ERROR PROPAGATION

in the

ROCKET-GRENADE EXPERIMENT

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INTRODUCTION

The purpose of this report is to give a coherent explanation of the method of determining the standard error present in the calculated values of temperature and winds in the layer between grenade pairs. For the sake of completeness, a basic review of the ray-tracing technique used in the NASA Rocket-Grenade Experiment is presented. To provide an introduction to the discussion on errors and their propagation, a brief explanation of the meaning of the term "standard error" is presented. The sources of error in the Rocket-Grenade Experiment are then carefully enumerated and classified as either uncertainties in experimental measurement or approximations in theoretical development. In this report, it is assumed that the temperature and winds in a given layer between grenade burst positions are in error due to (1) the uncertainty in reading the break times of the wave as it crosses each microphone in the array; and (2) the deviation of the layered model atmosphere from the true atmosphere. The effect of the other errors on the final standard error in temperature and winds will be considered in a later report.

The approximate method used to determine the standard error in temperature and winds is then discussed in detail. The report concludes with a presentation of the details of the mathematical development of the error analysis.

II

REVIEW OF RAY-TRACING TECHNIQUE IN THE ROCKET-GRENADE EXPERIMENT

Before beginning detailed discussion on the propagation of errors in ray-tracing, let us briefly review the ray-tracing technique itself as it is applied to the Rocket-Grenade Experiment. Figure 1 shows an edge-on view of the elements involved in the ray-tracing problem. The asterisks on the ground line represent the microphone locations (x_i, y_i, z_i); the fact that they have been drawn on a single line does not mean that their z-coordinates are all the same.

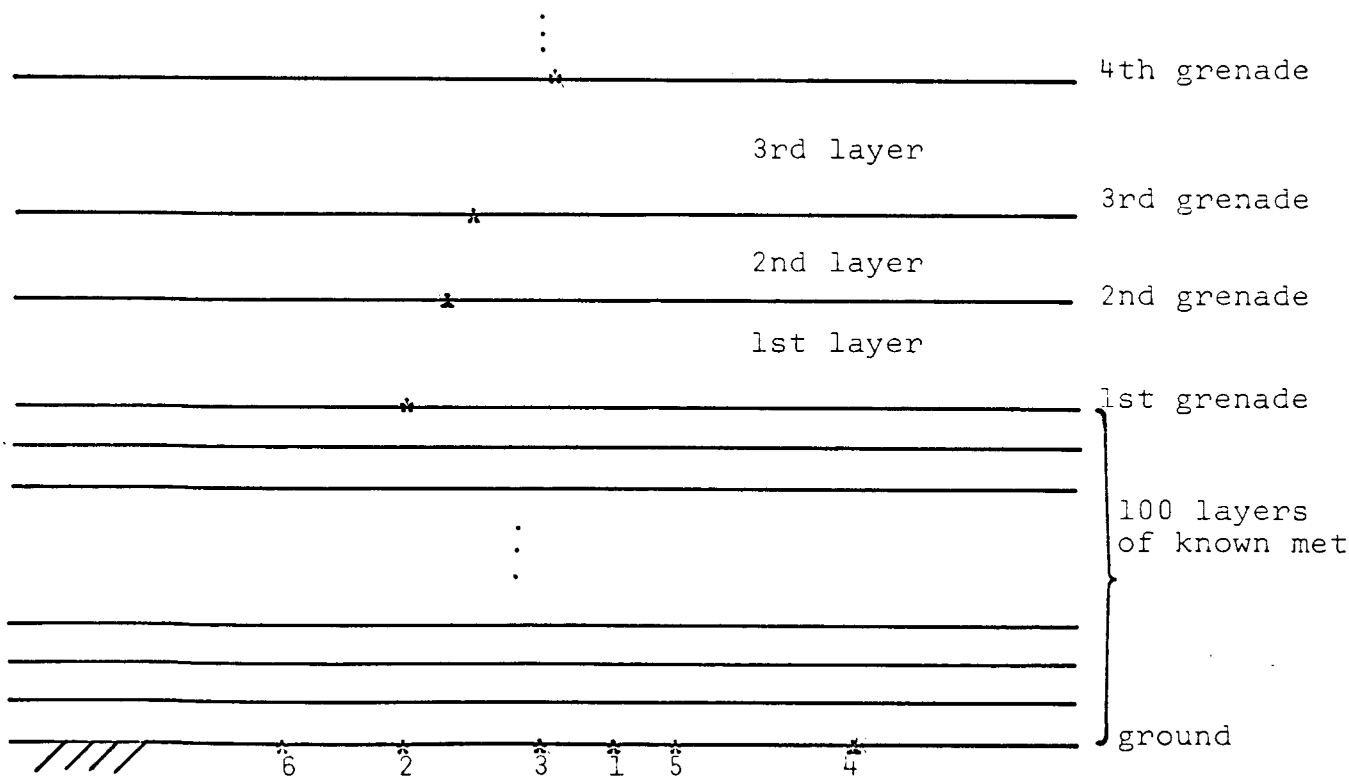

Figure 1. Ray-Tracing Problem. Horizontal lines are lines of constant height. Numbers below the ground line are microphone numbers.

The known meteorological data (hereafter referred to simply as "met") consist of temperature, wind speed, wind direction, pressure, and relative humidity; and is obtained from balloon or rocketsonde. The 100 layers of known met data are constructed in the following manner. The layer thickness is simply 1/100 of the altitude of the first grenade burst. Using a linear weighting method, we determine from the raw data a constant temperature, wind speed, etc. for each layer. It is tacitly assumed in constructing these layers that there are no horizontal gradients in either wind or temperature, and also that there are no vertical winds. The asterisks on the horizontal lines identifying the grenades denote the position (x,y,z) of the respective grenade bursts.

In analysis of the Rocket-Grenade Experiment, the layers between grenades are considered to have the same general properties as those of the known met, i.e., they are of constant temperature and winds, have no horizontal gradients, and deny the existence of vertical winds. The time and position of each grenade explosion along with the time at which the resulting acoustic wave strikes each microphone in the array are the fundamental measurements of the Rocket-Grenade Experiment. By computing the differences in travel time of the wave to each microphone in the array, one can employ the sound-ranging methods developed by E. A. Dean to determine the direction cosines of the associated ray as the wave sweeps the array.

For large arrays (such as the thirteen-microphone array at Wallops Island) Dean's method has been modified (1) to include a local speed of sound for each microphone, and (2) to solve for the radius of curvature of the wave as a least squares variable. With a known curvature, a unique sound-ranging solution for the direction cosines of the ray as the wave crosses the reference microphone in the array requires reading the break times on only three microphones. The modified solution treating the curvature as a variable requires four microphone readings for a unique solution. Since the break times can generally be read on more than three microphones, an overdetermined solution exists, enabling one to compute not only the direction cosines and travel time of the ray, but also the standard errors of these quantities.

The basic ideas involved in the ray-tracing process are the following:

(1) Given the azimuth and elevation of the ray at the ground, we can trace the ray-path back through the layers of known met by using equations developed from Snell's laws.

(2) Upon reaching the upper boundary of the final layer of known met, we can determine the winds and temperature in the layer between that boundary and the altitude of the next grenade burst by requiring the ray-path to terminate at the grenade burst position.

Now let us examine more closely the calculations which are involved. It is found that there are two "constants of the motion," so to speak, which are constant for each layer. They are ε and μ, defined as

$$\varepsilon = \beta_i/\alpha_i \tag{1}$$

$$\mu = \frac{c_i}{\alpha_i} - u_i - \varepsilon v_i \tag{2}$$

where β_i = y-direction cosine

α_i = x-direction cosine

c_i = sound speed

u_i = x-wind speed

v_i = y-wind speed

It is also true that

$$t_i = \frac{h_i}{\gamma_i c_i} \tag{3}$$

where t_i = travel time

h_i = layer thickness

γ_i = z-direction cosine

In all cases the subscript (i) refers to the ith layer. The constants ε and μ can be determined from the surface values.

$$\varepsilon = \beta_o/\alpha_o \qquad \mu = c_o/\alpha_o \text{ (no wind value)}$$

(The subscript zero indicates a surface value.) Since the sound speed (computed from the temperature) and wind components are known for each met layer, α_i and β_i can be computed from equations (1) and (2). If we have N layers of known met, and begin our ray-tracing at the origin, then the x and y displacement and travel time t in traversing those N layers is given by

$$x = \sum_{k=1}^{N} (\alpha_k c_k - u_k)\, t_k \tag{4}$$

$$y = \sum_{k=1}^{N} (\beta_k c_k - v_k)\, t_k \tag{5}$$

$$t = \sum_{k=1}^{N} \frac{h_k}{\gamma_k c_k}. \tag{6}$$

At this point, one must come to blows with a very real problem involved in the ray-tracing. If one takes the surface direction cosines of the ray from the first grenade and traces through the 100 layers of known met to the altitude of the first grenade, he will find invariably that the x, y, and t calculated from equations (4), (5), and (6) do not agree with the experimentally determined values for x, y, and t. The difference is in part a measure of the deviation of our 100-isolayer model atmosphere from the true continuous atmosphere the ray encounters in its actual path to the ground. In order to effectively remove this deviation, let us assume that the rays from all grenades take nearly the same path through these 100 layers. Then all ray-traced values will (at the top of the 100th layer) be in error by approximately Δx, Δy, and Δt, where

$$\Delta x = x \text{ (ray-traced)} - x \text{ (observed)} \tag{7}$$

$$\Delta y = y \text{ (ray-traced)} - y \text{ (observed)} \tag{8}$$

$$\Delta t = t \text{ (ray-traced)} - t \text{ (observed)} \tag{9}$$

where equations (7), (8), and (9) are evaluated for the first grenade. Thus, by shifting the grenade coordinates by Δx, Δy, and Δt, errors due to the model atmosphere are removed (to the approximation made above). As a footnote, note that the observed travel time includes the finite amplitude propagation (FAP) correction.

Finally, consider now the determination of the temperature and winds in a layer between the Nth and (N+1)th grenades, as shown in Figure 2. (After the met parameters between two successive grenades have been determined, that layer is regarded as a layer of known met.) By using the

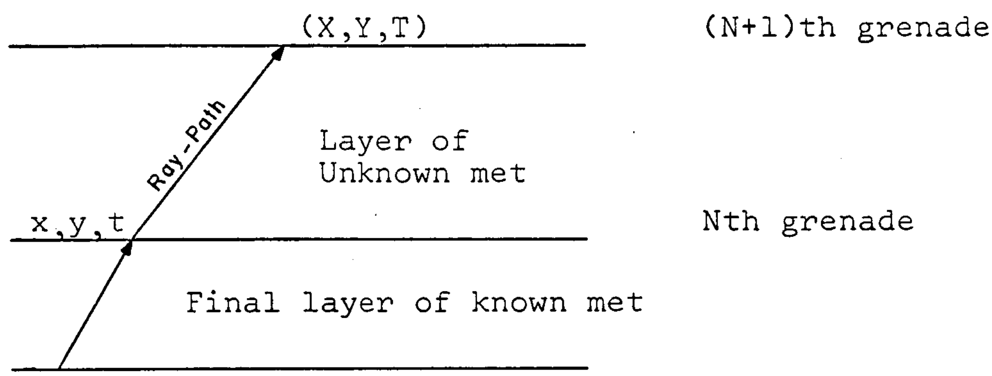

Figure 2. Ray path used for determination of unknown met.

ray-tracing techniques discussed previously, we can determine the position (x,y) and travel time (from the ground) for the ray from the (N+1)th grenade at the altitude of the Nth grenade. Since the ray must have emanated from the (N+1)th grenade, we determine the met parameters by forcing it to the position of the (N+1)th grenade. Mathematically, this can be accomplished by determining α, β, γ, c, u, and v for the layer between the Nth and (N+1)th grenades. If we denote X, Y, and T as the appropriate variables for the

(N+1)th grenade and let the layer thickness be h, the following six equations result:

$$\alpha^2 + \beta^2 + \gamma^2 = 1 \tag{10}$$

$$\beta = \varepsilon\alpha \tag{11}$$

$$c - \alpha u - \beta v - \alpha\mu = 0 \tag{12}$$

$$X - x = \psi = (\alpha c - u)\,\tau \tag{13}$$

$$Y - y = \eta = (\beta c - v)\,\tau \tag{14}$$

$$h = \gamma c \tau \tag{15}$$

where $$\tau = T - t \tag{16}$$

These six equations can be solved exactly, yielding, in order,

$$\alpha = \pm h \left[h^2(1 + \varepsilon^2) + (\tau\mu - \psi - \varepsilon\eta)^2\right]^{-\frac{1}{2}} \tag{17}$$

$$\beta = \varepsilon\alpha \tag{18}$$

$$\gamma = \sqrt{1 - \alpha^2 - \beta^2} \tag{19}$$

$$c = h/\gamma\tau \tag{20}$$

$$u = -\psi/\tau + \alpha c \tag{21}$$

$$v = -\eta/\tau + \beta c \tag{22}$$

where u and v are the x and y wind components respectively; the Kelvin temperature (T) is computed from the sound speed (c) in meters/second by

$$T = 0.002488\ c^2 \tag{23}$$

The proper root choice in (17) is that value of α which satisfies (12). Hence, this layer now becomes a layer of known met, and we advance to consider the ray from the next grenade. In this manner, the temperature and winds for the layer between each grenade pair can be determined.

III

SOURCES OF ERROR

After reviewing the basis of ray-tracing, we are now in a position to examine the sources of error present in the Rocket-Grenade Experiment and to discuss our method for computing a standard error for the temperature and winds in the layer between grenade pairs. To clarify future statements and to put our present discussion on a firm footing, let us define what is meant by standard error (σ) in reference to the Rocket-Grenade Experiment. When one uses the term standard error (standard deviation, normal error) in reference to a quantity, he immediately implies that repeated measurements of the given quantity form (nearly) a Gaussian (normal) distribution about some mean value. If one states, for example, that the travel time to the 5th microphone in the array is 17.32 seconds with a standard error of ±0.02 seconds, he means that if one could make a large number of determinations of the travel time (under identical experimental conditions), approximately 68% of these values would lie within 0.02 seconds of 17.32 seconds; further, 95% of the measurements would lie within two standard deviations of the mean, and 99.7% of the measurements would lie within three standard deviations of the mean. The quantity 3 σ is often referred to as the confidence interval because the likelihood of any single measurement occurring outside this interval is less than 0.3%. For this reason, many experimenters use the confidence interval as the error limit of a measurement.

In the Rocket-Grenade Experiment, one could as a gedanken exercise achieve repeated measurements of travel time by launching a great number of identical rockets and recording the break times of the waves from the explosion of the identical grenades (at identical times and positions) by a three-microphone array on the ground. Since many sets of the three break times necessary for a unique sound-ranging solution could be determined, one could compute the mean break times and their associated standard errors. To obtain the equivalent repeated measurements in a practical manner, the number of microphones in the array is increased beyond the three necessary for a unique sound-ranging solution. A least squares analysis is then applied to this overdetermined solution to yield a mean break time (relative to a reference microphone) and its associated standard error.

The various errors which may contribute to the final standard error in temperature and winds can be classified according to source as (1) errors inherent in the experimental measurements, and (2) errors resulting from approximations made in the course of the analysis. Those which result from uncertainties in experimental measurements are as follows:

(1) Surveying errors in determining the coordinates of each microphone in the array.

(2) Error in the determination of the burst time of the grenade and its position (x,y,z) at this time.

(3) The uncertainty in reading the break times of the wave as it crosses each microphone in the array. Providing more than three break times can be determined, one can perform a least squares analysis with the over-determined solution yielding standard errors in the break time of the wave and the direction cosines of its associated ray relative to the reference microphone in the array. It should be born in mind that these standard errors result directly from the uncertainty in the measurement of the break times.

Those errors which result from approximations in the course of the analysis are as follows:

(1) The error involved in forming the least squares operational equations. In the derivation of these equations, we assume a spherical wave emanates from a point source (the grenade or apparent grenade if the variable curvature is used) and travels through a homogeneous atmosphere before crossing the array. The assumption of a homogeneous atmosphere is partially compensated for in the large arrays by assigning a local speed of sound to each microphone. An additional approximation resulted when, to effect the least squares solution (complete with the required correlation coefficients), it was necessary to linearize the operational equations.

(2) The error resulting from the deviation of the 100-isolayer model atmosphere from the true atmosphere. This error is regarded as a systematic error which can be effectively removed from each grenade by a suitable shift of coordinates during the ray-tracing computation (See Sections II and IV).

(3) The error in computing the finite amplitude propagation (FAP) correction to the observed travel time. It should be realized first that the

expressions used to compute the FAP time correction are not exact, being based on Brode's numerical analysis of spherical blast waves in an isobaric, isothermal atmosphere. In addition, Otterman has introduced several necessary assumptions in adapting Brode's work to the Rocket-Grenade Expériment. The use of Otterman's expressions to compute the FAP time correction requires the knowledge of pressure and temperature at the point of explosion. Knowing the altitude of the explosion allows us to determine the temperature using the U. S. Standard Atmosphere, 1962. The pressure at the point of explosion is determined from the last known temperature-pressure pair (RAOB data) by use of equation I.2.10 - (4) of the U. S. Standard Atmosphere, 1962. The use of this equation involves assuming the validity of the hydrostatic equation, the perfect gas law, and assuming that within the altitude layer between the known and unknown pressures the gradient of the molecular-scale temperature with geopotential altitude is zero. Thus, the numerical value computed for the FAP time is in error not only due to the approximate nature of the theory but also due to the approximate values of pressure and temperature used to evaluate these theoretical expressions.

At the present time, exact information is not available regarding surveying errors or errors in the determination of the burst time and position of the grenade. The error in computing the FAP time correction to the observed travel time is thought to be a second order effect, since it is in effect a correction to a correction. Plans are now underway to investigate the effect of this error on the temperature and winds between grenade pairs. The remainder of this report will be concerned with a detailed discussion of the method by which the experimental uncertainties in reading the break times are propagated to yield standard errors in the temperature and winds between grenade pairs.

IV

METHOD OF ERROR PROPAGATION

Measurement of the break time of the wave from a grenade explosion on more than three microphones in the array produces an overdetermined sound-ranging solution. A least squares analysis using this overdetermined solution yields the direction cosines and break time of the ray (relative to a reference microphone) along with the standard errors of the quantities. The temperature and winds in a layer between grenade pairs depend through the ray-tracing analysis on the surface values of the direction cosines and travel times of the rays from each grenade; hence, standard errors in these surface values can be propagated through the ray-tracing analysis to yield standard errors in the computed values of temperature and winds in the layer. In this section, an explanation of the approximate method of error propagation is presented, along with a discussion of the coordinate shift technique for compensating for the deviation of the model atmosphere from the true atmosphere.

Consider the first layer of unknown met, the layer between the first and second grenades. For the first grenade, the ray tracing is performed (with the known values of direction cosines supplied by the sound-ranging data) through the 100 layers of known met data to the altitude of the first grenade. The ray-traced coordinates (x,y) and travel time are computed and the difference between these ray-traced values and their corresponding measured values calculated. Since the ray from the second grenade will make approximately the same error in traversing the 100 met layers as the one from the first, the measured x, y, and travel time of the second grenade are shifted by the differences calculated for the first grenade. The temperature and winds are then determined for the first layer as described in Section II. This shifting of coordinates of grenade two is our method of accounting for the deviation of the model atmosphere from the true atmosphere.

Now suppose one regarded the second grenade as the effective first one and ray-traced through now 101 layers of known met data. If one ray-traces through these layers with grenade two, he would miss its measured position (x,y) and travel time by some amount. The amount of that miss will be, however, exactly equal to the amount that the original coordinates and travel time of grenade two were shifted due to the initial error in the first grenade, i.e., Δx, Δy, Δt. Hence, the coordinates and travel time of the third grenade will be shifted by this same amount and its ray ray-traced to the altitude of the second grenade. The ray is then forced to its corrected position and the temperature and winds in the second layer determined. Then effectively the ray from the fourth grenade sees 102 layers of known met. Each grenade pair is then treated, as far as the analysis is concerned, in the same manner as the first pair. In this approximate method of analysis then, the standard error in determining the temperature and winds in each layer depends only on the standard error in the surface direction cosines and the travel time for the two grenades bounding this layer. The analysis thus proceeds by grenade pairs, with the grenade at the lower boundary of the layer being regarded as the effective first grenade and all lower grenade layers regarded as layers of known met.

Thus the salient features of the approximate error propagation method are the following:

(1) To account for the deviation of our model atmosphere from the true atmosphere, all grenade coordinates (x,y,t) are shifted by an amount equal to the deviation of the first grenade's ray-traced coordinates from its measured coordinates.

(2) All grenade pairs act the same as the first pair,

the only difference being the number of met layers below a given pair.

(3) To this approximation, the standard error in the temperature and winds in a given layer depends only on the standard error in the surface direction cosines and travel times for the pair of grenades bounding the layer. The theory of error propagation then enables one to calculate temperature and wind deviations in each successive layer, the analysis proceeding by grenade pairs with each grenade in turn becoming the effective first grenade. The next section describes the explicit mathematical steps required to implement this theory.

V

MATHEMATICAL DEVELOPMENT OF THE ERROR PROPAGATION

The computations detailed in this section are based on the discussion of the propagation of standard errors presented in Section 6, Chapter 10, Part 7 of the Handbook of Physics. It is pointed out there that the correct least squares analysis must provide not only the standard error for each variable, but also the correlation coefficient between each pair of variables; the correct computation of standard errors in derived quantities which depend on these least squares adjusted variables demands the knowledge of both their standard errors and associated correlation coefficients. We refer those not familiar with correlation coefficients to the reference cited.

Since the error propagation calculation proceeds pairwise with respect to grenades, let us consider the calculation of the standard deviations in temperature and winds in a general layer bounded below by a grenade whose variables shall be primed and above by a grenade whose variables are unprimed (Fig. 3). All surface values are denoted by the subscript zero. Let us denote by

σ_α = standard deviation in α

$r_{\alpha\beta}$ = correlation coefficient between α and β

There is no correlation between variables of different grenades, i.e., $r_{\alpha\alpha}{}' = 0$, $r_{\alpha\beta}{}' = 0$, etc.

______________________________ unprimed grenade

______________________________ primed grenade

N layers of known met

______________________________ ground

Figure 3. Consideration of a general layer of unknown met.

For the sound wave from each bounding grenade (primed and unprimed), there are three independent surface variables on which the temperature and winds in the layer depend. They are (1) the x-direction cosine, α_o, (2) the y-direction cosine, β_o, (3) the time at which the sound wave crosses the origin of the coordinate system, t_o. Since all of these are least squares adjusted values, we know their standard errors and pairwise correlation coefficients. Thus, in the layer between the primed and unprimed grenades, a function f can be written as

$$f = f(\alpha_o, \beta_o, t_o, \alpha_o', \beta_o', t_o') \qquad (24)$$

where f stands for the sound speed, x-wind, or y-wind. Using equation (10.30) from 7-157 of the Handbook of Physics, we can write the generalized law of the propagation of errors as

$$\sigma_f^2 = \sum_{i=1}^{N} \left(\frac{\delta f}{\delta x_i}\right)^2 \sigma_{x_i}^2 + \sum_{i,j}^{N} r_{ij} \left(\frac{\delta f}{\delta x_i}\right) \left(\frac{\delta f}{\delta x_j}\right) \sigma_{x_i} \sigma_{x_j} \qquad (25)$$

where $$f = f(x_1, x_2, x_3, \ldots, x_N) \qquad (26)$$

To adapt this equation to our case, we put N=6, and recall that $x_i \epsilon \left\{\alpha_o, \beta_o, t_o, \alpha_o', \beta_o', t_o'\right\}$. For purposes of clarity, the equation can be written as

$$\sigma_f^2 = (\frac{\delta f}{\delta\alpha_o})^2 \sigma^2_{\alpha_o} + (\frac{\delta f}{\delta\beta_o})^2 \sigma^2_{\beta_o} + (\frac{\delta f}{\delta t_o})^2 \sigma^2_{t_o} + (\frac{\delta f}{\delta\alpha_o'})^2 \sigma^2_{\alpha_o'}$$

$$+ (\frac{\delta f}{\delta\beta_o'})^2 \sigma^2_{\beta_o'} + (\frac{\delta f}{\delta t_o'})^2 \sigma^2_{t_o'} + 2 \left[\frac{\delta f}{\delta\alpha_o} \frac{\delta f}{\delta\beta_o} \sigma_{\alpha_o} \sigma_{\beta_o} r_{\alpha_o \beta_o}\right.$$

$$+ \frac{\delta f}{\delta\alpha_o} \frac{\delta f}{\delta t_o} \sigma_{\alpha_o} \sigma_{t_o} r_{\alpha_o t_o} + \frac{\delta f}{\delta t_o} \frac{\delta f}{\delta\beta_o} \sigma_{t_o} \sigma_{\beta_o} r_{t_o \beta_o}$$

$$+ \frac{\delta f}{\delta\alpha_o'} \frac{\delta f}{\delta\beta_o'} \sigma_{\alpha_o'} \sigma_{\beta_o'} r_{\alpha_o' \beta_o'} + \frac{\delta f}{\delta\alpha_o'} \frac{\delta f}{\delta t_o'} \sigma_{\alpha_o'} \sigma_{t_o'} r_{\alpha_o' t_o'}$$

$$\left. + \frac{\delta f}{\delta t_o'} \frac{\delta f}{\delta\beta_o'} \sigma_{t_o'} \sigma_{\beta_o'} r_{t_o' \beta_o'} \right] \qquad (27)$$

In this expression for σ_f^2 , all the standard deviations and correlation coefficients are known from the sound ranging data. Thus for each f we need to calculate six partial derivatives. Since f can be the sound speed, x-wind, or y-wind, we need to calculate 18 partial derivatives in all. The subject of the following is to show how these partial derivatives can be calculated.

The computations can be separated conveniently into two distinct groups: (1) The calculation of those quantities which are determined for the N layers of known met; (2) The calculation of those quantities which are computed solely for the final layer between the primed and unprimed grenades. We shall consider first those which are computed for the N layers of known met. All sums will be taken to be over all N layers of known met data.

In ray-tracing through the N layers of known met data the constants of the motion are

$$\mu = c_o/\alpha_o \tag{28}$$

$$\varepsilon = \beta_o/\alpha_o \tag{29}$$

The numerical values of these constants are determined by the observed surface values α_o , β_o , and c_o. In order to propagate the surface deviations in α_o , β_o , and t_o, we need to calculate the following partial derivatives.

$$\frac{\delta\alpha_n}{\delta\alpha_o},\ \frac{\delta\alpha_n}{\delta\beta_o},\ \frac{\delta\beta_n}{\delta\alpha_o},\ \frac{\delta\beta_n}{\delta\beta_o},\ \frac{\delta\gamma_n}{\delta\alpha_o},\ \frac{\delta\gamma_n}{\delta\beta_o},\ \frac{\delta t_n}{\delta\alpha_o},\ \frac{\delta t_n}{\delta\beta_o},$$

$$\frac{\delta x_n}{\delta\alpha_o},\ \frac{\delta x_n}{\delta\beta_o},\ \frac{\delta y_n}{\delta\alpha_o},\ \frac{\delta y_n}{\delta\beta_o}$$

t_n here is the travel time of the ray through the nth layer.

These differentials are written out in detail below.

$$\alpha_n = \alpha_o\ c_n/\ (c_o + \beta_o v_n + \alpha_o u_n\ ) \tag{30a}$$

$$\frac{\delta\alpha_n}{\delta\alpha_o} = \frac{\alpha_n}{\alpha_o}\left[1 - \frac{u_n\alpha_n}{c_n}\right] \tag{30b}$$

$$\frac{\delta\alpha_n}{\delta\beta_o} = -\ \frac{v_n\ \alpha_n^2}{\alpha_o\ c_n} \tag{30c}$$

$$\beta_n = \frac{\beta_o}{\alpha_o}\ \alpha_n \tag{31a}$$

$$\frac{\delta\beta_n}{\delta\alpha_o} = \varepsilon\left[\frac{\delta\alpha_n}{\delta\alpha_o} - \frac{\alpha_n}{\alpha_o}\right] \tag{31b}$$

$$\frac{\delta\beta_n}{\delta\beta_o} = \frac{\alpha_n}{\alpha_o} + \varepsilon\frac{\delta\alpha_n}{\delta\beta_o} \tag{31c}$$

$$\gamma_n = (1 - \alpha_n^2 - \beta_n^2)^{\frac{1}{2}} \tag{32a}$$

$$\frac{\delta\gamma_n}{\delta\alpha_o} = -\frac{1}{\gamma_n}\left[\alpha_n\frac{\delta\alpha_n}{\delta\alpha_o} + \beta_n\frac{\delta\beta_n}{\delta\alpha_o}\right] \tag{32b}$$

$$\frac{\delta\gamma_n}{\delta\beta_o} = -\frac{1}{\gamma_n}\left[\alpha_n\frac{\delta\alpha_n}{\delta\beta_o} + \beta_n\frac{\delta\beta_n}{\delta\beta_o}\right] \tag{32c}$$

$$t_n = \frac{h_n}{\gamma_n c_n} \tag{33a}$$

$$\frac{\delta t_n}{\delta\alpha_o} = -\frac{t_n}{\gamma_n}\frac{\delta\gamma_n}{\delta\alpha_o} \tag{33b}$$

$$\frac{\delta t_n}{\delta\beta_o} = -\frac{t_n}{\gamma_n}\frac{\delta\gamma_n}{\delta\beta_o} \tag{33c}$$

$$x_n = (\alpha_n c_n - u_n)\, t_n \tag{34a}$$

$$\frac{\delta x_n}{\delta \alpha_o} = \frac{x_n}{t_n} \frac{\delta t_n}{\delta \alpha_o} + t_n c_n \frac{\delta \alpha_n}{\delta \alpha_o} \tag{34b}$$

$$\frac{\delta x_n}{\delta \beta_o} = \frac{x_n}{t_n} \frac{\delta t_n}{\delta \beta_o} + t_n c_n \frac{\delta \alpha_n}{\delta \beta_o} \tag{34c}$$

$$y_n = (\beta_n c_n - v_n)\, t_n \tag{35a}$$

$$\frac{\delta y_n}{\delta \alpha_o} = \frac{y_n}{t_n} \frac{\delta t_n}{\delta \alpha_o} + t_n c_n \frac{\delta \beta_n}{\delta \alpha_o} \tag{35b}$$

$$\frac{\delta y_n}{\delta \beta_o} = \frac{y_n}{t_n} \frac{\delta t_n}{\delta \beta_o} + t_n c_n \frac{\delta \beta_n}{\delta \beta_o} \tag{35c}$$

If we denote

t = total calculated travel time through the N layers of known met

x = total calculated x-displacement

y = total calculated y-displacement

Then

$$x = \Sigma x_n \tag{36a}$$

$$y = \Sigma y_n \tag{36b}$$

$$t = \Sigma t_n \tag{36c}$$

$$\frac{\delta x}{\delta \alpha_o} = \Sigma \frac{\delta x_n}{\delta \alpha_o} \tag{37a}$$

$$\frac{\delta x}{\delta \beta_o} = \Sigma \frac{\delta x_n}{\delta \beta_o} \tag{37b}$$

$$\frac{\delta y}{\delta \alpha_o} = \Sigma \frac{\delta y_n}{\delta \alpha_o} \tag{37c}$$

$$\frac{\delta y}{\delta \beta_o} = \Sigma \frac{\delta y_n}{\delta \beta_o} \tag{37d}$$

$$\frac{\delta t}{\delta \beta_o} = \Sigma \frac{\delta t_n}{\delta \beta_o} \tag{37e}$$

$$\frac{\delta t}{\delta \alpha_o} = \Sigma \frac{\delta t_n}{\delta \alpha_o} \tag{37f}$$

In the previous enumeration of quantities which must be calculated for the layers of known met, no primes were used since these are calculated for all grenades. Observe also that the order of calculation is the order in which it must be carried out on the computer.

Suppose we have now ray-traced through the layers of known met and desire to determine the temperature, winds and their associated deviations in the layer bounded by the primed and unprimed grenades. The following notation will be used. The general remarks in the braces apply to both primed and unprimed variables.

t_o, t_o' — Arrival time of the sound wave at the coordinate origin. This is a least squares adjusted value, and hence has an associated standard deviation.

T, T' — Actual time of the grenade explosion. There is no standard deviation associated with this time.

$t = \Sigma t_n$

$t' = \Sigma t_n'$

$x = \Sigma x_n$

$x' = \Sigma x_n'$

$y = \Sigma y_n$

$y' = \Sigma y_n'$

(X,Y,Z), (X',Y',Z') — Measured coordinates of the grenades. These are regarded (as were T and T') as exact.

F, F' — Finite amplitude corrections for the grenades.

In the computer program, the sound wave from the first grenade is ray-traced through the 100 layers of known met to the altitude of the first grenade. Ray-traced values of x, y, and transit time are computed. The differences between these values and the measured values are computed as corrections which are then applied with equal weight to all other grenades. The corrections are the following:

$$\Delta x = x' - X' \tag{38}$$

$$\Delta y = y' - Y' \tag{39}$$

$$\Delta T = t' - (t_o' - T' + F') \tag{40}$$

The actual travel time $(t_o' - T' + F')$ is regarded as being in error only due to the error in determining t_o'.

In order to find the temperature and winds in the layer between the primed and unprimed grenades, we ray-trace the unprimed grenade's sound wave from the ground to the altitude of the primed grenade. Then we force the ray to go to the corrected value of the unprimed grenade's coordinates. Let X*, Y*, and T* be the corrected values of the unprimed grenade's coordinates.

$$X^* = X + \Delta x = X + x' - X' \tag{41}$$

$$Y^* = Y + \Delta y = Y + y' - Y' \tag{42}$$

The travel time from the ground to the unprimed grenade is $(t_o - T + F)$. In addition to that amount we must add ΔT. The travel time in the layer is then

$$\tau = t_o - T + F + \Delta T - t \tag{43a}$$

$$\tau = (t_o - t_o') + (t' - t)$$

$$+ (T' - T) + (F - F') \tag{43b}$$

Note that the quantities in the last two parentheses of (43b) are regarded as being exact. From the above it follows directly that

$$\psi = X^* - x = (X - X') + (x' - x) \tag{44}$$

$$\eta = Y^* - y = (Y - Y') + (y' - y) \tag{45}$$

Now the task is to find out how τ, ψ, and η vary with

α_o, β_o, t_o, α_o', β_o', t_o' . With these we can calculate how α varies with these surface variables and ultimately how c, u, and v vary with them. This is the ultimate goal.

$$\frac{\delta\tau}{\delta t_o} = 1 \qquad (46a) \qquad\qquad \frac{\delta\tau}{\delta t_o'} = -1 \qquad (47a)$$

$$\frac{\delta\tau}{\delta\alpha_o} = -\frac{\delta t}{\delta\alpha_o} \qquad (46b) \qquad\qquad \frac{\delta\tau}{\delta\alpha_o'} = \frac{\delta t'}{\delta\alpha_o'} \qquad (47b)$$

$$\frac{\delta\tau}{\delta\beta_o} = -\frac{\delta t}{\delta\beta_o} \qquad (46c) \qquad\qquad \frac{\delta\tau}{\delta\beta_o'} = \frac{\delta t'}{\delta\beta_o'} \qquad (47c)$$

$$\frac{\delta\psi}{\delta\alpha_o} = -\frac{\delta x}{\delta\alpha_o} \qquad (46d) \qquad\qquad \frac{\delta\psi}{\delta\alpha_o'} = \frac{\delta x'}{\delta\alpha_o'} \qquad (47d)$$

$$\frac{\delta\psi}{\delta\beta_o} = -\frac{\delta x}{\delta\beta_o} \qquad (46e) \qquad\qquad \frac{\delta\psi}{\delta\beta_o'} = \frac{\delta x'}{\delta\beta_o'} \qquad (47e)$$

$$\frac{\delta\eta}{\delta\alpha_o} = -\frac{\delta y}{\delta\alpha_o} \qquad (46f) \qquad\qquad \frac{\delta\eta}{\delta\alpha_o'} = \frac{\delta y'}{\delta\alpha_o'} \qquad (47f)$$

$$\frac{\delta\eta}{\delta\beta_o} = -\frac{\delta y}{\delta\beta_o} \qquad (46g) \qquad\qquad \frac{\delta\eta}{\delta\beta_o'} = \frac{\delta y'}{\delta\beta_o'} \qquad (47g)$$

Now let us return to the expression for α. Insertion of (28) and (29) into (17) yields that

$$\alpha = \frac{\alpha_o h}{B} \tag{48}$$

where

$$B \equiv \left[h^2(\alpha_o^2 + \beta_o^2) + (\tau c_o - \psi\alpha_o - \eta\beta_o)^2\right]^{\frac{1}{2}} \tag{49}$$

Denoting $\rho \varepsilon \left\{\alpha_o, \beta_o, t_o, \alpha_o', \beta_o', t_o'\right\}$, we see that

$$\frac{\delta\alpha}{\delta\rho} = \frac{h}{B} \, \delta_{\alpha_o \rho} - \frac{\alpha}{B} \, \frac{\delta B}{\delta\rho} \tag{50}$$

We need to know all the $\frac{\delta B}{\delta\rho}$. Straightforward differentiation shows that

$$\frac{\delta B}{\delta\alpha_o} = \frac{1}{B} \left[h^2\alpha_o + (\tau c_o - \psi\alpha_o - \eta\beta_o)(\alpha_o \frac{\delta x}{\delta\alpha_o} + \beta_o \frac{\delta y}{\delta\alpha_o} - c_o \frac{\delta t}{\delta\alpha_o} - \psi\right] \tag{51a}$$

$$\frac{\delta B}{\delta \beta_o} = \frac{1}{B}\left[\beta_o h^2 + (\tau c_o - \psi\alpha_o - \eta\beta_o)(\alpha_o \frac{\delta x}{\delta \beta_o} + \beta_o \frac{\delta y}{\delta \beta_o} - \eta - c_o \frac{\delta t}{\delta \beta_o})\right] \tag{51b}$$

$$\frac{\delta B}{\delta t_o} = \frac{c_o}{B} (\tau c_o - \psi\alpha_o - \eta\beta_o) \tag{51c}$$

$$\frac{\delta B}{\delta \alpha_o'} = \frac{1}{B}\left[(\tau c_o - \psi\alpha_o - \eta\beta_o)(c_o \frac{\delta t'}{\delta \alpha_o'} - \alpha_o \frac{\delta x'}{\delta \alpha_o'} - \beta_o \frac{\delta y'}{\delta \alpha_o'})\right] \tag{51d}$$

$$\frac{\delta B}{\delta \beta_o'} = \frac{1}{B}\left[(\tau c_o - \psi\alpha_o - \eta\beta_o)(c_o \frac{\delta t'}{\delta \beta_o'} - \alpha_o \frac{\delta x'}{\delta \beta_o'} - \beta_o \frac{\delta y'}{\delta \beta_o'})\right] \tag{51e}$$

$$\frac{\delta B}{\delta t_o'} = -\frac{c_o}{B} (\tau c_o - \psi\alpha_o - \eta\beta_o) = -\frac{\delta B}{\delta t_o} \tag{51f}$$

Thus,

$$\frac{\delta \alpha}{\delta \alpha_o} = \frac{h}{B} - \frac{\alpha}{B} \frac{\delta B}{\delta \alpha_o} \tag{52a}$$

$$\frac{\delta \alpha}{\delta \beta_o} = -\frac{\alpha}{B} \frac{\delta B}{\delta \beta_o} \tag{52b}$$

$$\frac{\delta \alpha}{\delta t_o} = -\frac{\alpha}{B} \frac{\delta B}{\delta t_o} = -\frac{\delta \alpha}{\delta t_o'} \tag{52c}$$

$$\frac{\delta\alpha}{\delta\alpha'_o} = - \frac{\alpha}{B} \frac{\delta B}{\delta\alpha'_o} \tag{52d}$$

$$\frac{\delta\alpha}{\delta\beta'_o} = - \frac{\alpha}{B} \frac{\delta B}{\delta\beta'_o} \tag{52e}$$

We need to compute now the derivatives with respect to $\rho\varepsilon\left\{\alpha_o, \beta_o, t_o, \alpha'_o, \beta'_o, t'_o\right\}$ for β, γ, c, u, and v. The calculations are straightforward and will be presented without discussion.

$$\beta = (\beta_o/\alpha_o)\,\alpha \tag{53a}$$

$$\gamma = (1 - \alpha^2 - \beta^2)^{\frac{1}{2}} \tag{54a}$$

$$\frac{\delta\beta}{\delta\alpha_o} = \varepsilon \frac{\delta\alpha}{\delta\alpha_o} - \varepsilon \frac{\alpha}{\alpha_o} \tag{53b}$$

$$\frac{\delta\gamma}{\delta t_o} = - \frac{1}{\gamma}\left[\alpha \frac{\delta\alpha}{\delta t_o} + \beta \frac{\delta\beta}{\delta t_o}\right] = - \frac{\delta\gamma}{\delta t'_o} \tag{54b}$$

$$\frac{\delta\beta}{\delta\beta_o} = \varepsilon \frac{\delta\alpha}{\delta\beta_o} + \frac{\alpha}{\alpha_o} \tag{53c}$$

$$\frac{\delta\gamma}{\delta\alpha_o} = - \frac{1}{\gamma}\left[\alpha \frac{\delta\alpha}{\delta\alpha_o} + \beta \frac{\delta\beta}{\delta\alpha_o}\right] \tag{54c}$$

$$\frac{\delta\beta}{\delta t'_o} = \varepsilon \frac{\delta\alpha}{\delta t'_o} = - \frac{\delta\beta}{\delta t_o} \tag{53d}$$

$$\frac{\delta\gamma}{\delta\beta_o} = - \frac{1}{\gamma}\left[\alpha \frac{\delta\alpha}{\delta\beta_o} + \beta \frac{\delta\beta}{\delta\beta_o}\right] \tag{54d}$$

$$\frac{\delta\beta}{\delta\alpha'_o} = \varepsilon \frac{\delta\alpha}{\delta\alpha'_o} \tag{53e}$$

$$\frac{\delta\gamma}{\delta\beta'_o} = - \frac{1}{\gamma}\left[\alpha \frac{\delta\alpha}{\delta\beta'_o} + \beta \frac{\delta\beta}{\delta\beta'_o}\right] \tag{54e}$$

$$\frac{\delta\beta}{\delta\beta_o'} = \varepsilon\frac{\delta\alpha}{\delta\beta_o'} \qquad (53f) \qquad \frac{\delta\gamma}{\delta\alpha'} = -\frac{1}{\gamma}\left[\alpha\frac{\delta\alpha}{\delta\alpha_o'} + \beta\frac{\delta\beta}{\delta\alpha_o'}\right] \qquad (54f)$$

$$c = \frac{h}{\gamma\tau} \qquad (55a)$$

$$\frac{\delta c}{\delta t_o'} = -c\left[\frac{1}{\gamma}\frac{\delta\gamma}{\delta t_o'} - \frac{1}{\tau}\right] = -\frac{\delta c}{\delta t_o} \qquad (55b)$$

$$\frac{\delta c}{\delta\alpha_o'} = -c\left[\frac{1}{\gamma}\frac{\delta\gamma}{\delta\alpha_o'} + \frac{1}{\tau}\frac{\delta t'}{\delta\alpha_o'}\right] \qquad (55c)$$

$$\frac{\delta c}{\delta\beta_o'} = -c\left[\frac{1}{\gamma}\frac{\delta\gamma}{\delta\beta_o'} + \frac{1}{\tau}\frac{\delta t'}{\delta\beta_o'}\right] \qquad (55d)$$

$$\frac{\delta c}{\delta\alpha_o} = -c\left[\frac{1}{\gamma}\frac{\delta\gamma}{\delta\alpha_o} - \frac{1}{\tau}\frac{\delta t}{\delta\alpha_o}\right] \qquad (55e)$$

$$\frac{\delta c}{\delta\beta_o} = -c\left[\frac{1}{\gamma}\frac{\delta\gamma}{\delta\beta_o} - \frac{1}{\tau}\frac{\delta t}{\delta\beta_o}\right] \qquad (55f)$$

$$u = \alpha c - \frac{\psi}{\tau} \qquad (56a)$$

$$\frac{\delta u}{\delta t_o} = \alpha \frac{\delta c}{\delta t_o} + c \frac{\delta \alpha}{\delta t_o} + \frac{\psi}{\tau^2} \qquad (56b)$$

$$\frac{\delta u}{\delta \alpha_o} = \alpha \frac{\delta c}{\delta \alpha_o} + c \frac{\delta \alpha}{\delta \alpha_o} + \frac{1}{\tau} \frac{\delta x}{\delta \alpha_o} - \frac{\psi}{\tau^2} \frac{\delta t}{\delta \alpha_o} \qquad (56c)$$

$$\frac{\delta u}{\delta \beta_o} = \alpha \frac{\delta c}{\delta \beta_o} + c \frac{\delta \alpha}{\delta \beta_o} + \frac{1}{\tau} \frac{\delta x}{\delta \beta_o} - \frac{\psi}{\tau^2} \frac{\delta t}{\delta \beta_o} \qquad (56d)$$

$$\frac{\delta u}{\delta t'_o} = \alpha \frac{\delta c}{\delta t'_o} + c \frac{\delta \alpha}{\delta t'_o} - \frac{\psi}{\tau^2} \qquad (56e)$$

$$\frac{\delta u}{\delta \alpha'_o} = \alpha \frac{\delta c}{\delta \alpha'_o} + c \frac{\delta \alpha}{\delta \alpha'_o} - \frac{1}{\tau} \frac{\delta x'}{\delta \alpha'_o} + \frac{\psi}{\tau^2} \frac{\delta t'}{\delta \alpha'_o} \qquad (56f)$$

$$\frac{\delta u}{\delta \beta'_o} = \alpha \frac{\delta c}{\delta \beta'_o} + c \frac{\delta \alpha}{\delta \beta'_o} - \frac{1}{\tau} \frac{\delta x'}{\delta \beta'_o} + \frac{\psi}{\tau^2} \frac{\delta t'}{\delta \beta'_o} \qquad (56g)$$

$$v = \beta c - {}^{\eta}/_{\tau} \qquad (57a)$$

$$\frac{\delta v}{\delta t_o} = c \frac{\delta \beta}{\delta t_o} + \beta \frac{\delta c}{\delta t_o} + {}^{\eta}/_{\tau^2} \qquad (57b)$$

$$\frac{\delta v}{\delta \alpha_o} = c \frac{\delta \beta}{\delta \alpha_o} + \beta \frac{\delta c}{\delta \alpha_o} + \frac{1}{\tau} \frac{\delta y}{\delta \alpha_o} - {}^{\eta}/_{\tau^2} \frac{\delta t}{\delta \alpha_o} \qquad (57c)$$

$$\frac{\delta v}{\delta \beta_o} = c\,\frac{\delta \beta}{\delta \beta_o} + \beta\,\frac{\delta c}{\delta \beta_o} + \frac{1}{\tau}\,\frac{\delta y}{\delta \beta_o} - {}^{\eta}/_{\tau^2}\,\frac{\delta t}{\delta \beta_o} \qquad (57d)$$

$$\frac{\delta v}{\delta t'_o} = c\,\frac{\delta \beta}{\delta t'_o} + \beta\,\frac{\delta c}{\delta t'_o} - {}^{\eta}/_{\tau^2} \qquad (57e)$$

$$\frac{\delta v}{\delta \alpha'_o} = c\,\frac{\delta \beta}{\delta \alpha'_o} + \beta\,\frac{\delta c}{\delta \alpha'_o} - \frac{1}{\tau}\,\frac{\delta y'}{\delta \alpha'_o} + {}^{\eta}/_{\tau^2}\,\frac{\delta t'}{\delta \alpha'_o} \qquad (57f)$$

$$\frac{\delta v}{\delta \beta'_o} = c\,\frac{\delta \beta}{\delta \beta'_o} + \beta\,\frac{\delta c}{\delta \beta'_o} - \frac{1}{\tau}\,\frac{\delta y'}{\delta \beta'_o} + {}^{\eta}/_{\tau^2}\,\frac{\delta t'}{\delta \beta'_o} \qquad (57g)$$

Since $c^2 = KT$

$$\frac{\delta T}{\delta c} = \frac{2T}{c}$$

$$\sigma_T = \frac{2T}{c}\,\sigma_c \qquad (58)$$

Thus to determine the standard deviation in the sound speed, x-wind, and y-wind in the layer between the primed and unprimed grenades, all one must do is to insert (55b-f), (56b-g), and (57b-g), respectively, into (27). To obtain the standard deviation in the temperature simply insert the numerical value of σ_c into (58).

If in addition to these deviations, it is desired to compute those of the wind speed and wind direction, one proceeds directly as follows. Denoting the wind speed by W,

$$W = (u^2 + v^2)^{\frac{1}{2}} \tag{59}$$

If as before we denote $\rho \varepsilon \left\{ \alpha_o, t_o, \beta_o, \alpha'_o, \beta'_o, t'_o \right\}$, then

$$\frac{\delta W}{\delta \rho} = \frac{1}{W} \left[u \frac{\delta u}{\delta \rho} + v \frac{\delta v}{\delta \rho} \right] \tag{60}$$

By allowing ρ to assume its six possible values, one obtains the necessary partial derivatives to insert into (27) to calculate the standard deviation in the wind speed.

If we call $\theta = \tan^{-1} \frac{v}{u}$, then the wind direction is some constant plus or minus θ. Thus we can compute the standard deviation in the wind direction by computing the needed partial derivatives of θ. If $\rho \varepsilon \left\{ \alpha_o, t_o, \beta_o, \alpha'_o, \beta'_o, t'_o \right\}$, then

$$\frac{\delta \theta}{\delta \rho} = \frac{1}{W^2} \left[-v \frac{\delta u}{\delta \rho} + u \frac{\delta v}{\delta \rho} \right] \tag{61}$$

The standard deviation in wind direction can be calculated by inserting the derivatives from (61) into (27).

VI

CONCLUSION

After reviewing the ray-tracing procedure used in the Rocket-Grenade Experiment, we enumerated the factors which contribute to the error in temperature and winds in the layer between grenade pairs. Of these factors, we considered the computed temperature and winds to be in error due to (1) the uncertainty in reading the break times of the wave as it crosses each microphone in the array; and (2) the deviation of the layered model atmosphere from the true atmosphere. The latter was regarded as a systematic error which can be approximately removed by a suitable shift in grenade coordinates. If a least squares sound-ranging solution can be performed, the uncertainties in break times will result in standard errors in the surface values of direction cosines and travel time of the ray from each grenade. An approximate method for propagating these surface standard errors to yield standard errors in the computed temperature and winds was then discussed. Finally, standard techniques of error propagation theory were employed to derive a lengthy set of mathematical expressions which facilitate the calculation of the required standard errors in temperature and winds. The expressions have been incorporated into the existing computer program; evaluation of the results indicates that this method of error analysis is valid.

DEFINITION OF NOTATIONS

$\left.\begin{array}{l} x \\ y \\ z \end{array}\right\}$ - cartesian coordinates with origin at center of microphone array

β - y direction cosine

α - x direction cosine

γ - z direction cosine

c - sound speed

t - travel time of sound

u - x wind speed

v - y wind speed

h - layer thickness

ε - constant of the motion ($\varepsilon = \beta/\alpha$)

μ - constant of the motion ($\mu = [c/\alpha] - u - \varepsilon v$)

Ψ - x displacement in layer

η - y displacement in layer

τ - sound travel time in layer

σ_a - standard deviation of a

r_{ab} - correlation coefficient between a and b

δ - Kronecker delta

REFERENCES

Brode, H. L. "Numerical Solution of Spherical Blast Waves", Journal of Applied Physics, 26:766 (1955).

Dean, E. A. "Ray-Tracing in a Layered Medium", pp. 104-110, Rocket Grenade Experiment, Final Report, June 1965. El Paso, Texas: Schellenger Research Laboratories.

DuMond, J. W. M. and E. Richard Cohen. "Calculation of Standard Errors and Correlation Coefficients," in the Handbook of Physics, ed. by Condon and Odishaw, New York: McGraw Hill Book Company, Inc., 1958.

Otterman, Joseph. "Finite - Amplitude Propagation Effect on Shock-Wave Travel Times from Explosions at High Altitudes", Journal of the Acoustical Society of America, 31: 470 (1959).