delta\Phi(s) \tag{3.104}$$

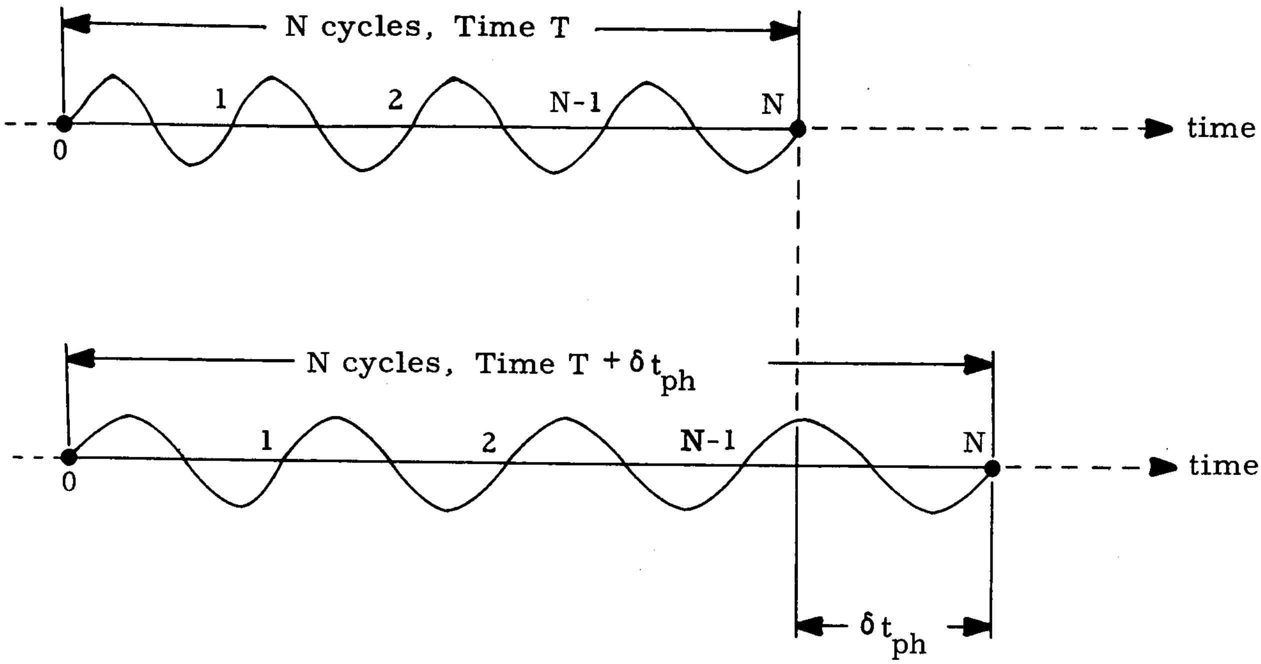
where the minus sign arises since a phase advance (i.e. $\delta\Phi(s) \geq 0$)

implies that the phase front has arrived at the observer earlier than it would have in the absence of coronal inhomogeneities, and the fluctuation in the time of flight is therefore negative. Usually δt_{ph} is not observed directly; this would seem to imply "tagging" individual phase fronts on a CW signal. However, δt_{ph} might be expected to be observed indirectly as a spectral broadening of a CW signal, since fluctuations in time of arrival of successive phase fronts result in a frequency modulation of the signal, and consequently a spectral broadening. We may analyze this effect as follows. Consider a segment of the reference (unperturbed) CW signal containing N complete cycles extending over a time interval T, as illustrated in the first of the two figures below. The frequency of this reference signal is therefore

$$f_1 = \frac{N}{T} .$$

Now suppose a perturbation occurs such that the time of arrival of phase front 'N' is delayed by δt_{ph} with respect to the time of arrival of phase front 'O', as illustrated in the second figure below. We may then write the frequency of this perturbed signal as

$$f_2 = \frac{N}{T + \delta t_{ph}} .$$



Now, for solar coronal disturbances we expect the variation of δt_{ph} with time to be slow, typically on the order of seconds (Slee (1959), Hewish and Dennison (1966)). This thus represents the order of magnitude of T. However, as will be shown in the numerical discussions of Chapter IV , we expect δt_{ph} to be typically less than 10^{-4} seconds. We may therefore assume

$$\frac{\delta t_{ph}}{T} << 1 \tag{3.105}$$

to obtain

$$\frac{f_2 - f_1}{f_1} = \frac{\delta t_{ph}}{T} \tag{3.106}$$

This expression may be generalized somewhat to read

$$\frac{\delta f}{f} = - \frac{d\delta t_{ph}}{dt} \tag{3.107}$$

where here δf signifies the fluctuation in CW frequency, and f the unperturbed CW frequency. Unfortunately it is difficult to pursue (3.107) further, since both the amplitude and the fluctuation rate of δt_{ph} are random functions of time. We shall, however, discuss the significance of δf in subsequent chapters. For completeness, we write, with the aid of equation (3.104),

$$\boxed{\frac{\delta f}{f} = \frac{\lambda}{2\pi c} \frac{d\delta\Phi}{dt}} \tag{3.108}$$

Having discussed the variations in the time necessary for a phase front to traverse a ray path, we now conclude the present chapter with a consideration of the fluctuations in the propagation time along a ray path of a signal pulse.

We call this fluctuation δt_{gr}. As was the case with $\delta\Phi(s)$ (see equation (3.100) and corresponding discussion) there are two distinct contributions to δt_{gr}. The first of these is readily found from equation (2.46), which yields

$$\left(\delta t_{gr}\right)_1 = \frac{1}{c}\, \delta \int_0^s \frac{ds'}{\mu(s')} = -\frac{1}{c} \int_0^s \frac{\delta\mu(s') + \left(\frac{d\mu_o}{dr}\right)' \delta r(s')}{\mu_o(s')^2}\, ds'$$

where in the integrand the value of $\delta\mu(s')$ is to be evaluated along the appropriate basic ray (see equation (3.9) and appropriate discussion). The sign above is negative since a positive variation of the refractive index implies an increase in the group velocity, and thus a decrease in the flight time of a pulse traversing a ray. The second factor contributing to δt_{gr} is the displacement, $\delta\varkappa(s)$, of a ray parallel to itself. The effect of a positive $\delta\varkappa(s)$ is, as may be readily seen from the illustration of page 64 , to lessen the ray transit time of a signal pulse. Thus

$$\left(\delta t_{gr}\right)_2 = -\frac{\delta\varkappa(s)}{\mu_o c} .$$

Combining $(\delta t_{gr})_1$ and $(\delta t_{gr})_2$ we obtain immediately

$$\boxed{\delta t_{gr} = -\frac{1}{c}\left\{\frac{\delta\varkappa(s)}{\mu_o(s)} + \int_o^s \frac{\left[\delta\mu(s') + \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\right]}{\mu_o(s')^2} ds'\right\}} \tag{3.109}$$

In the special case that $\mu_o \simeq$ constant equation (3.109) becomes

$$\boxed{\delta t_{gr} = -\frac{1}{c}\int_o^s \frac{\delta\mu(s')}{\mu_o(s')^2}\, ds'} \qquad \mu_o \simeq \text{const.} \tag{3.110}$$

It should be pointed out that equations (3.109) and (3.110) are expected to be of great use, since δt_{gr} may be directly observed, for example by noting the fluctuations in the relative times of arrival of a series of equally spaced (in time) pulses originating at some artificial source.

Summary

We have in the present chapter been concerned with the establishment of a basic formalism with which we may proceed in a general fashion to discuss scattering phenomena in a non-homogeneous, non-isotropically turbulent solar corona. To that end, we have successfully derived general expressions, in quadrature form, for the scattering parameters $\delta\varphi(s)$, $\delta\xi(s)$, $\delta\varkappa(s)$, $\delta\Phi(s)$, δt_{ph}, δf, and δt_{gr}, of interest for the assumed ray description. Their validity is limited only by the applicability of the first-order perturbation analysis used and by the assumption of spherical symmetry in the average corona. We have, however, said nothing thus far about the nature of the perturbing refractive index other than that it is small, which assumption forms the basis of the validity of the first order treatment.

The scattering parameters here derived have been shown to be identical with those found by Chandrasekhar (1952) in the limit of a homogeneous corona, μ_o = constant. They are considerably more general, though, in their validity when the assumption of coronal homogeneity is relaxed, i.e. when μ_o is allowed to be any function of r. Thus we have obtained a formalism which allows precise study of the effects of overall coronal refraction on scattering phenomena. The usefulness of this in the study of the scattering of signals from sources "external" to the corona is obvious, but it should be pointed out that the present formalism is also applicable to the study of the scattering of signals originating in the solar atmosphere itself, a problem which has never been adequately treated.

CHAPTER IV

Scattering by a Nearly Homogeneous Medium

Introduction

It is the purpose of this chapter to analytically discuss the statistical behavior of scattered rays in a medium for which the average component, μ_o, of the refractive index is sufficiently approximated by a constant that the basic rays, about which the scattering occurs, may be considered nearly linear. In this case important simplifications occur and under certain assumptions the equations determining the scattering parameters may be inverted, providing a determination of the properties of the scattering component of the medium from appropriate sets of observations of the scattering.

The Statistical Properties of the Medium

As presented in Chapter III, the refractive medium is considered to consist of two components: a local average component, $\mu_o(r)$, about which occur turbulent fluctuations, and the fluctuating component of the refractive index itself, $\delta\mu(\vec{r})$. The local average component is an exactly specifiable function of r, but the fluctuating component, on the other hand, is not exactly specifiable, but is known in only a statistical sense. In this paragraph we seek to discuss the statistical properties of the fluctuating component.

The basic significant statistical quantity by which we shall describe the fluctuating component of the refractive index is its spatial autocorrelation function, which will be written as

$$F = < \delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2) > \tag{4.1}$$

where, as in Chapter III, the angular brackets represent an ensemble average or, if the turbulent processes in question are ergodic, a time average. It is clear that we may write in general

$$F = F(\vec{r}_1; \vec{r}_2 - \vec{r}_1) \tag{4.2}$$

The first functional dependence of F, namely $(\vec{r}_1)$, specifies the local region of space "around which" the autocorrelation function is being calculated, while the second functional dependence, namely $(\vec{r}_2 - \vec{r}_1)$, represents the usual vector offset in the calculation of the autocorrelation function.

Now, equation (4.2) is completely general, no specification about the medium having been made, but as such it is of no use to us as its generality precludes useful analytic treatment. It will be useful therefore to extend the assumption of spherical symmetry from the average corona to the statistical properties of the turbulent component as well. In this case equation (4.2) takes the form

$$F = F(r_1; \vec{r}_2 - \vec{r}_1) \tag{4.3}$$

where $r_1 = |\vec{r}_1|$. This says that the autocorrelation function about any point on a sphere of radius r_1 is, apart from geometrical factors associated with the orientation of $\vec{r}_2 - \vec{r}_1$, the same as that about any other point on the same sphere. The geometrical dependence on $(\vec{r}_2 - \vec{r}_1)$ is still free, however.

Specification of the geometrical dependence of the autocorrelation function, associated with the functional dependence on $\vec{r}_2 - \vec{r}_1$, leads us to distinguish several interesting special cases. The first, and by far the simplest, is that of isotropic turbulence in which it is assumed that no mechanism exists which can maintain a preference for any particular direction in space. Clearly, then the dependence of the autocorrelation function on geometrical factors associated with the orientation of $\vec{r}_2 - \vec{r}_1$ must be null, and equation (4.3) then becomes

$$F = F(r_1; |\vec{r}_2 - \vec{r}_1|) . \tag{4.4}$$

A subtle point arises in this connection, however, which must be pointed out. In writing the autocorrelation function, equation (4.4), as a function of both r_1 and the magnitude of $\vec{r}_2 - \vec{r}_1$ we essentially say that once the point at $\vec{r}_1$, about which the autocorrelation function is to be calculated, is chosen one sees the same behavior, in a statistical sense, in all directions from that point. This is peculiar when we recall that we are allowing a radial gradient of the parameters governing the statistics to exist, and it indeed represents a contradiction, as will be now demonstrated.

It is clear that in calculating the correlation of equation (4.1) no preference is given to either point $\vec{r}_1$ or to point $\vec{r}_2$. The form of equation (4.2) does imply a preference, however, for point $\vec{r}_1$. It may be removed by requiring a compatibility condition:

$$F(\vec{r}_1; \vec{r}_2 - \vec{r}_1) = F(\vec{r}_2; \vec{r}_1 - \vec{r}_2) \tag{4.5}$$

which says simply that it doesn't matter whether the correlations are calculated about point $\vec{r}_1$ or about point $\vec{r}_2$, as long as they are calculated between points $\vec{r}_1$ and $\vec{r}_2$. If we restrict ourselves to a radial gradient of the parameters governing the scattering, and to a radial displacement, the above requirement takes the form

$$F\left\{r_1; (r_2 - r_1)\,\hat{r}\right\} = F\left\{r_2; -(r_2 - r_1)\,\hat{r}\right\} \tag{4.6}$$

where $\hat{r}$ is the unit vector in the radial direction. If we now impose the condition of isotropy, equation (4.4), on requirement (4.6) the latter becomes

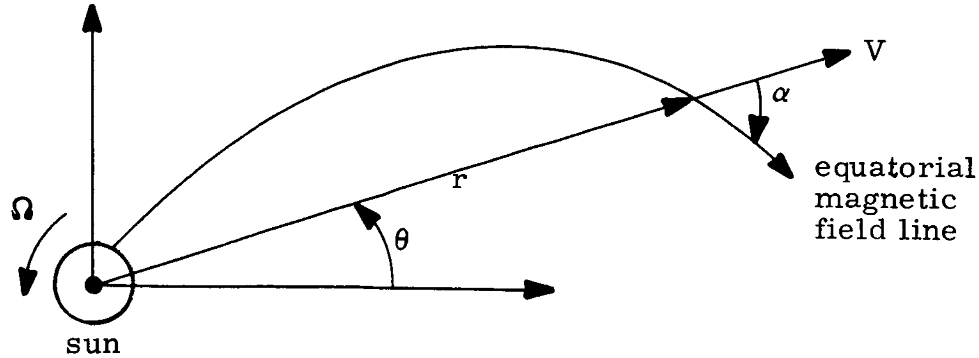
$$F\left\{r_1; |r_2 - r_1|\right\} = F\left\{r_2; |r_2 - r_1|\right\}. \tag{4.7}$$

Equation (4.7) is the form taken by the compatibility condition, equation (4.5) under the assumptions of isotropy and radial symmetry; it is clearly violated in virtue of the assumed radial gradient of the parameters governing the statistics, thus proving the aforementioned contradiction. Hence equation (4.4) is incorrect. However, it will be very nearly valid if the displacements over which appreciable correlations exist are small compared to the distances over which the statistical properties of the corona vary significantly. As the former lengths are on the order of 200 km (Hewish and Dennison (1966)) and the latter are on the order of $R_\odot$ we expect the contradiction discussed above to be of no importance, and equation (4.4) may be regarded as a valid form for a possible coronal correlation function.

A case more interesting than that of isotropy, however, is when the radial direction acquires some special significance. In the solar corona this can occur for several reasons. First, the fact of radial coronal out flow could in itself be sufficient to ascribe a preference to the radial direction and destroy isotropy. A considerably more likely mechanism, however, is the preference imparted to the direction of the coronal magnetic field in virtue of the relative enhancement of diffusion along a given field line with respect to that transverse to it (van de Hulst (1950), Hewish (1955, 1958), Högbom (1960), Hewish and Wyndham (1963), Erickson (1964)). As the magnetic field is frozen into the coronal plasma one expects the radial coronal out flow to produce a radial magnetic field. Solar rotation will cause the field lines to be curved, however. The greatest curvature occurs in the vicinity of the solar equatorial plane where the magnetic field takes the form of an Archimedian spiral with a pitch angle, α, given by

$$\tan \alpha = \frac{r\Omega}{V}$$

where Ω is the angular rate of the solar rotation, and V is the (constant) out flow velocity of the coronal plasma. These relationships are illustrated below.



The predicted results of Parker (1958, 1960a, 1960b), and recent experimental evidence, for example that of Ness and Wilcox (1965) indicate a value of approximately 50° for α near the orbit of the earth. This figure, coupled with the above equation, implies that within approximately 30 $R_{\odot}$ from the sun the magnetic field is very nearly radial, and the radial direction may then be preferred. In this case the dependence of the autocorrelation function, equation (4.3), on $(\vec{r}_2 - \vec{r}_1)$ can be separated into a dependence on the radial component of $(\vec{r}_2 - \vec{r}_1)$, namely $(r_2 - r_1)$, and on the "transverse" component, namely $\left[r_1^{\,2} (\varphi_2 - \varphi_1)^2 + r_1^{\,2} \sin^2 \varphi_1 \, (\theta_2 - \theta_1)^2 \right]^{1/2}$ for small $(\theta_2 - \theta_1)$ and $(\varphi_2 - \varphi_1)$. The autocorrelation function of equation (4.3) then takes the form

$$F = F\left\{ r_1 ; \, | r_2 - r_1 | , \left[r_1^{\,2} (\varphi_2 - \varphi_1)^2 + r_1^{\,2} \sin^2 \varphi_1 (\theta_2 - \theta_1)^2 \right]^{1/2} \right\} \tag{4.8}$$

The functional dependence in equation (4.8) of the correlation on the magnitude of the radial component of $\vec{r}_2 - \vec{r}_1$ does not satisfy the compatibility condition, (4.5), for the reasons discussed in connection with the isotropic case. As in that case, however, we shall assume the distances over which appreciable correlations occur to be small when compared with the distances over which the statistical properties of the corona vary significantly, in which event equation (4.8) may be regarded as valid.

A third case of interest is that in which the correlation function exhibits a three-fold preference for the r, θ, and φ directions. Within 30 $R_{\odot}$ the origin of the radial preference has already been suggested, but we here make plausible a difference in the behaviors of F in the θ and φ directions. If, as we have been assuming, the plane $\varphi = \pi/2$ is the solar equatorial plane (or, nearly, the plane of the ecliptic) we would expect to observe, in the solar equatorial plane, a difference in the behaviors of F in the θ and φ directions if the coronal out flow were not radial, but confined somewhat to either the equatorial plane or to the solar polar regions (Hewish and Wyndham (1963)). If this is the case the dependence of the autocorrelation function on the displacement $(\vec{r}_2 - \vec{r}_1)$ might (in the solar equatorial plane) be separated into a dependence on the radial component of $(\vec{r}_2 - \vec{r}_1)$, namely $(r_2 - r_1)$, and on the two "transverse" components, namely $\left[r_1^{\,2} (\varphi_2 - \varphi_1)^2 \right]^{1/2}$ and $\left[r_1^{\,2} (\theta_2 - \theta_1)^2 \right]^{1/2}$ for small $(\theta_2 - \theta_1)$ and $(\varphi_2 - \varphi_1)$. For correlations not in the vicinity of the solar equatorial plane complicating geometrical factors will appear; since most artificial sources of current interest are expected to be confined to the vicinity of the solar equatorial plane we shall neglect these complicating factors and consider only the simpler case. Then the autocorrelation function of equation (4.3) takes the form

$$F = F\left\{ r_1 ; \left| r_2 - r_1 \right| , \left| r_1 (\theta_2 - \theta_1) \right| , \left| r_1 (\varphi_2 - \varphi_1) \right| \right\}. \qquad (4.9)$$

Once again, as was the case with the correlation functions of equations (4.4) and (4.8), equation (4.9) does not satisfy the compatibility condition (4.5). We shall, however, assume that the correlation distances are small compared to the distances over which the statistical properties of the corona vary appreciably, in which event equation (4.9) shall be regarded as valid.

The fourth, and final, case we shall consider is that in which we drop (for the sake of simplicity) the difference of the statistical dependences on the θ and φ directions, and ascribe rather a special preference only to the direction of the magnetic field, but for the case in which the non-radial nature of the field lines is taken into account. In that case it is clear that the correlation function will be of a form very similar to that of equation (4.8), but with the field line direction replacing the radial direction. If the field lines may be regarded as linear over a (small) correlation distance a simple transformation of coordinates gives a correlation function of the form

$$F = F\left\{r_1; \left|(\cos\alpha)(r_2 - r_1) - (\sin\alpha)\, r_1 (\theta_2 - \theta_1)\right|, \right.$$
$$\left[r_1^{\,2} (\varphi_2 - \varphi_1)^2 + \Big((\sin\alpha)(r_2 - r_1)\right.$$
$$\left.\left. + (\cos\alpha)\, r_1 (\theta_2 - \theta_1)\Big)^2\right]^{1/2}\right\} \qquad (4.10)$$

where for the sake of simplicity we confine our attention to the <u>solar equatorial plane</u>, where the magnetic field lines are planar; for satellite sources this will be the case of interest. Finally, it should be remarked that, as has been the case with equations (4.4), (4.8), and (4.9), equation (4.10) does not satisfy the compatibility condition, equation (4.5). For the reasons discussed above we shall neglect this inconsistency.

To briefly recapitulate, equations (4.4), (4.8), (4.9), and (4.10) represent the forms taken by the spatial correlation function, equation (4.1), of the refractive index fluctuations of the medium for the following conditions

i) isotropy (local)

ii) only the radial direction acquires a special significance

iii) in the solar equatorial plane the r, θ, and φ directions acquire a special significance

iv) only the direction of the (curved) magnetic field acquires a special significance,

respectively. The parameters governing the statistics have been assumed to be functions of the coordinate r only. In all cases the correlation distances have been assumed small compared to the distances over which the parameters governing the statistics vary appreciably. Case (iv) has incorporated the additional restriction that the correlation lengths be much less than the distance over which the angle α changes appreciably.

Now equations (4.4), (4.8), (4.9) and (4.10) are in themselves of no future use to us, having served primarily to introduce the reader in a general way to the kind of medium we are considering. To proceed we must postulate explicit functional forms for the correlation functions. We begin with the simplest case, corresponding to equation (4.4), that of isotropic turbulence. We choose a correlation function of the form (Chandrasekhar (1952))

$$<\delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2)> = <\delta^2\mu>_{r_1} \exp\left[-\,|\vec{r}_2 - \vec{r}_1|^2/\tau_o(r_1)^2\right] \qquad (4.11)$$

where τ_o is the correlation length and $<\delta^2\mu>$ is a correlation amplitude; both τ_o and $<\delta^2\mu>$ are functions of r_1. Equation (4.11) is clearly of the form suggested by equation (4.4), but it is certainly not the only expression of that form. Other correlation functions have been suggested (Obukhov (1949), Liebermann (1951), Chernov (1960)) but the function of equation (4.11) possesses certain virtues, among which are analytical convenience and proper behavior near $|\vec{r}_2 - \vec{r}_1| = 0$ (Chernov (1960)). The only restriction we shall impose on the validity of equation (4.11) is that the correlation length be small:

$$\tau_o(r_1) \ll \left[\frac{1}{<\delta^2\mu>} \frac{d<\delta^2\mu>}{dr_1} \right]^{-1} \tag{4.12}$$

$$\tau_o(r_1) \ll \left[\frac{1}{\tau_o} \frac{d\tau_o}{dr_1} \right]^{-1} \tag{4.13}$$

The second case, corresponding to equation (4.8), is that in which only the radial direction acquires special significance. The correlation function chosen for this case is (c.f. equation (4.11))

$$<\delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2)> =$$

$$<\delta^2\mu> \exp\left\{ -\frac{\left[r_2 - r_1\right]^2}{a^2} - \frac{\left[r_1^{\,2} \sin^2\varphi_1(\theta_2 - \theta_1)^2 + r_1^{\,2}(\varphi_2 - \varphi_1)^2\right]}{b^2} \right\} \tag{4.14}$$

where (a) is the correlation length in the radial direction, (b) the correlation length in the "transverse" direction and $<\delta^2\mu>$ the correlation amplitude; quantities a, b, and $<\delta^2\mu>$ are all functions of r_1. Again, the only restriction on the validity of equation (4.14) is that the correlation lengths be small:

$$a(r_1),\ b(r_1) \ll \left[\frac{1}{a}\frac{da}{dr_1}\right]^{-1},\ \left[\frac{1}{b}\frac{db}{dr_1}\right]^{-1}$$

$$a(r_1),\ b(r_1) \ll \left[\frac{1}{<\delta^2\mu>}\ \frac{d<\delta^2\mu>}{dr_1}\right]^{-1}. \qquad (4.15)$$

The third case, corresponding to equation (4. 9), is that in which, in the vicinity of the solar equatorial plane, the r, θ, and φ directions acquire special significance. The correlation function chosen for this case is (c.f. equation (4.14))

$$<\delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2)> =$$

$$<\delta^2\mu> \exp\left\{-\frac{\left(r_2-r_1\right)^2}{a^2} - \frac{r_1^{\,2}\,(\theta_2-\theta_1)^2}{b^2} - \frac{r_1^{\,2}\,(\varphi_2-\varphi_1)^2}{d^2}\right\} \qquad (4.16)$$

where (a) is the correlation length in the radial direction, (b) the correlation length in the θ-direction, (d) the correlation length in the φ-direction, and $<\delta^2\mu>$ the correlation amplitude; quantities a, b, d and $<\delta^2\mu>$ are all functions of r_1. The restriction on the validity of equation (4.16) is

$$a, b, d \ll \left[\frac{1}{a}\frac{da}{dr_1}\right]^{-1},\ \left[\frac{1}{b}\frac{db}{dr_1}\right]^{-1},\ \left[\frac{1}{d}\frac{d(d)}{dr_1}\right]^{-1}$$

$$a, b, d \ll \left[\frac{1}{<\delta^2\mu>}\ \frac{d<\delta^2\mu>}{dr_1}\right]^{-1}.$$

Finally, the fourth case, corresponding to equation (4.10), is that in which the direction of the magnetic field acquires special significance. Restricting ourselves, for the sake of simplicity, to correlations in the vicinity of the solar equatorial plane we take a correlation function:

$$\langle \delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2)\rangle =$$

$$\langle \delta^2\mu \rangle_{r_1} \exp\left\{ - \frac{\left[(\cos\alpha)(r_2 - r_1) - (\sin\alpha)\, r_1(\theta_2 - \theta_1)\right]^2}{a^2} - \frac{r_1^{\,2}\,(\varphi_2 - \varphi_1)^2 + \left[(\sin\alpha)(r_2 - r_1) + (\cos\alpha)\, r_1(\theta_2 - \theta_1)\right]^2}{b^2} \right\} \tag{4.17}$$

where (a) is the correlation length in the direction of the magnetic field, (b) the correlation length in the "transverse" direction, $\langle \delta^2\mu \rangle$ the correlation amplitude, and α the angle formed by the radius vector and the magnetic field direction; quantities a, b, $\langle \delta^2\mu \rangle$, and α are all functions of r_1. In addition to restrictions (4.15) we impose here the additional condition

$$a(r_1),\ b(r_1) \ll \left[\frac{1}{\alpha}\,\frac{d\alpha}{dr_1}\right]^{-1}. \tag{4.18}$$

To recapitulate, equations (4.11), (4.14), (4.16) and (4.17) represent postulated functional forms of the coronal correlation functions for the four cases under consideration, subject only to the assumptions of spherical symmetry and smallness of the correlation lengths. Indeed, the worst assumptions made are with respect to the functional forms per se, which may at best be regarded as eminently reasonable approximations.

Before we proceed to utilize equations (4.11), (4.14), (4.16) and (4.17) in the subsequent scattering analyses, it behooves us at this point to briefly discuss the connection between $<\delta^2\mu>$ and the coronal electron density fluctuations. Returning to equation (2.51) we had for the refractive index of the corona

$$\mu^2 = 1 - \frac{\omega_p^2}{\omega^2} = 1 - \frac{4\pi e^2 n}{m\,\omega^2} \tag{4.19}$$

Taking a variation of this we obtain

$$\delta(\mu^2) = 2\mu\delta\mu = -\frac{\omega_p^2}{\omega^2}\,\frac{\delta n}{n} \tag{4.20}$$

which to first order gives

$$\delta\mu = \frac{-\omega_{po}^2}{2\mu_o\omega^2}\,\frac{\delta n}{n_o}\,. \tag{4.21}$$

Now, from equations (4.11), (4.14), (4.16) and (4.17) it is clear that in the four cases considered

$$<\delta^2\mu> = <\delta\mu(\vec{r}_1)^2> \tag{4.22}$$

Combination of equations (4.21) and (4.22) yields

$$<\delta^2\mu> = \frac{\omega_{po}^4}{4\mu_o^2\omega^4 n_o^2} < \delta^2 n > \tag{4.23}$$

where all quantities are to be regarded as functions of r_1 and where

$$< \delta^2 n > = < \delta n(r_1)^2 > \tag{4.24}$$

Taking the square root of equation (4.23) gives

$$(\delta\mu)_{r.m.s.} = \frac{\omega_{po}^2}{2\mu_o \omega^2 n_o} (\delta n)_{r.m.s.} \tag{4.25}$$

which is the desired result.

It should be finally remarked, somewhat parenthetically, before closing that in this section our discussion of the statistical properties of the coronal turbulence has of necessity relied on some rather broad generalizations and physical intuition. It should be pointed out, however, that a detailed discussion of the coronal turbulence per se is an important area for future study, but unfortunately quite beyond the scope of the present work.

The Case of Isotropic Turbulence

The scattering of radio rays by an isotropically turbulent medium of uniform average refractive index may be completely described through use of the equations of Chapter III for the five scattering parameters $\delta z(s)$, $\delta \xi(s)$, $\delta \varkappa(s)$, $\delta \Phi(s)$, and δt_{gr}. In the limit of $\mu_o \simeq$ constant, the underlying assumption of this chapter, we had (3.94), (3.99), (3.103), and (3.110))

$$\delta z(s) = \int_0^s \left[\int_0^{s'} \frac{\partial}{\partial z} \left(\frac{\delta\mu}{\mu_0} \right)'' ds'' \right] ds' \tag{4.26}$$

$$\delta \xi(s) = \int_0^s \left[\int_0^{s'} \frac{\partial}{\partial \xi} \left(\frac{\delta\mu}{\mu_0} \right)'' ds'' \right] ds' \tag{4.27}$$

$$\delta \varkappa(s) = 0$$

$$\delta \Phi(s) = -\frac{2\pi}{\lambda} \int_0^s \delta\mu(s')\, ds' \tag{4.28}$$

$$\delta t_{gr} = -\frac{1}{c} \int_0^s \frac{\delta\mu(s')}{\mu_0(s')^2}\, ds' \tag{4.29}$$

We are to be reminded that these relations were obtained from their more general representations by formally using μ_0 = constant, which implied that the basic rays were linear. However, we are reminded also that the expressions above for $\delta z(s)$ and $\delta\xi(s)$ will hold even if $\mu_0 \neq$ constant, the only requirement for validity being that the basic rays be sufficienly linear that the geometrical relations of equations (3.86), (3.87), (3.90) and (3.96) closely hold. That is, the basic ray must be so slightly curved as to at all times depart only slightly from its linear approximation. As outlined in Chapter II, if ψ is the (small) ray turning angle, then our condition will be satisfied if, for a distant source

$$(s-s_o) \tan\psi \left[\frac{1}{X}\frac{dX}{dr}\right] << 1$$

and if, for source and observer equally distant from the sun,

$$(s-s_o) \tan\frac{\psi}{2} \left[\frac{1}{X}\frac{dX}{dr}\right] << 1$$

Satisfaction of these criteria may be examined for small ψ with the aid of equation (2.30).

On the other hand, the equations above for $\delta\varkappa(s)$, $\delta\Phi(s)$, and $\delta t_{gr}(s)$ seem formally to require that $\mu_o \equiv$ constant (see equations (3.84), (3.102), and (3.109)). Now expressions (4.28) and (4.29) above differ from their corresponding general expressions ($\mu_o(r)$ not constant) by terms in $\delta\varkappa(s)$ and $\delta r(s)$ (see equations (3.102) and (3.109). But as inspection of equations (3.67), (3.82), and (3.84) shows both $\delta\varkappa(s)$ and $\delta r(s)$ are integrals of appropriate functions multiplied by $\partial\delta\mu/\partial\xi$, whereas in our approximate expressions, (4.28) and (4.29), for $\delta\Phi(s)$ and $\delta t_{gr}(s)$ only integrals over the basic ray of $\delta\mu(s)$ appear. Since the scale size of the turbulence is expected to be small compared to other lengths in the problem, on the order of 200 kilometers as suggested by Hewish and Dennison (1966), the refractive index gradient, $\partial\delta\mu/\partial\xi$, will be large and we might thus expect the contributions to $\delta\Phi(s)$ and $\delta t_{gr}(s)$ from the terms we have dropped in writing (4.28) and (4.29) to be large in comparison with the terms we have kept, even if $\mu_o(r)$ is nearly constant and the basic ray nearly linear. It is shown in the Appendix, however, that equation (4.28) for $\delta\Phi(s)$ is valid even if $\mu_o \neq$ constant as long as the basic ray is nearly linear and $\omega_p^{\ 2}/\omega^2 << 1$. In that case, to first order in $\omega_p^{\ 2}/\omega^2$ the two terms (in $\delta\varkappa(s)$ and $\delta r(s)$) which we have dropped in writing equation (4.28) identically cancel. We are, however, not so lucky with expression (4.29) for $\delta t_{gr}(s)$, for there the terms which we have dropped do not cancel; the largeness of the correction terms leads us to consider use of equation (4.29) further.

Now as shall be seen shortly we will not be interested in $\delta t_{gr}(s)$ per se, but rather in the mean square of $\delta t_{gr}(s)$, which we shall write as $< \delta t_{gr} \delta t_{gr} >$. We here distinguish two related quantities, however: the mean square of δt_{gr} which is actually observed (to be for the present denoted $< \delta t_{gr} \delta t_{gr} >_o$) and the mean square of $\delta t_{gr}(s)$ expected on the basis of our approximate relationship (4.29) (to be for the present denoted $< \delta t_{gr} \delta t_{gr} >_R$). Anticipating somewhat our notation, it is shown in the Appendix (equation (A.62)) that under certain weak restrictions we may write

$$< \delta t_{gr} \delta t_{gr} >_R = < \delta t_{gr} \delta t_{gr} >_o$$

$$- \frac{R^4}{c^2} \left(\frac{d\mu_o}{dr} \right)^2_{r=R} F(n, m) < \delta\psi \delta\psi >_R$$

where

R = perpendicular distance from the solar center to the basic ray

$n(r) \sim 1/r^n$

$b(r)$ = correlation distance transverse to the radial direction $\sim 1/r^{-m}$

$F(n,m)$ = a function of n, m defined by equations (A.62) and (A.63).

$< \delta\psi\delta\psi >_R$ = an observed quantity to be defined later in the chapter.

Now it is seen that if $n(r)$ is known, as we shall presume it to be, and if $< \delta t_{gr} \delta t_{gr} >_o$ and $< \delta\psi\delta\psi >_R$ have been determined by observation, then $< \delta t_{gr} \delta t_{gr} >_R$ may be found as a function of R. This implies therefore that calculations based on equation (4.29) will still be of use, for even if $< \delta t_{gr} \delta t_{gr} >_R$ calculated therefrom is not directly observable, it may still be found in a straightforward manner from observable quantities. We shall in what follows, therefore, confine our discussion to equation (4.29), taking advantage of its simplicity, but keeping in mind that $< \delta t_{gr} \delta t_{gr} >_R$ thereby calculated does not represent the values we expect to observe, but is rather simply determinable from the observational data as discussed above.

We may now proceed. We at this point introduce two new quantities related to $\delta z(s)$ and $\delta\xi(s)$, but in a sense more physically meaningful than they:

$$\delta\psi(s) \equiv \frac{d\delta\xi(s)}{ds} \qquad (4.30)$$

$$\delta\Omega(s) \equiv \frac{d\delta z(s)}{ds} \quad . \tag{4.31}$$

Since $\delta\xi(s)$ and $\delta z(s)$ represent the displacements in the ξ and z directions, respectively, of a perturbed ray from its corresponding basic ray, it is clear that the quantities $\delta\psi(s)$ and $\delta\Omega(s)$ are respectively the tangents of the two angles formed by the basic ray with the projections of the perturbed ray onto the plane of the basic ray and onto the plane normal to that plane and tangent to the ray at s; in our small scattering treatment the tangents of these angles will be the angles themselves. Implicit in this identification of $\delta\psi(s)$ and $\delta\Omega(s)$ with appropriate angles formed by a perturbed ray and its corresponding basic ray is the assumption that $\delta\varkappa(s) = 0$, which is true only if $\mu_o \equiv$ constant. However, if μ_o is only approximately constant, and therefore $\delta\varkappa(s) \neq 0$, it may be shown that the tangent corresponding, for example, to $\delta\psi(s)$ is given by

$$\frac{d\delta\xi(s)}{ds}\left(1 + \frac{d\delta\varkappa(s)}{ds}\right)^{-1}$$

showing that for the perturbation analysis of scattering considered here the effect of a non-zero $\delta\varkappa(s)$ is second order. The quantities $\delta\psi(s)$ and $\delta\Omega(s)$ defined by equations (4.30) and (4.31) will be therefore identified with the angles described above even if μ_o is only approximately constant.

Now the scattering parameters $\delta\psi(s)$ and $\delta\Omega(s)$ are of greater physical importance than are the parameters $\delta\xi(s)$ and $\delta z(s)$ for the reason that the former quantities are, apart from effects due to the radial gradient of scattering, the angular displacements in the sky of a point source undergoing scattering, and are therefore direct observables, while the latter two quantities are not directly observable and therefore of less interest. If $P(\delta\psi)d\delta\psi$ represent the probability that $\delta\psi$ lies between $\delta\psi$ and $\delta\psi + d\delta\psi$, and if we assume, as the central limit theorem indicates, that the effect of multiple scattering is to produce a normal distribution, we then have

$$P(\delta\psi) = \frac{1}{\sqrt{2\pi < \delta\psi\delta\psi >}} \; e^{-\frac{(\delta\psi)^2}{2<\delta\psi\delta\psi>}}$$

$$P(\delta\Omega) = \frac{1}{\sqrt{2\pi < \delta\Omega\delta\Omega >}} \; e^{-\frac{(\delta\Omega)^2}{2<\delta\Omega\delta\Omega>}}$$

where $<\delta\psi\delta\psi>$ and $<\delta\Omega\delta\Omega>$ are the mean-squares of $\delta\psi$ and $\delta\Omega$. If one considers the scattering as redistributing the received signal power from a point source over some non-zero solid angle, the half width of the power distribution is

$$1.35\sqrt{<\delta\psi\delta\psi>}$$

in the plane of the basic ray, and

$$1.35\sqrt{<\delta\Omega\delta\Omega>}$$

normal to that plane. Thus $<\delta\psi\delta\psi>$ and $<\delta\Omega\delta\Omega>$ are readily observable by noting the angular distribution at the observer of the signal energy of what would be in the unscattered case a point source. In what follows, then, our attention shall be principally directed to the new scattering parameters $\delta\psi(s)$ and $\delta\Omega(s)$.

For linear basic rays equations (4.26) and (4.27) yield, upon insertion into (4.30) and (4.31)

$$\delta\psi(s) = \int_0^s \frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_0}\right)' ds' \tag{4.32}$$

$$\delta\Omega(s) = \int_0^s \frac{\partial}{\partial z}\left(\frac{\delta\mu}{\mu_0}\right)' ds' . \tag{4.33}$$

It is clear that for the random processes considered $<\delta\psi> = <\delta\Omega> = 0$. Equations (4.32) and (4.33) also yield

$$<\delta\psi(s)\,\delta\psi(s)> = \int_0^s\int_0^s \left< \left(\frac{\partial\delta\mu}{\partial\xi}\right)' \left(\frac{\partial\delta\mu}{\partial\xi}\right)'' \right> \frac{ds'ds''}{\mu_0^2} \tag{4.34}$$

$$<\delta\Omega(s)\,\delta\Omega(s)> = \int_0^s\int_0^s \left< \left(\frac{\partial\delta\mu}{\partial z}\right)' \left(\frac{\partial\delta\mu}{\partial z}\right)'' \right> \frac{ds'ds''}{\mu_0^2} \tag{4.35}$$

where the indicated correlations are to be taken between points lying on the same basic ray, and where for simplicity in writing we have let $\mu_0 \equiv$ constant (since the distances over which μ_0 varies are very much less than the correlation distances we shall find that once the indicated correlations are calculated we may readily relax this condition to obtain an integral over the basic ray of μ_0^{-2}). For the case of isotropic turbulence it is apparent that

$$\delta\psi(s) = \delta\,\Omega(s)$$

and therefore in what follows we shall be concerned only with equation (4.34).

It yet remains for us to formally evaluate

$$\left\langle \left(\frac{\partial\delta\mu}{\partial\xi}\right)' \left(\frac{\partial\delta\mu}{\partial\xi}\right)'' \right\rangle \tag{4.36}$$

For the case of isotropic turbulence under consideration we refer to equation (4.11) where we had

$$<\delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2)> = <\delta^2\mu>\; e^{-|\vec{r}_2 - \vec{r}_1|^2/\tau_o^2}\,. \tag{4.37}$$

For a linear ray we can write in the ray coordinate system

$$|\vec{r}_2 - \vec{r}_1|^2 = (s_2 - s_1)^2 + (\xi_2 - \xi_1)^2 + (z_2 - z_1)^2 \tag{4.38}$$

which when inserted into equation (4.37) gives

$$<\delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2)> =$$

$$<\delta^2\mu>\; e^{-[(s_2 - s_1)^2 + (\xi_2 - \xi_1)^2 + (z_2 - z_1)^2]/\tau_o^2}\,. \tag{4.39}$$

Equation (4.39) yields immediately

$$\left(\frac{\partial\delta\mu(\vec{r}_1)}{\partial\xi_1}\right)\left(\frac{\partial\delta\mu(\vec{r}_2)}{\partial\xi_2}\right) =$$

$$\frac{\partial^2}{\partial\xi_1\partial\xi_2}\left\{<\delta^2\mu> e^{-[(s_2-s_1)^2+(\xi_2-\xi_1)^2+(z_2-z_1)^2]/\tau_o^2}\right\}$$

$$= \frac{2}{\tau_o^2} <\delta^2\mu> e^{-(s_2-s_1)^2/\tau_o^2} ; \quad \begin{matrix}\xi_2 = \xi_1 \\ z_2 = z_1\end{matrix} \tag{4.40}$$

where the final evaluation of the derivatives has been at $(\xi_2 - \xi_1) = 0$, $(z_2 - z_1) = 0$; this is to correspond to the integrands of equations (4.34) and (4.35) where the indicated correlations are between two points on the same basic ray.

If now equation (4.40) is inserted into equation (4.34) we have

$$<\delta\psi(s)\,\delta\psi(s)> = \int_o^s\int_o^s \left(\frac{2}{\tau_o^2}<\delta^2\mu>\right)' e^{-(s''-s')^2/(\tau_o')^2}\,\frac{ds'ds''}{\mu_o^2}\,. \tag{4.41}$$

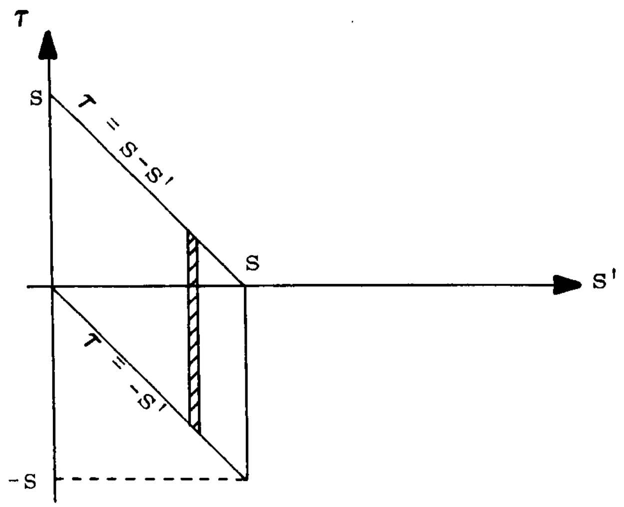
We define a new variable, τ ;

$$\tau \equiv s'' - s' \tag{4.42}$$

and equation (4.41) may be written

$$<\delta\psi\,\delta\psi> = \int_o^s\int_{-s'}^{s-s'} \left(\frac{2}{\tau_o^2}<\delta^2\mu>\right)' e^{-\tau^2/(\tau_o')^2}\,\frac{d\tau\, ds'}{(\mu_o)^2}\,. \tag{4.43}$$

Now in equations (4.41) and (4.43) both τ_o and $<\delta^2\mu>$ are to be construed as functions of s', and we may therefore do the integration over τ directly. Our area of integration is shown below:



It should be clear from the figure that if $\tau_o << s$ contributions to the integral over τ will be made only in a narrow band very close to the $\tau = 0$ axis, and we may therefore extend the τ limits of integration to $\tau = \pm \infty$. This approximation will be most valid indeed for we expect τ_o to be on the order of 200 km., and s to be on the order of 1 AU, implying

$$\frac{\tau_o}{s} \cong 10^{-6} . \qquad (4.44)$$

Equation (4.43) thus becomes

$$<\delta\psi\,\delta\psi> = \int_0^s \int_{-\infty}^{\infty} \left(\frac{2}{\tau_o^2} <\delta^2\mu>\right)' e^{-\tau^2/(\tau_o')^2} \frac{d\tau\, ds'}{\mu_o^2} \qquad (4.45)$$

which yields upon integration

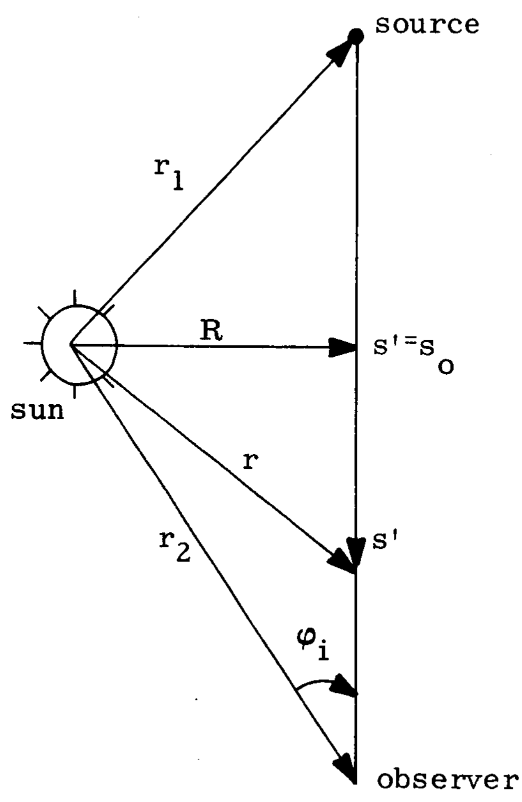
$$<\delta\psi\delta\psi> = 2\sqrt{\pi}\int_o^s \left(\frac{<\delta^2\mu>}{\tau_o}\right)' \frac{ds'}{\mu_o^2} \; . \tag{4.46}$$

Consider now the following ray geometry which will be appropriate under the conditions

$$r_1 < r_2 : \; |\sin\varphi_i| \leq \frac{r_1}{r_2}$$

$$r_1 \geq r_2 : \; |\varphi_i| \leq \pi/2$$

where r_1 is the (constant) distance of the source from the solar center, and r_2 is the (constant) distance of the observer from the solar center; the assumed constancy of r_1 and r_2 is a simplification appropriate for astronomical sources or artificial sources of sufficiently small orbital eccentricity, and for earth-based observers:



Clearly we have

$$ds' = \frac{rdr}{\pm\sqrt{r^2 - R^2}} \quad ; \quad s' \gtrless s_o \, . \tag{4.47}$$

Equation (4.46) becomes then

$$<\delta\psi\delta\psi> = 2\sqrt{\pi}\int_R^{r_1}+\int_R^{r_2}\left(\frac{<\delta^2\mu>}{\tau_o}\right)_r \frac{rdr}{\sqrt{r^2 - R^2}}\left(\frac{1}{\mu_o^2}\right) \tag{4.48}$$

where $<\delta^2\mu>$ and τ_o are now both to be considered functions of r; this corresponds to our assumption of coronal sphericity. Now, equation (4.48) is our desired result, expressing the mean square of the scattering parameter $\delta\psi$ as an appropriate integral over the basic ray of the parameters governing the coronal statistics. Only the following assumptions have been incorporated:

i) $\mu_o \cong$ constant, implying linearity of the basic rays.

ii) spherical symmetry

iii) isotropic turbulence, with a correlation function given by equation (4.11). This implies that equations (4.12) and (4.13) hold.

iv) $\tau_o \ll s$

v) appropriateness of the geometry shown on page 105.

Once again, it should be stated that the assumption that $\mu_o \equiv$ constant is not formally required for the foregoing derivation, the only requirement being that the basic rays be nearly linear. In the case where this latter requirement is obeyed, but $\mu_o = \mu_o(r)$, it is easy to show that

(4.48) becomes

$$<\delta\psi\delta\psi>_R = 2\sqrt{\pi}\int_R^{r_1} + \int_R^{r_2} \left(\frac{<\delta^2\mu>}{\mu_o^2\tau_o}\right)_r \frac{rdr}{\sqrt{r^2-R^2}} \tag{4.49}$$

as long as

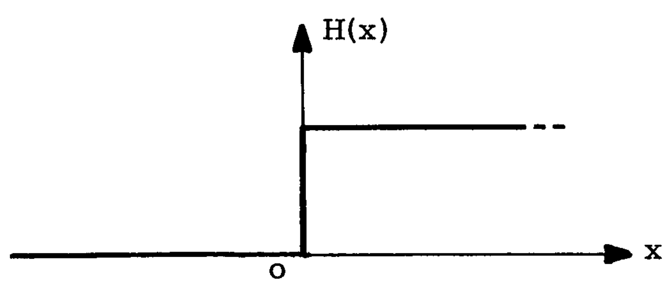
$$\tau_o \ll \left[\frac{1}{\mu_o} \quad \frac{d\mu_o}{dr}\right]^{-1} \tag{4.50}$$

In what follows we shall use this latter form.

Now, what we would like to do is invert equation (4.49) to give $\left(\frac{<\delta^2\mu>}{\mu_o^2\tau_o}\right)_r$ explicitly as a function of $<\delta\psi\delta\psi>_R$, which is an observable quantity. To this end we rewrite equation (4.49) in the form

$$\frac{<\delta\psi\delta\psi>_R}{2\sqrt{\pi}} = \int_R^{\infty} \frac{<\delta^2\mu>}{\mu_o^2\tau_o}\,_r \left[H(r_1-r)+H(r_2-r)\right]\frac{rdr}{\sqrt{r^2-R^2}} \tag{4.51}$$

where H is the Heaviside step function:



But equation (4.51) is an Abel integral equation with solution (see, for example, Hildebrand, "Methods of Applied Mathematics," Prentice-Hall (1952), §4.13)

$$\left(\frac{<\delta^2\mu>}{\mu_o^2\tau_o}\right)_r \left[H(r_1-r)+H(r_2-r)\right] =$$

$$-\frac{1}{\pi^{3/2}}\frac{d}{dr}\int_r^\infty <\delta\psi\,\delta\psi>_R \frac{r}{R}\frac{dR}{\sqrt{R^2-r^2}} \qquad (4.52)$$

if $r_1 \neq r_1(R)$ and $r_2 \neq r_2(R)$; these latter conditions imply that the source and observer maintain constant distances from the solar center, an assumption approximately valid for typical astronomical and satellite sources. Now, for the geometry shown on page 105 we will always have $R \leq r_1$ or r_2, whichever is smaller, implying that we shall be able to ascertain the parameters governing the coronal statistics only for $r \leq r_1$ or r_2, whichever is smaller. Equation (4.52) becomes then

$$\left(\frac{<\delta^2\mu>}{\mu_o^2\tau_o}\right)_r = \frac{-1}{2\pi^{3/2}}\frac{d}{dr}\int_r^\infty <\delta\psi\delta\psi>_R \frac{r}{R}\frac{dR}{\sqrt{R^2-r^2}} \qquad (4.53)$$

Thus equation (4.53) allows determination of the combination of the parameters governing the coronal statistics on the left-hand side <u>over a range of r equal to the range of R over which one has observational data</u> of $<\delta\psi\delta\psi>_R$. We have required only

i) spherical symmetry

ii) the constancy of the distances of the source and observer from the solar center.

The latter restriction will generally be satisfied by most sources of interest to an observer on the earth, but it should be pointed out that it becomes an unnecessary restriction in the event that source and observer lie beyond the region of the corona where the greater part of the scattering occurs. For an interplanetary electron density distribution specified by equation (2.37) it is expected that this will be the case for sources and observers approximately 1 A U from the solar center as long as $R \leq .5$ AU.

We turn now to a consideration of the fluctuations in the transit time of a signal pulse traversing the coronal medium. For the present case of μ_o = constant equation (4.29) is appropriate:

$$\delta t_{gr} = -\frac{1}{c}\int_o^s \frac{\delta\mu(s')}{\mu_o^2(s')}\, ds'$$

For the random processes we are considering it is clear that $<\delta t_{gr}> = 0$, but we can also obtain

$$<\delta t_{gr}\,\delta t_{gr}>_R = \frac{1}{c^2}\int_o^s\int_o^s \frac{<\delta\mu(s')\delta\mu(s'')>}{\mu_o^2(s')\mu_o^2(s'')}\, ds'ds''$$

where the correlation in the integrand is to be taken between two points lying on the same basic ray. If equation (4.39) for the correlation is inserted into the equation above, and the integration over $(d\tau)$ carried out, we obtain the following

$$<\delta t_{gr}\,\delta t_{gr}>_R = \frac{\sqrt{\pi}}{c^2}\int_o^s \left(\frac{<\delta^2\mu>\tau_o}{\mu_o^4}\right)' ds'$$

where we have implicitly taken the average refractive index to be sufficiently constant that approximation (4.50) holds. Now if the geometry of page 105 is appropriate, equation (4.47) may be used and we obtain

$$<\delta t_{gr}\,\delta t_{gr}>_{\mathbf{R}} = \frac{\sqrt{\pi}}{c^2}\left[\int_R^{r_1}+\int_R^{r_2}\right]\left(\frac{<\delta^2\mu>\tau_o}{\mu_o^{\,4}}\right)\frac{r\,dr}{\sqrt{r^2-R^2}} \qquad (4.54)$$

where, as before, $<\delta^2\mu>$, τ_o and μ_o are to be regarded as functions of r, corresponding to our assumption of spherical symmetry.

Now as discussed at the beginning of this section $<\delta t_{gr}\,\delta t_{gr}>_R$ may be found from the observable quantities $<\delta t_{gr}\,\delta t_{gr}>_o$ and $<\delta\psi\delta\psi>_R$, and is therefore in a sense an observable. It therefore behooves us, as in the discussion of $<\delta\psi\delta\psi>_R$, to invert equation (4.54) to obtain $(<\delta^2\mu>\tau_o/\mu_o^{\,4})_{\mathbf{r}}$ explicitly as a function of the "observable" $<\delta t_{gr}\,\delta t_{gr}>_R$. The proper inversion is (see discussion corresponding to equations (4.51)-(4.53))

$$\left(\frac{<\delta^2\mu>\tau_o}{\mu_o^{\,4}}\right)_r = \frac{-c^2}{\pi^{3/2}}\,\frac{d}{dr}\int_r^{\infty}<\delta t_{gr}\,\delta t_{gr}>_R\,\frac{r}{R}\,\frac{dR}{\sqrt{R^2-r^2}} \qquad (4.55)$$

Equation (4.55) allows us to determine $(<\delta^2\mu>\tau_o/\mu_o^{\,4})_r$ over a range of (r) equal to that range of (R) over which $<\delta t_{gr}\,\delta t_{gr}>_R$ has been observed.

Thus our problem is solved. Under the assumption of spherical symmetry equations (4.53) and (4.55) allow us to determine, for a range of (r) identical to that range of (R) for which observations have been made, the following two quantities:

i) $\left(\dfrac{<\delta^2\mu>}{\mu_o^2 \tau_o} \right)_r$

ii) $\left(\dfrac{<\delta^2\mu> \tau_o}{\mu_o^4} \right)_r$

Since $\mu_o(r)$ is know these two quantities allow us to determine $<\delta^2\mu>$ and $\tau_o(r)$, the two quantities characterizing the statistics of the medium and the ones we sought to investigate.

Now to complete this description of the scattering of radio waves by an isotropically turbulent corona we consider the phase variations (and thus the spectral broadening) induced by scattering. For the case μ_o = constant equation (4.28) gave

$$\delta\Phi(s) = -\frac{2\pi}{\lambda_o} \int_o^s \delta\mu(s')\, ds'$$

For the random processes we are considering $<\delta\Phi> = 0$. But the above equation yields also

$$<\delta\Phi(s)\,\delta\Phi(s)> = \left(\frac{2\pi}{\lambda_o}\right)^2 \int_o^s \int_o^s <\delta\mu(s')\,\delta\mu(s'')>\, ds'ds'' \qquad (4.56)$$

where the correlation in the integrand is to be taken between two points lying on the same basic ray. We may use equation (4.39) for the indicated correlation and carry out the integration over $(d\tau)$ to obtain

$$<\delta\Phi\delta\Phi> = \frac{4\pi^{5/2}}{\lambda_o^2} \int_o^s (<\delta^2\mu> \tau_o)'\, ds'$$

If the geometry of page 105 is appropriate equation (4.47) may be inserted into the above to yield

$$<\delta\Phi\delta\Phi>_R = \frac{4\pi^{5/2}}{\lambda_o^2} \int_R^{r_1} + \int_R^{r_2} \left(<\delta^2\mu> \tau_o\right)_r \frac{rdr}{\sqrt{r^2 - R^2}} \tag{4.57}$$

where, as before, both $<\delta^2\mu>$ and τ_o are to be regarded as functions of r, corresponding to our assumption of spherical symmetry. Equation (4.57) is the desired result, expressing the mean square of the scattering parameter $\delta\Phi(s)$ as an appropriate integral over the basic ray of the parameters governing the ray statistics.

But let us now inquire into the significance of $<\delta\Phi\delta\Phi>_R$ with regard to observability. As was discussed in Chapter III a variation in phase may be most conveniently interpreted as a frequency modulation of a CW signal, and thus a spectral broadening is implied. The variation in frequency, δf, of a CW signal of frequency f was shown in Chapter III to obey the relation (3.108)

$$\frac{\delta f}{f} = \frac{\lambda_o}{2\pi c} \frac{d\delta\Phi}{dt}$$

If we remember that we are dealing with random processes we get immediately $<\delta f> = 0$. But

$$<\delta f \delta f> = \frac{1}{(2\pi)^2} \left\langle \frac{d\delta\Phi}{dt} \frac{d\delta\Phi}{dt} \right\rangle$$

where we have used the relation

$$f\lambda_o = c$$

The question now is what to do with the time derivative of $\delta\Phi$ in the equation above. As a simple example consider a case where $\delta\Phi$ varies sinusoidally in time. Then

$$\delta\Phi = (\delta\Phi)_o \sin(2\pi f_{ph} t)$$

where f_{ph} represents the frequency of phase fluctuations. It is then easy to show that

$$\left\langle \frac{d\delta\Phi}{dt} \frac{d\delta\Phi}{dt} \right\rangle = \langle \delta\Phi\delta\Phi \rangle \left[(2\pi)^2 f_{ph}^{\;2} \right]$$

If, however, the variation of $\delta\Phi$ with time is not sinusoidal it appears that the above expression may be generalized to read

$$\left\langle \frac{d\delta\Phi}{dt} \frac{d\delta\Phi}{dt} \right\rangle = (2\pi)^2 \langle \delta\Phi\delta\Phi \rangle \langle f_{ph}^{\;2} \rangle$$

In that case the spectral broadening is related to the phase fluctuations according to

$$\frac{\langle \delta f \delta f \rangle}{\langle f_{ph}^{\;2} \rangle} = \langle \delta\Phi\delta\Phi \rangle \tag{4.58}$$

and with the aid of equation (4.57) we may write

$$\frac{\langle \delta f \delta f \rangle_R}{\langle f_{ph}^{\;2} \rangle_R} = \frac{4\pi^{5/2}}{\lambda_o^2} \int_R^{r_1} + \int_R^{r_2} \left(\langle \delta^2 \mu \rangle \tau_o \right)_r \frac{r dr}{\sqrt{r^2 - R^2}} \tag{4.59}$$

As indicated by the notation of (4.59) we expect $\langle f_{ph}^{\;2} \rangle$ to be in fact a function of the ray parameter R, since $\langle f_{ph}^{\;2} \rangle$ will be expected to depend on the scale size of the turbulence, τ_o, and the spectrum of the turbulent velocity distribution, appropriately averaged over a ray path. Now $< f_{ph}^2 >_R$ will not be easily observed directly, but may rather be inferred from measurements of $< \delta f \delta f >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$. For if we recall that for the linear basic rays we are considering $\mu_o \cong 1$ equations (4.54) and (4.59) readily yield

$$< f_{ph}^2 >_R = \frac{1}{\omega^2} \frac{< \delta f \delta f >_R}{< \delta t_{gr} \delta t_{gr} >_R} \tag{4.60}$$

Now from the foregoing discussion it is clear that $< f_{ph}^2 >_R$ has been introduced in heuristic fashion, and its interpretation must be therefore approximate. It seems reasonable to suggest, though, that

$$\sqrt{< f_{ph}^2 >_R} \quad \tau_o (r = R) \tag{4.61}$$

will be representative of the coronal velocity at $r = R$. Further discussion would necessitate a detailed discussion based on turbulent velocity spectra within the corona; this problem, although of exceeding great importance, is not our purpose here, and we content ourselves with the heuristic description presented.

Briefly, then, we have in this section considered in some detail the problem of the scattering of radio rays by a spherically symmetric, isotropically turbulent solar corona, for which $\mu_o \cong$ constant. The principal result is that we have successfully described a means by which $< \delta^2 \mu >$ and τ_o might be found, as functions of r, from the measurable quantities $< \delta\psi \, \delta\psi >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$. We have also described a means by which the heuristically introduced parameter $< f_{ph}^{2} >$ might be found from measurements of $< \delta f \delta f >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$, and have suggested how it might be related to the coronal velocities.

We proceed now to consider the case in which only the radial direction acquires special significance in destroying isotropy.

Anisotropic Turbulence with a Preference for the Radial Direction

It is the purpose of this section to discuss the scattering of radio rays by a turbulent solar corona describable by a homogeneous average component, $\mu_o \cong$ constant, and a randomly fluctuating component specifiable by a correlation function exhibiting a preference for only the radial direction. This is the situation we expect to find in the solar corona within 30 $R_{\odot}$, where the solar magnetic field can be expected to induce a preference for the radial direction.

Since we are here, as in the above discussion of the isotropically turbulent corona, considering scattering about nearly linear basic rays, the same relation for the scattering parameters will be used here as were used above, namely equations (4.28), (4.29), (4.32), and (4.33) for $\delta\Phi(s)$, δt_{gr}, $\delta\psi(s)$, and $\delta\Omega(s)$; these are the scattering parameters with which we shall be here concerned as they are very closely related to directly observable quantities. We had

$$\delta\psi(s) = \int_o^s \frac{\partial}{\partial\xi} \left(\frac{\delta\mu}{\mu_o}\right)' ds' \tag{4.62}$$

$$\delta\Omega(s) = \int_o^s \frac{\partial}{\partial z} \left(\frac{\delta\mu}{\mu_o}\right)' ds' \tag{4.63}$$

$$\delta\Phi(s) = -\frac{2\pi}{\lambda_o}\int_o^s \delta\mu(s')\,ds' \tag{4.64a}$$

$$\delta t_{gr} = -\frac{1}{C}\int_o^s \frac{\delta\mu(s')}{\mu_o(s')^2}\,ds' \tag{4.64b}$$

we will be interested in obtaining the mean squares of these quantities, and we therefore obtain from equations (4.62)-(4.64)

$$<\delta\psi\delta\psi> = \int_o^s\int_o^s \left\langle\left(\frac{\partial\delta\mu}{\partial\xi}\right)'\left(\frac{\partial\delta\mu}{\partial\xi}\right)''\right\rangle \frac{ds'ds''}{\mu_o^2} \tag{4.65}$$

$$<\delta\Omega\delta\Omega> = \int_o^s\int_o^s \left\langle\left(\frac{\partial\delta\mu}{\partial z}\right)'\left(\frac{\partial\delta\mu}{\partial z}\right)''\right\rangle \frac{ds'ds''}{\mu_o^2} \tag{4.66}$$

$$<\delta\Phi\delta\Phi> = \left(\frac{2\pi}{\lambda_o}\right)^2\int_o^s\int_o^s <\delta\mu(s')\,\delta\mu(s'')>\,ds'ds'' \tag{4.67a}$$

$$<\delta t_{gr}\,\delta t_{gr}> = \frac{1}{c^2}\int_o^s\int_o^s <\delta\mu(s')\,\delta\mu(s'')> \frac{ds'ds''}{\mu_o^4} \tag{4.67b}$$

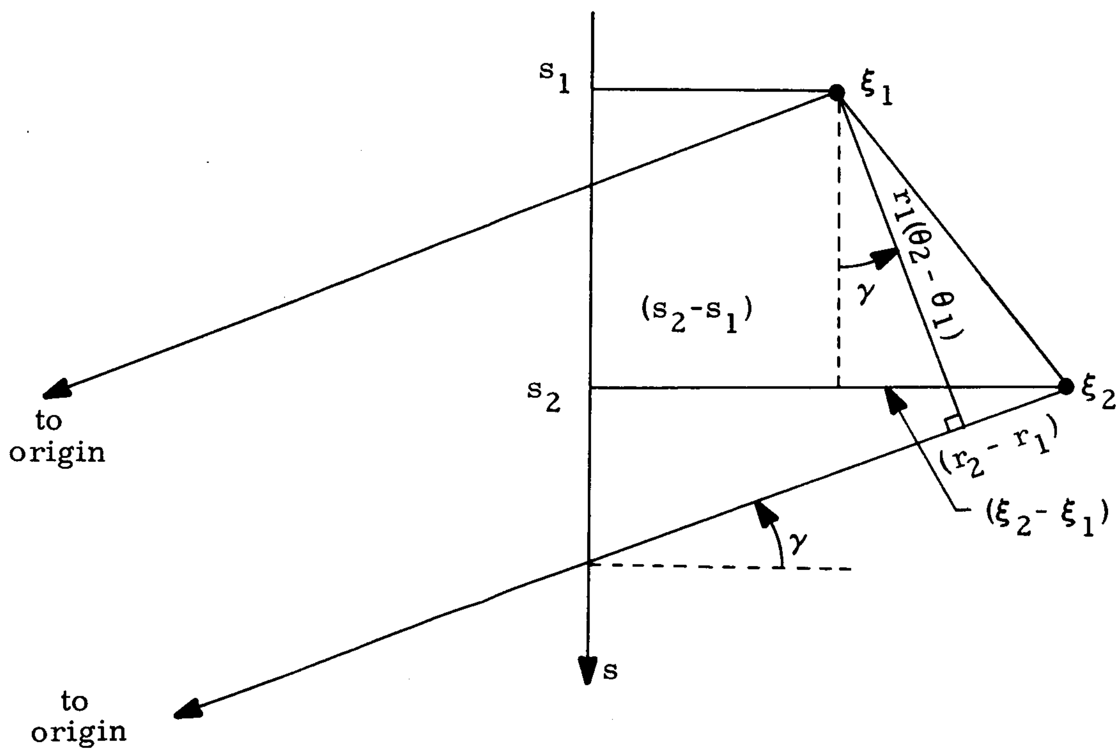
where all the indicated correlations are to be taken between points lying on the same basic ray, and where for the sake of simplicity in writing we have let μ_o = constant, a restriction which we have seen can be slightly violated and yet preserve the validity of this general discussion (see equation (4.50)). We should note that unlike the case of isotropic turbulence we here can not set $\delta\psi = \delta\Omega$, and equations (4.65) and (4.66) must be both considered.

To further discuss equations (4.65)-(4.67) it is necessary to introduce now explicit expressions for the correlations indicated therein. For the case where the statistical properties exhibit a preference for the radial direction we had (equation (4.14))

$$< \delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2) > =$$

$$< \delta^2\mu > \exp\left\{ - \frac{(r_2 - r_1)^2}{a^2} - \frac{\left[r_1{}^2 \sin^2 \varphi_1 (\theta_2 - \theta_1)^2 + r_1{}^2 (\varphi_2 - \varphi_1)^2 \right]}{b^2} \right\} \tag{4.68}$$

where quantities a, b, and $< \delta^2\mu >$ are functions of r_1 only. To be of value equation (4.68) must be written in terms of coordinates natural to the basic ray, namely, s, ξ, z (see equation (4.38)). Now since the case we are considering admits no preference with regard to the orientation of the $\varphi = 0$ axis we shall for convenience consider a basic ray lying wholly in the plane $\varphi_o = \pi/2$, and we can then set $\varphi_1 = \pi/2$ in equation (4.68) and draw the following figure illustrating the relationships between the (r, θ, φ) and (s, ξ, z) coordinate systems:



In drawing the figure it has been assumed that the correlation lengths (a) and (b) are sufficiently small so that for separations between points (2) and (1) over which statistical correlations are significant we may draw the two lines connecting points (2) and (1) to the origin as parallel. This means that (a) and (b) are small compared with the distance from the origin to the basic ray under consideration. As we expect (a) and (b) to be on the order of several hundred kilometers and $|\vec{r}_o|$ to be at least $R_\odot$ this seems to be a legitimate assumption. From the figure it is easily ascertained that, if we measure θ from the observer (so that $\theta_1 > \theta_2$).

$$
\begin{aligned}
-r_1(\theta_2-\theta_1) &= \cos\gamma\,(s_2-s_1) + \sin\gamma\,(\xi_2-\xi_1) \\
r_2-r_1 &= \cos\gamma\,(\xi_2-\xi_1) - \sin\gamma\,(s_2-s_1) \qquad (4.69) \\
r_1(\theta_2-\theta_1) &= z_2-z_1
\end{aligned}
$$

When these relations are inserted into equation (4.68) we have

$$
\begin{aligned}
&< \delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2) > \\
&= < \delta^2\mu >_{r_1} \exp\left\{ -(s_2-s_1)^2 \left(\frac{\sin^2\gamma}{a^2} + \frac{\cos^2\gamma}{b^2} \right) \right. \\
&\quad - (\xi_2-\xi_1)^2 \left(\frac{\cos^2\gamma}{a^2} + \frac{\sin^2\gamma}{b^2} \right) - \frac{(z_2-z_1)^2}{b^2} \\
&\quad \left. + 2\sin\gamma\cos\gamma\,(s_2-s_1)(\xi_2-\xi_1)\left(\frac{1}{a^2} - \frac{1}{b^2} \right) \right\} \qquad (4.70)
\end{aligned}
$$

where (a), (b), and $< \delta^2\mu >$ are functions only of r_1. We note that this reduces to equation (4.39) if a=b (isotropic turbulence), as it ought.

Now to obtain the correlations indicated in equations (4.65) and (4.66) we shall take the appropriate derivatives of equation (4.70), neglecting however all derivatives of (a), (b), and $< \delta^2\mu >$, corresponding to assumptions (4.15) and (4.16). We thus obtain

$$\left\langle \frac{\partial \delta\mu(\vec{r}_1)}{\partial \xi_1} \frac{\partial \delta\mu(\vec{r}_2)}{\partial \xi_2} \right\rangle = \frac{\partial^2}{\partial \xi_1 \partial \xi_2} < \delta\mu(\vec{r}_1)\, \delta\mu(\vec{r}_2) >$$

$$= < \delta^2 u > \left[2 \left(\frac{\cos^2 \gamma}{a^2} + \frac{\sin^2 \gamma}{b^2} \right) - \sin^2 2\gamma \left(\frac{1}{a^2} - \frac{1}{b^2} \right)^2 (s_2 - s_1)^2 \right] e^{-(s_2 - s_1)^2 \left(\frac{\sin^2 \gamma}{a^2} + \frac{\cos^2 \gamma}{b^2} \right)} \tag{4.71}$$

$$\left\langle \frac{\partial \delta\mu(\vec{r}_1)}{\partial z_1} \frac{\partial \delta\mu(\vec{r}_2)}{\partial z_2} \right\rangle = \frac{\partial^2}{\partial z_1 \partial z_2} < \delta\mu(\vec{r}_1)\, \delta\mu(\vec{r}_2) >$$

$$= \frac{2 < \delta^2 \mu >}{b^2} e^{-(s_2 - s_1)^2 \left(\frac{\sin^2 \gamma}{a^2} + \frac{\cos^2 \gamma}{b^2} \right)} \tag{4.72}$$

where we have evaluated the derivatives at $(\xi_2 - \xi_1) = 0$, $(z_2 - z_1) = 0$ to correspond to equations (4.65) and (4.66) where the correlations are taken between points on the same basic ray. Similarly, the same evaluation applied to equation (4.70) gives

$$< \delta\mu(\vec{r}_1)\, \delta\mu(\vec{r}_2) > =$$

$$< \delta^2 \mu > e^{-(s_2 - s_1)^2 \left(\frac{\sin^2 \gamma}{a^2} + \frac{\cos^2 \gamma}{b^2} \right)} \tag{4.73}$$

which is the form we shall insert into equation (4.67) for $< \delta\Phi\delta\Phi >$.

Equations (4.71)-(4.73) may be inserted into equations (4.65)-(4.67) yielding explicit integral forms for $<\delta\psi\,\delta\psi>$, $<\delta\Omega\delta\Omega>$, $<\delta\Phi\delta\Phi>$, and $<\delta t_{gr}\delta t_{gr}>$. As was done earlier in connection with equation (4.41), we may define a new variable $\tau = s'' - s'$ replacing s'' in the integrals and, if a, b << s, extend the limits of the integration over ($d\tau$) to $\pm\infty$. The integration over ($d\tau$) may be then carried out to yield (c.f. equations (4.46) and (4.59))

$$<\delta\psi\,\delta\psi> = 2\sqrt{\pi}\int_0^s \left(\frac{<\delta^2\mu>}{\mu_o^2}\,\frac{a}{b^2}\right)' \frac{1}{\left[1+\left(\frac{a^2}{b^2}-1\right)\cos^2\gamma\right]'^{\,3/2}}\,ds' \tag{4.74}$$

$$<\delta\Omega\delta\Omega> = 2\sqrt{\pi}\int_0^s \left(\frac{<\delta^2\mu>}{\mu_o^2}\,\frac{a}{b^2}\right)' \frac{1}{\left[1+\left(\frac{a^2}{b^2}-1\right)\cos^2\gamma\right]'^{\,1/2}}\,ds' \tag{4.75}$$

$$<\delta\Phi\delta\Phi> = \left(\frac{2\pi}{\lambda_o}\right)^2\sqrt{\pi}\int_0^s (<\delta^2\mu>a)' \frac{1}{\left[1+\left(\frac{a^2}{b^2}-1\right)\cos^2\gamma\right]'^{\,1/2}}\,ds' \tag{4.76a}$$

$$<\delta t_{gr}\delta t_{gr}> = \frac{\sqrt{\pi}}{c^2}\int_0^s \left(\frac{<\delta^2\mu>a}{\mu_o^4}\right)' \frac{1}{\left[1+\left(\frac{a^2}{b^2}-1\right)\cos^2\gamma\right]'^{\,1/2}}\,ds' \tag{4.76b}$$

If we now, as we did in the case of isotropic turbulence, assume the geometry shown on page 105 it is clear that

$$\cos\gamma = \frac{R}{r} \tag{4.77}$$

Then defining

$$A(r) = \frac{a^2}{b^2} - 1 \tag{4.78}$$

and utilizing equation (4.47), equations (4.74)-(4.76) become

$$< \delta\psi\,\delta\psi >_R = 2\sqrt{\pi}\left[\int_R^{r_1} + \int_R^{r_2}\right]\left(\frac{< \delta^2\mu >}{\mu_o^2\, b}\right)_r \frac{(1 + A)^{1/2}}{\left(1 + A\,\frac{R^2}{r^2}\right)^{3/2}} \frac{rdr}{\sqrt{r^2 - R^2}} \tag{4.79}$$

$$< \delta\Omega\delta\Omega >_R = 2\sqrt{\pi}\left[\int_R^{r_1} + \int_R^{r_2}\right]\left(\frac{< \delta^2\mu >}{\mu_o^2\, b}\right)_r \frac{(1 + A)^{1/2}}{\left(1 + A\,\frac{R^2}{r^2}\right)^{1/2}} \frac{rdr}{\sqrt{r^2 - R^2}} \tag{4.80}$$

$$< \delta\Phi\delta\Phi >_R = \frac{4\pi^{5/2}}{\lambda_o^2}\left[\int_R^{r_1} + \int_R^{r_2}\right](< \delta^2\mu > b)_r \frac{(1 + A)^{1/2}}{\left(1 + A\,\frac{R^2}{r^2}\right)^{1/2}} \frac{rdr}{\sqrt{r^2 - R^2}} \tag{4.81a}$$

$$< \delta t_{gr}\delta t_{gr} >_R = \frac{\sqrt{\pi}}{c^2}\left[\int_R^{r_1} + \int_R^{r_2}\right]\left(\frac{< \delta^2\mu > b}{\mu_o^4}\right)_r \frac{(1 + A)^{1/2}}{\left(1 + A\,\frac{R^2}{r^2}\right)^{1/2}} \frac{rdr}{\sqrt{r^2 - R^2}} \tag{4.81b}$$

where, corresponding to our assumption of spherical symmetry, $< \delta^2\mu >$, (b), and (A) are to be construed as functions of r. Equations (4.79)-(4.81) are the desired relationships, expressing the mean squares of the scattering parameters $\delta\psi$, $\delta\Omega$, $\delta\Phi$, and δt_{gr} as appropriate integrals over the relevant basic ray of the parameters governing the statistical properties of the coronal turbulence in the case where only the radial direction acquires special significance. In the limit of A = 0 equations

(4.79) to (4.81) reduce to equations (4.49), (4.58), and (4.60) as they ought. On the other hand, in the limit of very pronounced anisotropy, $A(r) \to \infty$, our equation above for $< \delta\Omega\delta\Omega >_R$ approaches that used by Vitkevich (1966) for radial inhomogeneities. Our two equations (4.79) and (4.80) represent a significant generalization of those used by him. The set of equations (4.79)-(4.81), appearing for the first time in this work, allow discussion of coronal radio scattering for a generally anisotropically turbulent solar corona.

In obtaining the above set of equations the following assumptions have been made:

i) $\mu_o \cong$ constant, implying approximate linearity of the basic rays

ii) spherical symmetry

iii) anisotropic turbulence, but with a preference for only the radial direction, with a correlation function of equation (4.14). This implies in turn that equations (4.15) hold.

iv) $a, b << s$

v) appropriateness of the geometry shown on page 117 implying $a, b << |\vec{r}_o|$

vi) appropriateness of the geometry shown on page 105

Now, as in the case of isotropic turbulence discussed earlier, we would seek to invert equations (4.79)-(4.81) to enable determination of the function A(r), b(r), and $< \delta^2 \mu >$ explicitly in terms of the observable quantities $< \delta\psi\, \delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, $< \delta\Phi\delta\Phi >_R$ and $< \delta t_{gr} \delta t_{gr} >$. Unfortunately, however, the distinctly unpleasant forms of equations (4.79)-(4.81) seem to preclude the possibility of obtaining formal inversions, and we must therefore resort to a number of approximations.

If A(r) is sufficiently large so that

$$A \frac{R^2}{r^2} >> 1 \tag{4.82}$$

over the range of (r) for which $< \delta^2\mu >$ is of sufficient magnitude to contribute significantly to the integrals of equations (4.79)-(4.81) we may write these as

$$R^3 < \delta\psi\, \delta\psi >_R = 4\sqrt{\pi} \int_R^\infty \left(r^3 \frac{< \delta^2\mu >}{\mu_o^2\, b} \right) \frac{1}{A(r)} \frac{r dr}{\sqrt{r^2 - R^2}} \tag{4.83}$$

$$R < \delta\Omega\delta\Omega >_R = 4\sqrt{\pi} \int_R^\infty \left(r \frac{< \delta^2\mu >}{\mu_o^2\, b} \right) \frac{r dr}{\sqrt{r^2 - R^2}} \tag{4.84}$$

$$R \frac{< \delta f \delta f >_R}{< f_{ph}^2 >_R} = \frac{8\pi^{5/2}}{\lambda_o^2} \int_R^\infty (r < \delta^2\mu > b) \frac{r dr}{\sqrt{r^2 - R^2}} \tag{4.85a}$$

$$R < \delta t_{gr} \delta t_{gr} > = \frac{2\sqrt{\pi}}{c^2} \int_R^\infty \left(\frac{r < \delta^2\mu > b}{\mu_o^4} \right) \frac{r dr}{\sqrt{r^2 - R^2}} \tag{4.85b}$$

where we have for simplicity extended the limits of integration to infinity; this is a legitimate thing to do as long as we are concerned about the values of A(r), b(r), and $< \delta^2\mu >_r$ only for a range of r equal to the range of R over which one has observational data of $< \delta\psi\, \delta\psi >$, $< \delta\Omega\delta\Omega >$, $< \delta f \delta f >$, and $< \delta t_{gr} \delta t_{gr} >$ (see equation (4.49) et seq.). The assumed largeness of A(r), equation (4.82), implies typically for our model corona

$$\frac{a(r)}{b(r)} \gtrsim 4 \tag{4.86}$$

i.e. the coronal turbulence exhibits a radial filamentary structure with a correlation length in the radial direction roughly four or more times that in the "transverse" direction. Now, for a corona in which the preference for the radial direction is maintained by a radial general magnetic field, one expects the enhancement of diffusion along the lines of force to produce a radial filamentary structure. This kind of structure has indeed been observed optically, and more recently by radio scattering measurements (Hewish (1958), Gorgolewski and Hewish (1960), Högbom (1960), Hewish and Wyndham (1963), Erickson (1964)). The radio observations, however, indicate correlation length ratios on the order of 2-3, making assumption (4.86) suspect. But since the existing radio measurements leading to the above values for the correlation length ratios are not of great accuracy, our assumption may well be of value. This may particularly be so in the lower coronal regions ($r < 5\ R_{\odot}$) where the magnetic field is strongest and where turbulent mixing will probably not have destroyed the preference for the radial direction due to the outward streaming of matter from localized regions of the photosphere and chromosphere (spicules). The optical observations of distinct radial coronal filaments in the lower regions of the corona support these notions. It must be emphasized, however, that the regions of the corona over which the assumption of large A(r) holds may well be larger than indicated here.

Equations (4.83)-(4.86), valid under the restriction that A(r) be large (equations (4.82) and (4.86)), are Abel integral equations and are directly soluble. Equations (4.84) and (4.85) upon inversion yield

$$r\frac{<\delta^2\mu>}{\mu_o^2 b} = \frac{-1}{2\pi^{3/2}}\frac{d}{dr}\int_r^\infty r\frac{<\delta\Omega\delta\Omega>_R}{\sqrt{R^2-r^2}}dR\,;\ \frac{a}{b}\gtrsim 4 \qquad (4.87)$$

$$r\frac{<\delta^2\mu>b}{\mu_o^4} = \frac{-c^2}{\pi^{3/2}}\frac{d}{dr}\int_r^\infty r\frac{<\delta t_{gr}\delta t_{gr}>_R}{\sqrt{R^2-r^2}}dR\,;\ \frac{a}{b}\gtrsim 4 \qquad (4.88)$$

These expressions readily yield both b(r) and $<\delta^2\mu>$ for a range of r identical to that range of R for which one has experimental measurements of $<\delta\Omega\delta\Omega>_R$ and $<\delta t_{gr}\delta t_{gr}>_R$. Equation (4.83) is the only one of our set containing A(r). When it is inverted and combined with equation (4.87) above one obtains the following expression for A(r):

$$A(r) = r^2\frac{\frac{d}{dr}\int_r^\infty r\frac{<\delta\Omega\delta\Omega>_R}{\sqrt{R^2-r^2}}dR}{\frac{d}{dr}\int_r^\infty r\frac{R^2<\delta\psi\,\delta\psi>_R}{\sqrt{R^2-r^2}}dR}\ ;\ \frac{a}{b}\gtrsim 4 \qquad (4.89)$$

Again, the range of r over which (4.89) is valid is that range of R over which one has observed values of $<\delta\psi\,\delta\psi>_R$ and $<\delta\Omega\delta\Omega>_R$. Thus, to sum, given appropriate observational measurements of $<\delta\psi\,\delta\psi>_R$, $<\delta\Omega\delta\Omega>_R$, and $<\delta t_{gr}\delta t_{gr}>_R$ equations (4.87)-(4.89) determine A(r), b(r), and $<\delta^2\mu>_r$ for the case where A(r) is large, $A(r)\frac{R^2}{r^2} \gg 1$ over the range of r for which significant contributions are made to the integrals of (4.79)-(4.81). For the model corona (2.37) this implies roughly $a/b > 4$. In practice the validity of this assumption could be checked by using equation (4.89) to determine $A(r)$ from the observational data and then determining whether the value so obtained is consistent with that assumption.

A second, but less interesting, situation for which we are able to obtain inversions of equations (4.79)-(4.81) is when

$$|A| \frac{R^2}{r^2} \ll 1 \tag{4.90}$$

This implies

$$.9 < \frac{a}{b} < 1.1 \tag{4.91}$$

a rather highly restrictive condition limiting our consideration only to very slightly anisotropic conditions. We shall pursue this case no further for two reasons: first, condition (4.91) is so highly restrictive that it implies anisotropic scattering so slight as to be probably not observable, and, second, even should the implied anisotropic scattering be observed it will be shown later in the chapter when we consider numerical examples that for only slightly anisotropic scattering the value of

$$\sqrt{\frac{<\delta\Omega\delta\Omega>_R}{<\delta\psi\,\delta\psi>_R}} \tag{4.92}$$

is, for all R, the value of

$$\frac{a(r = R)}{b(r = R)}$$

to within some 10%. Thus examination of the scattering anisotropy as measured by quantity (4.92) provides a short cut to determination of the correlation length ratio, a/b.

In the case where A(r) is neither large nor small we have been unable to obtain inversions to our original set of equations, (4.79)-(4.81). It is still possible, however, to obtain a significant amount of information from these relations. We begin with the following very important observation. For an average corona with electron density decreasing monotonically outward we expect most of the contributions to the integrals of (4.79)-(4.81) to occur in the vicinity of r = R. In that case it is apparent from the form of the integrals that the quantity (A) will be of little effect on the values of $< \delta\Omega\delta\Omega >_R$, $< \delta\Phi\delta\Phi >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$. Later in the chapter where we consider some explicit numerical examples it will be shown that for the model corona of (2.37) the effect of (A) on $< \delta\Omega\delta\Omega >_R$, $< \delta\Phi\delta\Phi >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$ will be to increase these quantities by no more than some 10-15%, and to that degree of accuracy then we may write equations (4.80)-(4.81) as

$$< \delta\Omega\delta\Omega >_R = 2\sqrt{\pi} \left[\int_R^{r_1} + \int_R^{r_2} \right] \left(\frac{< \delta^2\mu >}{\mu_o^2 \, b} \right)_r \frac{r\,dr}{\sqrt{r^2 - R^2}} \tag{4.93}$$

$$< \delta\Phi\delta\Phi >_R = \frac{4\pi^{5/2}}{\lambda_o^2} \left[\int_R^{r_1} + \int_R^{r_2} \right] (< \delta^2\mu > b)_r \frac{r\,dr}{\sqrt{r^2 - R^2}} \tag{4.94}$$

$$< \delta t_{gr} \delta t_{gr} >_R = \frac{\sqrt{\pi}}{c^2} \left[\int_R^{r_1} + \int_R^{r_2} \right] \left(\frac{< \delta^2\mu > b}{\mu_o^4} \right)_r \frac{r\,dr}{\sqrt{r^2 - R^2}} \tag{4.95}$$

If, as we have done before, we extend the limits of integration to infinity the equations above may be readily inverted to give

$$\left(\frac{< \delta^2\mu >}{\mu_o^2 \, b} \right)_r = - \frac{1}{2\pi^{3/2}} \frac{d}{dr} \int_r^{\infty} \frac{r}{R} < \delta\Omega\delta\Omega >_R \frac{dR}{\sqrt{R^2 - r^2}} \tag{4.96}$$

$$\left(\frac{< \delta^2\mu > b}{\mu_o^4} \right)_r = - \frac{c^2}{\pi^{3/2}} \frac{d}{dr} \int_r^{\infty} < \delta t_{gr} \delta t_{gr} >_R \frac{r}{R} \frac{dR}{\sqrt{R^2 - r^2}} \tag{4.97}$$

Since $\mu_o(r)$ is presumably known, these equations allow determination of $< \delta^2\mu >_r$ and b(r) for a range of (r) equal to that range of (R) for which $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$ have been observed. Thus having found $< \delta^2\mu >_r$ and b(r), we need now find only A(r), and we have another equation with which this may be done, namely (4.79) for $< \delta\psi \, \delta\psi >_R$. Let us examine this equation. Since we expect most of the contributions to the integral to occur in the vicinity of r = R, we inquire what would happen if all contributions were concentrated there. In that case it is not difficult to see that

$$\frac{a(r = R)}{b(r = R)} = \sqrt{\frac{< \delta\Omega\delta\Omega >_R}{< \delta\psi \, \delta\psi >_R}} \tag{4.98}$$

Now, since most of the contributions to the integral of (4.79) occur near r = R we expect equation (4.98) to be approximately true, and we may then use direct measurements of the quantity on the right-hand side to provide an estimate of the behavior of a/b with r. The error incurred by the procedure will depend on the electron distribution in the corona (a steeper radial gradient of the electron density will yield a smaller error, for then a greater part of the contributions to the integrals of equations (4.79) and (4.80) will appear near $r = R$), on the functional form of $b(r)$ (since $b(r)$ appears in the denominators of the integrands of (4.79) and (4.80) we expect our errors to be less for values of $b(r)$ which increase more rapidly with r) and on the functional form of a/b. These predictions are verified by the numerical calculations to be done subsequently; we show that if $a(r)/b(r) < 4.0$ (otherwise we would use approximation (4.89)) the error incurred in our estimate for a/b is less than some 30%. If this error is not acceptable an iteration procedure based on equation (4.98) as a first estimate may be employed.

Now, it may have occurred to the reader that we have thus far not utilized equation (4.81a) for $< \delta\Phi\delta\Phi >_R$; the reason for this is that we have found equations (4.79), (4.80), and (4.81b) sufficient to determine the parameters governing the statistical behavior of the corona, that is $< \delta^2\mu >_r$, $b(r)$, and $A(r)$. However, the fluctuations, $\delta\Phi$, in the phase of a signal will, it will be recalled, be observed as a frequency modulation, or line broadening, of that signal. For a CW signal it was suggested that the spectral broadening obeys

$$\frac{< \delta f \delta f >}{< f_{ph}^2 >} = < \delta\Phi\delta\Phi > \tag{4.100}$$

and, utilizing (4.81a), we may write

$$\frac{< \delta f \delta f >_R}{< f_{ph}^2 >_R} = \frac{4\pi^{5/2}}{\lambda_o^2} \left(\int_R^{r_1} + \int_R^{r_2} \right) (< \delta^2\mu > b)_r \frac{(1+A)^{1/2}}{\left(1 + A\frac{R^2}{r^2}\right)^{1/2}} \frac{r dr}{\sqrt{r^2 - R^2}} \tag{4.101}$$

If we recall that for linear basic rays $\mu_o \cong 1$ we may use equations (4.81b) and (4.101) to write

$$< f_{ph}^{\ 2} >_R = \frac{1}{\omega^2} \frac{< \delta f \delta f >_R}{< \delta t_{gr} \delta t_{gr} >_R} \tag{4.102}$$

Thus measurement of $< \delta f \delta f >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$ immediately provides an estimate for the heuristically introduced parameter $< f_{ph}^{\ 2} >_R$. Its interpretation in terms of local turbulent motions in the corona is not the subject of this work; we can only say that $\left(< f_{ph}^{\ 2} >_R\right)^{1/2}$ will be related in some way to an average over the basic ray of the velocities of turbulent "blobs" divided by some combination of the correlation lengths; for example, if the outflow is radial and the bulk of the scattering occurs in the vicinity of $r = R$, then the quantity

$$< f_{ph}^{\ 2} >_R \ a(r = R)$$

may be expected to be representative of the coronal outflow velocity at $r = R$. Of this we shall say no more.

To summarize, in this section we have considered in some detail the scattering of radio rays by an anisotropically turbulent solar corona of approximately constant average refractive index (implying nearly linear basic rays) for which the anisotropy exhibits a preference for the radial direction only. For the case of highly pronounced anisotropy we have succeeded in obtaining explicit expressions for the parameters governing the statistical properties of the medium in terms of the observed values of $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$. If, however, the anisotropy is not large we have been able to describe an approximate procedure for deducing A(r), b(r), and $< \delta^2\mu >$ from the observations. The discussion of the present section should prove of value in detailed observational studies of solar coronal turbulence when the assumptions of homogeneity (i.e. $\mu_o \cong$ constant) and radial preference hold. In this regard we should be here reminded that the values of $< \delta^2\mu >$ determined by the methods discussed in this and the previous section can be directly related to the statistical fluctuations of the coronal electron density through use of equation (4.23).

We proceed next to a discussion of scattering when the coronal turbulence exhibits in the solar equatorial plane a preference for the r, θ, and φ directions.

Anisotropic Turbulence with a Preference in the Solar Equatorial Plane for the r, θ, and φ Directions

In this section we wish to extend our discussion of the scattering of nearly linear radio rays to the case where the coronal turbulence exhibits, in the solar equatorial plane, a preference to the r, θ, and φ directions, where the equatorial plane (or, as is nearly the case, the ecliptic plane) is specified by $\varphi = \pi/2$, and contains the basic rays about which fluctuations occur. The preference for the radial direction is expected in virtue of the enhanced diffusion along magnetic field lines, which we expect to be radial within about 30 $R_\odot$. In the solar equatorial plane the distinction between the θ and φ directions might be expected if the coronal outflow were not strictly radial, but confined somewhat to either the solar equatorial plane or to the polar regions; this possibility has been suggested earlier by Hewish and Wyndham (1963). In this case we choose the correlation function of equation (4.16):

$$< \delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2) > =$$

$$< \delta^2\mu > \exp\left\{ - \frac{(r_2 - r_1)^2}{a^2} - \frac{r_1^{\,2}(\theta_2 - \theta_1)^2}{b^2} - \frac{r_1^{\,2}\,(\varphi_2 - \varphi_1)^2}{d^2} \right\} \qquad (4.103)$$

where a(r), b(r), and d(r) are the correlation lengths in the r, θ, and φ directions, respectively, and $< \delta^2\mu >$ is the correlation amplitude, also a function of r. Equation (4.103) is reasonable subject to

$$a, b, d \ll \left[\frac{1}{a}\frac{da}{dr}\right]^{-1}, \left[\frac{1}{b}\frac{db}{dr}\right]^{-1}, \left[\frac{1}{d}\frac{d(d)}{dr}\right]^{-1}$$

$$a, b, d \ll \left[\frac{1}{< \delta^2\mu >}\;\frac{d < \delta^2\mu >}{dr}\right]^{-1} \qquad (4.104)$$

Now since we are here, as in the above two sections, considering scattering about nearly linear basic rays the same expressions for the scattering parameters will be used here as were used above, namely equations (4.28), (4.29), (4.32), and (4.33) for $\delta\Phi(s)$, $\delta t_{gr}(s)$, $\delta\psi(s)$, and $\delta\Omega(s)$. We may then immediately write (see equations (4.65)-(4.67)):

$$< \delta\psi\,\delta\psi > = \int_0^s\!\!\int_0^s \left< \left(\frac{\partial\delta\mu}{\partial\xi}\right)' \left(\frac{\partial\delta\mu}{\partial\xi}\right)'' \right> \frac{ds'ds''}{\mu_o^{\,2}} \qquad (4.105)$$

$$< \delta\Omega\delta\Omega > = \int_0^s\!\!\int_0^s \left< \left(\frac{\partial\delta\mu}{\partial z}\right)' \left(\frac{\partial\delta\mu}{\partial z}\right)'' \right> \frac{ds'ds''}{\mu_o^{\,2}} \qquad (4.106)$$

$$< \delta\Phi\delta\Phi > = \left(\frac{2\pi}{\lambda_o}\right)^2 \int_0^s\!\!\int_0^s < \delta\mu(s')\delta\mu(s'') > ds'ds'' \qquad (4.107)$$

$$< \delta t_{gr} \delta t_{gr} > = \frac{1}{c^2} \int_0^s \int_0^s < \delta\mu(s') \, \delta\mu(s'') > \frac{ds' ds''}{\mu_o^4} \qquad (4.108)$$

where the indicated correlations are to be taken between points lying on the same basic ray, and where for simplicity in writing we have taken μ_o = constant, a restriction which will be shortly relaxed.

Now the correlations in the integrands of (4.105)-(4.108) are determined by the autocorrelation function (4.103), with the coordinate transform of equation (4.69). If we introduce a new variable $\tau = s'' - s'$ we may transform the integrals of equations (4.105)-(4.108) according to

$$\int_0^s \int_0^s ds'' ds' \longrightarrow \int_0^s \int_{-s'}^{s-s'} d\tau \, ds' \qquad (4.109)$$

If a, b, d << s we may then, as has been shown earlier, extend the limits of the integration over ($d\tau$) to $\pm \infty$ to obtain

$$\int_0^s \int_0^s ds'' ds' \longrightarrow \int_0^s \int_{-\infty}^{+\infty} d\tau \, ds' \qquad (4.110)$$

The integrations over ($d\tau$) in (4.105)-(4.108) may be then carried out directly. If we define the following quantities

$$A = \frac{a^2}{b^2} - 1 \qquad (4.111)$$

$$D = \frac{d^2}{b^2} \qquad (4.112)$$

we may then obtain (c.f. equations (4.79)-(4.81)):

$$\frac{<\delta\psi\delta\psi>_R}{2\sqrt{\pi}} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu>}{\mu_o^2 b}\right)_r \frac{(1+A)^{1/2}}{\left(1+A\frac{R^2}{r^2}\right)^{3/2}} \frac{rdr}{\sqrt{r^2-R^2}} \quad (4.113)$$

$$\frac{<\delta\Omega\delta\Omega>_R}{2\sqrt{\pi}} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu>}{\mu_o^2 b}\right)_r \frac{(1+A)^{1/2}}{\left(1+A\frac{R^2}{r^2}\right)^{1/2}} \frac{1}{D} \frac{rdr}{\sqrt{r^2-R^2}} \quad (4.114)$$

$$\frac{<\delta f\delta f>_R}{\frac{4\pi^{5/2}}{\lambda_o^2} <f_{ph}^2>_R} = \left[\int_R^{r_1} + \int_R^{r_2}\right] (<\delta^2\mu> b)_r \frac{(1+A)^{1/2}}{\left(1+A\frac{R^2}{r^2}\right)^{1/2}} \frac{rdr}{\sqrt{r^2-R^2}} \quad (4.115)$$

$$\frac{<\delta t_{gr}\delta t_{gr}>_R}{(\sqrt{\pi}/c^2)} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu> b}{\mu_o^4}\right)_r \frac{(1+A)^{1/2}}{\left(1+A\frac{R^2}{r^2}\right)^{1/2}} \frac{rdr}{\sqrt{r^2-R^2}}$$

(4.116)

We have implicitly introduced the assumption of spherical symmetry and have assumed that the geometry of page 105 is appropriate. We have also used the relation

$$\frac{<\delta f\delta f>}{<f_{ph}^2>} = <\delta\Phi\delta\Phi>$$

Equations (4.113)-(4.116) are the desired relationships, expressing the mean squares of the scattering parameters $\delta\psi$, $\delta\Omega$, δf and δt_{gr} as integrals over a basic ray of the parameters governing the statistics of the coronal turbulence, for the case when in the equatorial plane the r, θ, and φ directions acquire special significance. Only the following assumptions have been made:

i) $\mu_o \cong$ constant, implying approximate linearity of the basic rays

ii) spherical symmetry

iii) the basic rays be in the vicinity of the solar equatorial plane

iv) anisotropic turbulence with a preference, in the solar equatorial plane, for the r, θ, and φ directions, and a correlation function given by equation (4.16). This implies that equations (4.104) hold.

v) a, b, $d \ll s$

vi) appropriateness of the geometry of page 117 implying a, b, $d \ll |\vec{r}_o|$

vii) appropriateness of the geometry of page 105

Now we would like to be able to utilize equations (4.113)-(4.116) to determine the functions A(r), D(r), b(r), $\langle \delta^2\mu \rangle$, and $\langle f_{ph}^2 \rangle_R$ from observations of $\langle \delta\psi\delta\psi \rangle_R$, $\langle \delta\Omega\delta\Omega \rangle_R$, $\langle \delta f\delta f \rangle_R$, and $\langle \delta t_{gr}\delta t_{gr} \rangle_R$. Unfortunately we have five quantities we would like to determine and only four observational quantities; complete determination of the five quantities is thus impossible. Let us see what can be learned from (4.113)-(4.116) however. We begin by noting that since for the linear rays we are considering $\mu_o \cong 1$ we may ascertain $\langle f_{ph}^2 \rangle_R$ directly from equations (4.115) and (4.116):

$$< f_{ph}^{\;2} >_R = \frac{1}{\omega^2} \frac{< \delta f \delta f >_R}{< \delta t_{gr} \delta t_{gr} >_R} \tag{4.117}$$

Thus $< f_{ph}^{\;2} >_R$ may be determined. As mentioned earlier the interpretation of $< f_{ph}^{\;2} >_R$ in terms of the turbulent behavior of the medium is not our purpose here; we note only that it should be a function of the scale lengths and velocity spectrum of the turbulence.

Let us confine our attention to equations (4.113), (4.114), and (4.116) for $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$, and let us first recall what we found when D(r) = 1, the case of the last section. We there asserted, in anticipation of the numerical examples later in the present chapter, that the effect of a non-zero A(r) on $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$ is to increase these quantities by no more than some 10-15%, and that the effect of anisotropy may be roughly considered to manifest itself only in $< \delta\psi\delta\psi >_R$, such that

$$\frac{a(r = R)}{b(r = R)} = \sqrt{\frac{< \delta\Omega\delta\Omega >_R}{< \delta\psi\delta\psi >_R}} \tag{4.118}$$

Thus we were able to utilize $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$ to ascertain, via equations (4.96) and (4.97), $< \delta^2\mu >$ and (b) as functions of r, while A(r) could be found from $< \delta\Omega\delta\Omega >_R$ and $< \delta\psi\delta\psi >_R$ via equation (4.118). Now by analogy with the case just discussed we expect that in the present case also the effect of a non-zero A(r) will appear only in $< \delta\psi\delta\psi >_R$. We thus approximate equations (4.114) and (4.116) as

$$\frac{<\delta\Omega\delta\Omega>_R}{2\sqrt{\pi}} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu>}{\mu_o^2\, b}\right)_r \frac{1}{D(r)} \frac{r\,dr}{\sqrt{r^2 - R^2}} \tag{4.119}$$

$$\frac{<\delta t_{gr}\delta t_{gr}>_R}{(\sqrt{\pi}/c^2)} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu>b}{\mu_o^4}\right)_r \frac{r\,dr}{\sqrt{r^2 - R^2}} \tag{4.120}$$

These two equations contain the three unknowns $<\delta^2\mu>$, b(r), and D(r), and therefore do not form a complete system. By the same token, since D(r) appears in the equation for $<\delta\Omega\delta\Omega>_R$, we cannot utilize equation (4.118) to estimate A(r). To make any progress we must make some assumptions about the behavior of the coronal turbulence.

Now it should be clear that if we assume the functional form of the behavior with r of any one of the quantities $<\delta^2\mu>$, b(r), A(r) or D(r), then the functional forms of the behaviors with r of the remaining quantities may be found from $<\delta\Omega\delta\Omega>_R$, $<\delta\psi\delta\psi>_R$, and $<\delta t_{gr}\delta t_{gr}>_R$ via equations (4.113), (4.119), and (4.120). In general we will have no apriori information concerning the behavior of b(r) and D(r), and we shall always regard these as unknown quantities. However, let us first suppose that we have some knowledge, postulated or experimental, of the average electron density. Then with the help of equation (4.23) we can postulate the form of the behavior of $<\delta^2\mu>$ with r. Equations (4.119) and (4.120) may then be inverted to allow us to ascertain the behavior with r of b(r) and D(r); the proper inversion is (c.f. equations

(4.96) and (4.97)):

$$\frac{1}{b(r)D(r)} = -\frac{\mu_o^2}{<\delta^2\mu>}\frac{1}{2\pi^{3/2}}\frac{d}{dr}\int_r^\infty \frac{r}{R}<\delta\Omega\delta\Omega>_R \frac{dR}{\sqrt{R^2-r^2}}$$

(4.121)

$$b(r) = -\frac{\mu_o^4}{<\delta^2\mu>}\frac{c^2}{\pi^{3/2}}\frac{d}{dr}\int_r^\infty <\delta t_{gr}\delta t_{gr}>_R \frac{r}{R}\frac{dR}{\sqrt{R^2-r^2}}$$

(4.122)

from which we may determine the functional forms of b(r) and D(r) for a range of r equal to that range of R for which $<\delta\Omega\delta\Omega>_R$ and $<\delta t_{gr}\delta t_{gr}>_R$ have been found. Having thus found the forms of b(r) and D(r), the form of A(r) may be found as follows. If the bulk of the contributions to the integrals of equations (4.113) and (4.119) for $<\delta\psi\delta\psi>_R$ and $<\delta\Omega\delta\Omega>_R$ were concentrated near r = R we could then write (c.f. equation (4.118))

$$\frac{1}{\sqrt{D(r=R)}}\frac{a(r=R)}{b(r=R)} = \sqrt{\frac{<\delta\Omega\delta\Omega>_R}{<\delta\psi\delta\psi>_R}} \qquad (4.123)$$

We expect equation (4.123) to be closely correct, and we thus see that having found the form of D(r) we may use this relationship to find a(r)/b(r).

Alternatively, we may assume that we have some knowledge of the functional form of a(r)/b(r). Then equation (4.123) may be used to ascertain the form of D(r), and equations (4.121) and (4.122) can then be used to find the forms of the behavior with r of b(r) and $<\delta^2\mu>$.

In practice, however, it is expected that better results will be obtained by starting with some knowledge of the behavior of n(r), and thus $< \delta^2 \mu >$, and then deducing the behaviors of b(r), D(r), and a(r)/b(r).

It should be clear from the above discussion that if the functional form of $< \delta^2 \mu >$ is initially assumed known, then observational data of both $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$ are necessary in order to determine the form of D(r). Hewish and Wyndham (1963), however, sought to determine D(r) from measurement only of $< \delta\Omega\delta\Omega >_R$, and therefore found it necessary to assume not only the form of $< \delta^2 \mu >$, but also that of b(r); they took $b(r) \sim r$ and concluded that the solar corona exhibited non-radial outflow, i.e. $D(r) \neq 1$. However, as we shall see, the data cited by them is consistent with the radial outflow model if b(r) = constant beyond some 10 solar radii. The constancy of b(r) in that range is consistent with the results of Hewish and Dennison (1966), leading us to the tentative conclusion that existing radio scattering data supports the radial outflow model. All this serves to point out that care must be taken in the treatment of equations (4.113), (4.114), and (4.116).

One final point. The inversions of equations (4.113), (4.114), and (4.116) may be dealt with exactly if A(r) is sufficiently large so that

$$A \frac{R^2}{r^2} >> 1 \qquad (4.124)$$

over the range of (r) for which $< \delta^2 \mu >$ is of sufficient magnitude to contribute significantly to the integrals. For the model corona specified by equation (2.37) this implies typically

$$\frac{a(r)}{b(r)} \gtrsim 4$$

If (4.124) holds the proper inversions are (c.f. equations (4.87)-(4.89)):

$$\frac{1}{b(r)\,D(r)} = -\frac{\mu_o^2}{r<\delta^2\mu>}\,\frac{1}{2\pi^{3/2}}\,\frac{d}{dr}\int_r^\infty r\,\frac{<\delta\Omega\delta\Omega>_R}{\sqrt{R^2-r^2}}\,dR \qquad (4.125)$$

$$b\;(r) = -\frac{\mu_o^4}{r<\delta^2\mu>}\,\frac{c^2}{\pi^{3/2}}\,\frac{d}{dr}\int_r^\infty r\,\frac{<\delta t_{gr}\delta t_{gr}>_R}{\sqrt{R^2-r^2}}\,dR \qquad (4.126)$$

$$\frac{A(r)}{D(r)} = r^2\,\frac{\dfrac{d}{dr}\displaystyle\int_r^\infty r\,\frac{<\delta\Omega\delta\Omega>_R}{\sqrt{R^2-r^2}}\,dR}{\dfrac{d}{dr}\displaystyle\int_r^\infty r\,\frac{R^2<\delta\psi\delta\psi>_R}{\sqrt{R^2-r^2}}\,dR} \qquad (4.127)$$

These expressions readily yield the functional behavior of b(r), A(r), and D(r), if the form of $<\delta^2\mu>$ is presumed known, for a range of r equal to that range of R for which one has experimental measurements of $<\delta\Omega\delta\Omega>_R$, $<\delta\psi\delta\psi>_R$, and $<\delta t_{gr}\delta t_{gr}>_R$. They are valid subject to (4.124), an assumption which could be checked from the results of the expressions above.

To summarize, we have in this section considered the scattering of radio rays by an anisotropically turbulent solar corona of approximately constant average refractive index, implying nearly linear basic rays, for which the anisotropy exhibits a preference, in the solar equatorial plane, for the r, θ, and φ directions. Our discussion has followed closely that of the previous section, where we considered the situation of preference for the radial direction only, but with the modification that our system of equations is no longer complete, requiring us to assume some apriori knowledge of one of our unknown quantities $< \delta^2\mu >$, b(r), A(r), or D(r); in practice the quantity which will be assumed known will usually be $< \delta^2\mu >$. The primary usefulness of this discussion will be the investigation of the extent to which the coronal outflow is radially directed, under the circumstance that we know beforehand the behavior of one of the quantities governing the statistical behavior of the corona. This circumstance may occur to sufficient accuracy only with some difficulty, and one may instead have to content himself with the assumption of radial outflow and proceed as in the previous section to find, to the degree of accuracy of the assumption of radial outflow, the quantities $< \delta^2\mu >$, b(r), and A(r).

In the next section we shall digress somewhat to consider some of the ideas of this section for the special case when the quantities governing the statistics of the medium exhibit simple power law behavior.

Power Law Behavior

It is our purpose here to discuss some of the foregoing ideas in the circumstance that the quantities $< \delta^2\mu >$, b(r), a/b, and d/b all exhibit power law behavior. If the coronal electron density is supposed to vary as $1/r^n$ we see from equation (4.23) that we may then expect $< \delta^2\mu >$ to vary as $1/r^{2n}$. We shall therefore write

$$< \delta^2 u > = \frac{< \delta^2 \mu >_o}{\left(\frac{r}{R_\odot}\right)^{2n}} \qquad (4.128)$$

For the model coronal electron density specified by equation (2.58) we see that for $(r/R_\odot) \gtrsim 6$ we expect $n = 2$. In similar fashion we shall suppose b(r) to vary as r^m, and write therefore

$$b(r) = b_o R_\odot \left(\frac{r}{R_\odot}\right)^m \qquad (4.129)$$

We shall further suppose a(r)/b(r) to vary as r^α. Then if a/b is large we expect A(r) to vary as $r^{2\alpha}$, and thus we write

$$A(r) = A_o \left(\frac{r}{R_\odot}\right)^{2\alpha} \quad ; \; A(r) >> 1 \qquad (4.130)$$

Finally, we shall suppose d(r)/b(r) to vary as r^Δ, as suggested by Hewish and Wyndham (1963), and we may then write

$$D(r) = \left(\frac{r}{R_\odot}\right)^{2\Delta} \qquad (4.131$$

where it has been implicitly assumed that the coronal outflow is radial at the solar surface, but becomes confined somewhat to the solar equatorial plane ($\Delta < 0$) or to the polar regions ($\Delta > 0$) at greater distances. Equations (4.128)-(4.131) are the assumptions underlying this discussion.

We shall concern ourselves now with equations (4.113), (4.114), and (4.116) for the scattering parameters $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$. For our present purposes we shall assume that $r_1, r_2 >> R$, <u>or</u> that the bulk of the contributions to the integrals occur near r = R; we may then extend the upper limits of integration in (4.113), (4.114), and (4.116) to infinity. We shall furthermore confine our attention to those situations in which $A\left(R^2/r^2\right)$ is either small or large compared to unity over the range of (r) in which significant contributions to the integrals occur. It is then clear that we shall be concerned with integrals of the form

$$\int_R^\infty \frac{1}{r^n} \frac{r\,dr}{\sqrt{r^2 - R^2}}$$

where (n) need not be an integer. This integral may be readily evaluated; the result is

$$\int_R^\infty \frac{1}{r^n} \frac{r\,dr}{\sqrt{r^2 - R^2}} = \frac{\sqrt{\pi}}{2R^{n-1}} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \tag{4.132}$$

where $\Gamma(z)$ is the Gamma function. Now as $\Gamma(z)$ has no zeroes along the real axis, but poles at z = 0, -1, -2, ..., we require for convergence of (4.132) that $n \neq 1, -1, -3, -5 \ldots$ If, however, n = 1, -1, -3, -5 etc., we do not expect our scattering parameters to become infinite, for then we cannot justify extending the limits of integration of (4.113), (4.114), and (4.116) to infinity.

Now we may insert equations (4.128)-(4.131) into our equations for $<\delta\psi\delta\psi>_R$, $<\delta\Omega\delta\Omega>_R$, and $<\delta t_{gr}\delta t_{gr}>_R$ and, with the aid of equation (4.132), carry out the necessary integrations. If $A\frac{R^2}{r^2} \ll 1$ we obtain

$$< \delta\psi\delta\psi >_R = \frac{2\pi \dfrac{\Gamma\left(\frac{2n+m-1}{2}\right)}{\Gamma\left(\frac{2n+m}{2}\right)} \dfrac{< \delta^2\mu >_o}{b_o}}{\left(\frac{R}{R_\odot}\right)^{2n+m-1}} \tag{4.133}$$

$$< \delta\Omega\delta\Omega >_R = \frac{2\pi \dfrac{\Gamma\left(\frac{2(n+\Delta)+m-1}{2}\right)}{\Gamma\left(\frac{2(n+\Delta)+m}{2}\right)} \dfrac{< \delta^2\mu >_o}{b_o}}{\left(\frac{R}{R_\odot}\right)^{2(n+\Delta)+m-1}} \tag{4.134}$$

$$< \delta t_{gr} \delta t_{gr} >_R = \frac{\pi \dfrac{R_\odot^2}{c^2} \dfrac{\Gamma\left(\frac{2n-m-1}{2}\right)}{\Gamma\left(\frac{2n-m}{2}\right)} < \delta^2\mu >_o b_o}{\left(\frac{R}{R_\odot}\right)^{2n-m-1}} \tag{4.135}$$

where we have, since the basic rays are nearly linear, taken $\mu_o \cong 1$. If, on the other hand, we may take $A \frac{R^2}{r^2} >> 1$ we then have

$$< \delta\psi\delta\psi >_R = \frac{2\pi \dfrac{\Gamma\left(\frac{2(n+\alpha)+m-4}{2}\right)}{\Gamma\left(\frac{2(n+\alpha)+m-3}{2}\right)} \dfrac{< \delta^2\mu >_o}{A_o b_o}}{\left(\frac{R}{R_\odot}\right)^{2(n+\alpha)+m-1}} \tag{4.136}$$

$$< \delta\Omega\delta\Omega >_R = \frac{2\pi \dfrac{\Gamma\left(\frac{2(n+\Delta)+m-2}{2}\right)}{\Gamma\left(\frac{2(n+\Delta)+m-1}{2}\right)} \dfrac{< \delta^2\mu >_o}{b_o}}{\left(\frac{R}{R_\odot}\right)^{2(n+\Delta)+m-1}} \tag{4.137}$$

$$< \delta t_{gr} \delta t_{gr} >_R = \frac{\pi \dfrac{R_\odot^2}{c^2} \dfrac{\Gamma\left(\frac{2n-m-2}{2}\right)}{\Gamma\left(\frac{2n-m-1}{2}\right)} < \delta^2\mu >_o b_o}{\left(\frac{R}{R_\odot}\right)^{2n-m-1}} \tag{4.138}$$

where we have again taken $\mu_o \cong 1$. Equations (4.133)-(4.138) are our desired results for $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$ as functions of $(R/R_\odot)$ when the radial anisotropy is either small or large, and the power law relations of equations (4.128)-(4.131) hold. We see that when the coronal turbulence exhibits power law behavior, so also do the observed parameters. Thus if one finds power law behavior of $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$ then one may assume that $< \delta^2\mu >$, b(r), a/b, and (d/b) also behave according to simple power laws. We note also that a(r)/b(r) does not affect $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$; thus the power law behavior of these latter quantities implies simple power law behavior of $< \delta^2\mu >$, b(r), and d/b. Similarly, since the effect of d(r)/b(r) does not appear in $< \delta t_{gr} \delta t_{gr} >_R$ a power law dependence of this quantity alone implies power law behavior of $< \delta^2\mu >$ and b(r).

Now what may be learned from relations (4.133)-(4.138)? Let us first consider only $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$. From equations

(4.134), (4.135), (4.137), and (4.138) we see that the slopes, on a log-log plot,* of $<\delta\Omega\delta\Omega>_R$ and $<\delta t_{gr}\delta t_{gr}>_R$ vs $(R/R_\odot)$ are the same when $A\frac{R^2}{r^2}$ is large as when it is small. This seems to imply an independence of these slopes from the behavior of a/b. Now, as the numerical examples later in this chapter will indicate, we have reason to believe that the independence of the slopes of $<\delta\Omega\delta\Omega>_R$ and $<\delta t_{gr}\delta t_{gr}>_R$ from the behavior of a/b will hold true even when $A\frac{R^2}{r^2}$ is neither small nor large. Thus the slopes of $<\delta\Omega\delta\Omega>_R$ and $<\delta t_{gr}\delta t_{gr}>_R$ provide two relations for the three quantities m, n, and Δ:

$$2(n+\Delta) + m - 1 = - \text{slope} <\delta\Omega\delta\Omega>_R \quad (4.139)$$

$$2n - m - 1 = - \text{slope} <\delta t_{gr}\delta t_{gr}>_R \quad (4.140)$$

If the coronal outflow is assumed radial ($\Delta = 0$) these equations allow determination of (n) and (m), that is the functional behavior of n(r) and b(r). If, on the other hand, we presume to know the functional form of the electron density, i.e. if we know (n), these equations then enable us to determine (m) and (Δ). If we thereby find $\Delta \neq 0$ we may be led to question the radial outflow model. Now regardless of how we may choose to interpret the slopes of $<\delta\Omega\delta\Omega>_R$ and $<\delta t_{gr}\delta t_{gr}>_R$, once they are determined the appropriate values may be used in the arguments of the Gamma functions appearing in (4.134) and (4.135), <u>or</u> (4.137) and (4.138), and the values of $<\delta^2\mu>_o$ and b_o may be then readily found from the observational data. The only problem is

*Future references to slopes will imply a log-log plot.

whether to use the first pair of equations or the second, and this depends on whether we suppose $A\frac{R^2}{r^2}$ to be large or small. This may be ascertained by examination of $< \delta\psi\delta\psi >_R$, as we shall see. At any rate, the difference in the obtained values of $< \delta^2\mu >_o$ and b_o for the two cases is not great.

We now examine equations (4.133) and (4.136) for $< \delta\psi\delta\psi >_R$. We first note that if we presume $A\frac{R^2}{r^2} << 1$ we have

$$2n + m - 1 = - \text{slope} < \delta\psi\delta\psi >_R \tag{4.141}$$

a relation which, when combined with equations (4.139) and (4.140), enables us to determine m, n, and Δ. If, on the other hand, we presume $A\frac{R^2}{r^2} >> 1$ we may write

$$2(n + \alpha) + m - 1 = - \text{slope} < \delta\psi\delta\psi >_R \tag{4.142}$$

This equation is to be considered together with (4.139) and (4.140). If we assume the coronal outflow to be radial ($\Delta = 0$) we may use these three equations to find n, m, and α. Alternatively, we may presume to know the behavior of the coronal electron density, that is we may presume to know (n), and equations (4.139), (4.140), and (4.142) allow us to find m, Δ, and α. Finally, we may presume to know that a/b is constant over some range of (r) so that $\alpha = 0$, and our three equations will then enable us to ascertain n, m, and Δ. This latter assumption is of some importance, for the subsequent numerical examples indicate that if a/b is constant, then even if $A\frac{R^2}{r^2}$ is neither large nor small we may still employ equation (4.141) which, together with (4.139) and (4.140), allows determination of n, m, and Δ. Thus we have given some insight into how, with one of several possible appropriate assumptions, we might determine n, m, α, and Δ. But we still must answer how we can ascertain whether $A \frac{R^2}{r^2}$ is large, small, or neither. We can proceed as follows. We might first assume $A \frac{R^2}{r^2} >> 1$. Then from equations (4.136) and (4.137) and the observed values of $< \delta\psi\delta\psi >_R$ and $< \delta\Omega\delta\Omega >_R$ we may determine A_o. If the value so determined is consistent with $A \frac{R^2}{r^2} >> 1$, we have made the correct guess. Alternatively, we may assume $A \frac{R^2}{r^2} << 1$. Then equations (4.133) and (4.134) may each be used to determine $< \delta^2\mu >_o/b_o$. If the two values so obtained agree we have made the proper guess. If, however, neither of these assumptions yields consistency, then $A \frac{R^2}{r^2}$ is neither small nor large, and we must then content ourselves to use equation (4.123) to find A(r), or, alternatively, compare numerical calculations based on equations (4.113) and (4.114) with the actual data.

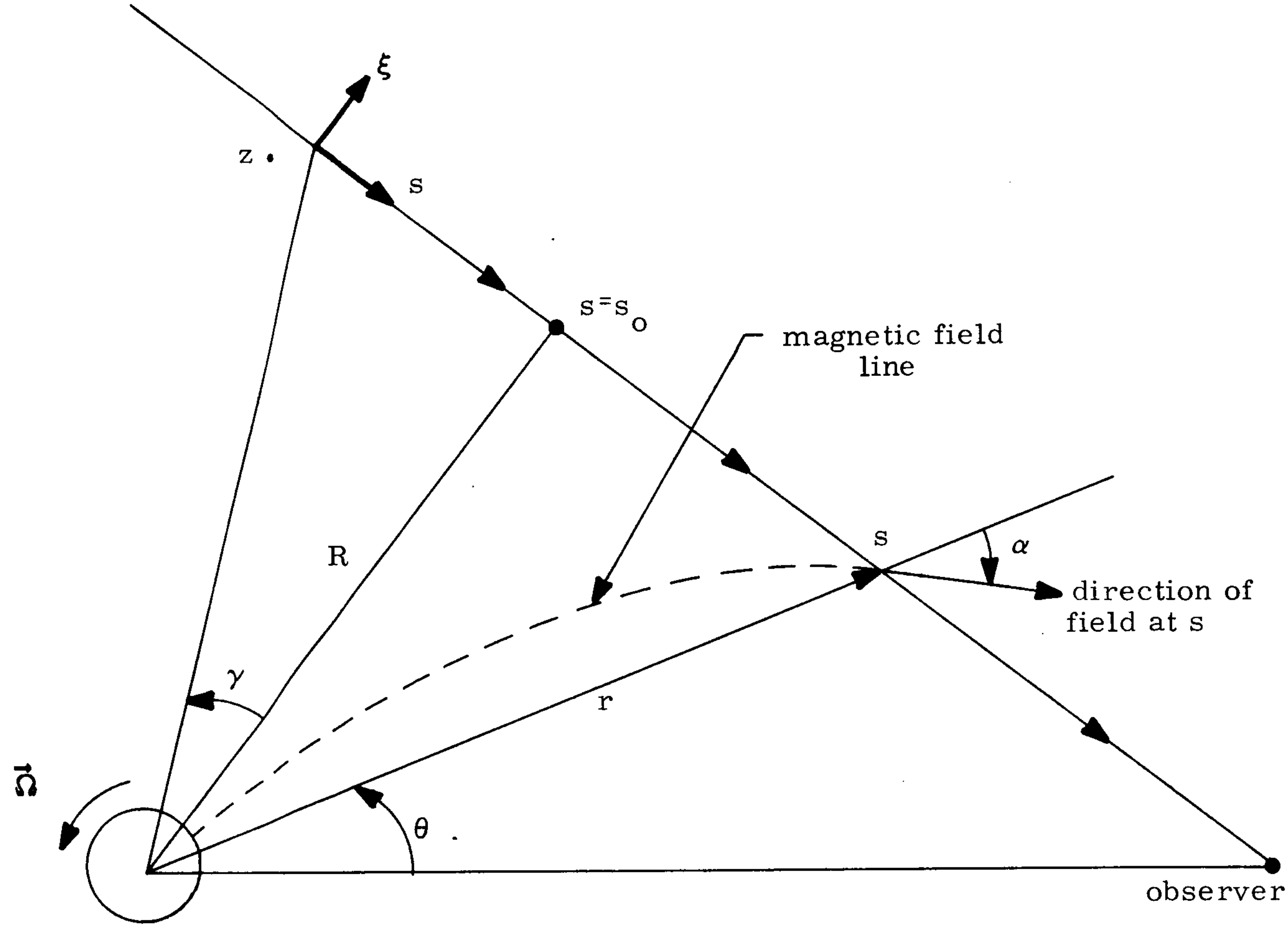
Thus we have in this section discussed the scattering of radio waves about nearly linear basic rays for the special case when the parameters governing the statistics of the medium exhibit simple power law behavior. We have found that then the scattering parameters $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr}\delta t_{gr} >_R$ can be expected to exhibit power law behavior also; we thus expect these parameters to obey a power law if $R \gtrsim 6 R_{\odot}$, where, for the model corona specified by equation (2.37), we expect $n(r) \sim 1/r^2$. We were then successful in understanding how from observations of $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, and $< \delta t_{gr}\delta t_{gr} >_R$ we might determine $< \delta^2\mu >$, b(r), a(r)/b(r), and d(r)/b(r). The value of this discussion should be apparent when we realize that the available data for the scattering of astronomical sources indicates power law behavior beyond some 10 solar radii (see, for example, Hewish and Wyndham (1963)).

We turn now to consider the effects of magnetic field curvature.

Magnetic Field Line Curvature: Solar Equatorial Plane

In this section we consider the forms taken by the scattering integrals when the curvature of the general solar magnetic field lines becomes important. At the outset of this chapter we suggested that the field is no longer radial beyond some 30 $R_\odot$. But how will this affect the radio scattering? It seems reasonable to suppose that if anisotropy in the turbulence is due to enhanced particle diffusion along the magnetic field direction, as is believed to be the case at least in the lower coronal regions where filamentary structures are optically visible and appear to be due to the general and local magnetic fields, then the coronal turbulence will exhibit a preference not for the radial direction, but for the direction of the (curved) magnetic field. Thus since the coronal turbulence can be expected to exhibit a preference for the direction of the non-radial field, we might expect to see this effect manifested in the scattering phenomena. It is this possibility we wish to examine in the present chapter. For simplicity we shall restrict our discussion to those situations where the basic rays may be regarded as lying in the vicinity of the solar equatorial plane; this provides the maximum effect and the simplest geometry.

We begin with the following geometry appropriate to our discussion:



With (α) and (θ) thus defined we may take as our appropriate statistical correlation function that of equation (4.17):

$$< \delta\mu(\vec{r}_1)\,\delta\mu(\vec{r}_2) > =$$

$$< \delta^2\mu >_{r_1} \exp\left\{-\frac{1}{a^2}\left[\cos\alpha\,(r_2 - r_1) - (\sin\alpha)\,r_1\,(\theta_2 - \theta_1)\right]^2 - \frac{r_1^{\,2}\,(\varphi_2 - \varphi_1)^2 + \left[\sin\alpha\,(r_2 - r_1) + (\cos\alpha)\,r_1\,(\theta_2 - \theta_1)\right]^2}{b^2}\right\} \qquad (4.143)$$

where (a) is the correlation length in the magnetic field direction, (b) the correlation length in the transverse direction, and $< \delta^2\mu >_{r_1}$ the correlation amplitude; these quantities are to be regarded as functions of (r_1), corresponding to the assumption of spherical symmetry. The validity of (4.143) implies satisfaction of conditions (4.15) and (4.18), as well as restriction to the solar equatorial plane.

Now equation (4.143) can be of use only after transformation from the (r, θ, φ) coordinate system to the (s, ξ, z) system. Equations (4.69) apply, and insertion of these into (4.143) yields)

$$
\begin{aligned}
&< \delta\mu(\vec{r}_1)\, \delta\mu(\vec{r}_2) > \\
&= < \delta^2\mu >_{r_1} \exp\Big\{ - (s_2 - s_1)^2 \left(\frac{\sin^2(\gamma - \alpha)}{a^2} + \frac{\cos^2(\gamma - \alpha)}{b^2} \right) \\
&- \left(\xi_2 - \xi_1\right)^2 \left(\frac{\cos^2(\gamma - \alpha)}{a^2} + \frac{\sin^2(\gamma - \alpha)}{b^2} \right) - \left(\frac{z_2 - z_1}{b} \right)^2 \qquad (4.144) \\
&+ 2 \sin(\gamma - \alpha) \cos(\gamma - \alpha)\, (s_2 - s_1)(\xi_2 - \xi_1) \left(\frac{1}{a^2} - \frac{1}{b^2} \right) \Big\}
\end{aligned}
$$

Now we note that this function is identical to that in equation (4.70), where the preference was for the radial direction, but with (γ) replaced by $(\gamma - \alpha)$. This is a convenient result, for it permits us to use all the results of the discussion where the preference was for the radial direction, but with (γ) replaced by $(\gamma - \alpha)$. Thus from equations (4.77) and (4.79)-(4.81) we readily obtain:

$$\frac{<\delta\psi\delta\psi>_R}{2\sqrt{\pi}} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu>}{\mu_o^2 b}\right)_r \frac{(1+A)^{1/2}}{\left(1 + A\cos^2(\gamma-\alpha)\right)^{3/2}} \frac{rdr}{\sqrt{r^2 - R^2}} \quad (4.145)$$

$$\frac{<\delta\Omega\delta\Omega>_R}{2\sqrt{\pi}} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu>}{\mu_o^2 b}\right)_r \frac{(1+A)^{1/2}}{\left(1 + A\cos^2(\gamma-\alpha)\right)^{1/2}} \frac{rdr}{\sqrt{r^2 - R^2}} \quad (4.146)$$

$$\frac{<\delta f \delta f>_R}{\frac{4\pi^{5/2}}{\lambda_o^2} < f_{ph}^2 >_R} = \left[\int_R^{r_1} + \int_R^{r_2}\right] (<\delta^2\mu> b)_r \frac{(1+A)^{1/2}}{\left(1 + A\cos^2(\gamma-\alpha)\right)^{1/2}} \frac{rdr}{\sqrt{r^2 - R^2}}$$

(4.147)

$$\frac{<\delta t_{gr}\delta t_{gr}>_R}{\sqrt{\pi}/c^2} = \left[\int_R^{r_1} + \int_R^{r_2}\right] \left(\frac{<\delta^2\mu> b}{\mu_o^4}\right)_r \frac{(1+A)^{1/2}}{\left(1 + A\cos^2(\gamma-\alpha)\right)^{1/2}} \frac{rdr}{\sqrt{r^2 - R^2}}$$

(4.148)

where, corresponding to our assumption of spherical symmetry, $<\delta^2\mu>$, (b), and (A) are functions of (r), and the geometry of page is assumed to apply. The angle (γ) is specified by

$$\gamma = \tan^{-1} \frac{s_o - s}{R} \quad (4.149)$$

while (α) may be any function of position along the basic ray. For a simple spherical outflow model, however, we may write

$$\alpha = \tan^{-1} \frac{r\Omega}{V} \quad (4.150)$$

(Parker (1958)) where V is the (constant) coronal outflow velocity, and Ω is the angular rate of rotation of the solar equatorial regions. Equations (4.145)-(4.148) are the desired results, expressing the mean squares of the scattering parameters $\delta\varphi$, $\delta\Omega$, δf, and δt_{gr} as integrals over a basic ray of the parameters governing the statistical properties of the coronal turbulence for the case where a non-radial solar magnetic field is effective in defining the turbulent anisotropy. They assume specifically the geometry of page 150 but if the basic ray is on the "other side" of the sun equations (4.145)-(4.150) all hold, but with (R) considered negative in (4.149) only.

Now we would seek to invert these equations to obtain $< \delta^2\mu >$, A, b(r), and α(r) as functions of the observed quantities $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, $< \delta f \delta f >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$. There is little we can do along this line however. If we note that $\mu_o \cong 1$ equations (4.147) and (4.148) yield

$$< f_{ph}{}^2 >_R = \frac{1}{\omega^2} \frac{< \delta f \delta f >_R}{< \delta t_{gr} \delta t_{gr} >_R} \tag{4.151}$$

a relation which could be of some use in examining coronal velocities if we knew the correlation length a(r). Any further statements we make will rely on what may be learned from the numerical examples of the next section. We shall there see that for a reasonable model of the magnetic field curvature the values of $< \delta\Omega\delta\Omega >_R$ and $< \delta t_{gr} \delta t_{gr} >_R$ appear not to be affected by anisotropy and therefore the approximate relations (4.96) and (4.97) may be used to find $< \delta^2\mu >$ and b(r). In addition, for $R/R_\odot$ less than about 100 it appears that $< \delta\psi\delta\psi >_R$ is not affected by anisotropy very much differently than when there is no field line curvature. Thus to within an accuracy of some 20% equation (4.98) may be used to find a/b. (If $A\, R^2/r^2 >> 1$ better accuracy may be obtained by use of equations (4.87)-(4.89)). However, it should be mentioned that for $R/R_{\odot} \lesssim 100$ the effect of the field line curvature seems to be to change the slope of $< \delta\psi\delta\psi >_R$ slightly indicating that $< \delta\psi\delta\psi >_R$ is not a good quantity to use in assessing the validity of the radial outflow model (see the previous two sections). More significant effects of the field curvature are observed beyond about 100 $R_{\odot}$, as shall be seen in the next section.

This case will be considered no further at present, but we shall return to it in the numerical examples of the next section.

Numerical Examples

We conclude the Chapter with a brief presentation of the results of machine evaluations of the scattering integrals of this chapter. We shall suggest several reasonable models for the behavior of $< \delta^2\mu >$, a(r)/b(r), and b(r), and discuss the resultant scattering in terms of the present data and the methods already proposed for deducing the parameters governing the coronal statistics from the scattering observations.

We present first the models used in the calculation, beginning with $< \delta^2\mu >$. We had earlier, equation (4.23),

$$< \delta^2\mu > = \frac{\omega_{po}^4}{4\mu_o^2\omega^4} \frac{< \delta^2 n >}{n^2} \qquad (4.152)$$

where

$$\omega_{po}^2 = \frac{4\pi e^2 n}{m} \qquad (4.153)$$

Now we expect $< \delta^2 n >$ to be some fraction of n^2:

$$< \delta^2 n > = \epsilon^2 n^2 \tag{4.154}$$

thus defining ϵ. Equation (4.152) becomes then

$$< \delta^2 \mu > = \epsilon^2 \frac{\omega_{po}^4}{4\mu_o^2 \omega^4} \tag{4.155}$$

We should mention that we expect ϵ to differ from unity if the root mean square of the local electron density fluctuations differ from the average density or, alternatively, if the density fluctuations do not fill all space, but are rather distributed "spottily" along a ray. Now for the average electron density we shall confine ourselves to the coronal model used throughout this work:

$$n(\rho) = 10^8 \left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2} \tag{4.156}$$

Equations (4.153), (4.155), and (4.156) define $< \delta^2 \mu >$.

We shall next specify a/b. The simplest situation is

$$\frac{a}{b} = \text{constant}$$

and several calculations shall be made for this case. However, we several times pointed out that we expect anisotropy to be the result of enchanced particle mobility along magnetic field lines. If we regard a statistical density fluctuation as appearing somewhere in the corona and then being carried outward by the general outflow, it then appears that the enhanced mobility along the field lines will result in a/b increasing with (r). How rapid will this increase be? If the correlation length in the direction of the magnetic field line is determined by the rapid motion of particles along the field we may then write approximately, regarding the corona as collisionless and the pressures parallel and perpendicular to the magnetic field as equal,

$$\frac{da}{dt} \simeq \sqrt{\frac{kT}{m_p}}$$

where the proton mass, m_p, appears since the faster, but less massive, electrons in the fluctuation are, over a Debye length, "linked" to the protons. Since the coronal Debye length is expected to be small compared to the fluctuation scale size the heavier protons will dominate the motion of a density fluctuation. If we now denote by V the velocity at which the fluctuation is carried outward we may write then

$$\frac{da}{dr} \simeq \frac{1}{V} \sqrt{\frac{kT}{m_p}} \qquad (4.157)$$

a relationship having meaning only if

$$V > \sqrt{\frac{kT}{m_p}}$$

we shall see that this relationship is satisfied for a reasonable coronal model. If we take $T = 10^6\ {}^{o}K$ and V in km/sec (4.157) becomes

$$\frac{da}{dr} \simeq \frac{90}{V} \qquad (4.158)$$

(Now we expect V to be on the order of several hundred km/sec (Parker (1960b)) showing that we may consider the inequality above to be satisfied.) Introducing the quantity

$$\rho = \frac{r}{R_\odot}$$

equation (4.158) becomes

$$\frac{da}{d\rho} = \frac{630 \times 10^5}{V}$$

If for V we take 100 km/sec, a reasonable value for the mass efflux above the solar surface, we obtain

$$\frac{da}{d\rho} = 630 \times 10^3 \text{ km} \tag{4.159}$$

Now denoting by (a_o) the value of a(r) at the solar surface equation (4.159) yields upon integration

$$a = 630 \times 10^3 (\rho - 1) + a_o \tag{4.160}$$

Thus we expect (a) to increase very rapidly with ρ, explaining the formation of coronal filaments very low in the corona. But what about a/b? Since we expect the transverse correlation length to be nearly constant, or to increase only slowly with ρ, we therefore expect a/b to increase rapidly with ρ also. Now as it is not our purpose here to model accurately the behavior of a/b (indeed, this is an important area for future research but not within the scope of this work) we shall merely suggest several forms for a/b which increase with ρ, which assume isotropy of fluctuations formed at the solar surface (i.e., $a(r = R_\odot) = b(r = R_\odot)$), and which are convenient for computation. We choose

$$\frac{a}{b} = .1 (\rho - 1) + 1 \tag{4.161}$$

$$\frac{a}{b} = \rho \tag{4.162}$$

$$\frac{a}{b} = 10\,(\rho - 1) + 1 \tag{4.163}$$

Thus we have made plausible values of a/b which increase with distance from the Sun. However, existing observations seem to indicate values of a/b which are either constant, or may perhaps decrease with ρ beyond some 10 solar radii (Hewish (1958), Gorgolewski and Hewish (1960), Hogbom (1960), Erickson (1964)). This seems to imply an instability of the "streamers" formed low in the corona. We therefore take for our numerical examples a number of convenient forms for a/b which decrease with increasing ρ, approaching unity (isotropy) as $\rho \to \infty$. We choose:

$$\frac{a}{b} = 1 + \frac{1}{\rho} \tag{4.164}$$

$$\frac{a}{b} = 1 + \frac{10}{\rho} \tag{4.165}$$

$$\frac{a}{b} = 1 + \frac{100}{\rho} \tag{4.166}$$

Equations (4.161)-(4.166) represent forms convenient for computation, and not proposed to represent properly the coronal behavior. We expect that the actual behavior will exhibit anisotropy rapidly increasing immediately above the solar surface but, due to instability, shortly decreasing to constant values of a/b at higher levels in the corona. The proper definition of this behavior is an area for further study, both theoretical and experimental. Theoretical studies of this point are essentially non-existent, as are high frequency scattering observations through the lower corona where, due to the very rapid initial increase of a/b, the instability might be expected to occur.

We next specify the kinds of behavior we might expect for the transverse correlation length, b(r). Hewish and Dennison (1966) observed a constant correlation length of 200 kilometers between R = .4 AU and R = .8 AU, and we shall take therefore as one of our examples

$$b(r) = 200 \text{ km} \tag{4.167}$$

Hewish and Dennison point out that this value is not much greater than the proton gyro-radius at those distances; we pursue this suggestion here. The root-mean-square proton gyro radius is

$$R_g = \frac{\sqrt{< v_1^2 >}}{e\,B/m_p} \tag{4.168}$$

Letting

$$\frac{1}{2}\, m < v_1^2 > = kT$$

we obtain

$$R_g = \frac{\sqrt{2kT}}{\dfrac{eB}{m_p^{1/2}}} \tag{4.169}$$

Now since the curvature of the general solar magnetic field lines is not great over most of the corona within the radius of the earth, we may to a good degree of approximation write

$$B = \frac{B_o}{\rho^2} \tag{4.170}$$

where B_o is the strength of the general solar magnetic field at the solar surface, and is on the order of one gauss. Combining equations (4.169) and (4.170) yields

$$R_g = \frac{\rho^2 \sqrt{2kT}}{\frac{e B_o}{m_p^{1/2}}} \tag{4.171}$$

The coronal temperature may be most simply specified by assuming that coronal heating maintains a constant temperature, T_o, out to $\rho = \rho_o$, and adiabatic expansion beyond. If we assume a specific heat ratio of 5/3 we obtain

$$R_g = \rho_o^{2/3} \frac{\sqrt{2kT_o}}{\frac{e B_o}{m_p^{1/2}}} \cdot \rho^{4/3} \tag{4.172}$$

Thus we obtain the important result that in the region of coronal adiabaticity $R_g \sim \rho^{4/3}$. Now if we take $T_o = 10^6$ °K and require, as indicated by direct satellite measurement, a temperature of 10^5 °K in the vicinity of the earth, we find then $\rho_o \cong 6$. Then with $B_o = 1$ gauss equation (4.172) yields

$$R_g = .045 \rho^{4/3} \text{ km} \tag{4.173}$$

At .6 AU ($\rho = 120$), where Hewish and Dennison found a correlation length of some 200 kilometers, we obtain $R_g = 27$ kilometers, a quantity smaller by a factor of about 7 than the observed correlation length;

it thus appears that Hewish & Dennison did not take into account the adiabatic variation of temperature. In spite of this discrepancy we shall retain the functional dependence on $\rho^{4/3}$, but to make our value coincide with that observed near .6 AU we shall take

$$b(\rho) = .30\,\rho^{4/3} \text{ km} \qquad (4.174)$$

The third example we shall choose for the transverse correlation length , $b(\rho)$, is a correlation length directly proportional to ρ, corresponding to the notion used by many workers that the turbulence behaves as, and corresponds to, the coronal rays or streamers. We will use

$$b(\rho) = 30\,\rho \text{ km} \qquad (4.175)$$

a figure corresponding to the scale size of the photospheric (ρ=1) microturbulence (Kuiper, ed., "The Sun," University of Chicago Press, (1953), pages 28, 175-176).

Having thus specified $< \delta^2\mu >$, a/b, and (b) as functions of ρ for a number of interesting cases, we need only specify $\alpha(r)$ to complete this prelude to the numerical calculations per se. For a simple coronal model we may write for the equatorial plane:

$$\alpha = \tan^{-1}\left(\frac{r\Omega}{V}\right) \qquad (4.176)$$

where Ω is the angular rate of the solar rotation and V the (constant) outflow velocity of the solar wind. We shall in our calculations take $\Omega = 14.38^{\circ}$/day and V = 300 km/sec.

To recapitulate briefly, we have specified the following quantities as prelude to the numerical calculations to follow. $< \delta^2\mu >$ has been specified as a function of ρ to correspond to the model average corona we have been using throughout this work, equation (2.49). The coronal anisotropy, a/b, has been chosen as: i) constant; ii) increasing with ρ, but isotropic at the solar surface; iii) decreasing as ρ increases, approaching isotropy as $\rho \rightarrow \infty$. Case (ii) is consistent with the enhanced mobility along the magnetic field of particles released at the solar surface, while case (iii) implies the existence of a mechanism which tends to destroy pronounced anisotropy. The transverse correlation length, (b), has been chosen as: i) constant, corresponding to the observations of Hewish and Dennison; ii) proportional to $\rho^{4/3}$, that is, proportional to the proton gyro-radius in an adiabatically expanding corona with an approximately radial magnetic field; iii) proportional to ρ, corresponding to the behavior of coronal rays and streamers. Finally, the magnetic field curvature has been specified for a constant solar wind velocity of 300 km/sec. Once again, we are reminded that the functions thus chosen are primarily for computational convenience, and are intended to bear only some suggestion of physical reality; this reflects the fact that a great deal of work yet remains to be done on the physics of the coronal turbulence per se; but this is not our purpose here and we proceed now with the numerical examples.

All our calculations will be based on equations (4.145)-(4.148) for $< \delta\psi\delta\psi >_R$, $< \delta\Omega\delta\Omega >_R$, $< \delta f \delta f >_R$, and $< \delta t_{gr} \delta t_{gr} >_R$; in all cases we shall consider both source and observer to be 1 AU from the sun, i.e. $r_1 = r_2 = 200 \ R_\odot$. We shall seek to see what values of the scattering parameters might be observed for the forms of $< \delta^2\mu >$, a/b, (b), and (α) specified above. In this connection we must recall that $< \delta t_{gr} \delta t_{gr} >_R$ does not represent an observed quantity; if $< \delta t_{gr} \delta t_{gr} >_o$ denotes the value of $< \delta t_{gr} \delta t_{gr} >$ we might actually expect to observe, we may write then

$$< \delta t_{gr} \delta t_{gr} >_o = < \delta t_{gr} \delta t_{gr} >_R + \frac{R^4}{c^2} \left(\frac{d\mu_o}{dr}\right)^2_{r=R} F(n, m) < \delta\psi\delta\psi >_R \qquad (4.177)$$

as has been derived in the Appendix (equations (A. 53), (A. 62), (A. 63)) for $n(r) \sim 1/r^n$ and $b(r) \sim r^m$. The function F(n, m) generally lies between 1 and 10 and is defined by equations (A. 62) and (A. 63). As is discussed in the Appendix, the final term can be quite large, depending on R and ω, and the values of $< \delta t_{gr} \delta t_{gr} >_R$ we calculate numerically here represent therefore a lower limit to the values of $< \delta t_{gr} \delta t_{gr} >$ we expect to observe. We calculate $< \delta t_{gr} \delta t_{gr} >_R$ numerically, though, rather than $< \delta t_{gr} \delta t_{gr} >_o$, since the former quantity is that used above in the body of this chapter.

Before presenting the numerical results we make one simplifying observation. In equations (4. 145)-(4. 148) we may, for the linear basic rays under consideration, neglect the frequency dependence of $\mu_o(r)$. Then the frequency dependence of the scattering integrals is due solely to that of $< \delta^2\mu >$, and by equation (4. 152) we may readily conclude:

$$\sqrt{<\delta\psi\delta\psi>_R} \sim \frac{1}{f^2\sqrt{b}}$$

$$\sqrt{<\delta\Omega\delta\Omega>_R} \sim \frac{1}{f^2\sqrt{b}}$$

$$\sqrt{<\delta t_{gr}\delta t_{gr}>_R} \sim \frac{\sqrt{b}}{f^2} \qquad (4.178)$$

$$\sqrt{<\delta f\delta f>_R} \sim \frac{\sqrt{b}}{f}$$

These relationships allow us to do the numerical calculations for one frequency only (100 MHz), the values for other frequencies being readily found from equations (4.178).

We begin by examining the scattering effects of coronal turbulence exhibiting a preference for the radial direction only ($\alpha = 0$), and uniform anisotropy (a/b = constant). Graphs 3-11 present values of the scattering parameters $\sqrt{<\delta\psi\delta\psi>_R}$, $\sqrt{<\delta\Omega\delta\Omega>_R}$, $\sqrt{<\delta t_{gr}\delta t_{gr}>_R}$, and $\sqrt{<\delta f\delta f>_R}$ for constant values of a/b ranging from 1 to 40. We may make the following observations:

i) The slopes of the calculated curves for $R/R_\odot > 6$ (where, according to equation (4.156), the electron density obeys a simple power law) are those expected on the basis of equations (4.139)-(4.141) with $n = 2$.

ii) A constant value of a/b does not affect, in the region where a power law behavior description is appropriate, the slope of $\sqrt{<\delta\psi\delta\psi>_R}$, in accordance with equation (4.142).

iii) Even for large values of a/b the effect of anisotropy is to increase $\sqrt{<\delta\Omega\delta\Omega>_R}$, $\sqrt{<\delta t_{gr}\delta t_{gr}>_R}$, and $\sqrt{<\delta f\delta f>_R}$ by only some

10-15%. This observation has been the basis of numerous approximations during the course of this chapter; for example, it formed the basis upon which equations (4.93)-(4.95) have been written.

Graphs 12-14 display, for a variety of (constant) values of a/b, the anisotropy in angle of arrival (i.e. $\sqrt{< \delta\Omega\delta\Omega >_R} / \sqrt{< \delta\psi\delta\psi >_R}$) to be expected; shown also for comparison are the values we would expect if equation (4.98) were strictly true. It is generally seen that:

iv) For values of a/b less than about 5 (otherwise we would use equation (4.89)) use of equation (4.98) to estimate a/b will result in less than about 25% error. This error becomes less the more rapidly $b(\rho)$ increases with ρ, in accordance with our expectations, for then a more significant part of the contributions to the integrals for $< \delta\psi\delta\psi >_R$ and $< \delta\Omega\delta\Omega >_R$ occurs in the vicinity of $r = R$, and this was the assumption upon which (4.98) was based.

Now how do the calculated values of Graphs 3-11 compare with the existing data? This is in general difficult to ascertain in view of the wide variabilities that occur in the observations in the course of a solar cycle. However, several remarks may be made. First, the root-mean-square fluctuations in ray angle of arrival seem to scale accurately as $1/f^2$ (Hewish (1958)), in accordance with equations (4.178). Second, simple power law behavior is observed for $R/R_\odot \gtrsim 10$, as might be expected from the electron density function of (4.156). The slopes of $\sqrt{< \delta\Omega\delta\Omega >_R}$ and $\sqrt{< \delta\psi\delta\psi >_R}$ generally seem to lie in the vicinity of -1.5 (Hewish and Wyndham (1963)), but show some steepening towards sunspot maximum (Hewish and Wyndham (1963), Erickson (1964)). The slope of -1.5 is consistent with a mean coronal electron density varying as $1/r^2$ and a constant correlation length, (b). Thus, since, according to Parker's solar wind model, we have reason to expect a $1/r^2$ dependence of the electron density for $R/R_\odot > 10$ it appears that the observations of Hewish and Dennison (1966) of a constant correlation length between .4 and .8 AU may in fact be extrapolated down to at least 10 solar radii. The apparent steepening towards solar maximum is not so readily explained, however. It may suggest that as solar activity increases the coronal structure becomes more filamentary, i.e. (b) becomes more nearly proportional to ρ. This notion is consistent with active regions on the sun producing far reaching coronal filaments through particle ejection. Alternatively, the apparent steepening may be due to a steeper radial gradient of the electron density. The $1/r^2$ dependence follows from the conservation of mass, nVr^2 = constant, when the solar wind velocity is constant. A constant V is expected on the basis of Parker's work for reasonable coronal models (Parker (1960b)). However, it is also a consequence of Parker's model that an increase in the coronal heating results in an increase in the distance from the sun of the transition from a region where V increases with (r) to the region where V is nearly constant. This transition occurs in the vicinity of $\rho = \rho_o$, that is, where the adiabatic expansion begins. Now from the conservation of mass, nVr^2 = constant, it is apparent that if V cannot be considered as constant, but increases with (r), then the density will exhibit a steeper radial gradient, resulting in the observed steepening of the fluctuations in angle of arrival. However, we found above that $\rho_o = 6$, and it is unlikely that even at solar maximum ρ_o could increase to such an extent as to cause the steepening in $\sqrt{<\delta\Omega\delta\Omega>_R}$

and $\sqrt{<\delta\psi\delta\psi>_R}$ to be observed as far out as some 100 solar radii. Finally, the observed steepening could be the result of the coronal outflow becoming more confined to the solar polar regions during solar maximum (see equations (4.131) and (4.139)), but this possibility seems unlikely. In short, it appears that the observed steepening of $\sqrt{<\delta\psi\delta\psi>_R}$ and $\sqrt{<\delta\Omega\delta\Omega>_R}$ during solar maximum can be best explained by assuming that the corona tends to become more filamentary during solar maximum, but further observations, particularly of the type discussed in this chapter, will be necessary before any definite conclusions may be drawn.

Finally, the available data for the fluctuations in angle of arrival seem generally to indicate an (ϵ) of about .1 for the case where b = 200 kilometers (Hewish (1958), Slee (1959), Högbom (1960), Erickson (1964)). This discrepancy may be due to a mean electron density less than that of equation (4.156), a correlation length greater than the 200 km. assumed, $<\delta^2 n>/n^2 < 1$, or to a spotty distribution of the regions of coronal turbulence. Which of these alternatives is true can only be answered by future observations of the type described.

At present no data exist for $<\delta t_{gr}\delta t_{gr}>_R$, but there is some data for $<\delta f \delta f>_R$ obtained from a solar occultation of Mariner IV (Goldstein (1967)). At $R/R_\odot = 4$ the signal bandwidth increased by some 5 Hz at 2295 MHz. Now from Graph 5 , with $\epsilon = .1$ to coincide with the data on fluctuations of angle of arrival, it is readily seen that if b = 200 km this line broadening implies

$$\sqrt{<f_{ph}^2>_{\frac{R}{R_\odot}=4}} = .05 \text{ sec.}^{-1} \quad . \qquad (4.179)$$

which we believe implies in turn that

$$\frac{V\ (r = 4\ R_\odot)}{a\ (r = 4\ R_\odot)} = .05\ \text{sec.}^{-1} \tag{4.180}$$

If for $V(r = 4\ R_o)$ we take 100 km/sec (Parker (1960b)) we find that (4.180) implies

$$\frac{a\ (r = 4\ R_\odot)}{b\ (r = 4\ R_\odot)} = 10 \tag{4.181}$$

a not impossible result. We shall pursue this no further, detailed measurements being unavailable and the connection between $< f_{ph}^{\,2} >$ and V being only postulated.

One final point will be made in connection with Graphs 3-11. From Graphs 4, 7, and 10, it appears that at $R/R_\odot = 2.0$ we may expect

$$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{< \delta t_{gr} \delta t_{gr} >_{R = 2R_\odot}} \lesssim 10^{-3}\ \text{sec.}$$

Let us see what this says about the general relativity experiment suggested by Shapiro (1964, 1966). If we take $\epsilon = .1$ and $f = 8350$ MHz we obtain

$$\sqrt{< \delta t_{gr} \delta t_{gr} >_{R = 2R_\odot}} \lesssim 1.4 \times 10^{-8}\ \text{sec.}$$

a figure well below the 1.6×10^{-4} sec. gravitational delay Shapiro expects to measure. However, $\sqrt{< \delta t_{gr} \delta t_{gr} >_R}$ is not the quantity which we expect to observe. But from equation (4.177) we see that at 8350 MHz and $R/R_{\odot} = 2.0$ we may expect $\sqrt{<\delta t_{gr}\delta t_{gr}>_{R=2R_{\odot}}}$ to closely represent the observed value, and we may therefore conclude that statistical fluctuations due to coronal inhomogeneities should, at 8350 MHz, not affect Shapiro's experiment. The effects at lower frequencies could, however, be appreciable.

Similarly, let us see to what extent fluctuations in the arrival time of a signal pulse may be expected to affect the Sunblazer experiment, where it is suggested that measurements of the relative delay between pulse signals on carriers of different frequency can be used to deduce the average coronal electron density (Harrington (1965)). For the model corona specified by equation (2.58) the relative delay due to the integrated electron density along the path may be shown for carrier signals at 100 and 300 MHz to be $(R/R_{\odot} > 1.5)$

$$\Delta t_{12}^{(e)} = \left\{ \frac{1.6}{(R/R_{\odot})^5} + \frac{.027}{(R/R_{\odot})} \right\} \text{ seconds}$$

Now from equation (4.177) and the sample calculations of Graphs 4, 7, and 10 it is evident that $\sqrt{<\delta t_{gr}\delta t_{gr}>_o}$ falls off with distance from the sun at least as rapidly as $\Delta t_{12}^{(e)}$, and if we can show that $\sqrt{<\delta t_{gr}\delta t_{gr}>_o} \ll \Delta t_{12}^{(e)}$ at some value of R near to the sun (say $2R_{\odot}$) we can then conclude that this condition will be satisfied everywhere and that the fluctuations should not affect the mean density determination. From equation (4.177) and the calculations displayed on Graphs 3 and 4, 6 and 7, and 9 and 10 we see that for f = 100 MHz and $R = 2R_{\odot}$

$$\sqrt{<\delta t_{gr}\delta t_{gr}>_o} = 2 \times 10^{-2} \text{ seconds} \quad (m=0)$$

where we have taken $\epsilon = .1$ to correspond to the data on fluctuations in angle of arrival. But at $R/R_\odot = 2$ we find $\Delta t_{12}^{(e)} = 6.5 \times 10^{-2}$ seconds; we may thus with reasonable certainty conclude that fluctuations in the time of arrival of signal pulses should not, for $R/R_\odot > 2$, $f_1 = 100$ MHz, and $f_2 = 300$ MHz, interfere with the measurement of the relative delay between pulse signals on carriers of different frequency.

We proceed now with further numerical examples, and examine the case of anisotropic scattering with a preference for the radial direction, but with non-constant values of a/b. For this purpose we employ the functions of equations (4.161)-(4.166) for a(r)/b(r). We plot on Graphs 15-17 only $<\delta\psi\delta\psi>_R$, as we expect the other scattering parameters to be only slightly affected by the anisotropy. Perhaps of greater use to us are Graphs 18-20 where is displayed, for a variety of functional forms of (a/b), the expected anisotropy in angle of arrival (i.e. $\sqrt{<\delta\Omega\delta\Omega>_R} / \sqrt{<\delta\psi\delta\psi>_R}$); shown also for comparison are the values we would expect if equation (4.98) were strictly true. It appears that for most cases where equation (4.89) would be inappropriate use of equation (4.98) to estimate a(r)/b(r) should not result in errors greater than some 25%; accuracy increases when $b(\rho)$ increases with ρ.

We proceed finally to examine numerically, on the basis of equations (4.145)-(4.148), the effects, in the solar equatorial plane, of curvature of the general solar magnetic field lines. The geometrical behavior of the field lines in the equatorial plane may be specified by equation (4.176) with $\Omega = 14.38^{o}$/day and, typically, V = 300 km/sec. For convenience we shall do the calculation only for a constant value of (b) (this seems reasonable since the field line curvature becomes important at distances from the sun where Hewish and Dennison found a constant correlation length; this may not be correct near solar maximum, however) and for a variety of constant values of a/b (this too is reasonable since the available data indicate a constant value of a/b ~ 2-4 beyond some 10 solar radii (Erickson (1964))). We shall still consider the source and observer to be 1 AU from the sun. Graphs 21-23 display, for $R/R_{\odot} > 10$ (we do not expect to see significant effects of field curvature within 10 solar radii), the scattering parameters for a variety of (constant) values of a/b. We may conclude:

v) $< \delta\Omega\delta\Omega >_R$, $< \delta t_{gr} \delta t_{gr} >_R$, and $< \delta f \delta f >_R$ are not appreciably affected by the field curvature.

vi) The effect of field line curvature on $< \delta\psi\delta\psi >_R$ is not noticeable below about 100 solar radii, except perhaps for a slight reduction in the steepness of its slope.

vii) Some effects of the field curvature may be seen beyond 100 solar radii, but these are probably too slight to be observable.

It appears, therefore, that we must conclude that for the coronal models we find it reasonable to consider the effects of solar magnetic field curvature on radio scattering will probably not be observable.

This concludes the numerical examples. We have, for a number of reasonable coronal models, successfully demonstrated the validity of the assumptions which have formed the basis of the analytical discussions of this chapter. We have also, within the limits of the present data, successfully correlated our results with the available observations of scattering. We found the existing data to be consistent, beyond some 10 solar radii, with an mean electron density varying as $1/r^2$ and a generally constant transverse correlation length. We also found, however, that near solar maximum the transverse correlation length may become more nearly proportional to distance from the sun, indicating that the corona then becomes more filamentary in structure.

Appendix I

In Chapter IV we stated, but did not prove, that under certain weak restrictions the complete expressions, (3.102) and (3.109) for the scattering parameters $\delta\Phi(s)$ and $\delta t_{gr}(s)$ reduce to the much simpler expressions (4.28) and (4.29) when the coronal refractive index is sufficiently constant to allow us to regard the basic rays about which occur perturbations as nearly linear. We wish here to discuss this in more detail.

We begin with equation (3.102) for $\delta\Phi(s)$:

$$\delta\Phi(s) = \frac{2\pi}{\lambda}\left\{\mu_o(s)\,\delta\varkappa(s) - \int_o^s \left[\delta\mu(s') + \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\right] ds'\right\} \qquad (A.1)$$

We wish specifically to examine the terms which we have dropped in Chapter IV, namely

$$D_\Phi(s) = \mu_o(s)\,\delta\varkappa(s) - \int_o^s \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\, ds' \qquad (A.2)$$

We start by evaluating $\delta\varkappa(s)$ under the conditions of nearly linear rays, and nearly constant refractive index. We had, equation (3.84),

$$\delta\varkappa(s) = 2C\int_o^s \left[\int_o^{s'}\left(\mu_o \frac{d(r_o^2)}{ds}\right)'' \frac{\partial}{\partial\xi}\left(\frac{\delta\mu}{\mu_o}\right)'' ds''\right]\left(\frac{1}{\mu_o(s')} - \frac{1}{\mu_o(s)}\right)\frac{ds'}{\left\{\mu_o\left[\frac{d(r_o^2)}{ds}\right]^2\right\}'}$$

(A.3)

Now we have shown that if the refractive index is sufficiently constant that the basic rays are nearly linear, we may write (equation 3.96)

$$\frac{d(r_o^2)}{ds} = 2\,(s-s_o) \tag{A.4}$$

where $s_o = r_i \cos \varphi_i$. Inserting equation (A.4) into (A.3) yields

$$\delta x(s) \cong C \int_o^s \left[\int_o^{s'} \mu_o''(s''-s_o)\, \frac{\partial}{\partial \xi} \left(\frac{\delta\mu}{\mu_o}\right)'' ds'' \right] \left(\frac{1}{\mu_o'} - \frac{1}{\mu_o(s)} \right) \frac{1}{\mu_o'}\, \frac{ds'}{(s'-s_o)^2} \tag{A.5}$$

We must now decide what form to take for the refractive index, μ_o. Our restriction to nearly linear basic rays implies, via equation (2.51),

$$\frac{\omega_p^2}{\omega^2} \ll 1 \tag{A.6}$$

From Graph 24 it is apparent that this is valid at 25 MHz beyond about 3 solar radii, while at 300 MHz it is valid beyond about 1.5 solar radii. The smallness of ω_p^2/ω^2 will lead us to consider only terms up to first order in that quantity. The question is then what form to take for $\omega_p^2 \sim n(r)$. For simplicity we shall confine our attention to the coronal region beyond about 6 solar radii where, according to equation (2.58) we may take

$$\omega_p^2 \sim n(r) \sim \frac{1}{r^2} \tag{A.7}$$

Thus we consider the average refractive index to be of the form

$$\mu_o^2 = 1 - \frac{\varepsilon}{r^2} \tag{A.8}$$

$$\frac{\varepsilon}{r^2} << 1$$

If now we insert equation (A.8) into equation (A.5) for $\delta\varkappa(s)$, and if we retain only terms to first order in ω_p^2/ω^2, we obtain

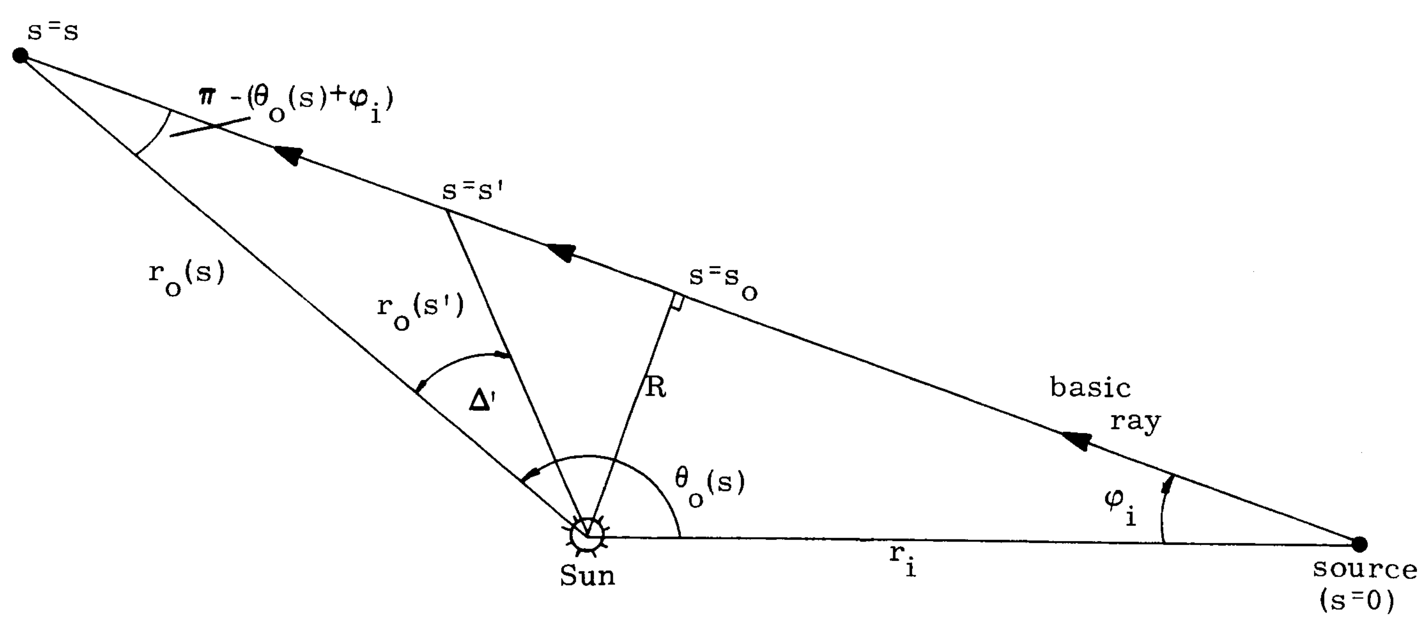
$$\delta\varkappa(s) \cong \frac{C\varepsilon}{2}\int_o^s \left[\int_o^{s'} (s''-s_o)\left(\frac{\partial\delta\mu}{\partial\xi}\right)'' ds''\right]\left(\frac{1}{r_o^2(s')} - \frac{1}{r_o^2(s)}\right)\frac{ds'}{(s'-s_o)^2} \tag{A.9}$$

Inverting the order of integration we obtain

$$\delta\varkappa(s) \cong \frac{C\varepsilon}{2}\int_o^s \left[\int_{s''}^{s}\left(\frac{1}{r_o^2(s')} - \frac{1}{r_o^2(s)}\right)\frac{ds'}{(s'-s_o)^2}\right](s''-s_o)\left(\frac{\partial\delta\mu}{\partial\xi}\right)'' ds''$$

(A.10)

Now consider the geometry appropriate to the present discussion of nearly linear rays:



We may write then

$$r_o^2(s') = r_o^2(s) + (s-s')^2 + 2\,r_o(s)(s-s')\cos(\theta_o(s) + \varphi_i) \qquad (A.11)$$

Inserting equation (A.11) into (A.10), and carrying out the integration over (ds'), we obtain

$$\delta\varkappa(s) \cong \frac{C\mathcal{E}}{2}\int_o^s \left\{ \left(\frac{1}{R^2} - \frac{1}{r_o^2(s)}\right)\left(\frac{s-s'}{s-s_o}\right) - \frac{(s'-s_o)}{R^3}\ \Delta'(s')\left(\frac{\partial\delta\mu}{\partial\xi}\right)' ds' \right. \qquad (A.12)$$

where $\Delta'(s')$ is defined by the figure. This is our desired approximate expression for $\delta\varkappa(s)$, valid to first order in ω_p^2/ω^2, derived from (3.102) under only the assumptions of equations (A.4) and (A.8). It is thus appropriate for nearly linear rays greater than about 6 solar radii distant from the solar center.

We proceed now to evaluate the remaining term in our expression (A.2) for $D_\Phi(s)$, namely

$$\int_o^s \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\, ds'$$

We begin by writing from equations (3.74) and (3.76)

$$\delta r(s) = r_o \frac{d\theta_o}{ds}\,\delta\xi(s) + \frac{dr_o}{ds}\,\delta\varkappa(s)$$

or

$$\delta r(s) = r_o \frac{d\theta_o}{ds}\,\delta\xi(s)\left[1 + \frac{\frac{dr_o}{ds}\,\delta\varkappa(s)}{r_o\frac{d\theta_o}{ds}\,\delta\xi(s)}\right] \qquad (A.13)$$

It is useful to examine the second term in the square brackets of (A.13). For nearly linear basic rays this term becomes, using equations (3.98) and (A.12).

$$\frac{\frac{dr_o}{ds}\,\delta\varkappa(s)}{r_o\frac{d\theta_o}{ds}\,\delta\xi(s)} \cong \frac{\frac{\varepsilon}{2}\,\frac{C}{R^2(s-s_o)}\,\frac{dr_o}{ds}}{r_o\frac{d\theta_o}{ds}} \times$$

$$\frac{\int_o^s \left\{(s-s') - \frac{(s'-s_o)(s-s_o)}{R}\,\Delta'(s')\right\}\left(\frac{\partial\delta\mu}{\partial\xi}\right)'\,ds'}{\int_o^s (s-s')\left(\frac{\partial\delta\mu}{\partial\xi}\right)'\,ds'}$$

where we have noted that for most rays of interest

$$\frac{1}{R^2} >> \frac{1}{r_o^{\,2}(s)}$$

and that $\mu_o(s) \cong 1$. Now the two integrals in the expression above will be of the same order of magnitude (they will be identical if the bulk of the scattering occurs in the vicinity of $s'=s_o$, as we expect to be the case in virtue of the assumed $1/r^2$ dependence of the electron density) and we therefore have

$$\frac{\frac{dr_o}{ds}\,\delta\varkappa(s)}{r_o\frac{d\theta_o}{ds}\,\delta\xi(s)} \cong \frac{\frac{\varepsilon}{2}\,\frac{C}{R^2(s-s_o)}\,\frac{dr_o}{ds}}{r_o\frac{d\theta_o}{ds}} \tag{A.14}$$

But by equation (2.13) we have for the present case of small refractive effects

$$C \cong r_o^2(s)\ \frac{d\theta_o}{ds}$$

and (A.14) becomes then

$$\frac{\frac{dr_o}{ds}\,\delta\varkappa(s)}{r_o\,\frac{d\theta_o}{ds}\,\delta\xi(s)} = \frac{\varepsilon}{2}\ \frac{r_o(s)}{R^2(s-s_o)}\ \left(\frac{dr_o}{ds}\right) \tag{A.15}$$

But from the geometry of page 175 we see that

$$\left(\frac{dr_o}{ds}\right) = \frac{s-s_o}{r_o(s)}$$

and (A.15) becomes

$$\frac{\frac{dr_o}{ds}\,\delta\varkappa(s)}{r_o\,\frac{d\theta_o}{ds}\,\delta\xi(s)} = \frac{1}{2}\ \frac{\varepsilon}{R^2} \ll 1 \tag{A.16}$$

the inequality following from equation (A.8). Thus if (A.16) is combined with (A.13) we obtain the convenient result

$$\delta r(s) \cong r_o\ \frac{d\theta_o}{ds}\ \delta\xi(s) \tag{A.17}$$

Inserting into (A.17) equation (3.98), and letting $\mu_o \cong 1$, we obtain

$$\delta r(s) \cong r_o(s)\ \frac{d\theta_o}{ds} \int_o^s (s-s')\left(\frac{\partial\delta\mu}{\partial\xi}\right)' ds' \tag{A.18}$$

From the geometry of page 175 it is easy to show that

$$r_o(s)\,\frac{d\theta_o}{ds} = \frac{R}{r_o(s)} \tag{A.19}$$

and we obtain then

$$\delta r(s) \cong \frac{R}{r_o(s)} \int_o^s (s-s')\left(\frac{\partial\delta\mu}{\partial\xi}\right)' ds' \tag{A.20}$$

Now, to proceed with our consideration of the second term in $D_\Phi(s)$, we must still evaluate $d\mu_o/dr$. From equation (A.8) we find immediately

$$\frac{d\mu_o}{dr} \cong \frac{\mathcal{E}}{r^3} \tag{A.21}$$

to first order in $\omega_p^{\;2}/\omega^2$.

We may now evaluate the second term in $D_\Phi(s)$, equation (A.2). Using equations (A.20) and (A.21) we obtain

$$\int_o^s \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\, ds' \cong \mathcal{E}R \int_o^s \left[\int_o^{s'} (s'-s'')\left(\frac{\partial\delta\mu}{\partial\xi}\right)'' ds''\right] \frac{ds'}{r_o^{\,4}(s')} \tag{A.22}$$

Inverting the order of integration yields

$$\int_o^s \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\, ds' \cong \mathcal{E}R \int_o^s \left[\int_{s''}^s (s'-s'')\,\frac{1}{r_o^{\,4}(s')}\, ds'\right]\left(\frac{\partial\delta\mu}{\partial\xi}\right)'' ds'' \tag{A.23}$$

If we write, from the geometry of page 175

$$r_o(s') = \sqrt{R^2 + (s'-s_o)^2}$$

and carry out the integration over (ds'), equation (A.23) becomes

$$\int_o^s \left(\frac{d\mu_o}{dr}\right)' \delta r(s')\, ds' \cong$$

$$\frac{R\varepsilon}{2} \int_o^s \left\{ \left(\frac{1}{R^2} - \frac{1}{r_o^2(s)} \right) \frac{s-s'}{s-s_o} - \frac{(s'-s_o)}{R^3} \Delta'(s') \right\} \left(\frac{\partial \delta\mu}{\partial \xi}\right)' ds' \tag{A.24}$$

This is our desired result, to <u>first order in ω_p^2/ω^2</u>, derived under only the assumptions of equations (A.4) and (A.8). It is thus appropriate for nearly linear rays lying beyond about 6 solar radii from the solar center.

We are now in a position to evaluate $D_\Phi(s)$, for if equations (A.12) and (A.24) are inserted into (A.2) we obtain immediately

$$D_\Phi(s) = \frac{\varepsilon}{2} \left(\mu_o(s)\, C - R\right) \int_o^s \{ \quad \} \left(\frac{\partial \delta\mu}{\partial \xi}\right)' ds' \tag{A.25}$$

But since our analysis is valid only to first order in ω_p^2/ω^2 we let $\mu_o(s) = 1$ in equation (A.25). Similarly, in virtue of equation (2.13) and (A.19) we obtain

$$C = r_o^2(s) \left(\frac{d\theta_o}{ds}\right)_s = R \tag{A.26}$$

Thus to our delight equation (A. 25) becomes to first order in ω_p^2/ω^2

$$\boxed{D_\Phi(s) = 0} \tag{A. 27}$$

We have thus shown that equation (4. 28) for $\delta\Phi(s)$ is valid subject to only the following restrictions:

i) approximate linearity of the basic rays, equation (A. 4)

ii) $\mu_o^2 = 1 - \omega_p^2/\omega^2$, with $\omega_p^2/\omega^2 \ll 1$

iii) $n(r) \sim 1/r^2$

Equation (A. 27) is of great importance for the reasons outlined in Chapter IV.

We examine next the validity of equation (4. 29) for $\delta t_{gr}(s)$. The correct expression, equation (3. 109), is

$$\delta t_{gr}(s) = -\frac{1}{c}\left\{\frac{\delta\varkappa(s)}{\mu_o(s)} + \int_o^s \frac{\delta\mu(s') + \left(\frac{d\mu_o}{dr}\right)' \delta r(s')}{\mu_o^2(s')} ds'\right\} \tag{A. 28}$$

In obtaining equation (4. 29) we have dropped the following terms:

$$D_t(s) = \frac{\delta\varkappa(s)}{\mu_o(s)} + \int_o^s \left(\frac{d\mu_o}{dr}\right)' \delta r(s') \frac{ds'}{\mu_o^2(s')} \tag{A. 29}$$

From equations (A. 12) and (A. 24) we obtain, to first order in ω_p^2/ω^2,

$$D_t(s) \cong R\mathcal{E} \int_o^s \left\{ \left(\frac{1}{R^2} - \frac{1}{r_o^2(s)} \right) \frac{s-s'}{s-s_o} - \frac{(s'-s_o)}{R^3} \Delta'(s') \right\} \left(\frac{\partial \delta\mu}{\partial \xi} \right)' ds'$$

(A.30)

Unlike the case of $D_\Phi(s)$ the two terms in $D_t(s)$ do not cancel, but rather add. Thus a correction term must be added to equation (4.29) for $\delta t_{gr}(s)$, and we have then

$$\delta t_{gr}(s) = -\frac{1}{c} \int_o^s \frac{\delta\mu(s')}{\mu_o^2(s')} ds' - \frac{1}{c} D_t(s) \quad \text{(A.31)}$$

The correction term may be large in virtue of the derivative of $\delta\mu$ in $D_t(s)$.

Let us examine the effects of the new term in some detail. We begin by writing (A.30) in the form

$$D_t(s) = \frac{\mathcal{E}(s-s_o)}{R\, r_o^2(s)} \int_o^s \left\{ (s-s') - \frac{r_o^2(s)(s'-s_o)}{R(s-s_o)} \Delta' \right\} \left(\frac{\partial \delta\mu}{\partial \xi} \right)' ds'$$

(A.32)

where we have used

$$r_o^2(s) = R^2 + (s-s_o)^2 \quad \text{(A.33)}$$

Inserting now (A.32) into (A.31) yields

$$< \delta t_{gr} \delta t_{gr} > \; = \; < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{2\varepsilon(s-s_o)}{c^2 R r_o^2(s)} \int_o^s \int_o^s I(s') \left< \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \delta\mu(s'') \right> ds' ds'' \tag{A.34}$$

$$+ \frac{\varepsilon^2 (s-s_o)^2}{c^2 R^2 r_o^4(s)} \int_o^s \int_o^s I(s')\, I(s'') \left< \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \left(\frac{\partial \delta\mu}{\partial \xi}\right)'' \right> ds' ds''$$

where for convenience in writing we have let

$$I(s') = \left\{ (s-s') - \frac{(s'-s_o)\, r_o^2(s)}{R\,(s-s_o)} \, \Delta' \right\} \tag{A.35}$$

We recall that $< \delta t_{gr} \delta t_{gr} >_R$ is that value of $< \delta t_{gr} \delta t_{gr} >$ calculated in Chapter IV from the approximate expression (4.29). Now to evaluate equation (A.34) we need to know the forms taken by the correlations within the integrands. If we assume an anisotropically turbulent medium with $< \delta\mu(\vec{r}_1)\, \delta\mu(\vec{r}_2) >$ specified by equation (4.70) (preference for only the radial direction) we may readily obtain

$$\left< \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \delta\mu(s'') \right> =$$

$$- < \delta^2 \mu > \sin 2\gamma \left(\frac{1}{a^2} - \frac{1}{b^2}\right) \tau \, e^{-\tau^2 \left(\frac{\sin^2\gamma}{a^2} + \frac{\cos^2\gamma}{b^2}\right)} \tag{A.36}$$

while $\left< \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \right>$ is given by equation (4.71); $\tau = s'' - s'$ and (a) and (b) are the radial and transverse correlation lengths, respectively. These expressions are to be inserted into equation (A.34) and the integrations over ds' and ds'' carried out. However a simple transformation of variables allows us to write

$$\int_0^s \int_0^s ds''ds' \rightarrow \int_0^s \int_{-s'}^{s-s'} d\tau\, ds' \tag{A.37}$$

If the correlation lengths are much less than (s) we may modify the limits on the $d\tau$ integration such that

$$\int_0^s \int_0^s ds''ds' \rightarrow \int_0^s \int_{-\infty}^{+\infty} d\tau\, ds' \tag{A.38}$$

We note now a useful occurrence. If we insert equation (A.36) into (A.34) and modify our variables of integration according to (A.38), and then perform the integration over ($d\tau$) the first integral will render zero, since (A.36) is an odd function of τ. (A.34) becomes then

$$< \delta t_{gr} \delta t_{gr} > = < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{\varepsilon^2 (s-s_o)^2}{c^2 R^2 r_o^4(s)} \int_0^s \int_0^s I(s')\, I(s'') \left< \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \left(\frac{\partial \delta\mu}{\partial \xi}\right)'' \right> ds'ds'' \tag{A.39}$$

Now transform the variables (s') and (s'') to (s') and (τ = s'' - s'); it is then an easy matter to show that as long as the correlation lengths are small compared with R we may write equation (A.39) as

$$< \delta t_{gr} \delta t_{gr} > = < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{\varepsilon^2 (s-s_o)^2}{c^2 R^2 r_o^4(s)} \int_0^s \int_0^s I^2(s') \left< \left(\frac{\partial \delta\mu}{\partial \xi}\right)' \left(\frac{\partial \delta\mu}{\partial \xi}\right)'' \right> ds'ds'' \tag{A.40}$$

If for the correlation we use equation (4.71), transform variables according to (A.38), and carry out the integration over $(d\tau)$ we obtain

$$< \delta t_{gr} \delta t_{gr} > = < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{2\sqrt{\pi}\varepsilon^2 (s-s_o)^2}{c^2 R^2 r_o^4(s)} \int_o^s \left\{ (s-s') - \frac{(s'-s_o) r_o^2(s)}{R(s-s_o)} \Delta' \right\}^2 \frac{< \delta^2 \mu >}{b} \frac{(1+A)^{1/2}}{\left(1 + A \frac{R^2}{r_o^2(s')}\right)^{3/2}} ds'$$

(A.41)

This expression is correct subject only to the restrictions that the correlation lengths be much smaller than either (s) or (R); we expect these conditions to be fully satisfied for the cases of interest.

Let us look at the integrand in (A.41). For the spherically symmetric situation we are considering we expect the terms outside of the curly brackets to be even functions of $(s'-s_o)$. If we now suppose Δ' and $(s-s')$ to vary slowly over the values of $(s'-s_o)$ for which significant contributions to the integral exist, a not too restrictive condition in virtue of the assumed $1/r^2$ dependence of n(r) (implying a $1/r^4$ dependence of $< \delta^2 \mu >$), we may then expand the curly brackets in the integrand of (A.41) and drop the term odd in $(s'-s_o)$ to obtain

$$< \delta t_{gr} \delta t_{gr} > = < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{\varepsilon^2 (s-s_o)^2}{c^2 R^2 r_o^4(s)} \, 2\sqrt{\pi} \int_o^s \left\{ (s-s')^2 + \frac{r_o^4(s)}{R^2 (s-s_o)^2} (s'-s_o)^2 (\Delta')^2 \right\} \times$$

$$\frac{< \delta^2 \mu >}{b} \frac{(1+A)^{1/2}}{\left(1 + A \frac{R^2}{r_o^2(s')}\right)^{3/2}} ds' \qquad \text{(A.42)}$$

Now since we have already assumed that (s-s') and $\Delta(s')$ vary slowly over the range of $(s'-s_o)$ for which significant contributions to the integral occur we must for consistency replace $(s-s')^2$ and $\Delta(s')$ in (A.42) with their values at $s'=s_o$, the point around which the contributions to the integrals occur. Thus equation (A.42) becomes

$$< \delta t_{gr} \delta t_{gr} > = < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{\varepsilon^2 (s-s_o)^2}{c^2 R^2 r_o^4(s)} \left\{ < \delta\psi\delta\psi >_R (s-s_o)^2 \left[1 - \frac{r_o^4(s)}{(s-s_o)^4} \left(\tan^{-1} \frac{s-s_o}{R} \right)^2 \right] \right. \tag{A.43}$$

$$\left. + \frac{r_o^4(s)}{R^2 (s-s_o)^2} \left(\tan^{-1} \frac{s-s_o}{R} \right)^2 2\sqrt{\pi} \int_0^s r_o^2(s') \frac{< \delta^2 \mu >}{b} \frac{(1+A)^{1/2}}{\left(1 + A \frac{R^2}{r_o^2(s')} \right)^{3/2}} ds' \right\}$$

where we have noted that

$$(s'-s_o)^2 = r_o^2(s') - R^2 \tag{A.44}$$

and have used equation (4.79) for $< \delta\psi\delta\psi >_R$. Equation (A.43) is the desired result expressing the important fact that even when the coronal refractive index is very nearly uniform and the basic rays are very nearly linear, the mean square of the fluctuations of times of arrival of signal pulses can differ significantly from that which would be calculated on the basis of a strictly uniform refractive index. To be useful, however, equation (A.43) should be cast in a form containing only $< \delta\psi\delta\psi >_R$, an observable quantity. To that end we consider the ratio

$$\frac{2\sqrt{\pi}\int_0^s r_o^2(s')\,\frac{<\delta^2\mu>}{b}\,\frac{(1+A)^{1/2}}{\left(1+A\,\frac{R^2}{r_o^2(s')}\right)^{3/2}}\,ds'}{2\sqrt{\pi}\int_0^s \frac{<\delta^2\mu>}{b}\,\frac{(1+A)^{1/2}}{\left(1+A\,\frac{R^2}{r_o^2(s')}\right)^{3/2}}\,ds'} \tag{A.45}$$

If we assume that in both integrals of (A.45) the quantity $A(R^2/r_o^2(s'))$ varies slowly over the range of (s') for which the bulk of the contributions to the integrals occur, we may then write this ratio as

$$\frac{\int_0^s r_o^2(s')\,\frac{<\delta^2\mu>}{b}\,ds'}{\int_0^s \frac{<\delta^2\mu>}{b}\,ds'} \tag{A.46}$$

We now note that for the present case of $n(r) \sim 1/r^2$ we have $<\delta^2\mu> \sim 1/r^4$; we shall for convenience assume that the correlation length (b) also exhibits a simple power law behavior: $b \sim r^m$. Then if we write

$$ds' = \frac{rdr}{\sqrt{r^2 - R^2}}$$

and assume that we may extend the path of integration to $\pm\infty$ the ratio (A.46) becomes

$$\frac{\int_R^\infty \frac{1}{r^{(2+m)}} \frac{rdr}{\sqrt{r^2 - R^2}}}{\int_R^\infty \frac{1}{r^{(4+m)}} \frac{rdr}{\sqrt{r^2 - R^2}}} \tag{A.47}$$

Evaluation of the integrals by use of Gamma functions gives the ratio as

$$R^2 \left(\frac{m+2}{m+1}\right) \tag{A.48}$$

Inspection of expressions (A.45) and (A.48) then allow us to write

$$2\sqrt{\pi} \int_o^s r_o^2(s') \frac{<\delta^2\mu>}{b} \frac{(1 + A)^{1/2}}{\left(1 + A \frac{R^2}{r_o^2(s')}\right)^{3/2}} ds' = R^2\left(\frac{m+2}{m+1}\right) < \delta\psi\delta\psi >_R \tag{A.49}$$

where equation (4.79) has again been used for $< \delta\psi\delta\psi >_R$. Inserting (A.49) into equation (A.43) gives

$$< \delta t_{gr} \delta t_{gr} > \cong < \delta t_{gr} \delta t_{gr} >_R$$

$$+ \frac{\varepsilon^2}{c^2 R^2} \left[\left(\frac{s-s_o}{r_o(s)}\right)^4 + \frac{1}{(m+1)} \left(\tan^{-1} \frac{s-s_o}{R}\right)^2 \right] < \delta\psi\delta\psi >_R \tag{A.50}$$

Now from equations (A.8) and (2.41) we note that

$$\frac{\varepsilon^2}{R^4} = \left(\frac{\omega_p}{\omega}\right)^4_{r=R} \tag{A.51}$$

But it is easy to show that when $n(r) \sim 1/r^2$

$$\left(\frac{\omega_p}{\omega}\right)^4_{r=R} = R^2 \left(\frac{d\mu_o}{dr}\right)^2_{r=R} \tag{A. 52}$$

Combining equations (A. 51) and (A. 52) allows us to write (A. 50) in the more meaningful form:

$$\boxed{\begin{aligned} &< \delta t_{gr} \delta t_{gr} >_o \cong < \delta t_{gr} \delta t_{gr} >_R \\ &+ \frac{1}{c^2} R^4 \left(\frac{d\mu_o}{dr}\right)^2_{r=R} \left[\left(\frac{s-s_o}{r_o(s)}\right)^4 + \frac{1}{(m+1)} \left(\tan^{-1} \frac{s-s_o}{R}\right)^2 \right] < \delta\psi\delta\psi >_R \end{aligned}} \tag{A. 53}$$

Equation (A. 53) is the result desired, relating the observable quantities $< \delta t_{gr} \delta t_{gr} >_o$ and $< \delta\psi\delta\psi >_R$ to the quantity $< \delta t_{gr} \delta t_{gr} >_R$ utilized extensively in the scattering discussion of Chapter IV. It is valid subject only to the following restrictions:

i) approximate linearity of the basic rays (A. 4)
ii) $n(r) \sim 1/r^2$
iii) only terms to first order in $\omega_p^{\,2}/\omega^2$ have been considered
iv) a, b << s (see equation (4. 44))
v) anisotropic turbulence with an autocorrelation function specified by (4. 70) (preference for only the radial direction)
vi) a, b $\lesssim$ R
vii) (s-s'), Δ(s'), $A(R^2/r_o^{\,2}(s'))$ vary slowly over the range of $(s'-s_o)$ for which significant contributions to the scattering integrals occur
viii) $b \sim r^m$

Thus we have successfully determined the difference between the values of $< \delta t_{gr} \delta t_{gr} >$ which we would actually expect to observe, and those calculated on the basis of the approximate expression (4. 29). This difference can be considerable, and it might at first appear that the discussion of Chapter IV based on equation (4. 29) is invalid. However, if the average electron density is known equation (A. 53) may be used to deduce $< \delta t_{gr} \delta t_{gr} >_R$ from observed values of $< \delta\psi\delta\psi >$ and $< \delta t_{gr} \delta t_{gr} >$, and that quantity may be then employed as discussed extensively in Chapter IV. When the simplicity of equation (4. 29) is considered in relation to the complexity of the equations of this chapter, the usefulness of equation (A. 53) becomes quite clear.

How large will the difference between $< \delta t_{gr} \delta t_{gr} >_o$ and $< \delta t_{gr} \delta t_{gr} >_R$ be? We shall write (A. 53) as

$$< \delta t_{gr} \delta t_{gr} >_o = < \delta t_{gr} \delta t_{gr} >_R (1 + G(R)) \qquad \text{(A. 54)}$$

where G(R) is defined by this expression and is

$$G(R) = \frac{R^4}{c^2} \left(\frac{d\mu_o}{dr}\right)^2_{r=R} \left[\left(\frac{s-s_o}{r_o(s)}\right)^4 + \frac{1}{(m+1)} \left(\tan^{-1} \frac{s-s_o}{R}\right)^2 \right] \frac{< \delta\psi\delta\psi >_R}{< \delta t_{gr} \delta t_{gr} >_R}$$

(A. 55)

Now we note that if the bulk of the scattering occurs in the vicinity of $s' = s_o$ equations (4. 80) and (4. 81b) allow us to write

$$\frac{< \delta\Omega\delta\Omega >_R}{< \delta t_{gr} \delta t_{gr} >_R} \simeq \frac{2c^2}{b^2(r=R)} \qquad \text{(A. 56)}$$

But as has been discussed in Chapter IV we may also write approximately

$$\frac{<\delta\Omega\delta\Omega>_R}{<\delta\psi\delta\psi>_R} = \left[\frac{a(r=R)}{b(r=R)}\right]^2 \qquad (A.57)$$

Combining equations (A. 55)-(A. 57) yields

$$G(R) = \frac{2R^4}{a^2(r=R)}\left(\frac{d\mu_o}{dr}\right)^2_{r=R}\left[\left(\frac{s-s_o}{r_o(s)}\right)^4 + \frac{1}{(m+1)}\left(\tan^{-1}\frac{s-s_o}{R}\right)^2\right] \qquad (A.58)$$

Now if we take, according to equation (2.37),

$$n(r) = \frac{10^6\, R_\odot^{\;2}}{r^2}\ \mathrm{cm}^{-3} \qquad (A.59)$$

equation (A. 58) becomes

$$G(R) \cong \frac{1.6 \times 10^3}{\left(\frac{f}{100}\right)^4\left(\frac{R}{R_\odot}\right)^2\left(\frac{a}{200}\right)^2}\left[\left(\frac{s-s_o}{r_o(s)}\right)^4 + \frac{1}{(m+1)}\left(\tan^{-1}\frac{s-s_o}{R}\right)^2\right] \qquad (A.60)$$

where here (f) is in megacycles and (a) is in kilometers. Equations (A. 60) and (A. 54) allow a computational estimation of the difference between the value of $<\delta t_{gr}\delta t_{gr}>$ actually expected and that found by equation (4. 29). For example, if we consider a frequency of 100 MHz and a correlation length of 200 kilometers we find for a distant source $\left(\tan^{-1}\frac{s-s_o}{R} \cong \pi/2\right)$ that G(R) lies between 55. 5 (m=0; or 35. 7, m=1) at R = 10 $R_\odot$ and . 555 (m=0; or . 357, m=1) at R = 100 $R_\odot$. Thus it appears that the values observed for $<\delta t_{gr}\delta t_{gr}>$ should at 100 MHz be significantly higher than those calculated on the basis of equation (4.29). However, (A.60) indicates that if the frequency and/or the correlation length is increased the correction factor [1 + G(R)] will tend toward unity. For example, at 400 MHz G(R) varies between .217 (m=0; or .14, m=1) at R = 10 $R_\odot$ and .00217 (m=0; or .0014, m=1) at R = 100 $R_\odot$.

Thus these results are important, and we must ask what happens when n(r) does not vary as $1/r^2$? This will be the case for rays passing close to the sun, for from equation (2.58) we see that

$$n(r) \sim \frac{1}{r^6} \; ; \; 1.5 < \rho < 6$$
$$n(r) \sim \frac{1}{r^{16}} \; ; \; \rho < 1.5 \tag{A.61}$$

If we carry out an analysis similar to that presented here, but for $n(r) \sim 1/r^n$, we may obtain with some difficulty the following expression, which is to be compared with equation (A.53):

$$\begin{aligned} <\delta t_{gr}\delta t_{gr}> &= <\delta t_{gr}\delta t_{gr}>_R \\ &+ \frac{1}{c^2} R^4 \left(\frac{d\mu_o}{dr}\right)^2_{r=R} \left(\frac{4}{n^2}\right) \left\{ \left(g_n - 1\right)^2 - 2 g_n \left(g_n - 1\right) \frac{2(n-1)+m}{2(n-1)+m-1} \right. \\ &+ g_n^{\,2} \left(\frac{2(n-1)+m}{2(n-1)+m-1}\right) \left(\frac{2(n-2)+m}{2(n-2)+m-1}\right) \\ &\left. \frac{(n-1)^2(n-3)^2 \ldots (1)}{(n-2)^2(n-4)^2 \ldots (4)} \left(\frac{1}{m+1}\right) \left(\frac{\pi}{2}\right)^2 \right\} <\delta\psi\delta\psi>_R \end{aligned} \tag{A.62}$$

where

$$g_n = \left\{ \left(\frac{n-1}{n-2}\right) + \left(\frac{n-1}{n-2}\right)\left(\frac{n-3}{n-4}\right) + \ldots \frac{n-2}{2} \text{ terms} \right\} \qquad (A.63)$$

The derivation of equation (A.62) has required the following additional assumptions:

ix) n even

x) $R << r_o(s)$

Assumption (ix) would appear to be satisfied for a real case in virtue of equation (2.58). Assumption (x) would also seem to be generally valid since the exponent 'n' does not depart from (2) until one is close to the sun. It should be mentioned in addition, however, that for a corona with a density distribution of the form of (2.58) a ray which passes sufficiently close to the sun to be in a region where $n(r) \sim 1/r^6$, say, will <u>also</u> be in the region where $n(r) \sim 1/r^2$. Since (A.62) has been derived under the assumption that the ray lies wholly in a region where $n(r) \sim 1/r^n$, its use bears the implicit restriction that

xi) that part of the ray for which the bulk of the scattering occurs lies wholly in a region of $1/r^n$ density dependence.

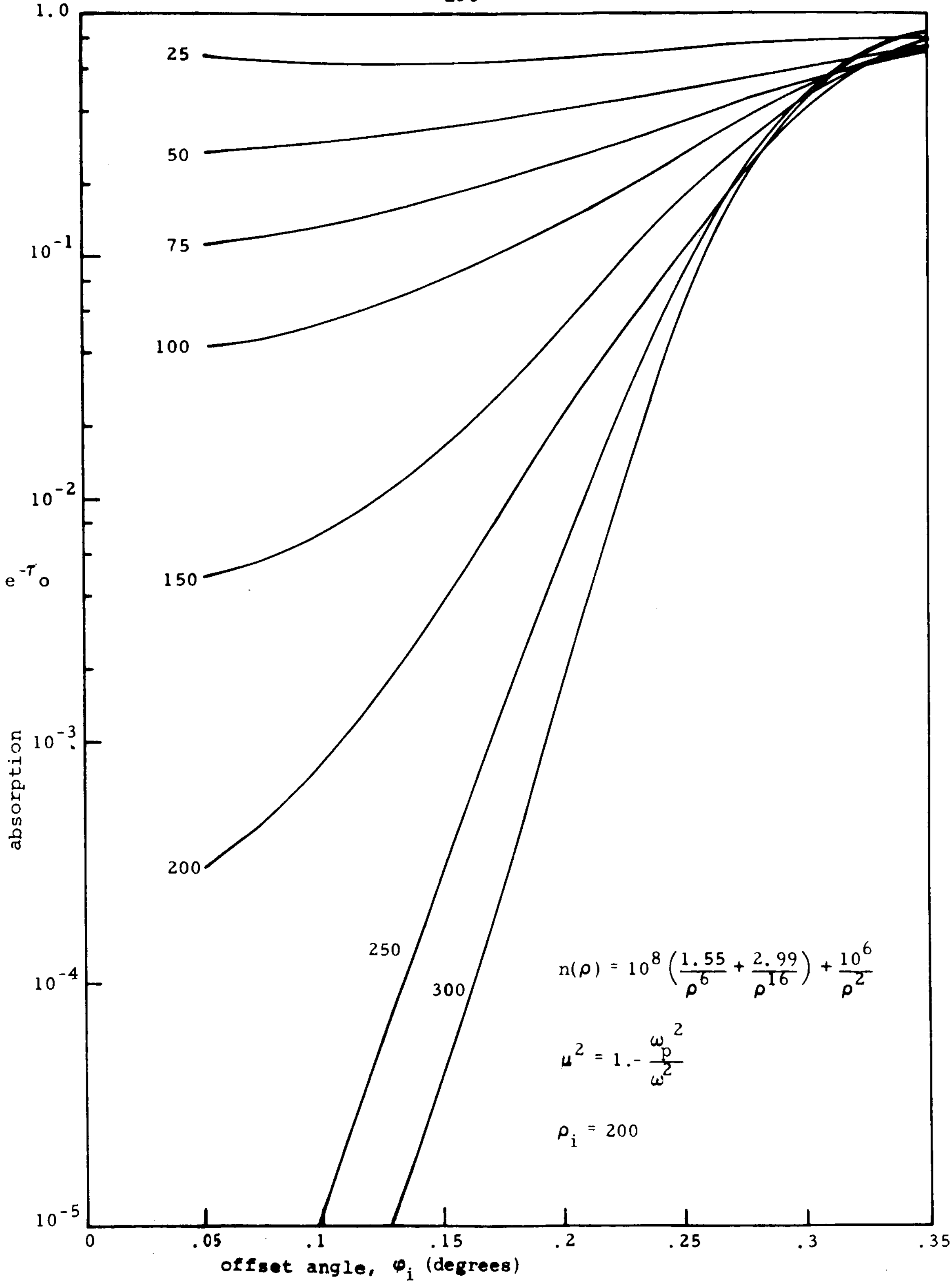
When conditions (i) - (xi) are satisfied we may use equations (A.62) and (A.63) to find G(R) (equation (A.54)) when $n(r) \sim 1/r^n$. For example, it may be readily found that when n = 6, G(R) is roughly one third of its value when n = 2.

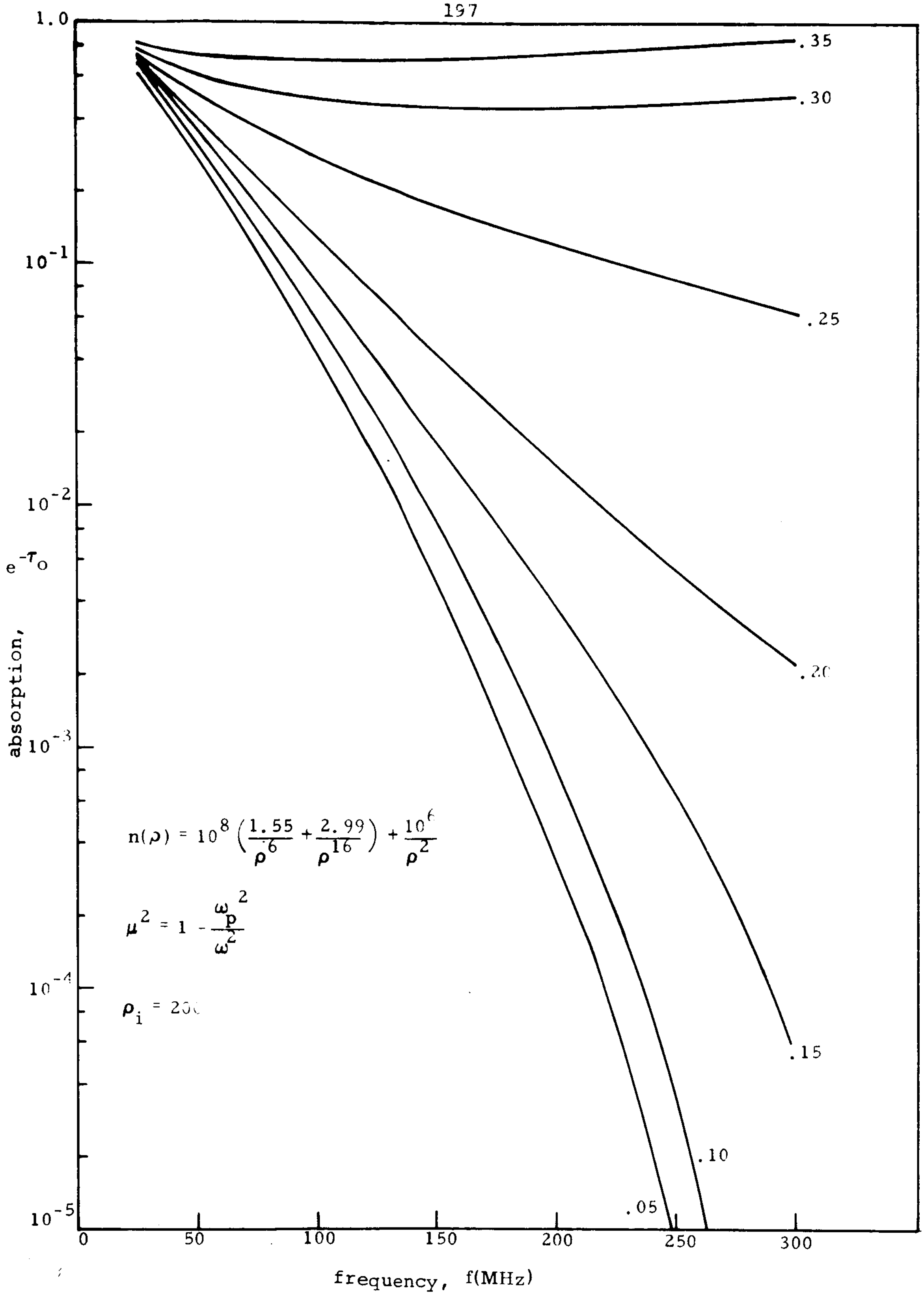
In closing one final point must be made. The utility of equations (A.53) ($n(r) \sim 1/r^2$) and (A.62) ($n(r) \sim 1/r^n$) lies in their allowing us to determine $< \delta t_{gr} \delta t_{gr} >_R$, the quantity of most use to us as discussed in Chapter IV, from observations of $< \delta t_{gr} \delta t_{gr} >$ and $< \delta\psi\delta\psi >$. The only problem is that this determination necessitates knowledge of (m) ($b(r) \sim r^m$), whereas the determination of b(r) necessitates, as has been discussed in Chapter IV, knowledge of $< \delta t_{gr} \delta t_{gr} >_R$. In practice, then, one would assume a value for (m) with which $< \delta t_{gr} \delta t_{gr} >_R$ would be found via equation (A. 53) or (A. 62), and the behavior of $< \delta t_{gr} \delta t_{gr} >_R$ so determined would then enable us to check for consistency; if a discrepancy were found a new value for (m) would be chosen. This problem would, of course, be eliminated if (R) and/or (f) were such that $G(R) << 1$.

Graphs 1 and 2

Graph 1: Absorption as a function of ray offset angle at various frequencies.

Graph 2: Absorption as a function of frequency at various ray offset angles.


1.0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$e^{-\tau_o}$
absorption
25
50
75
100
150
200
250
300
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$\mu^2 = 1. - \frac{\omega_p^2}{\omega^2}$
$\rho_i = 200$
0
.05
.1
.15
.2
.25
.3
.35
offset angle, φ_i (degrees)

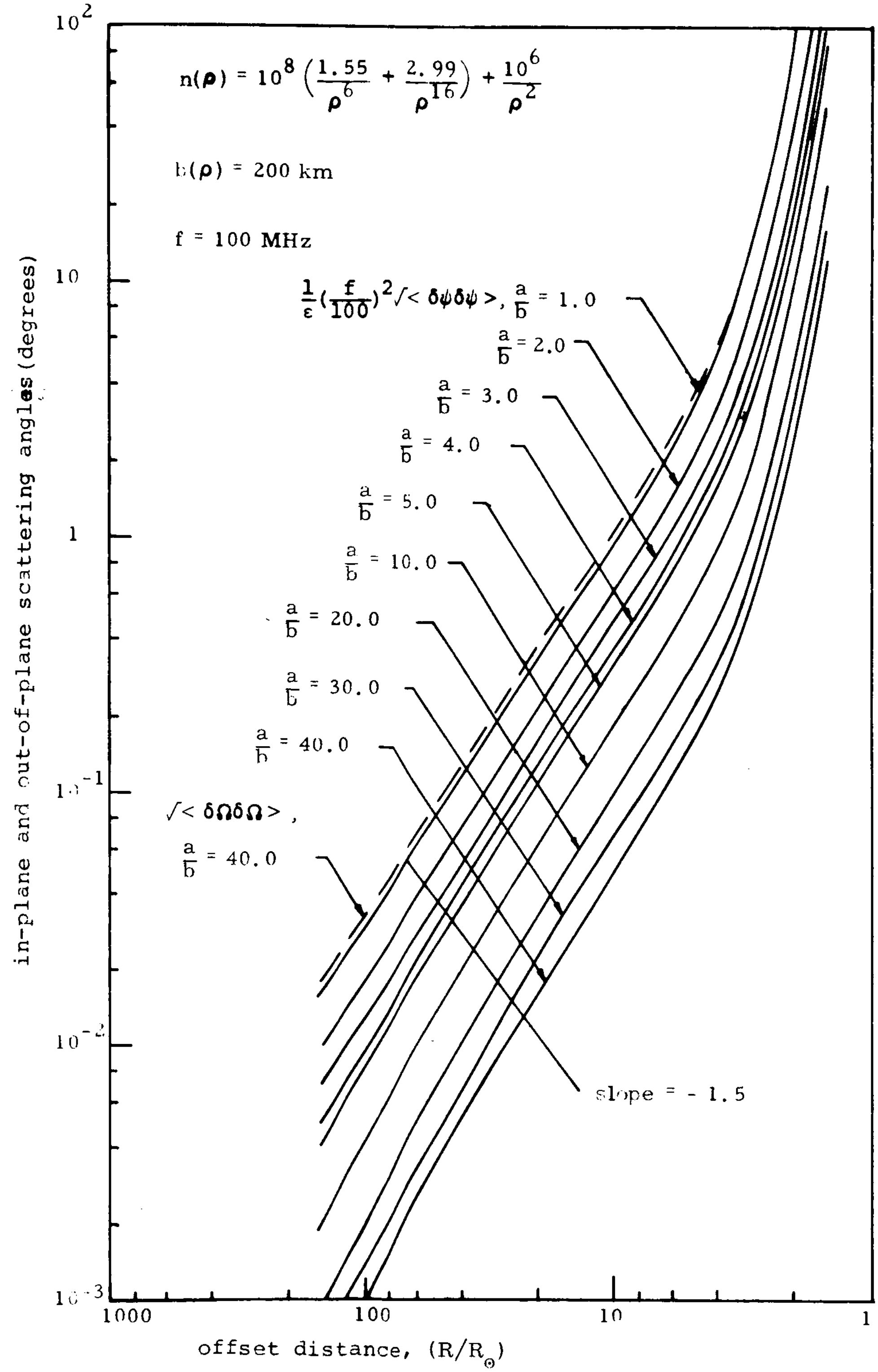

1.0
10^-1
10^-2
10^-3
10^-4
10^-5
absorption, e^-τo
.35
.30
.25
.20
.15
.10
.05
n(ρ) = 10^8 (1.55/ρ^6 + 2.99/ρ^16) + 10^6/ρ^2
μ^2 = 1 - ωp^2/ω^2
ρi = 20
0
50
100
150
200
250
300
frequency, f(MHz)

Graphs 3, 4, 5

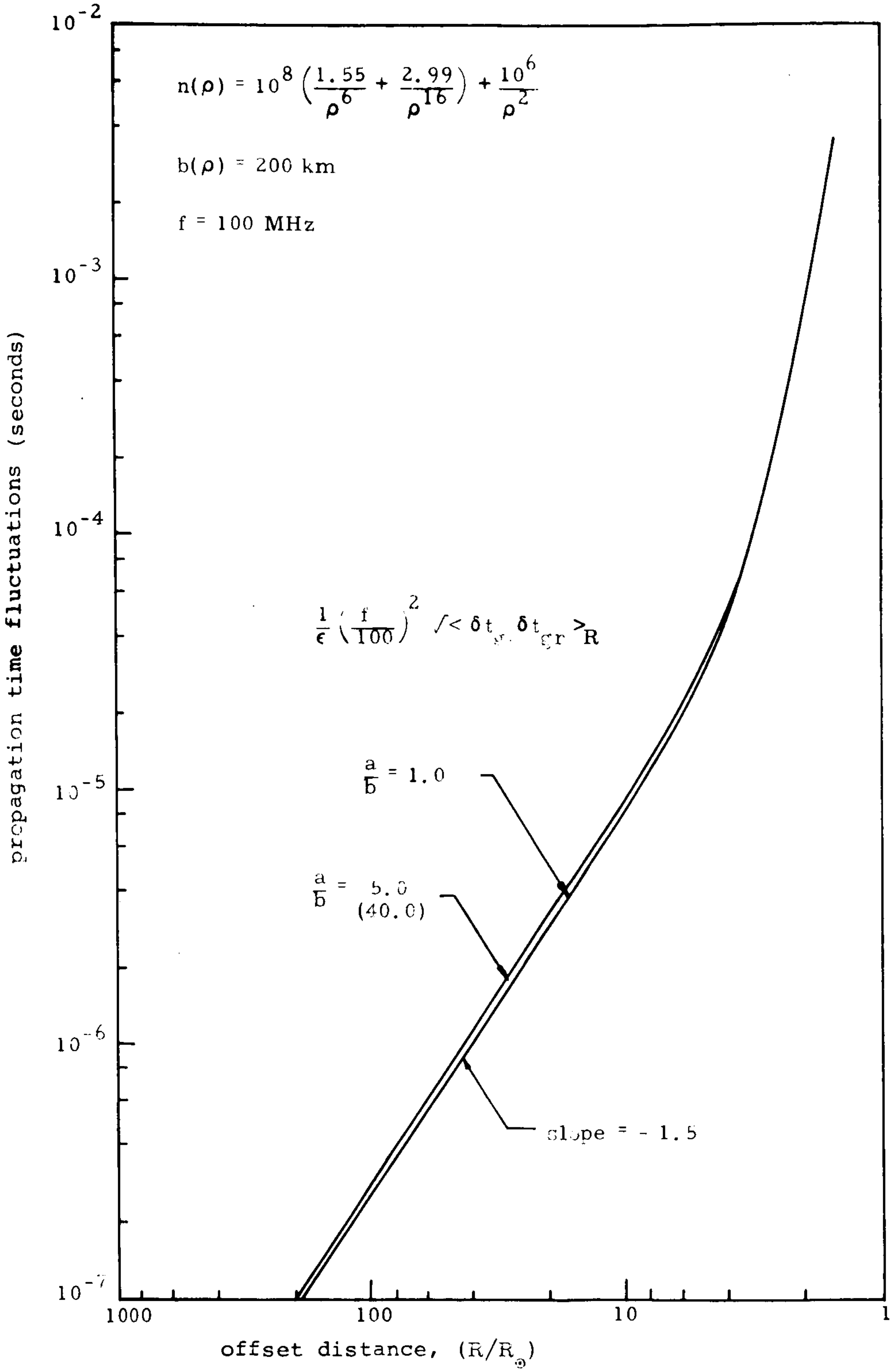
Graph 3: The in-plane and out-of-plane scattering angles vs. path offset distance for constant anisotropy ratios and a transverse correlation length of 200 km.

Graph 4: The fluctuations in pulse signal propagation times vs. path offset distance for constant anisotropy ratios and a transverse correlation length of 200 km.

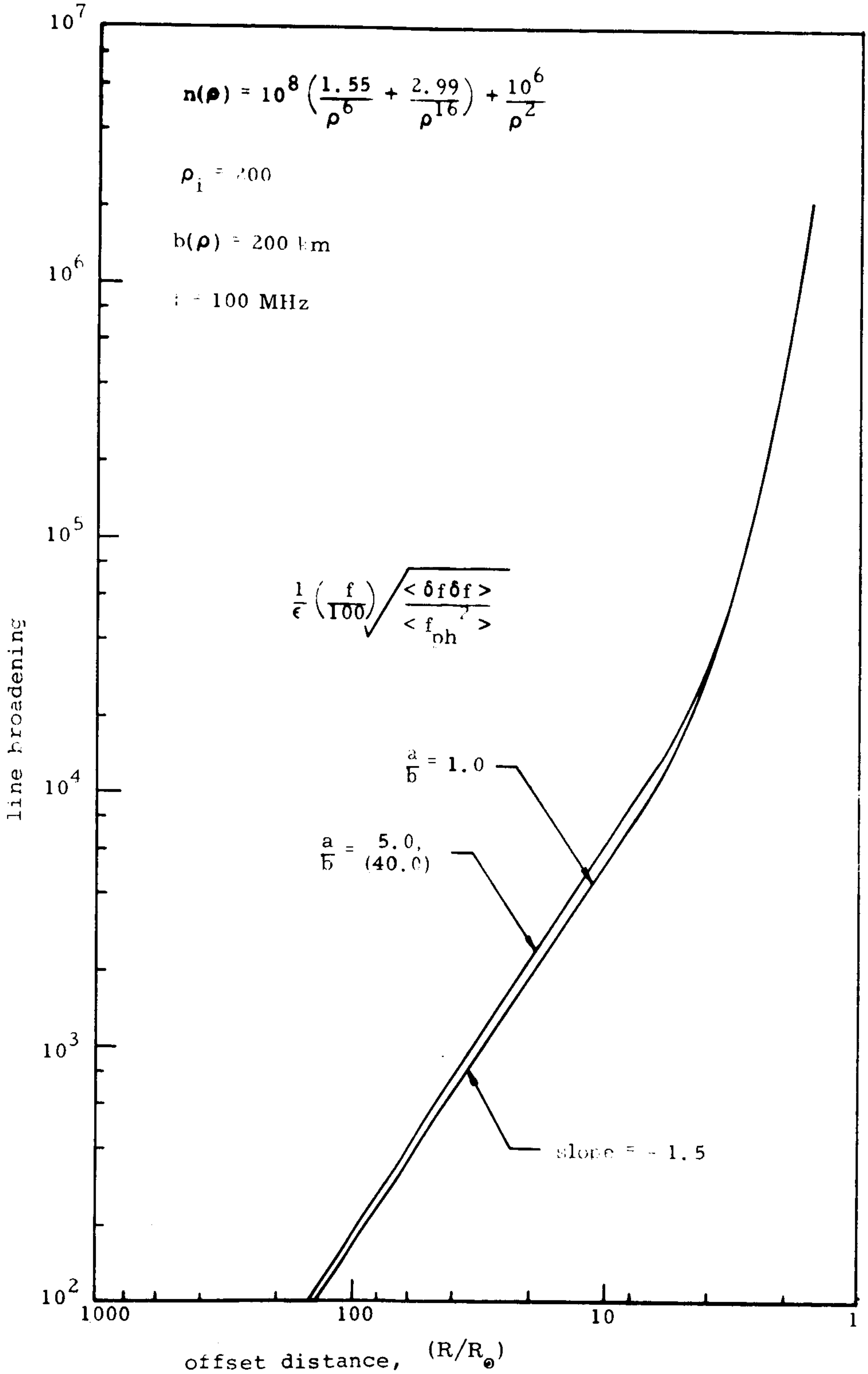
Graph 5: The line broadening vs. path offset distance for constant anisotropy ratios and a transverse correlation length of 200 km.



$n(\rho) = 10^8 \left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho)$ = 200 km
f = 100 MHz
$\frac{1}{\varepsilon}(\frac{f}{100})^2\sqrt{<\delta\psi\delta\psi>}$, $\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 2.0
$\frac{a}{b}$ = 3.0
$\frac{a}{b}$ = 4.0
$\frac{a}{b}$ = 5.0
$\frac{a}{b}$ = 10.0
$\frac{a}{b}$ = 20.0
$\frac{a}{b}$ = 30.0
$\frac{a}{b}$ = 40.0
$\sqrt{<\delta\Omega\delta\Omega>}$,
$\frac{a}{b}$ = 40.0
slope = - 1.5
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
1000
100
10
1
in-plane and out-of-plane scattering angles (degrees)
offset distance, $(R/R_\odot)$



$n(\rho) = 10^8 \left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = 200$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta t_{gr}\,\delta t_{gr}>_R}$
$\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 5.0 (40.0)
slope = - 1.5
propagation time fluctuations (seconds)
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
1000
100
10
1
offset distance, $(R/R_\odot)$



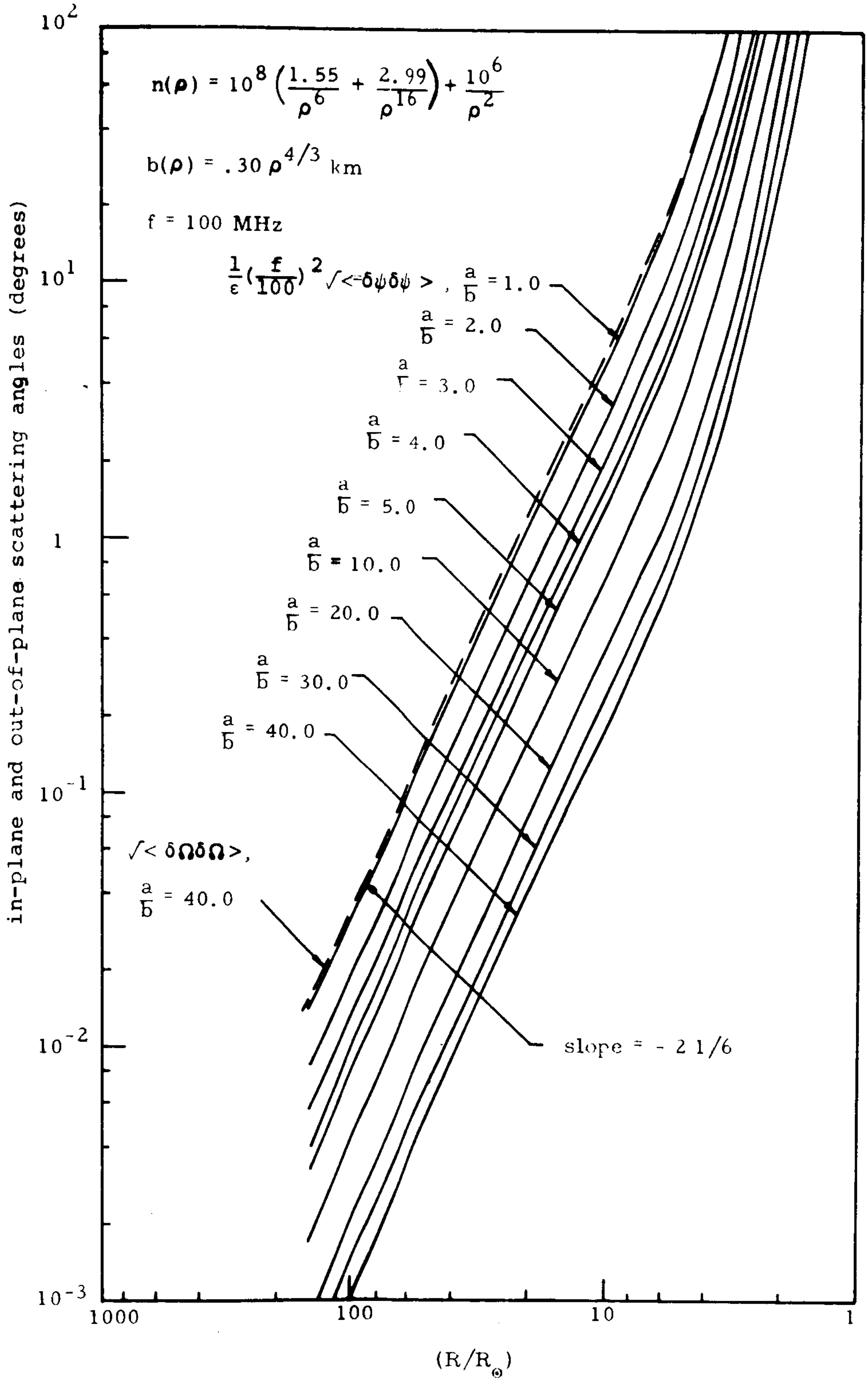
$n(\rho) = 10^8 \left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
ρ_i = 200
$b(\rho)$ = 200 km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)\sqrt{\frac{<\delta f \delta f>}{<f_{ph}^2>}}$
$\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 5.0, (40.0)
slope = - 1.5
10^7
10^6
10^5
10^4
10^3
10^2
line broadening
1000
100
10
1
offset distance, $(R/R_\odot)$

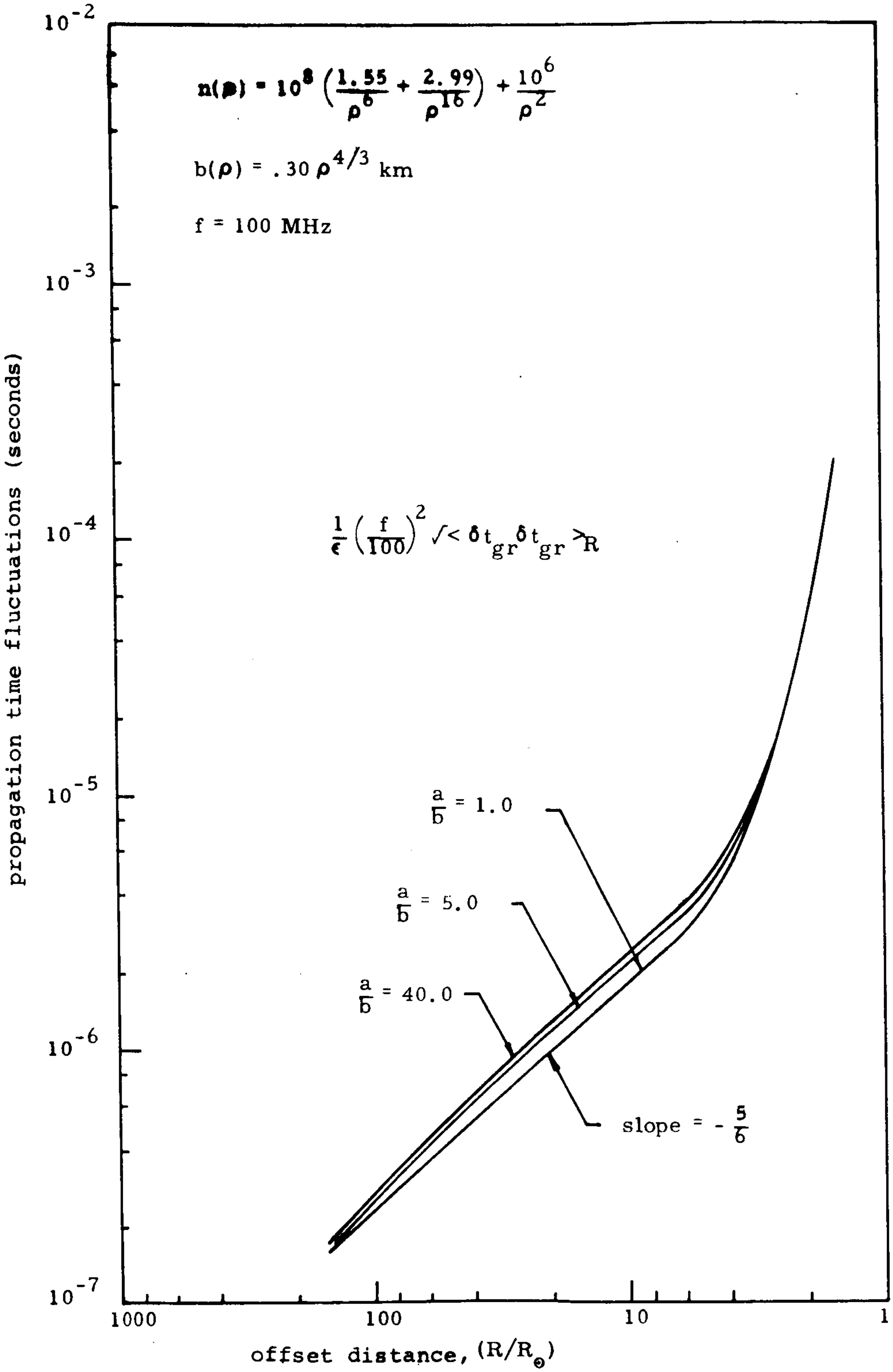
Graphs 6, 7, 8

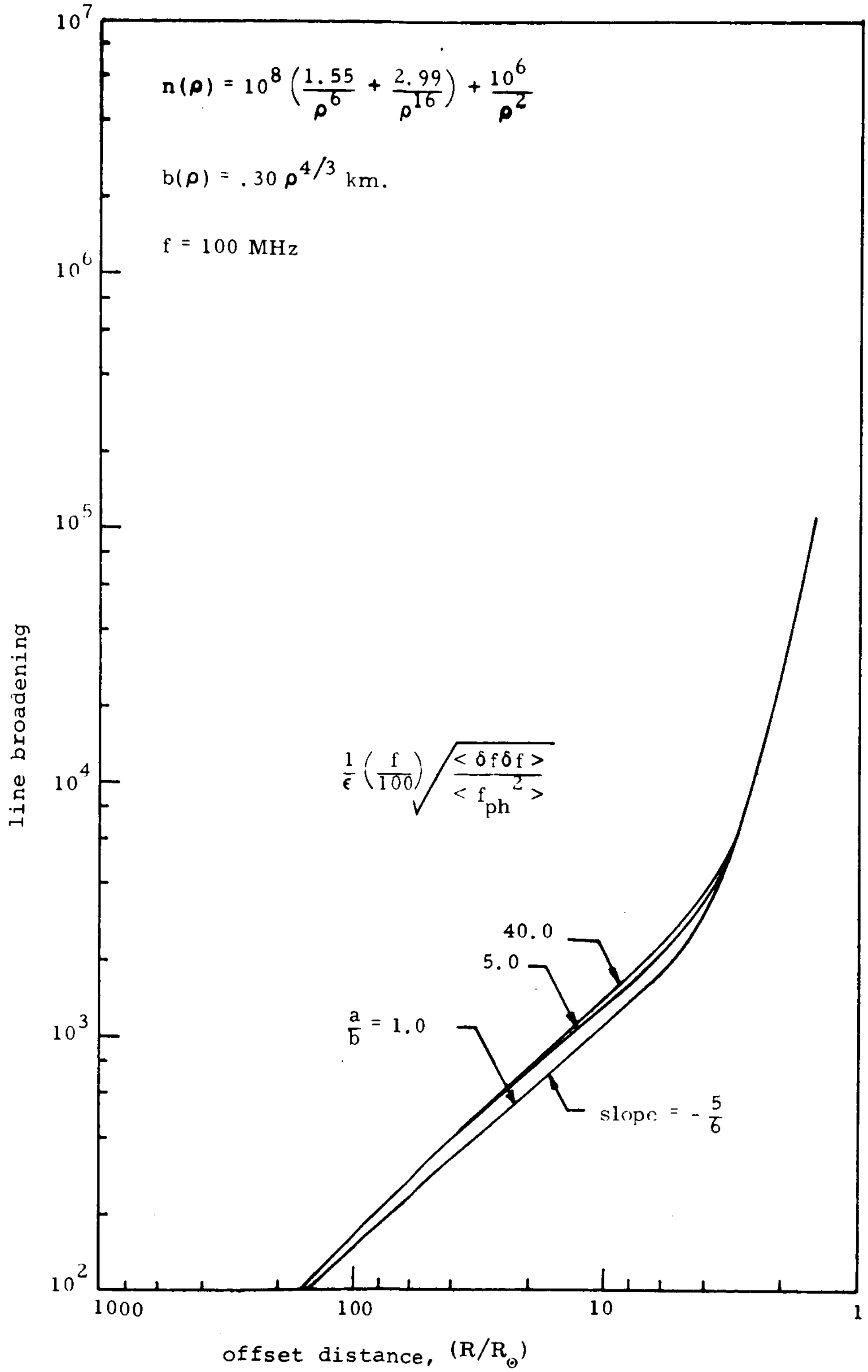
Graph 6: The in-plane and out-of-plane scattering angles vs. path offset distance for constant anisotropy ratios and a transverse correlation distance $b(\rho) = .30\rho^{4/3}$ km.

Graph 7: The fluctuations in pulse signal propagation times vs. path offset distance for constant anisotropy ratios and a transverse correlation distance $b(\rho) = .30\rho^{4/3}$km.

Graph 8: The line broadening vs. path offset distance for constant anisotropy ratios and a transverse correlation distance $b(\rho) = .30\rho^{4/3}$ km.


$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = .30\,\rho^{4/3}$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta\psi\delta\psi>}$, $\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 2.0
$\frac{a}{b}$ = 3.0
$\frac{a}{b}$ = 4.0
$\frac{a}{b}$ = 5.0
$\frac{a}{b}$ = 10.0
$\frac{a}{b}$ = 20.0
$\frac{a}{b}$ = 30.0
$\frac{a}{b}$ = 40.0
$\sqrt{<\delta\Omega\delta\Omega>}$,
$\frac{a}{b}$ = 40.0
slope = - 2 1/6
in-plane and out-of-plane scattering angles (degrees)
10^2
10^1
1
10^{-1}
10^{-2}
10^{-3}
1000
100
10
1
$(R/R_\odot)$


$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = .30\,\rho^{4/3}$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{\langle \delta t_{gr} \delta t_{gr} \rangle_R}$
$\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 5.0
$\frac{a}{b}$ = 40.0
slope = $-\frac{5}{6}$
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
propagation time fluctuations (seconds)
1000
100
10
1
offset distance, $(R/R_\odot)$



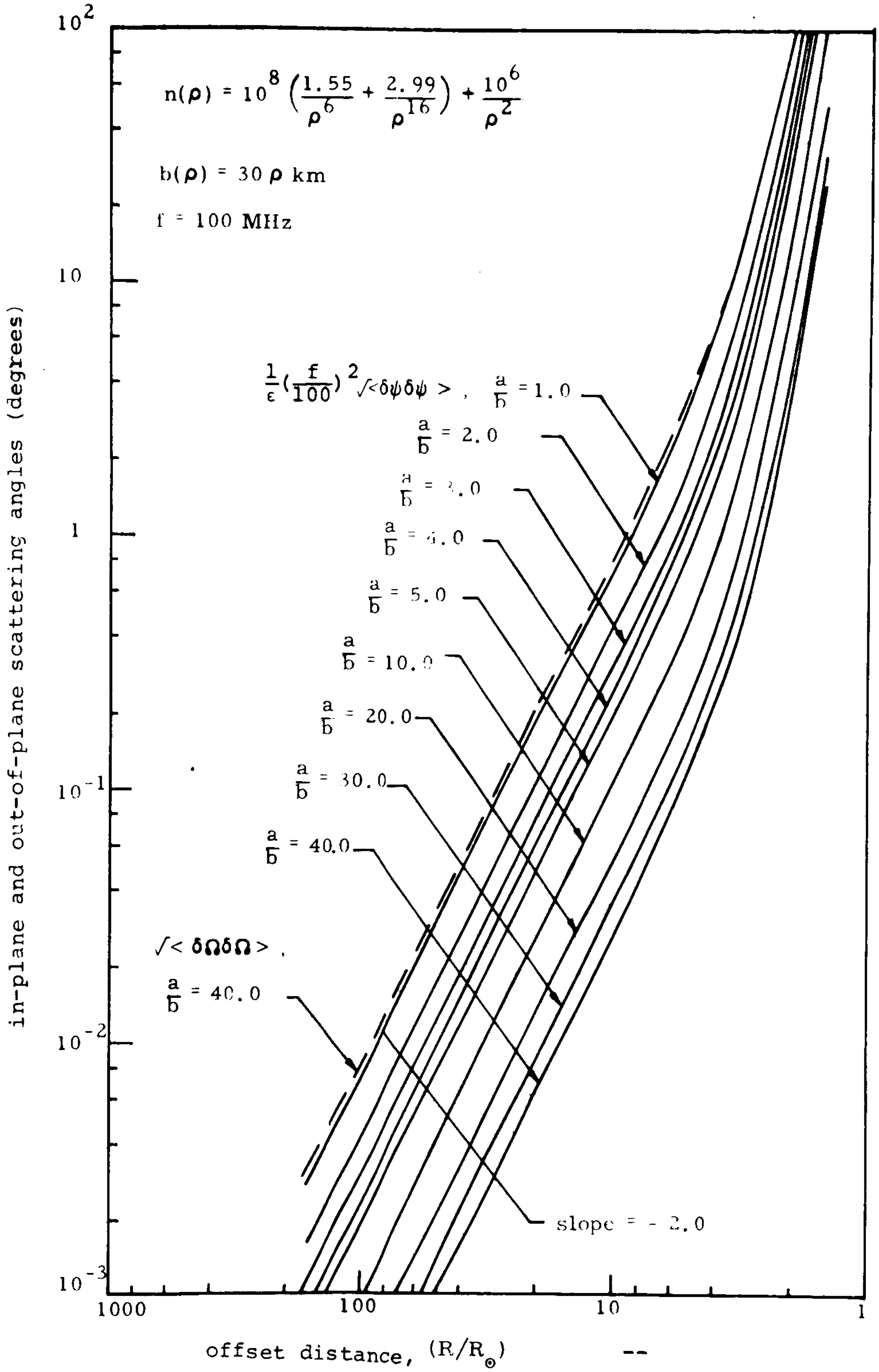
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = .30\,\rho^{4/3}$ km.
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)\sqrt{\frac{<\delta f \delta f>}{<f_{ph}^2>}}$
40.0
5.0
$\frac{a}{b}$ = 1.0
slope = $-\frac{5}{6}$
line broadening
10^7
10^6
10^5
10^4
10^3
10^2
1000
100
10
1
offset distance, $(R/R_\odot)$

Graphs 9, 10, 11

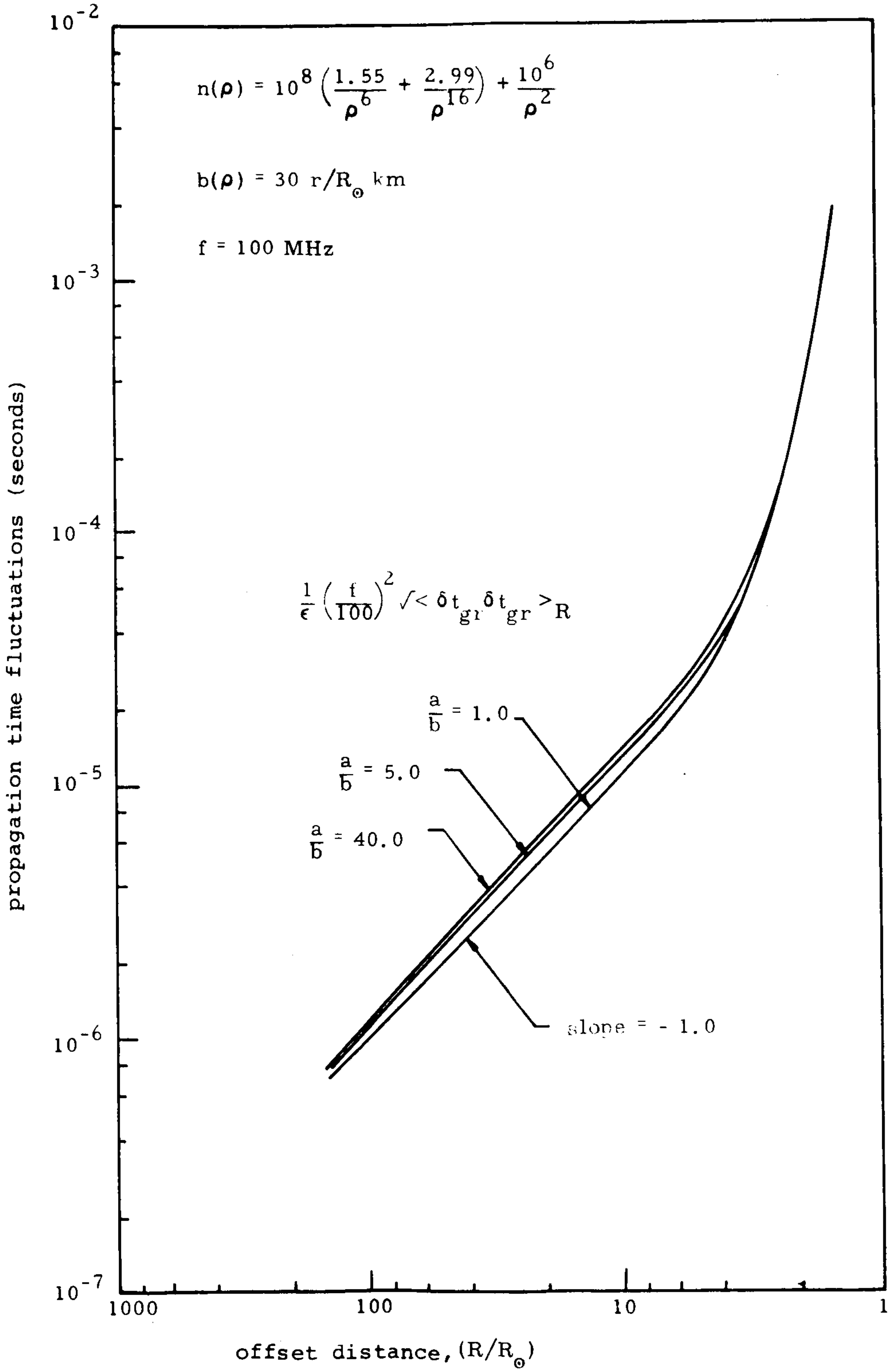
Graph 9: The in-plane and out-of-plane scattering angles vs. path offset distance for constant anisotropy ratios and a transverse correlation distance $b(\rho) = 30\rho$ km.

Graph 10: The fluctuations in pulse signal propagation times vs. path offset distance for constant anisotropy ratios and a transverse correlation distance $b(\rho) = 30\rho$ km.

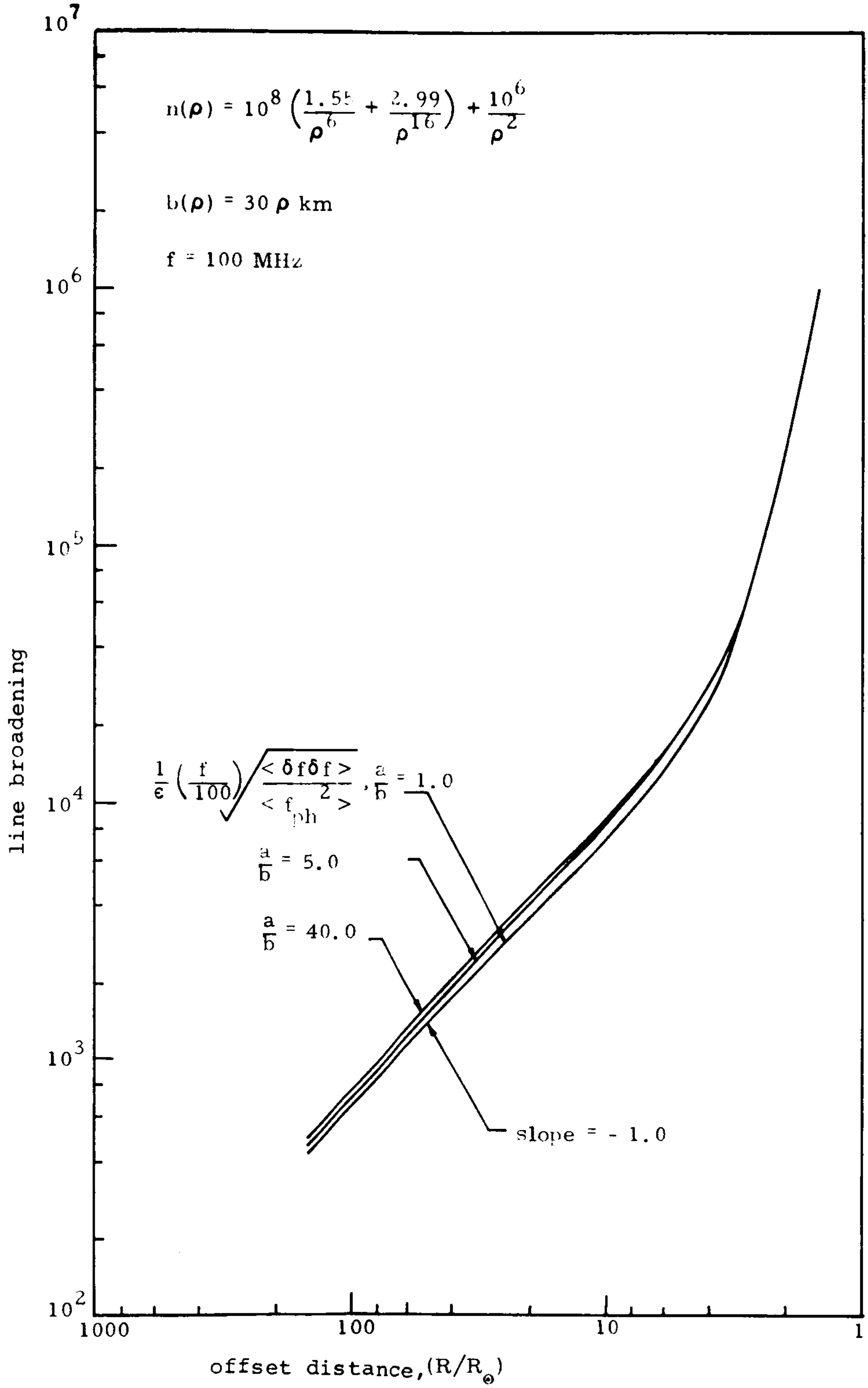
Graph 11: The line broadening vs. path offset distance for constant anisotropy ratios and a transverse correlation length $b(\rho) = 30\rho$ km.



n(ρ) = 10^8 (1.55/ρ^6 + 2.99/ρ^16) + 10^6/ρ^2
b(ρ) = 30 ρ km
f = 100 MHz
1/ε (f/100)^2 √<δψδψ> , a/b = 1.0
a/b = 2.0
a/b = 5.0
a/b = 20.0
a/b = 40.0
√< δΩδΩ > ,
a/b = 40.0
slope = - 2.0
in-plane and out-of-plane scattering angles (degrees)
offset distance, (R/R⊙)
10^2
10
1
10^-1
10^-2
10^-3
1000
100
10
1



$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = 30\ r/R_\odot$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta t_{gr}\delta t_{gr}>_R}$
$\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 5.0
$\frac{a}{b}$ = 40.0
slope = - 1.0
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
propagation time fluctuations (seconds)
1000
100
10
1
offset distance, $(R/R_\odot)$



$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = 30\,\rho$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)\sqrt{\frac{<\delta f \delta f>}{<f_{ph}^2>}}$, $\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 5.0
$\frac{a}{b}$ = 40.0
slope = - 1.0
line broadening
10^7
10^6
10^5
10^4
10^3
10^2
1000
100
10
1
offset distance, $(R/R_\odot)$

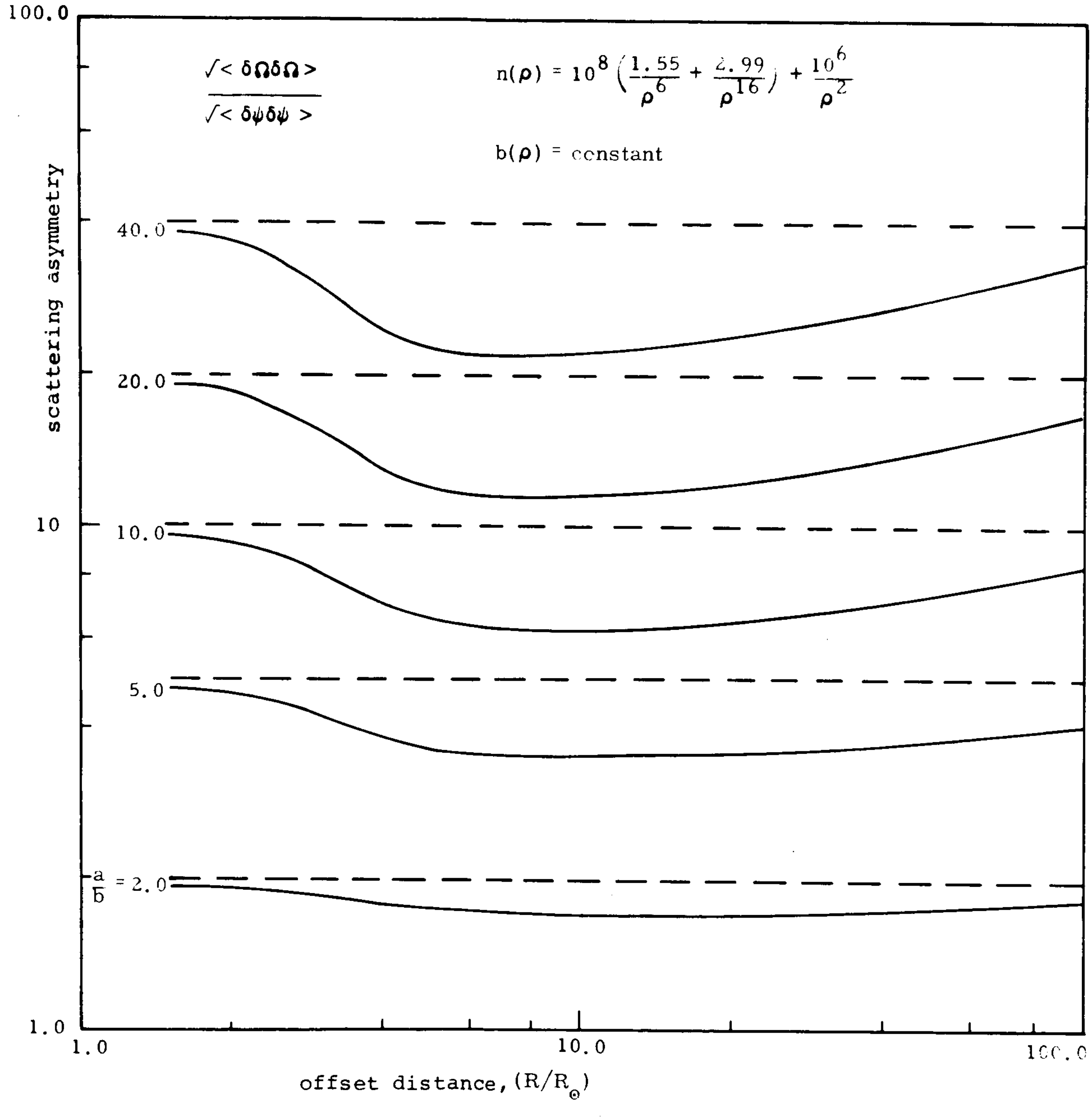
Graphs 12, 13, 14

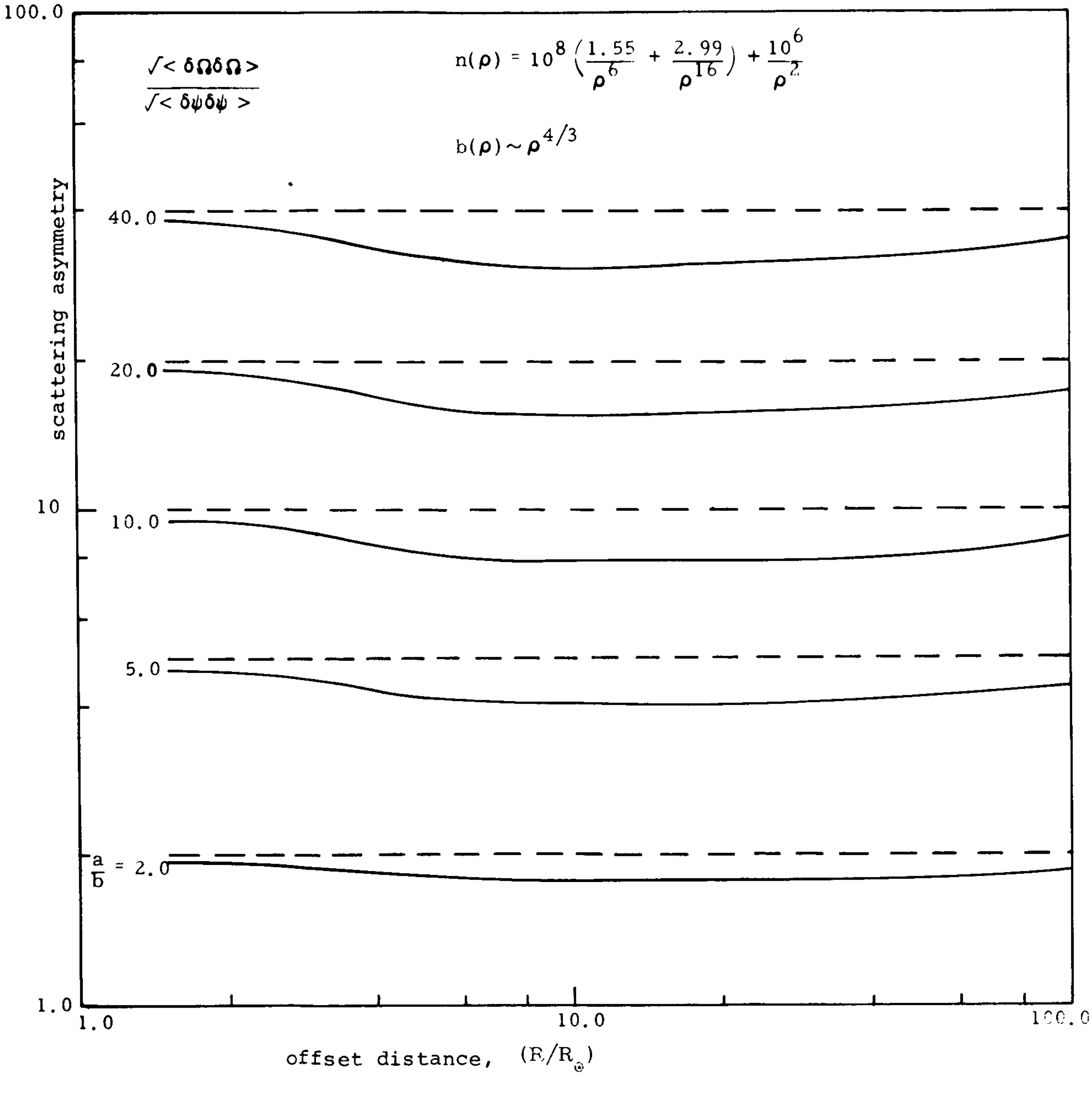
The scattering asymmetry (the ratio of the out-of-plane r.m.s. scattering angle to the in-plane r.m.s. scattering angle) vs. path offset distance for constant anisotropy ratios. The solid lines represent the numerically calculated values while the dashed lines are those values which would be observed were (4.98) strictly true.

Graph 12: Transverse correlation length = constant

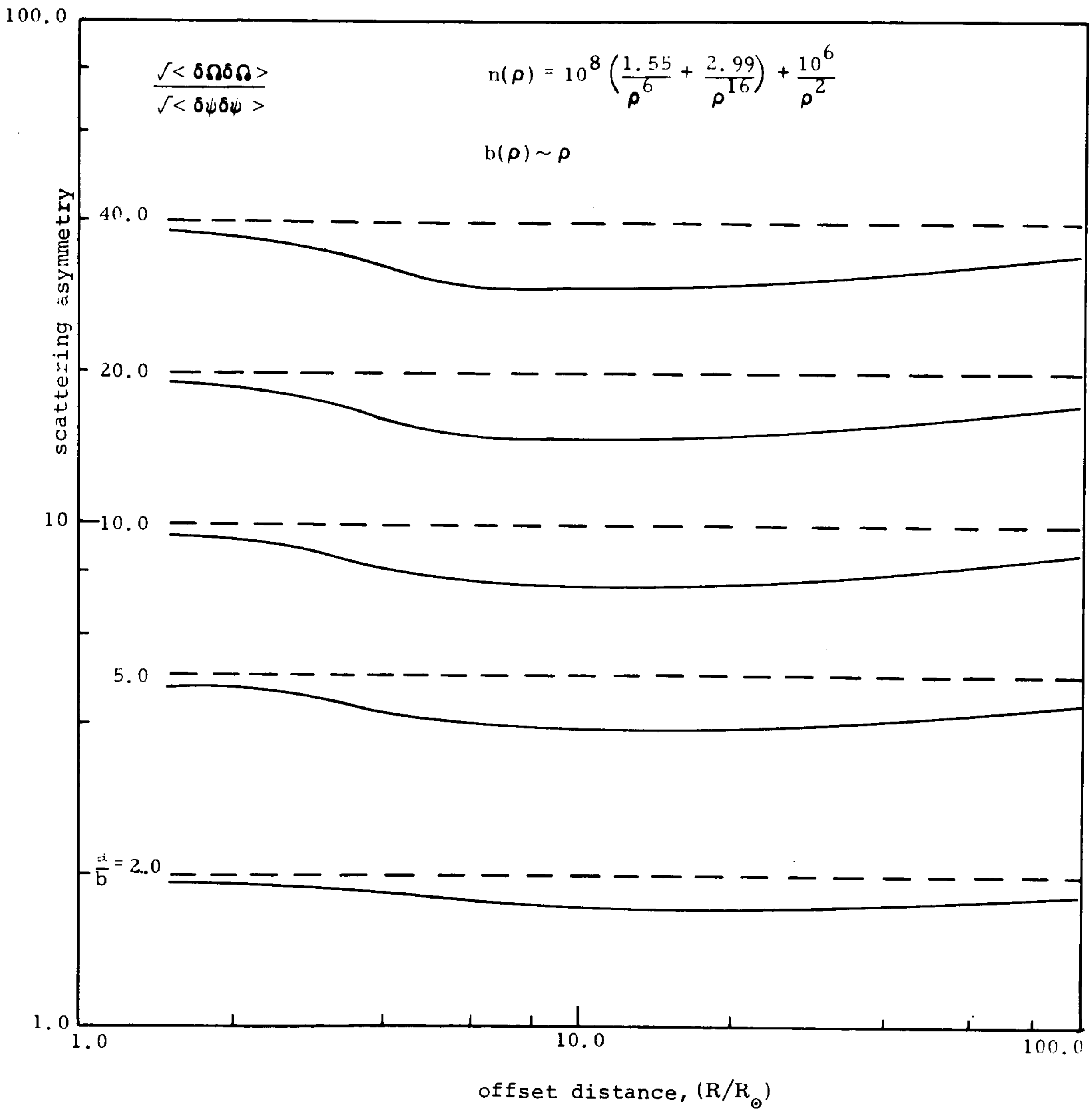
Graph 13: Transverse correlation length $\sim \rho^{4/3}$

Graph 14: Transverse correlation length $\sim \rho$


$\frac{\sqrt{<\delta\Omega\delta\Omega>}}{\sqrt{<\delta\psi\delta\psi>}}$
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho)$ = constant
scattering asymmetry
100.0
10
1.0
40.0
20.0
10.0
5.0
$\frac{a}{b}$ = 2.0
1.0
10.0
100.0
offset distance, $(R/R_\odot)$



100.0
10
1.0
$\frac{\sqrt{<\delta\Omega\delta\Omega>}}{\sqrt{<\delta\psi\delta\psi>}}$
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) \sim \rho^{4/3}$
scattering asymmetry
40.0
20.0
10.0
5.0
$\frac{a}{b}$ = 2.0
1.0
10.0
100.0
offset distance, $(R/R_\odot)$


$\frac{\sqrt{<\delta\Omega\delta\Omega>}}{\sqrt{<\delta\psi\delta\psi>}}$
$n(\rho) = 10^8 \left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) \sim \rho$
scattering asymmetry
100.0
40.0
20.0
10
10.0
5.0
$\frac{a}{b} = 2.0$
1.0
1.0
10.0
100.0
offset distance, $(R/R_\odot)$

Graphs 15, 16, 17

The in-plane scattering angle vs. path offset distance for various functional forms of the anisotropy ratio a/b.

Curve 1: $a/b = 1$

2: $a/b = .1(\rho - 1) + 1$

3: $a/b = \rho$

4: $a/b = 10(\rho - 1) + 1$

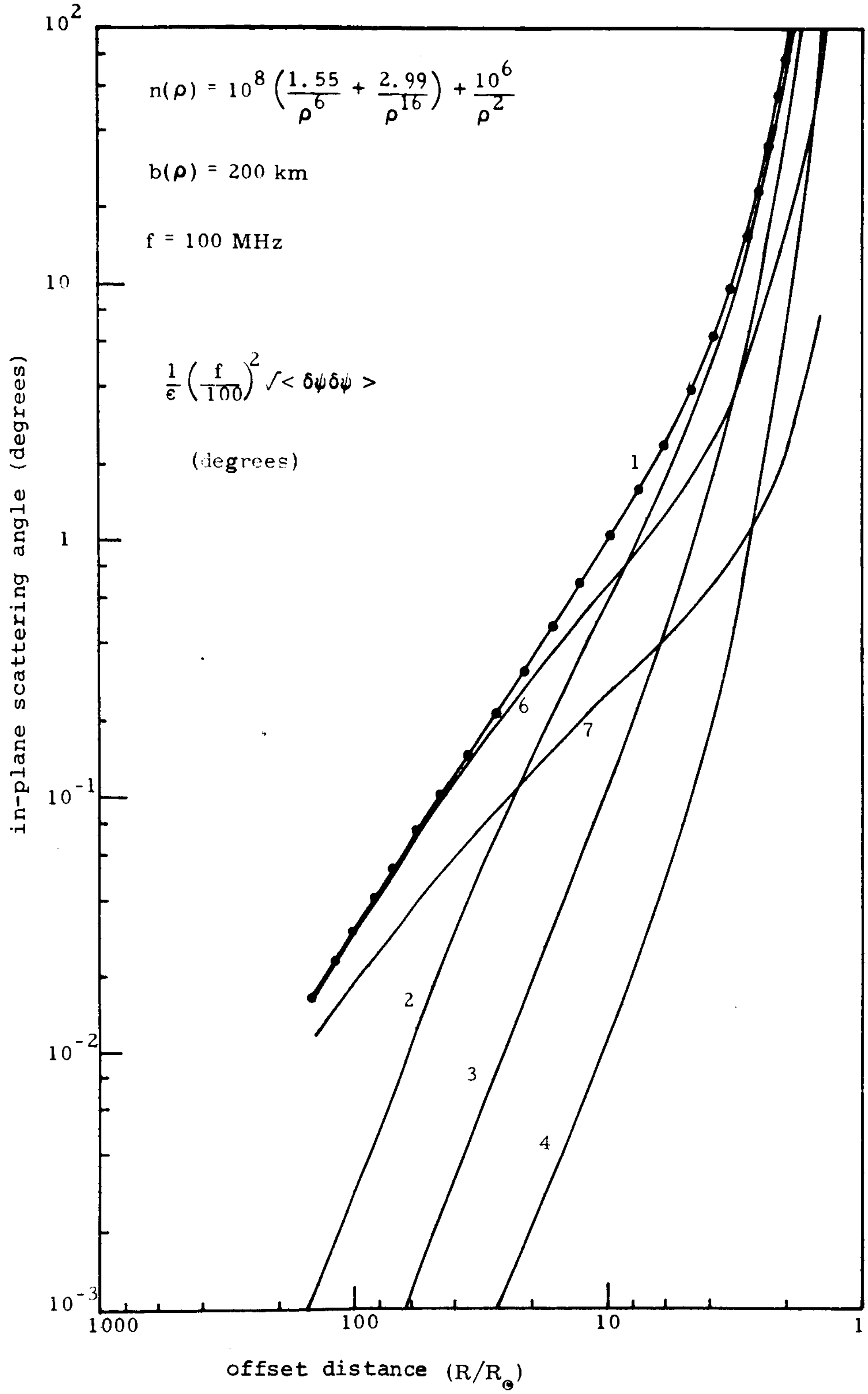
6: $a/b = 1 + 10/\rho$

7: $a/b = 1 + 100/\rho$

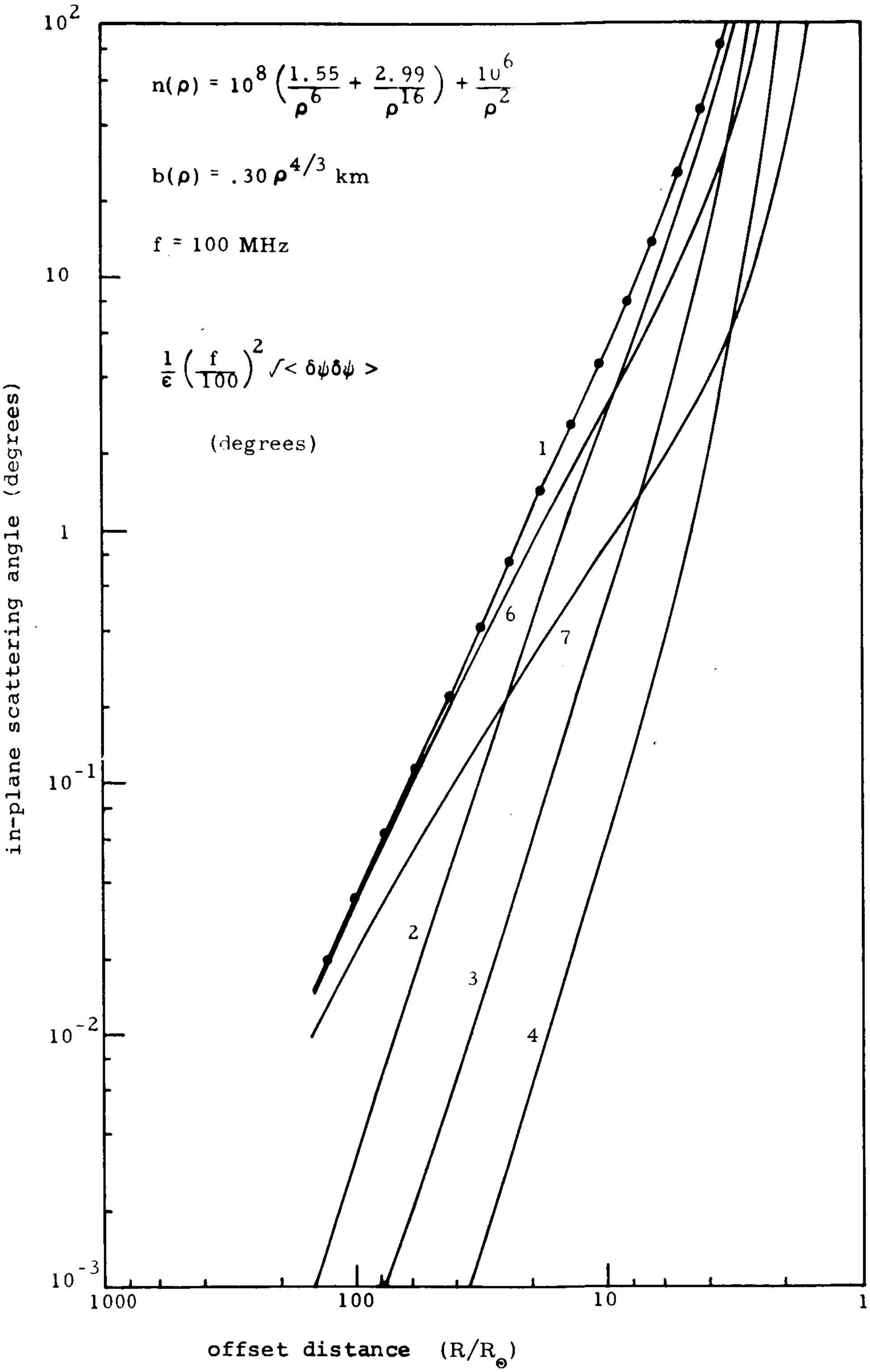
Graph 15: Transverse correlation length = 200 km

Graph 16: Transverse correlation length = $.30\rho^{4/3}$ km

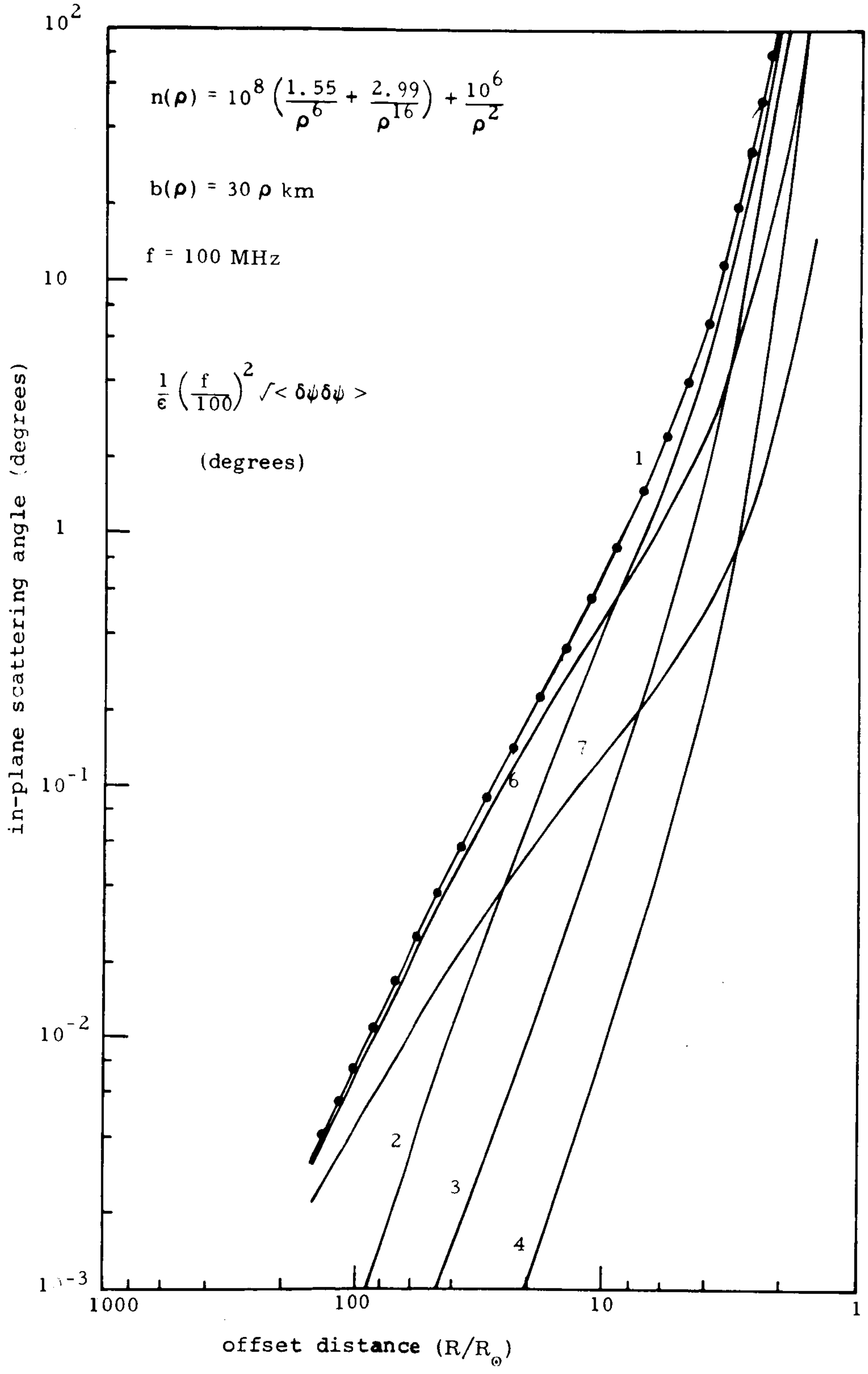
Graph 17: Transverse correlation length = 30ρ km



$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho)$ = 200 km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta\psi\delta\psi>}$
(degrees)
1
2
3
4
6
7
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
1000
100
10
1
in-plane scattering angle (degrees)
offset distance $(R/R_\odot)$



$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = .30\,\rho^{4/3}$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta\psi\delta\psi>}$
(degrees)
1
6
7
2
3
4
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
1000
100
10
1
in-plane scattering angle (degrees)
offset distance ($R/R_\odot$)



$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = 30\ \rho$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta\psi\delta\psi>}$
(degrees)
1
2
3
4
6
7
10^2
10
1
10^{-1}
10^{-2}
10^{-3}
1000
100
10
1
in-plane scattering angle (degrees)
offset distance ($R/R_\odot$)

Graphs 18, 19, 20

The scattering asymmetry (the ratio of the out-of-plane r.m.s. scattering angle to the in-plane r.m.s. scattering angle) vs. path offset distance for various functional forms of the anisotropy ratio a/b. The solid lines represent the numerically calculated values while the dashed lines are those values which would be observed were (4.98) strictly true.

Curve 2: $a/b = .1(\rho - 1) + 1$

3: $a/b = \rho$

4: $a/b = 10(\rho - 1) + 1$

5: $a/b = 1 + 1/\rho$

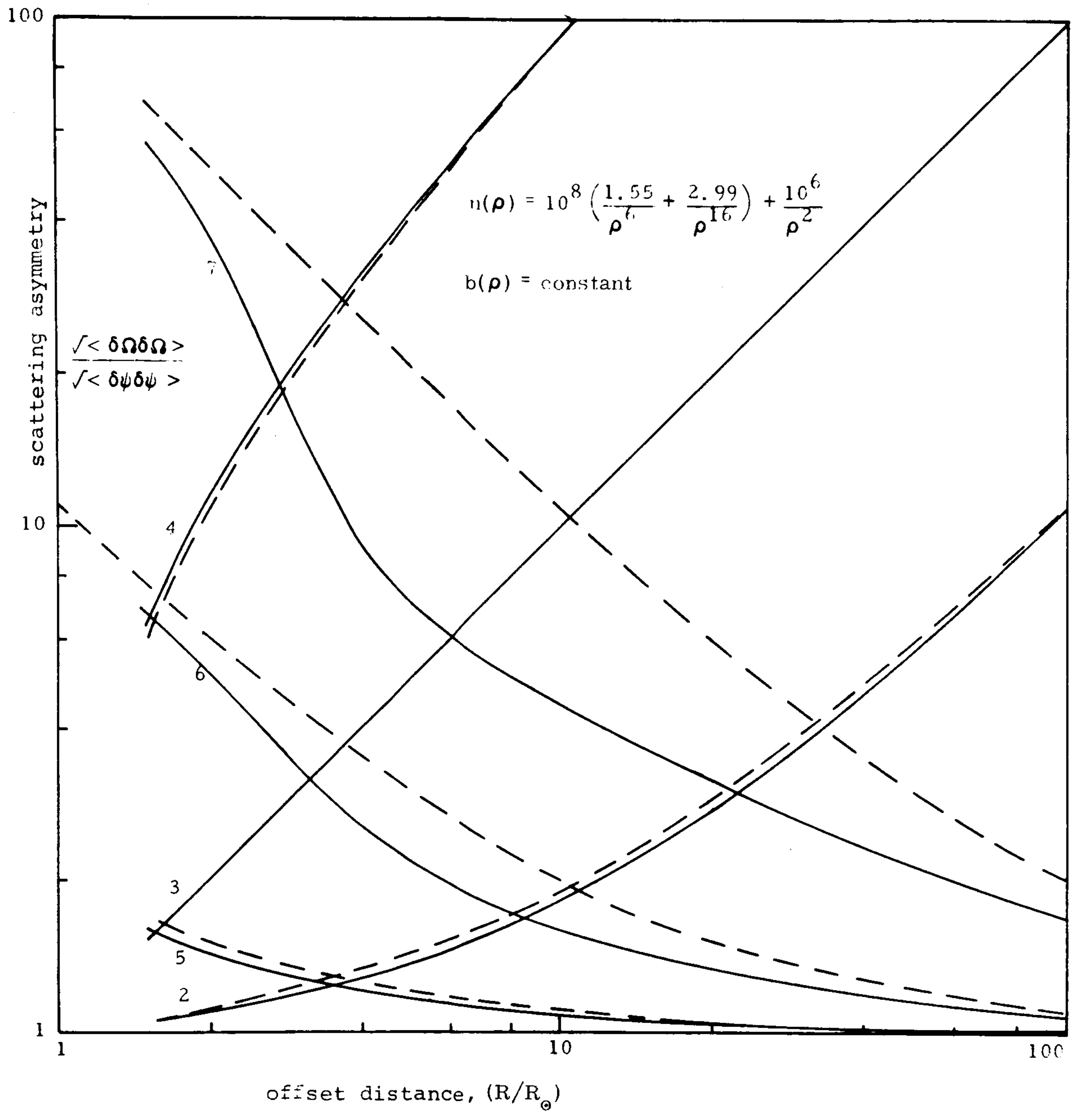
6: $a/b = 1 + 10/\rho$

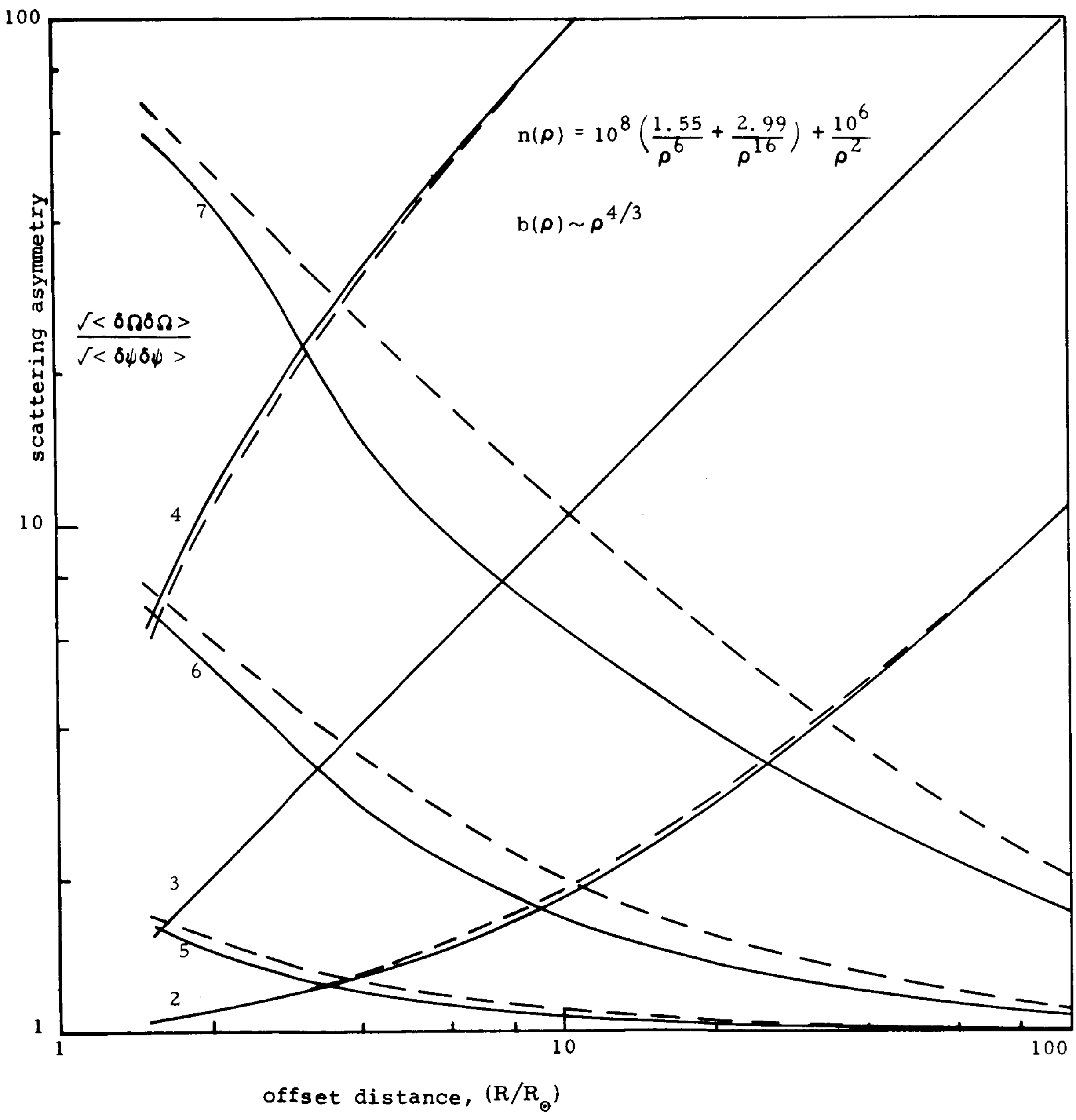
7: $a/b = 1 + 100/\rho$

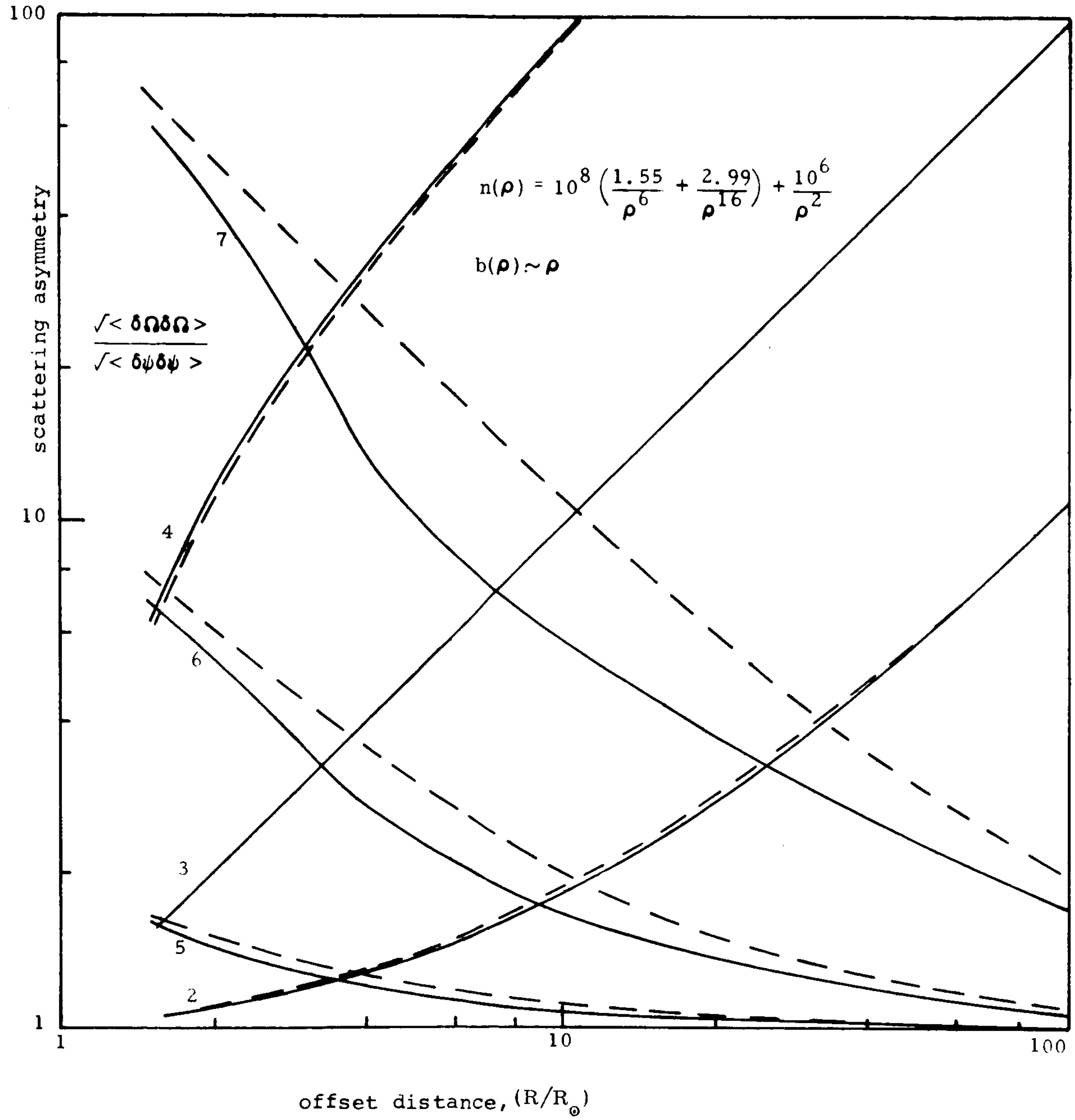
Graph 18: Transverse correlation length = constant

Graph 19: Transverse correlation length $\sim \rho^{4/3}$

Graph 20: Transverse correlation length $\sim \rho$


100
10
1
1
10
100
scattering asymmetry
√< δΩδΩ >
√< δψδψ >
n(ρ) = 10^8 (1.55/ρ^6 + 2.99/ρ^16) + 10^6/ρ^2
b(ρ) = constant
7
4
6
3
5
2
offset distance, (R/R⊙)


100
10
1
1
10
100
scattering asymmetry
$\frac{\sqrt{<\delta\Omega\delta\Omega>}}{\sqrt{<\delta\psi\delta\psi>}}$
offset distance, $(R/R_{\odot})$
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) \sim \rho^{4/3}$
7
4
6
3
5
2


100
10
1
1
10
100
scattering asymmetry
offset distance, (R/R$_\odot$)
$\frac{\sqrt{<\delta\Omega\delta\Omega>}}{\sqrt{<\delta\psi\delta\psi>}}$
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) \sim \rho$
7
4
6
3
5
2

Graphs 21, 22, 23

The scattering parameters for the case of a curved general solar magnetic field in the solar equatorial plane. The angle between the magnetic field and the radial direction is

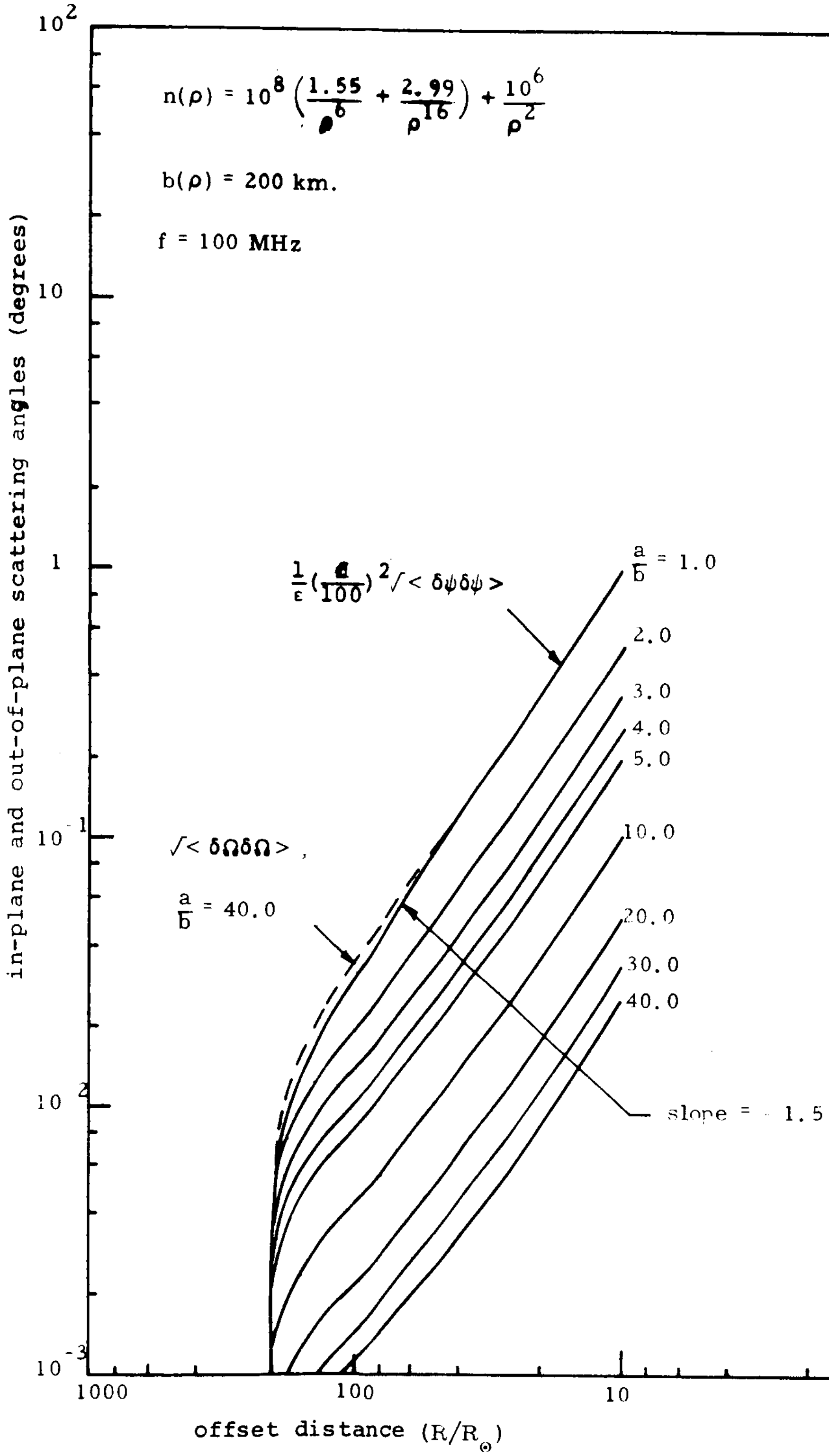
$$\alpha = \tan^{-1} \frac{r\Omega}{V}$$

where (Ω) is the solar rotation rate (14.38°/day) and (V) is the (constant) solar wind velocity (300 km/sec). The transverse correlation length is 200 km.

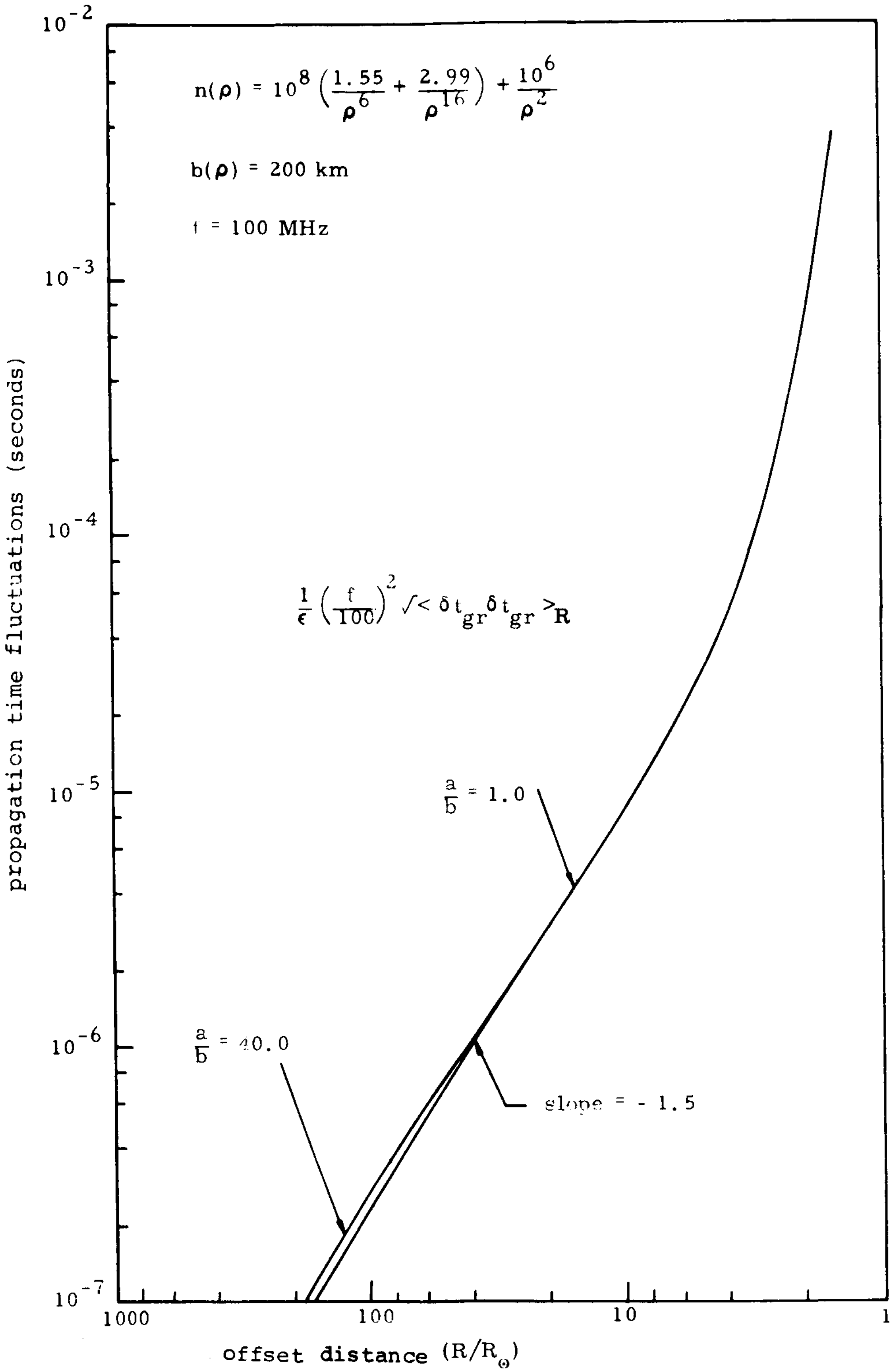
Graph 21: The in-plane and out-of-plane scattering angles vs. path offset distance for constant anisotropy ratios.

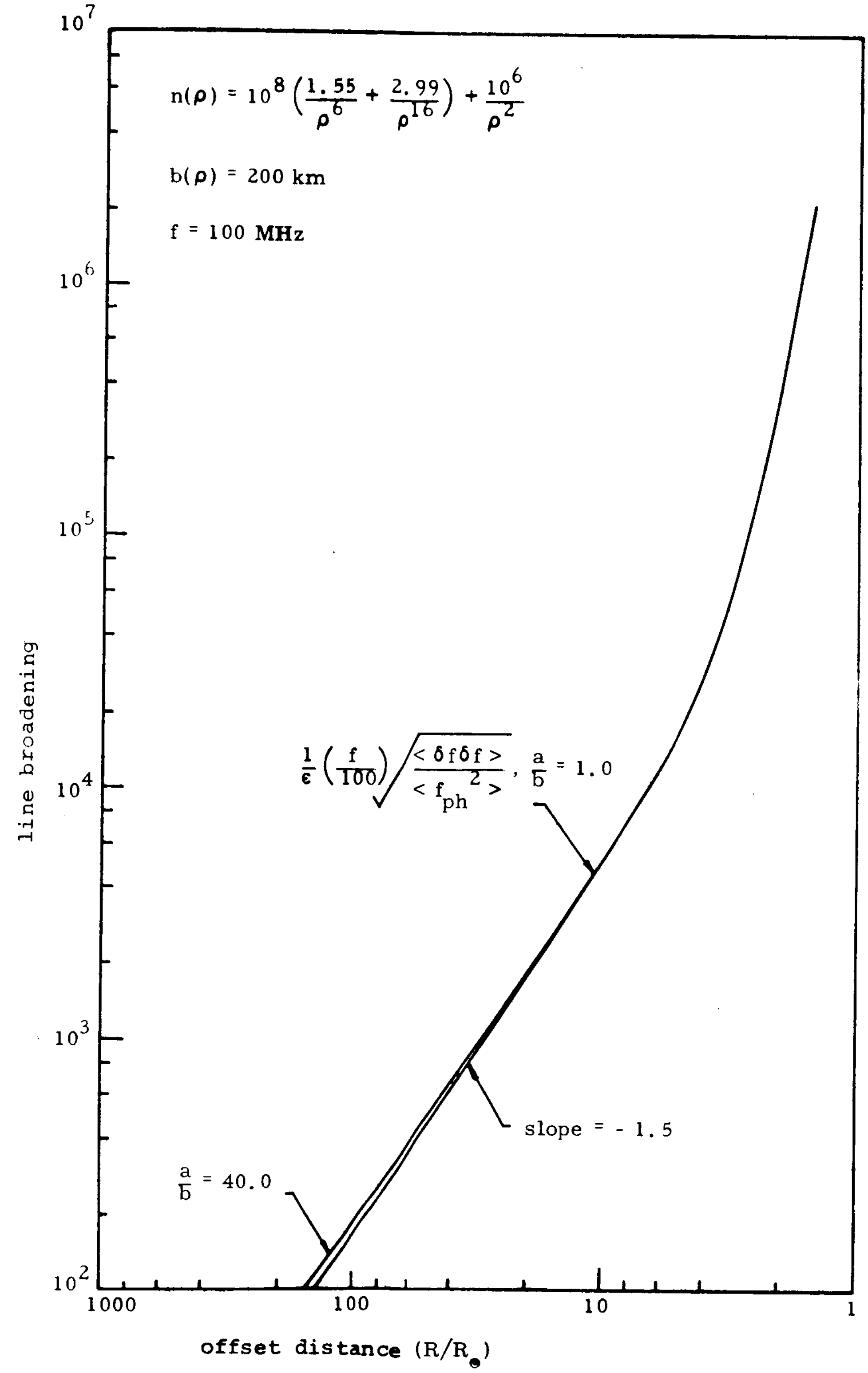
Graph 22: The fluctuations in pulse signal propagation times vs. path offset distance for constant anisotropy ratios.

Graph 23: The line broadening vs. path offset distance for constant anisotropy ratios.



n(ρ) = 10^8 (1.55/ρ^6 + 2.99/ρ^16) + 10^6/ρ^2
b(ρ) = 200 km.
f = 100 MHz
1/ε (ε/100)^2 √< δψδψ >
a/b = 1.0
2.0
3.0
4.0
5.0
10.0
20.0
30.0
40.0
√< δΩδΩ > ,
a/b = 40.0
slope = - 1.5
in-plane and out-of-plane scattering angles (degrees)
10^2
10
1
10^-1
10^-2
10^-3
1000
100
10
1
offset distance (R/R⊙)

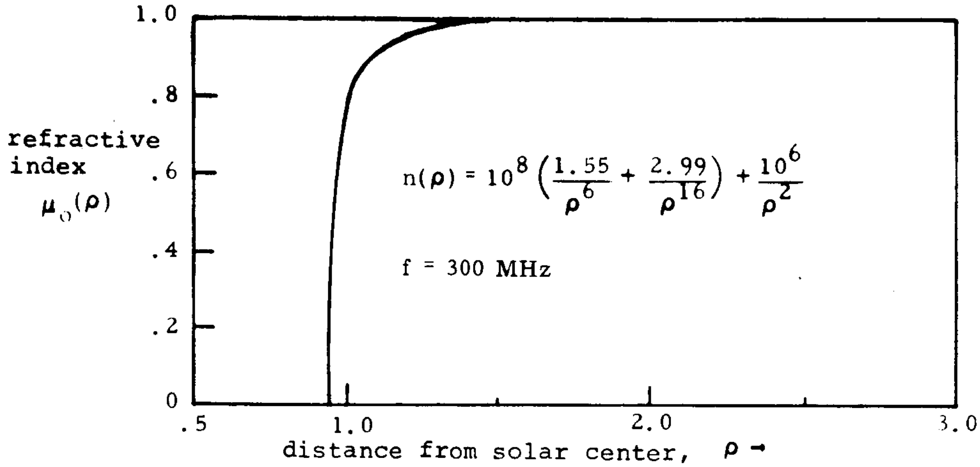

$n(\rho) = 10^8 \left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho)$ = 200 km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)^2 \sqrt{<\delta t_{gr}\delta t_{gr}>_R}$
$\frac{a}{b}$ = 1.0
$\frac{a}{b}$ = 40.0
slope = - 1.5
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
1000
100
10
1
propagation time fluctuations (seconds)
offset distance ($R/R_\odot$)

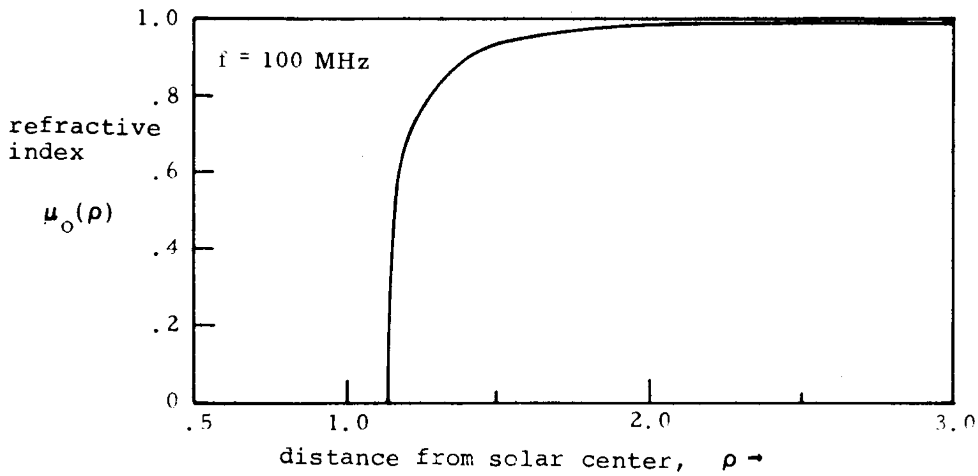


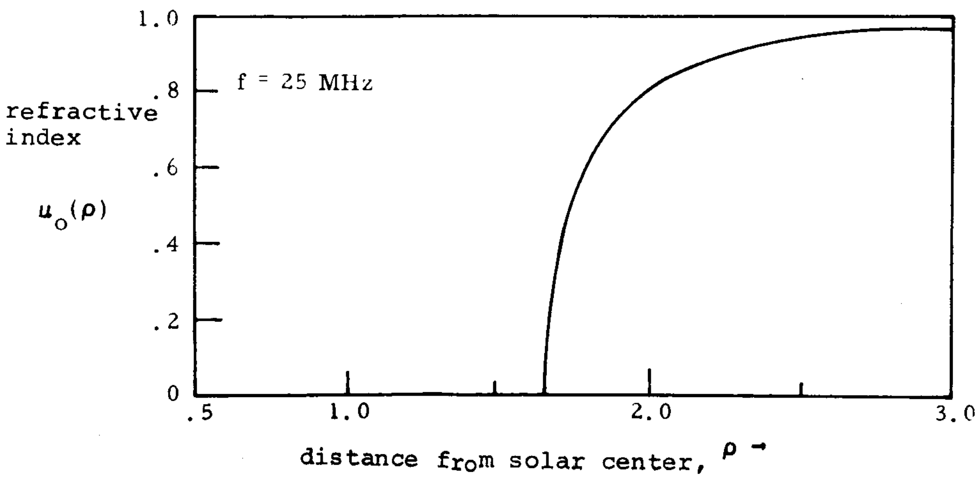
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
$b(\rho) = 200$ km
f = 100 MHz
$\frac{1}{\epsilon}\left(\frac{f}{100}\right)\sqrt{\frac{<\delta f \delta f>}{<f_{ph}^2>}}$, $\frac{a}{b} = 1.0$
slope = - 1.5
$\frac{a}{b} = 40.0$
10^7
10^6
10^5
10^4
10^3
10^2
line broadening
1000
100
10
1
offset distance $(R/R_\odot)$

Graph 24

The average refractive index as a function of distance from the solar center for various frequencies.


1.0
.8
refractive index
$\mu_o(\rho)$
.6
.4
.2
0
$n(\rho) = 10^8\left(\frac{1.55}{\rho^6} + \frac{2.99}{\rho^{16}}\right) + \frac{10^6}{\rho^2}$
f = 300 MHz
.5
1.0
2.0
3.0
distance from solar center, $\rho \rightarrow$


1.0
f = 100 MHz
.8
refractive index
$\mu_o(\rho)$
.6
.4
.2
0
.5
1.0
2.0
3.0
distance from solar center, $\rho \rightarrow$


1.0
f = 25 MHz
.8
refractive index
$\mu_o(\rho)$
.6
.4
.2
0
.5
1.0
2.0
3.0
distance from solar center, $\rho \rightarrow$

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