A MACROSCOPIC QUASI-LINEAR THEORY OF THE GARDEN-HOSE INSTABILITY*

Ronald C. Davidson

Department of Physics, University of California, Berkeley, California

and

Heinrich J. Völk†

Department of Aeronautics and Astronautics,

Massachusetts Institute of Technology, Cambridge, Massachusetts

ABSTRACT

The quasi-linear stabilization of the garden-hose instability is discussed from a macroscopic point of view with closure in the fluid model obtained by neglecting the effects of heat flow. In order to keep the problem well-posed mathematically finite Larmor radius corrections to the conventional growth rate are retained, which leads to a natural cut-off in the growth rate for sufficiently large wave-number.

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† Present address: Max Planck Institute for Extraterrestrial Physics, 8046, Garching bei München, Germany.

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I. INTRODUCTION

The quasi-linear theory of the stabilization of the garden-hose instability[1] was first developed within a kinetic (Vlasov) framework by Shapiro and Shevchenko[2] and more recently by Sagdeev and Galeev[3] and also Kennel and Sagdeev.[4] Since the garden-hose instability serves as one of the simplest examples in which the principle diffusion mechanism is the adiabatic (rather than resonant) interaction between waves and particles,[3] it is of interest to ascertain whether or not an adequate description of stabilization is contained within the framework of a fluid model. Such may be expected to be the case since the instability is hydrodynamic in nature with the growth rate depending only on gross plasma properties and not on the detailed structures of the electron or ion distribution functions. An example where such a theory should be of particular interest, is the well-known hydromagnetic theory of the solar wind,[5] which disregards the possible pressure anisotropy altogether, although with increasing distance from the sun, a particle should acquire an ever larger ratio of $v_{\parallel}/v_{\perp}$ with respect to the solar magnetic field direction. As a further example in which resonant wave-particle interactions are unimportant and the primary diffusion mechanism is the adiabatic interaction between waves and particles, Völk[6] has recently developed a macroscopic quasi-linear theory of the symmetric two-stream instability. In this article we follow a similar procedure for a garden-hose-unstable plasma and in fact demonstrate that the results of the kinetic theory of Shapiro and Shevchenko are recovered in a fluid description.

The physical situation under consideration is the propagation of low frequency waves parallel to a uniform, external magnetic field, $\underset{\sim}{B}_0$, in circumstances where the perturbing electric field, magnetic field, and fluid velocities lie in the plane perpendicular to $\underset{\sim}{B}_0$. It is well known that for the case of isotropic particle pressures this gives rise to pure oscillatory Alfvén waves; however, for sufficiently large average particle pressure parallel to $\underset{\sim}{B}_0$, $\langle P_\parallel \rangle_\Sigma$, compared to average particle pressure perpendicular to $\underset{\sim}{B}_0$, $\langle P_\perp \rangle_\Sigma$, the wave perturbations are purely growing. The conventional growth rate for wave-number k, $\gamma^0(k)$, is given by

$$(\gamma^0(k))^2 = k^2 v_A^2 \frac{\left\{ \frac{4\pi}{B_0^2} (\langle P_\parallel \rangle_\Sigma - \langle P_\perp \rangle_\Sigma) - 1 \right\}}{1 + \frac{v_A^2}{c^2}} , \qquad (1.1)$$

where v_A is the Alfven speed $(B_0^2/4\pi\langle\rho\rangle)^{1/2}$, $\langle\rho\rangle$ the average mass density, and $\langle P_\parallel \rangle_\Sigma$ and $\langle P_\perp \rangle_\Sigma$ denote the summation over species of the parallel and perpendicular pressures, i.e. $\sum_j \langle P_{\parallel j} \rangle$ and $\sum_j \langle P_{\perp j} \rangle$ respectively.

We assume that initially

$$\langle P_\parallel \rangle_\Sigma > \langle P_\perp \rangle_\Sigma + \frac{B_0^2}{4\pi} , \qquad (1.2)$$

and consider in Section II the time evolution of a uniformly turbulent ensemble of such garden-hose-unstable plasmas within the framework of a multi-species fluid model. Closure of the moment equations is obtained by neglecting the effects of heat flow. The small parameters of the analysis are the ratios of growth rate to Larmor frequency, and Larmor radius to parallel wavelength, i.e.,

$$\left|\frac{\gamma_k}{\Omega_j}\right| \ll 1, \text{ and } \left|\frac{kv_{THj}}{\Omega_j}\right| \ll 1 . \qquad (1.3)$$

In addition, it is assumed that the wave disturbances grow a negligible amount in the time it takes for a thermal particle to transverse a wavelength,

$$\left|\frac{\gamma_k}{kv_{THj}}\right| \ll 1 . \qquad (1.4)$$

Under these conditions of weak instability a quasi-linear analysis is applicable in determining the reaction of the plasma to the unstable electromagnetic field fluctuations. In particular, for purposes of describing stabilization, the slow reaction of the average particle stresses $\langle P_{\|}\rangle_{\Sigma}$ and $\langle P_{\perp}\rangle_{\Sigma}$ is of considerable interest. To this end a coupled system of equations is obtained describing the time evolution of $\langle P_{\|}\rangle_{\Sigma}$, $\langle P_{\perp}\rangle_{\Sigma}$ and the spectral energy density, $\psi^{\delta B}(k,t)$, in the magnetic field fluctuations, where

$$\langle \delta \underset{\sim}{B}(k_1,t)\cdot\delta \underset{\sim}{B}(k_2,t)\rangle \equiv \psi^{\delta B}(k_1,t)\delta(k_1+k_2)$$

for uniformly turbulent situations. The averaging procedure used throughout (where averages are denoted by $\langle\ \rangle$) is with respect to a spatially uniform ensemble and has been described in detail elsewhere.[7,8] It should be noted, however, that insofar as the calculations presented in this article include only the interaction of modes with themselves and not the effects of three- (or higher) wave coupling or nonlinear wave-particle interactions, these ensemble averages may also be viewed as spatial averages.[4] The coupled system (given by Eqs. (2.32)-(2.34)) which describes the time evolution correct to $O(1/\Omega_j^{\,2})$ of the quantities $\langle P_{\|}\rangle_{\Sigma}$, $\langle P_{\perp}\rangle_{\Sigma}$ and $\psi^{\delta B}$, is in agreement with the corresponding results of Shapiro and Shevchenko[2] based on a kinetic model. The reaction of the particle stresses to the unstable electromagnetic field fluctuations is such that the parallel (perpendicular) pressure decreases (increases) in time until

$$\langle P_{\|}\rangle_{\Sigma} = \langle P_{\perp}\rangle_{\Sigma} + \frac{B_0^{\,2}}{4\pi} \qquad (1.5)$$

and the system passes to a marginally stable state. In addition, it should be noted that the results of the kinetic theory are recovered in a fluid model in which closure is achieved by the neglect of heat flow. This is in contradistinction to the macroscopic theory of the symmetric two-stream instability[6] where the closure assumptions have an appreciable influence on the time-asymptotic behavior.

There is an additional point of mathematical interest which arises with regard to this instability. We remind the reader that the garden-hose problem with growth rate given by Eq. (1.1) is ill-posed mathematically.[9,10] The divergent behavior of $\gamma^0(k)$ as $|k| \to \infty$ in general precludes the existence of the inverse spatial Fourier transforms of the fluctuations for $t > 0$. To this end we retain finite Larmor radius corrections to $\gamma^0(k)$ in the analysis of Section II. This leads to a natural cut-off in the growth-rate for $|k| \sim |k_0|$ where k_0 is given by Eq. (2.20), thus keeping the problem well-posed mathematically. Although this point was overlooked by Shapiro and Shevchenko in the kinetic theory of the garden-hose instability[2] we hasten to add that their analysis is readily amended to give the same modified growth rate and cut-off k_0. It is assumed that such corrections have been made when comparing the macroscopic and kinetic theories of stabilization.

II. THEORY

(A) The Fluid Model

An exact consequence of taking velocity moments of the Vlasov equation for the j'th species distribution function f_j is the chain of equations advancing the density n_j, mean velocity $\underset{\sim}{v}_j$, and particle stresses $\underset{\approx}{P}_j, \ldots$, i.e.,

$$\frac{\partial}{\partial t} n_j + \underset{\sim}{\nabla} \cdot (n_j \underset{\sim}{v}_j) = 0 \ , \tag{2.1}$$

$$\frac{\partial}{\partial t} (n_j \underset{\sim}{v}_j) + \underset{\sim}{\nabla} \cdot (n_j \underset{\sim}{v}_j \underset{\sim}{v}_j) = - \frac{1}{m_j} \underset{\sim}{\nabla} \cdot \underset{\approx}{P}_j + \frac{n_j q_j}{m_j} \left(\underset{\sim}{E} + \frac{\underset{\sim}{v}_j \times \underset{\sim}{B}}{c} \right) , \tag{2.2}$$

$$\frac{\partial}{\partial t} \underset{\approx}{P}_j + \underset{\sim}{\nabla} \cdot (\underset{\approx}{Q}_j + \underset{\sim}{v}_j \underset{\approx}{P}_j) + \underset{\approx}{P}_j \cdot \underset{\sim}{\nabla} \underset{\sim}{v}_j + (\underset{\sim}{\nabla} \underset{\sim}{v}_j)^T \cdot \underset{\approx}{P}_j$$

$$= \frac{q_j}{m_j c} (\underset{\approx}{P}_j \times \underset{\sim}{B} - \underset{\sim}{B} \times \underset{\approx}{P}_j) \ , \qquad (2.3)$$

$$\vdots$$

where q_j and m_j are the charge and mass respectively associated with the j'th species; $\underset{\approx}{P}_j$ and the heat flow tensor $\underset{\approx}{Q}_j$ are defined relative to the mean velocity of the j'th species, for example

$$\underset{\approx}{P}_j \equiv m_j \int f_j (\underset{\sim}{v} - \underset{\sim}{v}_j)(\underset{\sim}{v} - \underset{\sim}{v}_j) d\underset{\sim}{v} \ .$$

In Eq. (2.3) the notation $(\ )^T$ denotes diadic transpose. The electric and magnetic fields $\underset{\sim}{E}$ and $\underset{\sim}{B}$ in Eqs. (2.1)-(2.3) evolve self consistently through Maxwell's equations,

$$\underset{\sim}{\nabla} \times \underset{\sim}{E} = - \frac{1}{c} \frac{\partial}{\partial t} \underset{\sim}{B} \ , \qquad (2.4)$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{B} = \frac{4\pi}{c} \sum_j n_j q_j \underset{\sim}{v}_j + \frac{1}{c} \frac{\partial}{\partial t} \underset{\sim}{E} \ , \qquad (2.5)$$

with initial conditions

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{B} = 0 \text{ and } \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = \sum_j 4\pi \ n_j q_j \ .$$

In relation to Eqs. (2.1)-(2.5) we consider a uniformly turbulent ensemble of garden-hose-unstable plasmas and write each of the quantities $\underset{\sim}{v}_j$, $\underset{\approx}{P}_j$ and n_j as an ensemble average (denoted by $\langle \ \rangle$) plus a fluctuation.[7,8,11] For example n_j is written as

$$n_j = \langle n_j \rangle + \delta n_j$$

where $\langle \delta n_j \rangle = 0$ and $\langle n_j \rangle$ is independent of position. Upon taking the average of the equation of continuity, Eq. (2.1), it follows from the spatial uniformity of the ensemble that $(\partial/\partial t)\langle n_j\rangle = 0$. Since the average density of the j'th species is time-independent, if $\sum_j \langle n_j\rangle q_j = 0$ initially it remains so for all times. In relation to the stabilization of the garden-hose instability, the evolution of the particle stresses, $\langle \underset{\approx}{P}_j\rangle$, will be of considerable interest. By taking the average of Eq. (2.3) this is seen to be given by

$$\frac{\partial}{\partial t}\langle \underset{\approx}{P}_j\rangle + \langle \delta\underset{\approx}{P}_j\cdot\underset{\sim}{\nabla}\delta\underset{\sim}{v}_j\rangle + \langle(\underset{\sim}{\nabla}\delta\underset{\sim}{v}_j)^T\cdot\delta\underset{\approx}{P}_j\rangle$$
$$= \frac{q_j}{m_j c}\left\{\langle\delta\underset{\approx}{P}_j\times\delta\underset{\sim}{B}\rangle - \langle\delta\underset{\sim}{B}\times\delta\underset{\approx}{P}_j\rangle\right\}$$
$$= \frac{q_j}{m_j c}\left\{\langle\underset{\approx}{P}_j\rangle\times\langle\underset{\sim}{B}\rangle - \langle\underset{\sim}{B}\rangle\times\langle\underset{\approx}{P}_j\rangle\right\} \tag{2.6}$$

where the divergence term on Eq. (2.3) is absent in Eq. (2.6) because of spatial uniformity of the ensemble. We see from Eq. (2.6) that there will be a rapid variation of the stresses $\langle \underset{\approx}{P}_j\rangle$ on the Ω_j^{-1} time scale (where $\Omega_j \equiv q_j B_0/m_j c$) unless

$$\langle\underset{\approx}{P}_j\rangle \times \langle\underset{\sim}{B}\rangle = \langle\underset{\sim}{B}\rangle \times \langle\underset{\approx}{P}_j\rangle \ . \tag{2.7}$$

Assuming relation (2.7) holds, it then follows that the form of $\langle \underset{\approx}{P}_j\rangle$ is given by

$$\langle\underset{\approx}{P}_j\rangle = \langle P_{\perp j}\rangle(\underset{\approx}{I} - \underset{\sim}{n}\underset{\sim}{n}) + \langle P_{\parallel j}\rangle\underset{\sim}{n}\underset{\sim}{n} \ , \tag{2.8}$$

where $\underset{\approx}{I}$ is the unit diadic and $\underset{\sim}{n}$ a unit vector in the direction of the uniform external magnetic field $\underset{\sim}{B}_0 = \langle\underset{\sim}{B}\rangle$. Equation (2.8) is just the usual statement of isotropic pressure in the plane perpendicular to $\underset{\sim}{B}_0$. From Eqs. (2.6) and (2.7) the (slow) reaction of $\langle \underset{\approx}{P}_j\rangle$, and hence $\langle P_{\perp j}\rangle$ and $\langle P_{\parallel j}\rangle$, is determined from

$$\frac{\partial}{\partial t}\langle \underset{\approx}{P}_j\rangle + \langle \delta\underset{\approx}{P}_j\cdot\underset{\sim}{\nabla}\delta\underset{\sim}{v}_j\rangle + \langle(\underset{\sim}{\nabla}\delta\underset{\sim}{v}_j)^T\cdot\delta\underset{\approx}{P}_j\rangle$$
$$= \frac{q_j}{m_j c}\{\langle\delta\underset{\approx}{P}_j \times \delta\underset{\sim}{B}\rangle - \langle\delta\underset{\sim}{B} \times \delta\underset{\approx}{P}_j\rangle\} \ . \tag{2.9}$$

Insofar as three-(or higher-) wave processes and the nonlinear interaction between waves and particles are negligible, the fluctuations $\delta\mathbf{P}_j$, $\delta\mathbf{v}_j$ and $\delta\mathbf{B}$ appearing in Eq. (2.9) are to be obtained from the linearized versions of Eqs. (2.1)-(2.5). To this end we subtract from each equation its average giving a dynamical system of equations describing the time evolution of the fluctuations. Truncation is obtained by omitting the heat flow tensor $\mathbf{Q}_j$ from the analysis. It may be readily demonstrated from Eqs. (2.2) and (2.5) that if the average velocity $\langle\mathbf{v}_j\rangle$ and electric field $\langle\mathbf{E}\rangle$ are initially zero they remain so for all later times. Furthermore one can show that Eq. (2.7) is preserved in time if it is fulfilled initially and if the excited wave-spectrum is axially symmetric with regard to the direction of $\mathbf{B}_0$. For the case of a non-axially symmetric spectrum we refer to the arguments given elsewhere[12] which readily apply in this case, because of our assumptions (1.3). In the following we shall assume these initial symmetry properties to hold. Upon taking the spatial Fourier transform of the linearized equations for the fluctuations and assuming that the time variation of $\delta n_j(k,t)$, $\delta\mathbf{v}_j(k,t)$, $\delta\mathbf{P}_j(k,t)$, $\delta\mathbf{E}(k,t)$ and $\delta\mathbf{B}(k,t)$ is given by $\exp(\int^t s(k,t')dt')$,[13] where in general $s(k,t) = -i\omega(k,t)+\gamma(k,t)$, it follows that

$$\delta n_j = 0 \,, \tag{2.10}$$

$$s\delta\mathbf{v}_j = \frac{-i\mathbf{k}\cdot\delta\mathbf{P}_j}{m_j\langle n_j\rangle} + \frac{q_j}{m_j}\left(\delta\mathbf{E} + \frac{\delta\mathbf{v}_j \times \mathbf{B}_0}{c}\right) , \tag{2.11}$$

$$\begin{aligned} s\delta\mathbf{P}_j &+ i(\langle\mathbf{P}_j\rangle\cdot\mathbf{k}\,\delta\mathbf{v}_j + \delta\mathbf{v}_j\,\mathbf{k}\cdot\langle\mathbf{P}_j\rangle) \\ &= \frac{q_j}{m_j c}\{\delta\mathbf{P}_j \times \mathbf{B}_0 - \mathbf{B}_0 \times \delta\mathbf{P}_j \\ &+ \langle\mathbf{P}_j\rangle \times \delta\mathbf{B} - \delta\mathbf{B} \times \langle\mathbf{P}_j\rangle\} \,, \end{aligned} \tag{2.12}$$

$$s\,\delta\underset{\sim}{B} = -ic\,\underset{\sim}{k} \times \delta\underset{\sim}{E}\ , \tag{2.13}$$

and

$$i\underset{\sim}{k} \times \delta\underset{\sim}{B} = \frac{4\pi}{c}\sum \langle n_j\rangle q_j \delta\underset{\sim}{v}_j + \frac{s}{c}\,\delta\underset{\sim}{E}\ . \tag{2.14}$$

For simplicity of notation the argument (k,t) of the fluctuations and s(k,t) has been suppressed in writing Eqs. (2.10)-(2.14). In addition, $\langle \underset{\approx}{P}_j\rangle$ has the form given by Eq. (2.8). We also remind the reader that in the present analysis the perturbations $\delta\underset{\sim}{v}_j$, $\delta\underset{\sim}{E}$ and $\delta\underset{\sim}{B}$ have been taken to lie in the plane perpendicular to $\underset{\sim}{B}_0$ with wave-vector $\underset{\sim}{k}$ parallel to $\underset{\sim}{B}_0$. Thus, as stated by Eq. (2.10), there is no charge separation associated with the perturbations and the field fluctuations are totally electromagnetic in nature. It is convenient to define the quantity $\delta\underset{\sim}{P}_j$ by

$$\delta\underset{\sim}{P}_j \equiv n\cdot\delta\underset{\approx}{P}_j\ , \tag{2.15}$$

and introduce the notation $\delta A_\pm$

$$\delta A_\pm \equiv \delta A_1 \pm i\delta A_2$$

where $\delta\underset{\sim}{A} = \delta A_1\underset{\sim}{e}_1 + \delta A_2\underset{\sim}{e}_2$ (with $\underset{\sim}{e}_1\cdot\underset{\sim}{e}_2 = 0$) lies in the plane perpendicular to $\underset{\sim}{B}_0$ and may represent any of the fluctuations $\delta\underset{\sim}{E}$, $\delta\underset{\sim}{B}$, $\delta\underset{\sim}{v}_j$ or $\delta\underset{\sim}{P}_j$. After some straightforward algebra $\delta v_{j\pm}$ and $\delta P_{j\pm}$ may be written in terms of $\delta E_\pm$ from Eqs. (2.11)-(2.14) as

$$\delta v_{j\pm} = \frac{q_j}{m_j}\,\delta E_\pm\ \frac{\left((s\pm i\Omega_j) + \frac{k^2}{\langle n_j\rangle m_j s}\left(\langle P_{\parallel j}\rangle - \langle P_{\perp j}\rangle\right)\right)}{\left((s\pm i\Omega_j)^2 + \frac{k^2}{\langle n_j\rangle m_j}\langle P_{\parallel j}\rangle\right)}\ , \tag{2.16}$$

and

$$\delta P_{j\pm} = \frac{-ik(q_j/m_j)}{(s\pm i\Omega_j)}\, \delta E_\pm \left\{ \frac{1}{s}\left(\langle P_{\perp j}\rangle - \langle P_{\| j}\rangle\right) \right.$$

$$\left. + \langle P_{\| j}\rangle \frac{\left((s\pm i\Omega_j) + \frac{k^2}{\langle n_j\rangle m_j s}\left(\langle P_{\| j}\rangle - \langle P_{\perp j}\rangle\right)\right)}{\left((s\pm i\Omega_j)^2 + \frac{k^2}{\langle n_j\rangle m_j}\langle P_{\| j}\rangle\right)} \right\} . \qquad (2.17)$$

In addition, from Eqs. (2.11)-(2.14) the dispersion relation determining $s(k,t)$ adiabatically in time (through the slow time variations of $\langle P_{\| j}\rangle$ and $\langle P_{\perp j}\rangle$) is given by

$$0 = s^2 + c^2k^2 + \sum_j \omega_j^2 \frac{\left(s(s\pm i\Omega_j) + \frac{k^2}{\langle n_j\rangle m_j}\left(\langle P_{\| j}\rangle - \langle P_{\perp j}\rangle\right)\right)}{\left((s\pm i\Omega_j)^2 + \frac{k^2}{\langle n_j\rangle m_j}\langle P_{\| j}\rangle\right)} , \qquad (2.18)$$

where $\omega_j^2 \equiv 4\pi\langle n_j\rangle q_j^2/m_j$.

Relations (2.16) and (2.17) may now be expanded in terms of the small parameters of the problem, $|s/\Omega_j|$ and $|kv_{THj}/\Omega_j|$, and the slow reaction of the particle stresses (given by Eq. (2.9)) determined in terms of the field fluctuations or more specifically in terms of the spectral energy density associated with the field fluctuations. Let us first however direct our attention to the dispersion relation given by Eq. (2.18).

(B) The Dispersion Relation

The upper (lower) sign in Eq. (2.18) corresponds to waves with left- (right-) hand polarization. If Eq. (2.18) is expanded in the small parameters $|s/\Omega_j|$ and $|kv_{THj}/\Omega_j|$ and only terms to $O(1/\Omega_j^2)$ are retained, the dispersion relation reduces to $s^2 \cong (\gamma^0)^2$ where the growth rate γ^0 is given in Eq. (1.1). As discussed in the introduction the use of s to this accuracy constitutes an ill-posed mathematical problem since $\gamma^0 \to +\infty$ for large $|k|$. If however, we retain $O(1/\Omega_j^3)$ in the dispersion relation this difficulty is overcome since the finite Larmor radius corrections give a growth rate which goes to zero for large $|k|$. In particular, Eq. (2.18) may be written approximately as

$$s^2 \pm 2is\,\gamma^0(k/k_0)-(\gamma^0)^2 \cong 0\ , \tag{2.19}$$

where

$$k_0 \cong \frac{2(\gamma^0/k)\left(1+\sum_j \frac{\omega_j^2}{\Omega_j^2}\right)}{\sum_j \left\{\frac{\omega_j^2}{\Omega_j^3}\left((\gamma^0/k)^2+\frac{1}{\langle n_j\rangle m_j}\left(2\langle P_{\perp j}\rangle-3\langle P_{\parallel j}\rangle\right)\right)+O\left(\frac{1}{\Omega_j^4}\right)\right\}} \tag{2.20}$$

With the neglect of $O(1/\Omega_j^4)$ terms Eq. (2.19) predicts a growth rate γ and real component ω to the frequency (where $s = -i\omega+\gamma$) given by

$$\gamma^2 \cong (\gamma^0)^2(1-k^2/k_0^2)\ , \quad |k| \lesssim |k_0|\ , \tag{2.21}$$

and

$$\omega = \pm\gamma^0(k/k_0), \quad |k| \lesssim |k_0|\ . \tag{2.22}$$

Consequently, γ reduces to γ^0 for $|k| \ll |k_0|$, passes through a maximum at $|k| = \sqrt{3}/2\ |k_0|$, and goes to zero for $|k| = |k_0|$. In an order of magnitude estimate from Eq. (2.20) k_0 may be scaled to the ion larmor $\rho_{ion} \sim v_{THi}/\Omega_i$. For $v_A \ll c$ we obtain

$$(k_0\rho_{ion})^2 \cong \frac{4\left(\langle P_\parallel\rangle_\Sigma-\langle P_\perp\rangle_\Sigma-\frac{B_0^2}{4\pi}\right)}{\langle n_i\rangle m_i\, v_{THi}^2}\ ,$$

which is small compared to unity in light of the assumption of small growth rate, Eq. (1.4). Thus the maximum garden-hose unstable wavenumber is considerably less than the inverse ion larmor radius ρ_{ion}^{-1}. Insofar as $O(1/\Omega_j^4)$ terms are negligible Eq. (2.19) gives a pure oscillation and zero growth rate for $|k| \gtrsim |k_0|$, that is to say, $\gamma \cong 0$ for $|k_0| \lesssim |k| \ll \rho_{ion}^{-1}$. We may now proceed to describe the long wave-length stabilization process within the context of the modified growth rate given by Eq. (2.21). The consequence of retaining finite Larmor radius corrections to the usual dispersion relation has been to introduce a cut-off k_0 in the growth rate and make the problem well-posed mathematically. In view of Expression (2.20) it should be noted that any argument whereby $\gamma^0(k,t) \to 0$ asymptotically in time involves a corresponding decrease to zero of the maximum garden-hose-unstable wave-number k_0, that is to say a shrinking to zero volume of the unstable domain in k-space.

(C) The Stabilization Process

In determining from Eq. (2.9) the reaction of the particle stresses $\langle \underset{\approx}{P}_j \rangle$ to the unstable fluctuations we expand $\delta P_{j\pm}$ and $\delta v_{j\pm}$ in the small parameters $|s/\Omega_j|$ and $|kv_{THj}/\Omega_j|$. This readily gives

$$\delta P_{j\pm} \cong (-ik)\, \frac{q_j}{m_j}\, \delta E_\pm \left\{ \frac{\langle P_{\perp j}\rangle - \langle P_{\parallel j}\rangle}{i\Omega_j s} + \frac{\langle P_{\perp j}\rangle - 2\langle P_{\parallel j}\rangle}{\Omega_j^{\,2}} + 0\left(\frac{1}{\Omega_j^{\,3}}\right)\right\} \qquad (2.23)$$

and

$$\delta v_{j\pm} \cong \frac{q_j}{m_j}\, \delta E_\pm \left\{ \frac{1}{\pm i\Omega_j} + \frac{s^2 - \frac{k^2}{\langle n_j\rangle m_j}\left(\langle P_{\parallel j}\rangle - \langle P_{\perp j}\rangle\right)}{s\Omega_j^{\,2}} + 0\left(\frac{1}{\Omega_j^{\,3}}\right)\right\}. \qquad (2.24)$$

Taking the double outer product of Eq. (2.9) with $\underset{\sim}{nn}$ gives the time rate of change of $\langle P_{\parallel j}\rangle$. After some straightforward algebra making use of Eq. (2.13) this simply yields

$$\frac{\partial}{\partial t} \langle P_{\parallel j}\rangle = \sum_{P} \frac{q_j}{m_j c} \iint dk_1 dk_2\; e^{i(k_1+k_2)z}$$

$$\times \left(\frac{-ick_1}{s(k_1,t)}\right) \langle \delta E_+(k_1,t)\delta P_{j-}(k_2,t) + \delta E_-(k_1,t)\delta P_{j+}(k_2,t)\rangle \qquad (2.25)$$

where $\sum_P$ denotes summation over right- and left-hand modes of polarization. The quantities $\delta P_{j\pm}$ appearing in Eq. (2.25) may be rewritten in terms of $\delta E_\pm$ by means of Expression (2.23), i.e.

$$\frac{\partial}{\partial t}\langle P_{\parallel j}\rangle = \sum_P \frac{q_j^2}{m_j^2}\iint dk_1 dk_2\, e^{i(k_1+k_2)z}$$
$$\times \frac{k_1 k_2}{s(k_1,t)}\left\{\frac{\langle P_{\perp j}\rangle-\langle P_{\parallel j}\rangle}{\pm i\Omega_j s(k_2,t)} + \frac{2\langle P_{\parallel j}\rangle-\langle P_{\perp j}\rangle}{\Omega_j^2} + O\left(\frac{1}{\Omega_j^3}\right)\right\}$$
$$\times \langle \delta E_+(k_1,t)\delta E_-(k_2,t) + \delta E_-(k_1,t)\delta E_+(k_2,t)\rangle \tag{2.26}$$

The average in Eq. (2.26) is simply related to the spectral energy density, $\Psi^{\delta E}(k,t)$, in the electric field fluctuations through

$$\langle \delta E_+(k_1,t)\delta E_-(k_2,t)+\delta E_-(k_1,t)\delta E_+(k_2,t)\rangle = 2\langle \delta\mathbf{E}(k_1,t)\cdot\delta\mathbf{E}(k_2,t)\rangle$$
$$\equiv 2\Psi^{\delta E}(k_1,t)\delta(k_1+k_2)\ , \tag{2.27}$$

the $\delta(k_1+k_2)$ factor in Eq. (2.27) being a manifestation of the spatial uniformity of the ensemble. Upon using the Maxwell equation (2.13) and the symmetry property $s(-k,t) = s^*(k,t)$, the spectral energy density $\Psi^{\delta E}(k,t)$ may be related to the energy density in the magnetic field fluctuations through

$$\frac{|s|^2}{c^2k^2}\Psi^{\delta B} = \Psi^{\delta E}\quad . \tag{2.28}$$

Neglecting $O(1/\Omega_j^3)$ terms in Eq. (2.26) and noting that the first term vanishes in the summation over polarizations, Eq. (2.26) may be written in terms of $\Psi^{\delta B}(k,t)$ $(= \Psi^{\delta B}(-k,t))$ as

$$\frac{\partial}{\partial t}\langle P_{\parallel j}\rangle = (2\langle P_{\perp j}\rangle-4\langle P_{\parallel j}\rangle)\int dk\ \gamma(k,t)\,\frac{\Psi^{\delta B}(k,t)}{B_0^2}\ . \tag{2.29}$$

Similarly, by taking the double outer product of Eq. (2.9) with $\frac{1}{2}(\mathbf{I}-\mathbf{nn})$, the evolution of $\langle P_{\perp j}\rangle$ may be shown to be given by

$$\frac{\partial}{\partial t}\langle P_{\perp j}\rangle = \sum \iint dk_1 dk_2 \; e^{i(k_1+k_2)z}$$

$$\times\left\{\frac{-ik_1}{2}\langle \delta v_+(k_1,t)\delta P_-(k_2,t)+\delta v_-(k_1,t)\delta P_+(k_2,t)\rangle\right.$$

$$\left.+\frac{ik_1 q_j/m_j}{s(k_1,t)}\langle \delta E_+(k_1,t)\delta P_-(k_2,t)+\delta E_-(k_1,t)\delta P_+(k_2,t)\rangle\right\} \; . \qquad (2.30)$$

Neglecting $O(1/\Omega_j^{\;3})$ terms as before Eq. (2.30) readily reduces to

$$\frac{\partial}{\partial t}\langle P_{\perp j}\rangle = \langle P_{\parallel j}\rangle \int dk \; \gamma(k,t)\,\frac{\Psi^{\delta B}(k,t)}{B_0^{\;2}} \; . \qquad (2.31)$$

In writing (2.29) and (2.31) we have dropped the summation over polarization notation since $\gamma(k,t)$ is the same for both right- and left-hand waves. The spectral density $\Psi^{\delta B}(k,t)$ includes both polarizations.

In view of the expression for the growth rate, Eq. (1.1), or the more accurate version, Eq. (2.21), the evolution of the total parallel and perpendicular pressures, $\langle P_\parallel\rangle_\Sigma$ and $\langle P_\perp\rangle_\Sigma$, is of special interest. Within the accuracy of Eqs. (2.29) and (2.31) we have that

$$\frac{\partial}{\partial t}\langle P_\parallel\rangle_\Sigma = \left(2\langle P_\perp\rangle_\Sigma - 4\langle P_\parallel\rangle_\Sigma\right)\int dk \; \gamma(k,t)\,\frac{\Psi^{\delta B}(k,t)}{B_0^{\;2}} \; , \qquad (2.32)$$

and

$$\frac{\partial}{\partial t}\langle P_\perp\rangle_\Sigma = \langle P_\parallel\rangle_\Sigma \int dk \; \gamma(k,t)\,\frac{\Psi^{\delta B}(k,t)}{B_0^{\;2}} \; . \qquad (2.33)$$

These must be solved in conjunction with

$$\frac{\partial}{\partial t}\Psi^{\delta B}(k,t) = 2\gamma(k,t)\Psi^{\delta B}(k,t) \; , \qquad (2.34)$$

describing the volution of the spectral energy density in the magnetic-field fluctuations.

The stabilization process may be simply summarized as follows. By hypothesis γ is initially positive for the range of wave-number under consideration, as is the energy density $\Psi^{\delta B}(k,t)$. Thus, as long as $\int dk\ \gamma(k,t)\Psi^{\delta B}(k,t)$ is non-zero, it follows from Eqs. (2.32) and (2.33) that

$$\frac{\partial}{\partial t} \langle P_{\|} \rangle_{\Sigma} < 0 \ ,$$

and

$$\frac{\partial}{\partial t} \langle P_{\perp} \rangle_{\Sigma} > 0 \ .$$

That is to say, the reaction of the particle stresses to the unstable electromagnetic field fluctuations is such as to cause the parallel (perpendicular) pressure to decrease (increase) monotonically with increasing time. In light of the definition of growth rate given by Eqs. (2.21) and (1.1), this is in the direction of stabilization. The time-asymptotic state predicted by Eqs. (2.32)-(2.34) is thus one for which

$$\gamma(k,t \to \infty) \to 0 \ , \tag{2.35}$$

and

$$\langle P_{\|}(\infty) \rangle_{\Sigma} = \langle P_{\perp}(\infty) \rangle_{\Sigma} + \frac{B_0^{\ 2}}{4\pi} \ . \tag{2.36}$$

As previously indicated this stabilization process involves a concurrent shrinking of the unstable domain of k-space to zero volume, i.e. $k_0(t \to \infty) \to 0$. From Eq. (2.34) the energy density $\Psi^{\delta B}$ begins to grow (from non-zero initial value) in the initially unstable region of k-space. As time proceeds, the growth rate decreases and the unstable region shrinks in volume; finally as $t \to \infty$ we are left with a stationary spectrum of magnetic field fluctuations, $\Psi^{\delta B}(k,\infty)$.

Such are the qualitative features of the time development and time-asymptotic state. With certain approximation methods,[2,3,4] and energy conservation relations[2] associated with Eqs. (2.32)-(2.34), a more detailed quantitative description may also be given. As stated in the introduction, in obtaining the coupled system (2.32)-(2.34) we have recovered the corresponding results of Shapiro and Shevchenko[2] based on a kinetic model. Moreover this has been done within a fluid framework which achieves closure by omitting the heat flow tensor $\underset{\approx}{Q}$ from the analysis. In conclusion, we remind the reader that in the analysis presented here mode coupling effects have been assumed negligible on the time-scale in which linear stabilization takes place. Once $\gamma(k,t) \rightarrow 0$ however, such higher nonlinear effects will become important and cause further change in the spectral energy density $\Psi^{\delta B}$.

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11. This procedure should be used with some caution. In general, the energy conservation law is not fulfilled exactly, if the fluctuations are treated adiabatically. The correct procedure is to define averages of momentum, total kinetic energy, and density,[6] instead of those of velocity, pressure and density as is done here. Under our assumptions (1.3) and (1.4), however, the difference in the expressions for the total energy is of higher order for the two decomposition schemes. Thus, we have adopted the second, simpler, procedure here.

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13. The quantity $s(k,t)$ is allowed to vary adiabatically in time through the slow time variation of $\langle \underset{\approx}{P}_j \rangle$ given by Eq. (2.9).