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Finite Difference Forms Containing Derivatives of Higher Order

by

Manfred Reimer
Visiting Research Assistant Professor
Computer Science Center

ABSTRACT

The use of finite difference methods for the numerical treatment of initial value problems depends on two concepts, namely stability and degree of approximation. The latter can be improved significantly if derivatives of higher order are used. In this report, the stability problem is solved for such generalized finite difference schemes.

Finite Difference Forms Containing Derivatives of Higher Order.[2]

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1. Introduction

Let us consider finite difference forms L defined by

$$(1.1)\qquad Ly = \sum_{\nu=0}^{k} \left\{ a_\nu^{(0)} y_\nu + \sum_{\lambda=r}^{r+m-1} h^\lambda a_\nu^{(\lambda)} y_\nu^{(\lambda)} \right\}$$

where

$$y_\nu^{(\lambda)} = y^{(\lambda)}(x + \nu h),\ h > 0 .$$

Here k,r,m are assumed to be positive integers, and the $a_\nu^{(\lambda)}$ are real numbers, $a_k^{(0)} \neq 0$.

The forms (1.1) are related to the differential equation

$$(1.2)\qquad y^{(r)} = f(x,y)$$

which, under some restrictions (high degree, stability), can be solved numerically by means of them with good success. (See Dahlquist [2].)

Let $\mathcal{P}_n$ denote the class of all real polynomials of degree not exceeding n. Let p be defined by

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(1.3) $\quad Ly \begin{cases} \equiv 0 & \text{for all } y \in \mathcal{P}_{p+r-1}, \\ \not\equiv 0 & \text{for one } y \in \mathcal{P}_{p+r}. \end{cases}$

We call p the <u>degree</u>, k the <u>order</u> and m the <u>rank</u> of the difference form. Often it is convenient to describe the difference form (1.1) by means of the polynomials

$$\rho_\lambda(s) = \sum_{\nu=0}^{k} a_\nu^{(\lambda)} s^\nu \quad (\lambda = 0;\ r, r+1, \ldots, r+m-1)$$

which are called the <u>generating</u> polynomials of L.

<u>Definition</u> (Stability).

L is called stable if and only if $p \geq 1$, all zeros of ρ_0 are situated on the disk $|s| \leq 1$ and none of the zeros on the unit circle has a multiplicity exceeding r.

For a discussion of the significance of this definition, see Dahlquist [2], Ch. 2; let us remark here only that it is indispensable to admit zeros of multiplicity r on the unit circle since otherwise there are no stable forms at all (see section 3).

With regard to numerical applications, those forms are of particular interest which are stable and of as high a degree as possible. Dahlquist already determined this maximum degree and all stable forms realizing it in the cases m = 1 and m = 2, r = 1 or 2. We are going to generalize his results for - with some restriction - arbitrary m. The basis for our approach will be a new characterization of the degree p (see (2.2)).

Finally we mention that (1.1) contains two remarkable cases of degeneration: For $(k, r, m) = (0,1,\infty)$ we obtain Taylor's series and for $m = 0$, $\rho_o = (s-1)^k$ the k-th difference of y.

2. Characterization of the Degree

L being linear, we can define p as follows (see (1.3)): Ly vanishes identically on a basis of $\mathcal{P}_{p+r-1}$, but there is a value $x = x_o$ such that Ly does not vanish at this point for one (and hence for any) polynomial of degree exactly p + r. Without restriction of generality, we may assume $x_o = 0$. Thus we are led to the following characterization of p:

$$(2.1) \qquad Lx^{\mu}\big|_{x=0} \begin{cases} = 0 \text{ for } \mu = 0,\ 1,\ p+r-1, \\ \neq 0 \text{ for } \mu = p+r. \end{cases}$$

On the other hand we get from (1.1)

$$Lx^{\mu}\big|_{x=0} = h^{\mu} \sum_{\nu=0}^{k} \left\{ a_{\nu}^{(0)} \nu^{\mu} + \sum_{\lambda=r}^{r+m-1} a_{\nu}^{(\lambda)} \cdot \mu(\mu-1)\cdots(\mu-\lambda+1)\cdot \nu^{\mu-\lambda} \right\}.$$

Now let

$$D = s\frac{d}{ds}$$

and define D^λ for $\lambda < 0$ by zero. Then we get

$$Lx^\mu\big|_{x=0} = h^\mu\Big[D^\mu \rho_o + \sum_{\lambda=r}^{r+m-1} \mu(\mu-1)\cdots(\mu-\lambda+1)\cdot D^{\mu-\lambda}\rho_\lambda\Big]_{s=1}.$$

Being aware of this and combining the inequality and the equalities of (2.1) linearly we find that p can be characterized also by means of the relations

$$(2.2)\qquad \Big[\pi(D)\rho_o + \sum_{\nu=r}^{r+m-1} \pi^{(\lambda)}(D)\rho_\lambda\Big]_{s=1} \begin{cases} = 0 \text{ for all } \pi \in \mathcal{P}_{p+r-1}, \\ \neq 0 \text{ for one } \pi \in \mathcal{P}_{p+r}. \end{cases}$$

It shall be seen that this characterization offers some advantage over the original definition (1.3). We shall need some rules concerning the operator D; they are listed below, for details see Reimer [4], p. 373 f.

Let f,g,u,v be polynomials. If

$$u(s) = \sum_{\nu=0}^{n} a_\nu s^\nu$$

and the polynomial u* is defined by

$$u^*(s) = s^n u(s^{-1}),$$

then the following formulas hold:

$$(2.3)\qquad f(D)u = \sum_{\nu=0}^{n} f(\nu)\, a_\nu s^\nu,$$

$$(2.4)\qquad f(D)\,(uv) = \sum_{\nu=0}^{n} a_\nu s^\nu f(D+\nu)v(s),$$

$$(2.5)\qquad [f(D)u]^* = f(n-D)u^*.$$

In particular, because of (2.3)

$$f(D)u = g(D)u \text{ if } f \equiv g \bmod \omega \tag{2.6}$$

is valid if ω is the polynomial

$$\omega(x) = x(x-1)\cdots(x-n).$$

The basis for section 5 is a theorem which we state here without proof:

Theorem 2.1.

Let u be a real polynomial with degree exactly n and with no zeros outside the unit circle. Assume f to be a real polynomial satisfying

$$f(n-s) = (-1)^{\iota} f(s)$$

with some integer ι. Let

$$\Delta^{\lambda} f(\nu) > 0$$

for all integers ν, λ with

$$0 \leq \nu \leq n-\lambda,\ 0 \leq \lambda \leq n,\ \lambda \equiv \iota \bmod 2.$$

Then the polynomial $f(D)u$ has no real zero outside the interval $[-1,1]$, and $(-1)^{\mu}$, μ an integer, is one of its zeros if and only if

$$u^* = (-1)^{\mu n+\iota+1} \cdot u.$$

This theorem is a specialization of theorem 3.3 of Reimer [4].

3. Construction of Degree $p \geq m(k+1)$

Because of (2.2), $p \geq 0$ is valid if and only if each of the polynomials

$$D^{o}\rho_{o}, D'\rho_{o}, \dots, D^{r-1}\rho_{o}$$

vanishes for $s = 1$. This is true if and only if

(3.1) $\rho_o(s) = (s-1)^r \cdot \varphi(s), \quad \varphi \in \mathcal{P}_{k-r}$,

which in turn implies the restriction

(3.2) $1 \leq r \leq k$

for r. Therefore the following statement holds:

Theorem 3.1. If $r > k$ then L is unstable.

Henceforth let $p \geq 0$, i.e., let (3.1) be valid. We define the index set

$$\mathfrak{J} = \{(\nu,\mu) | \nu,\mu \text{ integers}; \ 0 \leq \nu \leq k, \ 0 \leq \mu \leq m-1\}.$$

For each $(i,j) \in \mathfrak{J}$ we define q_{ij} to be the (as well known, unique) solution of the interpolation problem

(3.3) $q_{ij} \in \mathcal{P}_{m(k+1)-1}; \quad q_{ij}^{(\mu)}(\nu) = \begin{cases} 1 \text{ for } (\nu,\mu) = (i,j), \\ 0 \text{ for } (i,j) \neq (\nu,\mu) \in \mathfrak{J}. \end{cases}$

Then we define Q_{ij} to be the (unique) solution of the initial value problem

$$(3.4) \qquad Q_{ij}^{(r)} = q_{ij};\ Q_{ij}(\tfrac{k}{2}) = Q'_{ij}(\tfrac{k}{2}) = \ldots = Q_{ij}^{(r-1)}(\tfrac{k}{2}) = 0.$$

Together with the polynomials

$$1, x, x^2, \ldots, x^{r-1},$$

the Q_{ij} obviously form a basis for the space $\mathcal{P}_{m(k+1)+r-1}$. Thus it follows from (2.2) and (3.1) that $p \geqq m(k+1)$ is valid if and only if

$$\left[Q_{ij}(D)\rho_o + \sum_{\mu=0}^{m-1} Q_{ij}^{(r+\mu)}(D)\rho_{r+\mu}\right]_{s=1} = 0 \text{ for all } (i,j) \in \mathfrak{J}$$

which is, because of (3.4), (3.3) and (2.3), the same as

$$(3.5) \qquad a_i^{(r+j)} = -\sum_{\nu=0}^{k} Q_{ij}(\nu)\, a_\nu^{(0)} \text{ for all } (i,j) \in \mathfrak{J}.$$

Hence, given a polynomial ρ_o satisfying (3.1), all other generating polynomials are well defined by the condition $p \geq m(k+1)$:

Theorem 3.2.

Let $1 \leq r \leq k$ and ρ_o be a nonzero polynomial with $s = 1$ as zero of multiplicity not less than r. Then there is one and only one difference form (1.1) with degree $p \geq m(k+1)$ and with ρ_o as its generating polynomial.

Remark 1. If ρ_o is chosen so that the stability condition holds, then Theorem 3.2 states the existence of exactly one stable form L with ρ_o as its generating polynomial and with degree $p \geq m(k+1)$. Thus the set of all stable forms of such a degree is a (k-r)-parameter family.

In section 5 we shall examine how far the degree of a stable form can really exceed the value m(k+1).

Remark 2. A difference form (1.1) is called open, if

$$a_k^{(r)} = a_k^{(r+1)} = \ldots = a_k^{(r+m-1)} = 0.$$

It then defines an extrapolatory difference method. If we replace k by k-1 in the definition of $\mathfrak{F}$ above and likewise $\mathcal{P}_{m(k+1)-1}$ by $\mathcal{P}_{mk-1}$ in (3.3), then the conclusion already used above yields

Theorem 3.3.

Let $1 \leq r \leq k$ and ρ_o be a nonzero polynomial with s = 1 as zero of multiplicity not less than r. Then there is one and only one open difference form (1.1) with $p \geqq m \cdot k$ and with ρ_o as its generating polynomial.

The next section places some lemmas at our disposal and may be skipped by readers being not interested in details.

4. Lemmas

Within the interval $-1 < x < k + 1$ we define the following functions, which are related to the beta function:

$$(4.1) \quad B(x) = B(x+1, k+1-x) = \int_0^1 t^x (1-t)^{k-x}\, dt,$$

$$(4.2) \quad F_{mj}(x) = (x - \tfrac{k}{2})^j B^m(x) \quad (j=0,1,\ldots).$$

Because of $B(k-x) = B(x)$, obviously

$$(4.3) \quad F_{mj}^{(\lambda)}(k-x) = (-1)^{j+\lambda} F_{mj}^{(\lambda)}(x) \quad (\lambda, j=0,1,\ldots)$$

is valid. We are going to prove the

Lemma 4.1. Let j and λ be integers and $0 \leq j \leq \lambda$, $\lambda \equiv j \bmod 2$. Then

$$F_{mj}^{(\lambda)} > 0 \quad (-1 < x < k + 1)$$

holds.

Proof. Because of (4.3) it is sufficient to prove the statement only for $\frac{k}{2} \leq x < k+1$, this is done by induction. From (4.1) it follows that

$$(4.4) \quad B^{(i)}(x) = \int_{\frac{1}{2}}^{1} \left\{ t^x (1-t)^{k-x} + (-1)^i (1-t)^{k-x} \right\} \left(\ln \frac{t}{1-t} \right)^i dt$$

and thus

$$(4.5) \quad B^{(i)}(x) \underset{(=)}{\geq} 0 \text{ for } \frac{k}{2} \leq x < k+1, \; i=0,1,\ldots,$$

where equality is even excluded if $i \equiv 0 \bmod 2$. Under the assumptions of the Lemma, we get from (4.2)

$$F_{1j}^{(\lambda)}(x) = \sum_{\mu=0}^{j} \binom{\lambda}{\mu} B^{(\lambda-\mu)}(x) \left(\frac{d}{dx}\right)^{\mu} \left(x-\frac{k}{2}\right)^{j},$$

and thus together with (4.5) that

$$(4.6) \quad F_{mj}^{(\lambda)}(x) \underset{(=)}{>} 0 \text{ for } \frac{k}{2} \leq x < k+1$$

is valid at least if $m = 1$, equality being excluded if $\lambda \equiv j \bmod 2$. On the other hand, we obtain from (4.2)

$$F_{m+1,j}^{(\lambda)} = \sum_{\mu=0}^{\lambda} \binom{\lambda}{\mu} B^{(\mu)} F_{mj}^{(\lambda-\mu)}$$

and thus by means of (4.5), (4.6)

$$F_{m+1,j}^{(\lambda)}(x) \geqq B(x) \cdot F_{mj}^{(\lambda)}(x) \underset{(=)}{>} 0,$$

equality on the right-hand side being excluded again if $\lambda \equiv j \bmod 2$. Thus validity of (4.6) for $m = 1$ yields validity for all $m \geqq 1$.

For convenience we introduce the next lemma although it is essentially equivalent to theorem 2.1:

<u>Lemma 4.2.</u>

Let H be a real and $(k-r)$-times continuously differentiable function on $o \leq x \leq k-r$. Suppose that

$$H(k-r-x) = (-1)^{\iota} H(x)$$

$$H^{(\lambda)}(x) > 0 \text{ for } 0 \leq x \leq k-r, \ \lambda \equiv \iota \bmod 2$$

is valid for some integer ι and let P be any polynomial such that

$$P(\nu) = (-1)^{\mu\nu} H(\nu) \text{ for } \nu = 0, 1, \ldots, k - r$$

for some integer μ . If φ is a real polynomial of degree exactly $k - r$ with no zeros outside the unit circle, then

$$[P(D)\varphi]_{s=1} = 0$$

is valid if and only if

$$\varphi^* = (-1)^{\mu(k-r)+\iota+1} \cdot \varphi .$$

<u>Proof.</u> Let $\hat{H}$ be the interpolation polynomial associated with H and the point set $\{0,1,\ldots, k-r\}$. The assumptions of the lemma imply that

$$\hat{H}(k-r-x) = (-1)^{\iota} \hat{H}(x),$$

$$\Delta^{\lambda} \hat{H}(\nu) > 0 \text{ for } 0 \leq \nu \leq k-r-\lambda,\ 0 \leq \lambda \leq k-r,\ \lambda \equiv \iota \bmod 2.$$

On the other hand, it follows easily from (2.3) that

$$[P(D)\varphi]_{s=1} = [\hat{H}(D)\varphi]_{s=(-1)^{\mu}}.$$

The last three relations yield the statement of the lemma, if theorem 2.1 is applied.

Finally we prove a lemma concerning the rate of growth of the function B(x):

<u>Lemma 4.3.</u>

Let $\epsilon > 0$ be an arbitrary number. Then there is a number $\gamma > 1$ such that

$$B(x'') \geqq B(x') \cdot \gamma^{\epsilon^2}$$

for all $x', x'' \in (-1, k+1)$ with

$$\left|x'' - \frac{k}{2}\right| \geqq \left|x' - \frac{k}{2}\right| + \epsilon \; .$$

Proof. Because of $B(k-x) = B(x)$ it is sufficient to prove the statement under the additional condition

$$\frac{k}{2} \leq x' < x'' < k+1; \qquad x'' \geqq x' + \epsilon.$$

If we take into account that

$$\ln \frac{t}{1-t} \geqq 2\,(2t-1) \text{ for } \frac{1}{2} \leq t < 1,$$

then (4.4) implies the estimate

$$\frac{1}{2} B'(x) \geqq \mathbf{B}(x+2,\; k+1-x) - \mathbf{B}(x+1,\; k+2-x)$$

for $\frac{k}{2} \leq x < k+1$, and if we recall the relation

$$\Gamma(u+v) \cdot \mathbf{B}(u,v) = \Gamma(u) \cdot \Gamma(v),$$

we obtain finally

$$\frac{B'(x)}{B(x)} \geqq \frac{4}{k+2}\left(x - \frac{k}{2}\right) \text{ for } \frac{k}{2} \leq x < k+1.$$

Integration from x' to x'' now yields

$$\ln \frac{B(x'')}{B(x')} \geqq \frac{2}{k+2}\left[\left(x'' - \frac{k}{2}\right)^2 - \left(x' - \frac{k}{2}\right)^2\right] \geqq \frac{2\,\epsilon^2}{k+2}$$

and the statement of the lemma is evident.

5. Maximum Degree

Throughout this section, the difference form (1.1) is assumed to be stable and of degree

$$p = m(k + 1) + d, \ d \geq 0.$$

(See theorem 3.2, remark 1.) Because of (2.2), d can be characterized by the fact that for $j = 0, 1, \ldots, d$ there are polynomials π with degree exactly $m(k+1)+r+j$ such that the left-hand side of (2.2) vanishes for $j = 0, 1, \ldots, d - 1$, but not for $j = d$.

We are free in the choice of the polynomials π and therefore proceed as follows: Let

$$\omega(x) = x(x - 1)\cdots(x - k)$$

and let $q \not\equiv 0$ be an - as yet arbitrary - polynomial with degree exactly j. Define π to be the solution of the initial value problem

$$(5.1) \qquad \pi^{(r)} = q\,\omega^{m}; \ \pi(\tfrac{k}{2}) = \pi'(\tfrac{k}{2}) = \cdots = \pi^{(r-1)}(\tfrac{k}{2}) = 0.$$

Then, obviously, π has the degree $m(k + 1) + r + j$ and satisfies

$$(5.2) \qquad \pi^{(\lambda)} \equiv 0 \bmod \omega \text{ for } \lambda = r, r + 1, \ldots, r + m - 1.$$

Because of (2.2) and (2.6), d has therefore the property that

$$\left[\pi(D)\rho_o\right]_{s=1}$$

vanishes for $j = 0, 1, \ldots, d-1$ but not for $j = d$. Let

(5.3) $$P(x) = \Delta^r \pi(x);$$

then, using (3.1) and (2.4), we get

$$\left[\pi(D)\rho_o\right]_{s=1} = \left[P(D)\varphi\right]_{s=1}$$

and d can be characterized by

(5.4) $$\left[P(D)\varphi\right]_{s=1} \begin{cases} = 0 & \text{for } j = 0,1,\ldots, d-1, \\ \neq 0 & \text{for } j = d. \end{cases}$$

In order to apply lemma 4.2, we need some information concerning P. From (5.3) and (5.1) it follows that P has the representation

$$P(x) = \int_0^r q(x+t)\cdot\omega^m(x+t)\cdot\Phi(t)\,dt$$

where $\Phi = \Phi_r$ denotes the Peano-kernel belonging to the operator Δ^r (see e.g. Davis [3] p.70). Note that

(5.5) $$\Phi(0) = \Phi(r) = 0;\ \Phi(t) = \Phi(r-t) > 0 \text{ for } 0 < t < r$$

is valid. On the other hand, ω^m can be represented in the form

$$\omega^m(x) = c\cdot B^m(x)\cdot\sin^m \pi x \quad (-1 < x < k+1),$$

where B has been defined by (4.1) and $c \neq 0$ is a constant (see Reimer [4] , p.383). Now set

$$q(x) = c^{-1} (x - \frac{k}{2})^j ;$$

then from (4.2) we finally obtain

$$(5.6) \qquad P(x) = \int_0^r F_{mj}(x+t) \cdot \sin^m \pi(x+t) \cdot \Phi(t)dt \qquad (-1 < x < k+1-r).$$

Now introduce the function

$$(5.7) \qquad H(x) = \int_0^r F_{mj}(x+t) \cdot \sin^m \pi t \cdot \Phi(t)dt \qquad (-1 < x < k+1-r).$$

Then (4.3) and (5.5) yield

$$(5.8) \qquad H(k-r-x) = (-1)^{m(r+1)+j} H(x)$$

and because of (5.6) and (5.7), it follows that

$$(5.9) \qquad P(\nu) = (-1)^{m\nu} H(\nu) \qquad (\nu = 0,1,\ldots, k-r).$$

At first assume that m is even and j = 0 or 1. Then the kernel of the integral in (5.7) is non-negative and lemma 4.1 yields

$$(5.10) \quad H^{(\lambda)}(x) > 0 \text{ for } 0 \leq x \leq k-r, \ \lambda \equiv j \bmod 2, \ \lambda \geq 0.$$

Because of (5.8) to (5.10), H and P satisfy the assumptions of lemma 4.2 with $\iota = j$, $\mu = m$. Thus

$$(5.11) \qquad \left[P(D)\varphi\right]_{s=1} = 0$$

holds for $j = 0$ if and only if

$$\varphi^* = -\varphi.$$

But because of $\varphi^*(1) = \varphi(1)$, this condition implies $\varphi(1) = 0$ in contradiction to the stability condition. Therefore (5.11) does not hold for $j = 0$ and from (5.4) it follows that $d = 0$. Altogether we have proved the

<u>Theorem 5.1.</u> Let L be stable and m even. Then $p \leq m(k+1)$.

Henceforth, we assume that m is odd and $j = 0$ or 1. Now the kernel of the integral in (5.7) changes in general its sign and the arguments become more complicated, as Dahlquist already observed in the case $m = 1$. The two cases $r = 1$ and $r = 2$, which he was able to treat and which are of particular interest in the numerical applications, can again be investigated completely.

At first assume $r = 1$. The kernel of (5.7) is still non-negative so that (5.10) is valid again. By the same arguments as above, we see that (5.11) is valid if and only if

$$(5.12) \qquad \varphi^* = (-1)^{k+j}\varphi.$$

Now let $r = 2$. The kernel of (5.7) then changes its sign at $t = 1$; however, from (5.7) and (5.5) it follows that

$$-H(x) = \int_0^1 \int_{-t}^{t} F'_{mj}\ (x + 1 + u) \cdot \sin^m \pi t \cdot \Phi(t)\ du\,dt.$$

Thus application of lemma 4.1 yields

$$(5.13) \quad -H^{(\lambda)}(x) > 0 \text{ for } 0 \leq x \leq k - r,\ \lambda \equiv j + 1 \bmod 2,\ \lambda \geq 0.$$

Now because of (5.8), (5.9) and (5.13), $-H$ and $-P$ satisfy the assumptions on H and P in lemma 4.2 provided we choose $\iota = j + 1$ and $\mu = m$, and hence (5.11) is valid again if and only if (5.12) is.

Now let $r = 1$ or 2. Since (5.12) cannot be valid for $j = 0$ as well as for $j = 1$, because of $a_k \neq 0$, we get from (5.4) the estimate $d \leq 1$. On the other hand, $d = 1$ is valid if and only if

$$\varphi^* = (-1)^k \varphi.$$

Because of $\varphi(1) \neq 0$, this equality can be realized by a stable difference form again if and only if k is even. We thus have proved

Theorem 5.2. Let L be stable and m be odd, r = 1 or 2. Then

$$p \leq m(k+1) + \begin{cases} 1 \text{ if } k \text{ is even,} \\ 0 \text{ if } k \text{ is odd.} \end{cases}$$

If k is even then p attains its maximum value m(k + 1) + 1 if and only if

$$\rho_o^* = (-1)^r \rho_o.$$

Note that the last equality is equivalent with the relation $\varphi^* = \varphi$.

If m = 1, r = 3 or 4 then there are stable difference forms, as Dahlquist [2], p.30, has shown, with a degree strictly exceeding the bound given in theorem 5.2. Examples like these can be regarded as exceptions, as the following theorem shows.

Theorem 5.3.

Let (k,r) be any pair of positive integers with $1 \leq r \leq k$. Then a number $m_o(k,r)$ exists with the following property: If L is stable and m is odd, $m > m_o(k,r)$, then

$$p \leq m(k+1) + \begin{cases} 1 \text{ if } k \text{ is even,} \\ 0 \text{ if } k \text{ is odd.} \end{cases}$$

If k is even, then p attains its maximum value m(k + 1) + 1 if and only if

$$\rho_o^* = (-1)^r \rho_o.$$

<u>Proof.</u> As we saw in the proof of theorem 5.2, it is sufficient to prove the equivalence of (5.11) and (5.12) for $j = 0$ and $j = 1$. Thus, let $1 \leq r \leq k$ and j be one of these numbers and moreover, let m be odd. With

$$\hat{q}(x) = (x - \frac{k}{2})^j \tag{5.14}$$

it then follows from (5.7) and (4.2) that

$$H(\nu) = H_m(\nu) = \int_0^r \hat{q}(\nu+t) \cdot B^m(\nu+t) \cdot \sin^m \pi t \cdot \Phi(t) \cdot dt \tag{5.15}$$

for $\nu = 0, 1, \ldots, k - r$. From (4.1) and (4.5) we obtain

$$0 < B(\frac{k}{2}) \leq B(x) \leq B(k) \quad (0 \leq x \leq k). \tag{5.16}$$

Hence we can select a number

$$\varepsilon \in (0, \frac{1}{4}) \tag{5.17}$$

such that

$$B(k) \cdot \sin^m \pi t \leq \frac{1}{2} B(\frac{k}{2}) \quad \text{for } 0 \leq t \leq \varepsilon. \tag{5.18}$$

Now set

$$I_m(\lambda, \nu) = \int_{\lambda-1}^{\lambda} \hat{q}(t) \cdot B^m(t) \cdot |\sin \pi t|^m \cdot \Phi(t-\nu) dt \tag{5.19}$$

for all pairs (λ, ν) of integers satisfying

(5.20) $1 \leq \lambda \leq k;\ \lambda - r \leq \nu \leq \lambda - 1.$

In the following, (λ_1, ν_1) and (λ_2, ν_2) are assumed to be two such pairs of integers, satisfying the additional condition

(5.21) $$|\lambda_1 - \frac{k+1}{2}| < |\lambda_2 - \frac{k+1}{2}|.$$

This condition guarantees that

(5.22) $$|t_2 - \frac{k}{2}| > |t_1 - \frac{k}{2}| \text{ if } t_i \in [\lambda_i - 1 + \epsilon, \lambda_i - \epsilon] \text{ for } i = 1, 2$$

and that the intervals

$$(\frac{k-1}{2}, \frac{k+1}{2}) \quad \text{and} \quad [\lambda_2 - 1 + \epsilon,\ \lambda_2 - \epsilon]$$

have no point in common so that

(5.23) $$\hat{q}(t) \geq \hat{q}\ (\frac{k+1}{2}) > 0 \text{ for } t \in [\lambda_2 - 1 + \epsilon,\ \lambda_2 - \epsilon]$$

is valid.

We now decompose I_m into

(5.24) $$I_m(\lambda, \nu) = R_m(\lambda, \nu) + S_m(\lambda, \nu)$$

with

$$S_m(\lambda, \nu) = \int_{\lambda-1+\epsilon}^{\lambda-\epsilon} \hat{q}(t) \cdot B^m(t) \cdot |\sin \pi\, t|^m \cdot \Phi(t-\nu).$$

From (5.19), (5.24) it follows that

(5.25) $|R_m(\lambda_i, \nu_i)| \leq 2 \cdot C_1 \ [\frac{1}{2} B(\frac{k}{2})]^m$ for $i = 1,2$

where

$$C_1 = \hat{q}(k) \cdot \max_{0 \leq t \leq r} \Phi(t) > 0.$$

Moreover, using (5.22) and lemma 4.3, we get

$$|S_m(\lambda_1, \nu_1)| \leq C_1 \cdot \delta^m \int_{\lambda_2 - 1 + \epsilon}^{\lambda_2 - \epsilon} B^m(t) |\sin \pi t|^m dt$$

where $\delta < 1$ is a number depending only on ϵ. On the other hand, we obtain with the help of (5.23) and (5.5) the estimate

$$|S_m(\lambda_2, \nu_2)| \geq C_2 \int_{\lambda_2 - 1 + \epsilon}^{\lambda_2 - \epsilon} B^m(t) |\sin \pi t|^m dt$$

where

$$C_2 = \hat{q}\left(\frac{k+1}{2}\right) \cdot \min_{\epsilon \leq t \leq r - \epsilon} \Phi(t) > 0.$$

Together the two last estimates yield

$$|S_m(\lambda_1, \nu_1)| \leq C_3 \ \delta^m |S_m(\lambda_2, \nu_2)|$$

where $C_3 = C_1 \ C_2^{-1} > 0$. Thus we can derive from (5.24) the inequality

$$|I_m(\lambda_1, \nu_1)| \leq |R_m(\lambda_1, \nu_1)| + C_3 \ \delta^m \left\{ |I_m(\lambda_2, \nu_2)| + |R_m(\lambda_2, \nu_2)| \right\}.$$

Moreover, using (5.16), (5.17) and (5.23) we get

$$|I_m(\lambda_2,\nu_2)| \geq C_2 \cdot B^m(\tfrac{k}{2}) \cdot \int_{1/4}^{3/4} \sin^m \pi t \, dt \geq \tfrac{1}{2} C_2 [2^{-\frac{1}{2}} B(\tfrac{k}{2})]^m .$$

The last two inequalities and (5.25) finally imply

$$\left|\frac{I_m(\lambda_1,\nu_1)}{I_m(\lambda_2,\nu_2)}\right| \leq 4\, C_3 \cdot 2^{-\frac{m}{2}} + C_3\, \delta^m \{1 + 4\, C_3 \cdot 2^{-\frac{m}{2}}\}$$

and thus

$$(5.26) \qquad \lim \frac{I_m(\lambda_1,\nu_1)}{I_m(\lambda_2,\nu_2)} = 0 \qquad (m \to \infty,\ m \equiv 1 \bmod 2).$$

Now, let λ, ν be non-negative integers satisfying

$0 \leq \nu \leq k - r - \lambda$. From (5.15) and (5.19) it follows that

$$(5.27) \qquad \Delta^\lambda H_m(\nu) = \sum_{\mu=1}^{r} \sum_{\varkappa=0}^{\lambda} \binom{\lambda}{\varkappa} (-1)^{\lambda+\varkappa+\mu+1} \cdot I_m(\nu+\mu+\varkappa,\ \nu+\varkappa).$$

In order to apply theorem 2.1 we investigate the difference

(5.27) under the additional condition

$$(5.28) \qquad \lambda \equiv r + j + 1 \bmod 2.$$

Since it follows from (5.8) by (5.28) that

$$(5.29) \qquad \Delta^\lambda H_m(k - r - \lambda - \nu) = \Delta^\lambda H_m(\nu),$$

it is obviously sufficient to consider the case

$$\frac{k-r-\lambda}{2} \leq \nu \leq k - r - \lambda .$$

If $2\nu > k-r-\lambda$, then the sum in (5.27) is dominated by its element for $(\mu, \varkappa) = (r, \lambda)$. To be more precise: because of (5.26) and (5.28) we have

$$(5.30) \qquad \Delta^{\lambda} H_m(\nu) \sim (-1)^{r+1} I_m(\nu + r + \lambda, \nu + \lambda)$$

for $m \to \infty$, $m \equiv 1 \bmod 2$.

If $2\nu = k - r - \lambda$, then, using the symmetries occuring within the integrand of (5.19) (see (4.1), (5.5) and (5.23)), we find

$$(-1)^j I_m(\nu + 1, \nu) = I_m(\nu + r + \lambda, \nu + \lambda).$$

Thus, the two dominating elements in the right-hand side of (5.27) equal one another and therefore

$$(5.31) \qquad \Delta^{\lambda} H_m(\nu) \sim 2\cdot(-1)^{r+1} \cdot I_m(\nu + r + \lambda, \nu + \lambda)$$

is valid in this case for $m \to \infty$, $m \equiv 1 \bmod 2$.

The right-hand sides of (5.30) and (5.31) are positive. Therefore, since we are only dealing with a finite number of pairs, (λ, ν), an integer $m_o(k,r)$ can be found such that

$$(-1)^{r+1} \Delta^{\lambda} H_m(\nu) > 0 \text{ for } m > m_o(k,r),\ m \equiv 1 \bmod 2$$

holds for all pairs (λ, ν) with $0 \leq \nu \leq k - r - \lambda$, which satisfy (5.28).

Now let $m > m_o(k,r)$, $m \equiv 1 \mod 2$ and assume $\hat{H}(x)$ is the interpolation polynomial associated with the function $(-1)^{r+1}H(x)$ and the point set $\{0,1,\ldots, k - r\}$. Then obviously

$$\Delta^\lambda \hat{H}(\nu) > 0 \text{ for } 0 \leq \nu \leq k - r - \lambda;\ \lambda \equiv r + j + 1 \mod 2$$

and

$$\hat{H}(k - r - x) = (-1)^{r+j+1} \hat{H}(x)$$

hold (see (5.8)). Finally as a consequence of (5.9), (2.3) we have

$$[P(D)\varphi]_{s=1} = [\hat{H}(D)\varphi]_{s=-1}.$$

From the last three relations, it follows by theorem 2.1 that the statements (5.11) and (5.12) are equivalent. Thus theorem 5.3 has been proved.

6. References

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