Reports of the Department of Geodetic Science
Report No. 86

LEAST SQUARES ADJUSTMENT OF SATELLITE OBSERVATIONS FOR SIMULTANEOUS DIRECTIONS OR RANGES

PART 1 of 3: FORMULATION OF EQUATIONS

by

Edward J. Krakiwsky and Allen J. Pope

Prepared for
National Aeronautics and Space Administration
Washington, D. C.
Contract No. NSR 36-008-033
OSURF Project No. 1997

The Ohio State University
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PREFACE

This project is under the direction of Professor Ivan I. Mueller of the Department of Geodetic Science, The Ohio State University. Project manager is Jerome D. Rosenberg, Geodetic Satellites, Code SAG, NASA Headquarters, Washington, D. C. The contract is administered by the Office of Grants and Research Contracts, Office of Space Science and Applications, NASA Headquarters, Washington, D. C.

The report was written by Edward J. Krakiwsky, Graduate Research Associate. The equations presented herein were developed chiefly by Allen J. Pope while pursuing graduate studies at the Department of Geodetic Science.

Reports related to NASA Contract No. NSR 36-008-033 and published to date are the following:

<u>The Determination and Distribution of Precise Time</u>, Report No. 70 of the Department of Geodetic Science, The Ohio State University.

<u>Proposed Optical Network for the National Geodetic Satellite Program</u>, Report No. 71 of the Department of Geodetic Science, The Ohio State University.

<u>Preprocessing Optical Satellite Observations</u>, Report No. 82 of the Department of Geodetic Science, The Ohio State University.

<u>Least Squares Adjustment of Satellite Observations for Simultaneous Directions or Ranges</u>, Part 1 of 3: Formulation of Equations, Report No. 86 of the Department of Geodetic Science, The Ohio State University.

Quarterly Progress Reports, Numbers 1, 2, 3, 4, 5, 6.

ABSTRACT

The purpose of the report is to formulate the equations of the rigorous least squares adjustment of satellite observations for simultaneous directions or ranges. These equations are necessary for the development of computer programs documented in a separate report [Krakiwsky et al, in press].

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1. INTRODUCTION

1.1 Definition of Problem and Statement of Method

The problem is to tie remote ground stations together in the same geodetic coordinate system by use of satellite observations. Two major methods are available [Mueller, 1964, p. 145]: the orbital (short and long arc) methods and the space triangulation (trilateration) method. The solution presented here is developed about the latter.

In the space triangulation (trilateration) method satellites are observed simultaneously from groups of known and unknown ground stations, thus permitting a purely geometric solution. The main characteristic of this method is that orbital elements are not required. If the satellite positions are needed they can be computed from the preliminary coordinates of the ground stations and the observations themselves.

The optical observations are assumed to be in the true topocentric celestial system as preprocessed by [Hotter, 1967], while the topocentric ranging data is freed of systematic errors as explained in [Gross, in prep.]. The time system is UT1 as explained in [Preuss, 1966]. It should be noted that optical and range data are adjusted separately in this development.

This publication contains all the equations necessary for the computer programming in [Krakiwsky et al., in press].

1.2 Definition of Coordinate Systems

Two distinct types of coordinate systems have been used here:

(a) The terrestrial (average or instantaneous) system.

(b) The celestial (true) system.

The following summary of these systems assumes right-handed rectangular coordinates with axes numbered according to Figure 1-1. A further stipulation is that the centre of the coordinate system coincides with the centre of gravity of the earth.

<u>Average Terrestrial (X)</u>

(a) 3-axis directed toward the average north terrestrial pole as defined by the International Polar Motion Service (I.P.M.S.), commonly known as the average pole of 1900-05.

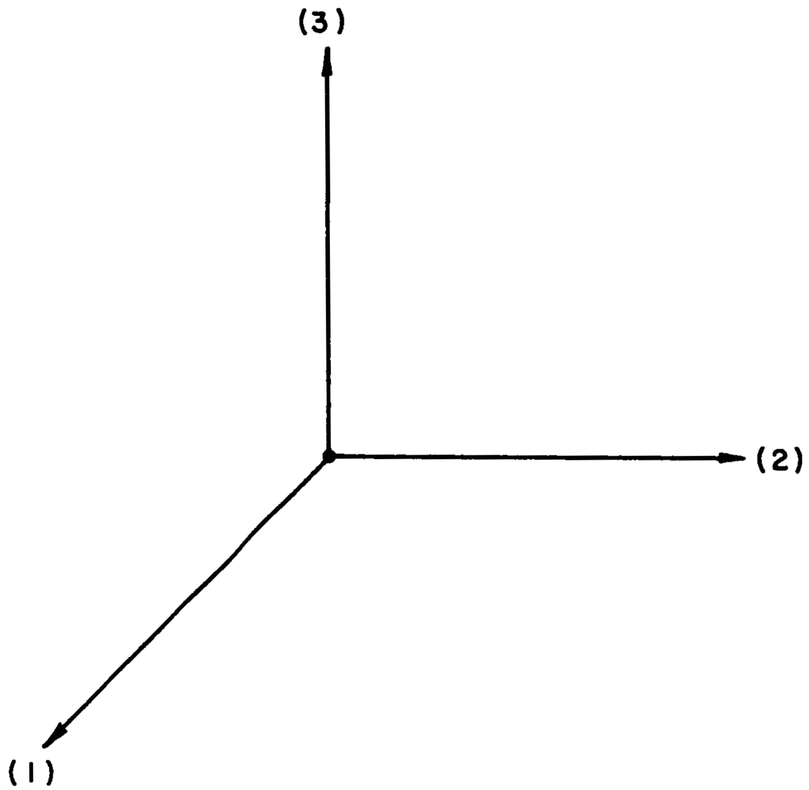

FIGURE 1-1. NUMBERING OF COORDINATE AXES.

(b) 1-3 plane parallel to the mean Greenwich astronomic meridian as defined by the Bureau International de l'Heure (B.I.H.).

This system is the ultimate goedetic coordinate system.

Instantaneous Terrestrial (Y)

(a) 3-axis directed toward the instantaneous rotation axis of the earth (true celestial pole), the coordinates of which are given by the I.P.M.S. with respect to the average pole of 1900-05.

(b) 1-3 plane contains the point where the mean Greenwich astronomic meridian intersects the true equator of date.

This coordinate system is used as the intermediate connection between the terrestrial and celestial coordinate systems.

True Celestial (Z)

(a) 3-axis equivalent to 3-axis of instantaneous terrestrial system (true celestial pole).

(b) 1-axis directed toward the true vernal equinox of date.

These and still other coordinate systems are discussed in detail in [Veis, 1963] and [Mueller, in press].

1.3 Transformation from the True Celestial to the Average Terrestrial System

Transformation between terrestrial and celestial coordinate systems becomes necessary in the case that topocentric directions to satellites are obtained by photographing the satellite against a background of stars. After corrections for the physical effects such as differential refraction and abberation, shimmer, etc. [Mueller, 1964, pp. 309-317; Hotter, 1967] have been applied, the resulting topocentric right ascension and declination form the purely geometric ground to satellite vector

$$\vec{z} = \begin{vmatrix} \cos\delta \ \cos\alpha \\ \cos\delta \ \sin\alpha \\ \sin\delta \end{vmatrix} \quad . \qquad 1\text{-}1$$

The above vector is rotated into the average terrestrial coordinate system since it is in this coordinate system that the adjustment takes place.

Transformation is first made into the instantaneous terrestrial system (see Fig. 1-2). This transformation is a function of a single finite rotation through the Greenwich apparent sidereal time (GAST) (see Section 1.31). A vector, $\vec{z}$, in the true celestial system is

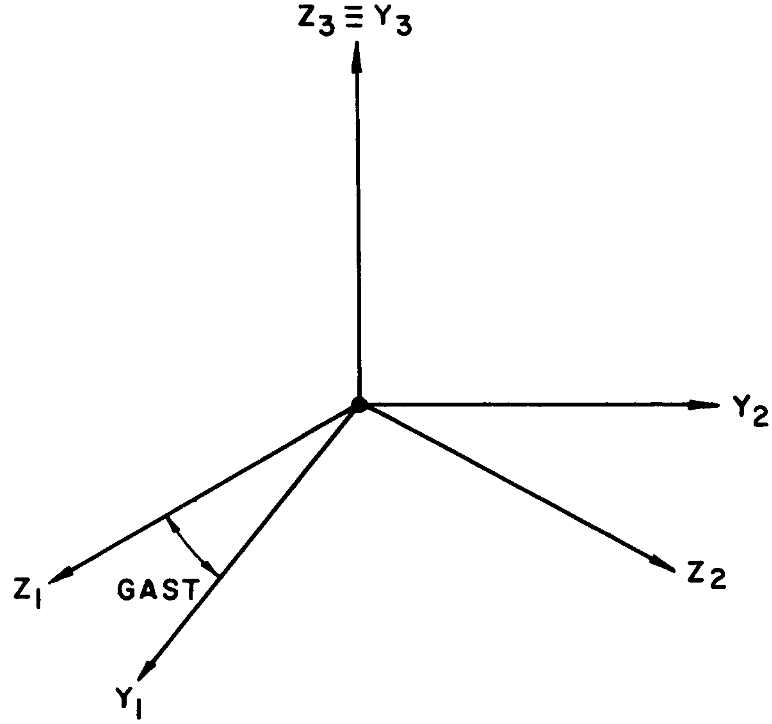

FIGURE 1-2. TRUE CELESTIAL AND INSTANTANEOUS TERRESTRIAL COORDINATE SYSTEMS.

rotated into the instantaneous terrestrial system by the following equation:

$$\vec{Y} = R_3 \text{ (GAST) } \vec{Z} \quad , \qquad 1\text{-}2$$

where $\vec{Y}$ is the resulting vector in the instantaneous terrestrial system and R_3 (GAST) is a 3 x 3 matrix that expresses a rotation about the 3 axis by the amount GAST, namely:

$$R_3 \text{ (GAST)} = \begin{vmatrix} \cos \text{(GAST)} & \sin \text{(GAST)} & 0 \\ -\sin \text{(GAST)} & \cos \text{(GAST)} & 0 \\ 0 & 0 & 1 \end{vmatrix} \qquad 1\text{-}3$$

Next the vector $\vec{Y}$ is rotated from the instantaneous terrestrial (Y) to the average terrestrial (X) system (see Figure 1-3). This transformation is a function of two rotations through the x and y coordinates of the instantaneous terrestrial pole (see Section 1.32). Mathematically,

$$\vec{X} = R_2 \text{ (-x) } R_1 \text{ (-y) } \vec{Y} \quad , \qquad 1\text{-}4$$

where $\vec{X}$ is the resulting vector in the average terrestrial coordinate system; R_1(-y) and R_2(-x) are 1-axis and 2-axis rotations through -y and -x. Since the x and y values are computed on the assumption that they are differential

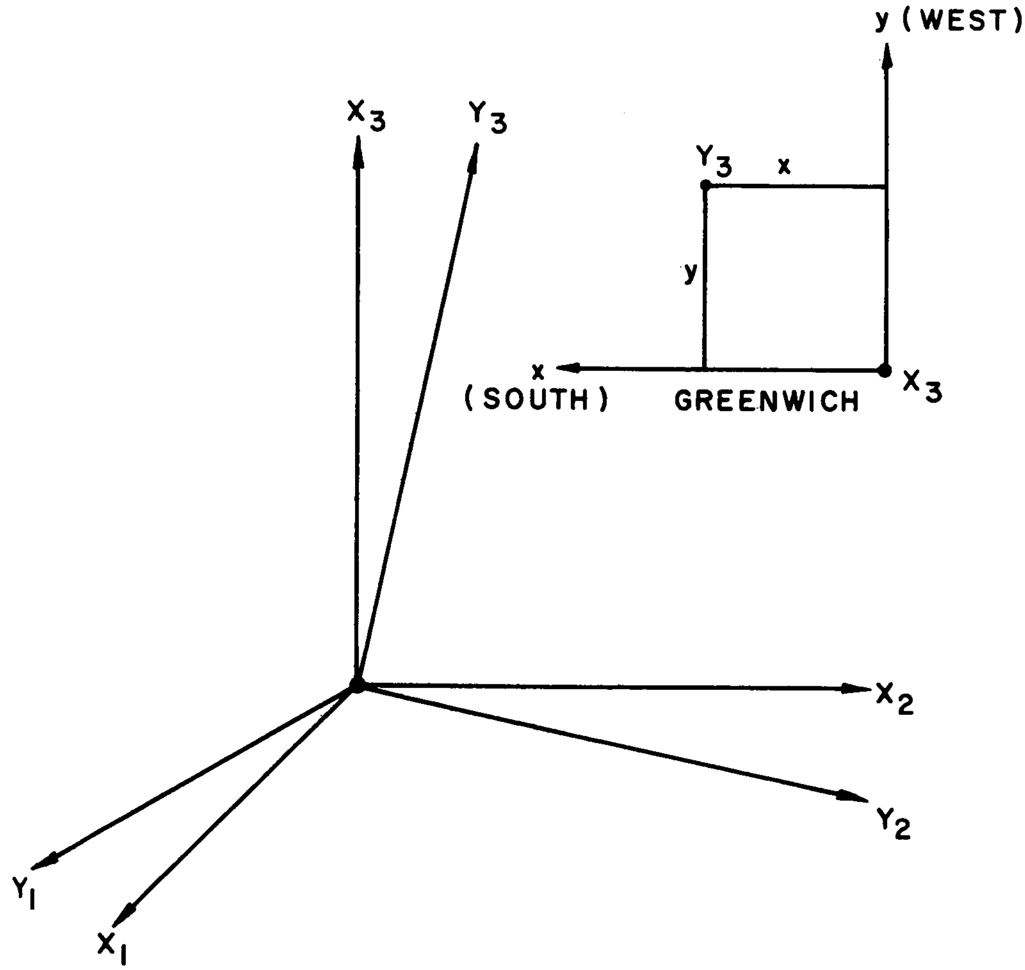

FIGURE 1-3. INSTANTANEOUS AND AVERAGE TERRESTRIAL COORDINATE SYSTEMS.

[Preuss, 1966, p.72], the finite rotations are replaced by differential rotations and Equation 1-4 is reduced to

$$\vec{X} = \begin{vmatrix} 1 & 0 & x \\ 0 & 1 & -y \\ -x & y & 1 \end{vmatrix} \vec{Y} \qquad 1\text{-}5$$

by ommitting products of xy. Thus the transformation from the true celestial to the average terrestrial coordinate system is achieved by combining the rotations expressed in Equations 1-2 and 1-4, namely:

$$\vec{X} = R_2(-x)\ R_1(-y)\ R_3(\mathrm{GAST})\ \vec{Y}, \qquad 1\text{-}6$$

and after considering Equation 1-5, the matrix form is

$$\vec{X} = S\ \vec{Y}, \qquad 1\text{-}7$$

where

$$S = \begin{vmatrix} \cos(\mathrm{GAST}); & \sin(\mathrm{GAST}); & x \\ -\sin(\mathrm{GAST}); & \cos(\mathrm{GAST}); & -y \\ -x\cos(\mathrm{GAST})-y\sin(\mathrm{GAST}); & -x\sin(\mathrm{GAST})+y\cos(\mathrm{GAST}); & 1 \end{vmatrix}. \qquad 1\text{-}8$$

1.31 Computation of the Greenwich Apparent Sidereal Time

The GAST is computed in four steps (Figure 1-4).

(1) The Greenwich Mean Sidereal Time (GMST) at 0^h UT1 is computed by means of Newcomb's formula [Expl. Supp., 1961, p. 75] as

$$\text{GMST at } 0^h \text{ UT1} = 6^h\ 38^m\ 45\overset{s}{.}836 + + 864\ 0184\overset{s}{.}542\ T_u + 0\overset{s}{.}0929\ T_u^2 \quad , \qquad 1\text{-}9$$

where T_u is the number of Julian centuries of 36525 days of universal time elapsed since January 0.5, 1900. The value of T_u is equal to the Julian date of the epoch of observation minus 2415020.0 (the Julian date of January 0.5, 1900.) divided by 36525.

(2) The UT1 time (interval) is converted to a mean sidereal interval by multiplying the former by the factor of 1.0027379093.

(3) GMST = GMST at 0^h UT1 + mean sidereal time interval. 1-10

(4) GMST is converted to GAST by adding the equation of the equinox, Δt [Expl. Supp., 1961, pp. 43 and 75].

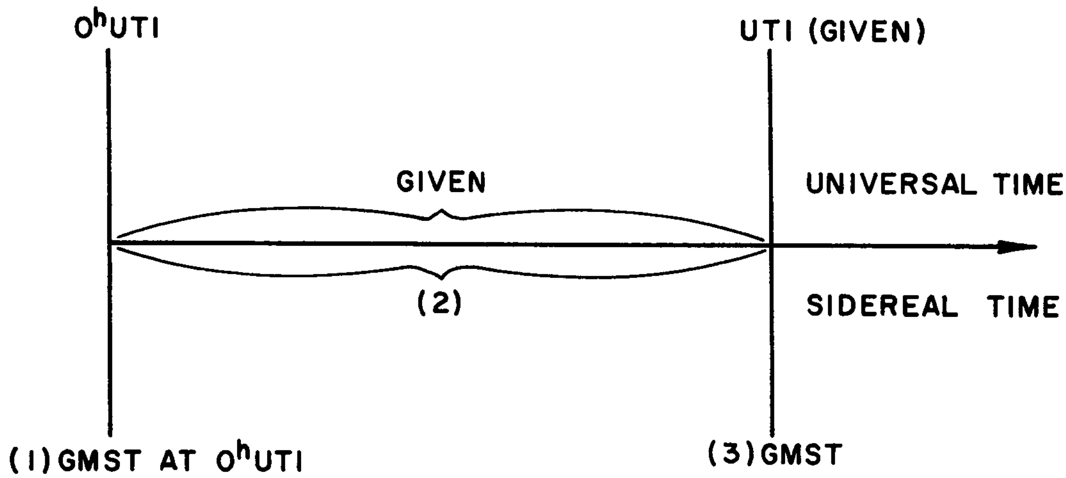

FIGURE 1-4. CONVERSION FROM UT1 TO GAST.

$$GAST = GMST + \Delta t, \qquad 1\text{-}11$$

where

$$\Delta t = \Delta\psi \cos \varepsilon, \qquad 1\text{-}12$$

where $\Delta\psi$ is the nutation in longitude and ε is the obliquity of the ecliptic at the epoch of observation.

Computation of the Obliquity (ε)

The computation of ε is made according to

$$\varepsilon = \varepsilon_m + \Delta\varepsilon, \qquad 1\text{-}13$$

where ε_m is the mean obliquity at the epoch of observation, and $\Delta\varepsilon$ is the nutation in obliquity. ε_m is computed by the following equation

[Expl. Supp., 1961, p. 98]:

$$\varepsilon_m = 23^\circ\ 27'\ 08\overset{''}{.}26 - 46\overset{''}{.}845\ T - 0\overset{''}{.}0059\ T^2 + 0\overset{''}{.}00181\ T^3, \qquad 1\text{-}14$$

where T is the number of Julian Centuries from the fundamental epoch of 1900 January $0\overset{d}{.}5$ ET.

The Computation of the Nutation in Obliquity ($\Delta\varepsilon$) and Nutation in Longitude ($\Delta\psi$)

The values of $\Delta\psi$ and $\Delta\varepsilon$ are computed according to the numerical series developed by E. W. Woolard [Expl.

Supp., 1961, pp. 44-45]. The programming procedure used is as follows [Allen, 1966, p. 19]:

(a) A vector with 5 elements is formed by solving Equations 1-15 to 1-19.

$$l = 296^{\circ}\!.10460\,8 + 13^{\circ}\!.06499\,24465d + 0^{\circ}\!.00068\,90D^2 + 0^{\circ}\!.00000\,0295D^3 \qquad 1\text{-}15$$

$$l' = 358^{\circ}\!.47583\,3 + 0^{\circ}\!.98560\,02669d - 0^{\circ}\!.00001\,12D^2 - 0^{\circ}\!.00000\,0068D^3 \qquad 1\text{-}16$$

$$F = 11^{\circ}\!.25088\,9 + 13^{\circ}\!.22935\,04490d - 0^{\circ}\!.00024\,07D^2 - 0^{\circ}\!.00000\,0007D^3 \qquad 1\text{-}17$$

$$D = 350^{\circ}\!.73748\,6 + 12^{\circ}\!.19074\,91914d - 0^{\circ}\!.00010\,76D^2 + 0^{\circ}\!.00000\,0039D^3 \qquad 1\text{-}18$$

$$\Omega = 259^{\circ}\!.18327\,5 - 0^{\circ}\!.05295\,39222d + 0^{\circ}\!.00015\,57D^2 + 0^{\circ}\!.00000\,0046D^3 \qquad 1\text{-}19$$

where

$$d = 1000\ D = 36525\ T, \qquad 1\text{-}20$$

and where T is the number of Julian Ephemeris Centuries of 36525 days from the fundamental epoch (1900 January $0^{d}\!.5$ ET = J.E.D. 241 5020.0) to the epoch of the observation.

Table 1-1. Series for the Nutation

Period (days)	l	l'	F	D	Ω	LONGITUDE (Δψ) Coefficient of sine argument	Unit = 0.0001"	OBLIQUITY (Δε) Coefficient of cosine argument	
6798					+1	-172327	-173.7T	+92100	+9.1T
3399					+2	+ 2088	+ 0.2T	- 904	+0.4T
1305	-2		+2		+1	+ 45		- 24	
1095	+2		-2			+ 10			
6786		-2	+2	-2	+1	- 4		+ 2	
1616	-2		+2		+2	- 3		+ 2	
3233	+1	-1		-1		- 2			
183			+2	-2	+2	- 12729	- 1.3T	+ 5522	-2.9T
365		+1				+ 1261	- 3.1T		
122		+1	+2	-2	+2	- 497	+ 1.2T	+ 216	-0.6T
365		-1	+2	-2	+2	+ 214	- 0.5T	- 93	+0.3T
178			+2	-2	+1	+ 124	+ 0.1T	- 66	
206	+2			-2		+ 45			
173			+2	-2		- 21			
183		+2				+ 16	- 0.1T		
386		+1			+1	- 15		+ 8	
91		+2	+2	-2	+2	- 15	+ 0.1T	+ 7	
347		-1			+1	- 10		+ 5	
200	-2			+2	+1	- 5		+ 3	
347		-1	+2	-2	+1	- 5		+ 3	
212	+2			-2	+1	+ 4		- 2	
120		+1	+2	-2	+1	+ 3		- 2	
412	+1			-2		- 3			
13.7			+2		+2	- 2037	- 0.2T	+ 884	-0.5T
27.6	+1					+ 675	+ 0.1T		
13.6			+2		+1	- 342	- 0.4T	+ 183	
9.1	+1		+2		+2	- 261		+ 113	-0.1T
31.8	+1			-2		- 149			
27.1	-1		+2		+2	+ 114		- 50	
14.8				+2		+ 60			
27.7	+1				+1	+ 58		- 31	
27.4	-1				+1	- 57		+ 30	
9.6	-1		+2	+2	+2	- 52		+ 22	

Table 1-1. (Cont'd)

Period (days)	ARGUMENT Multiple of l	l'	F	D	Ω	LONGITUDE (Δψ) Coefficient of sine argument Unit = 0".0001	OBLIQUITY (Δ ε) Coefficient of cosine argument
9.1	+1		+2		+1	- 44	+ 23
7.1			+2	+2	+2	- 32	+ 14
13.8	+2					+ 28	
23.9	+1		+2	-2	+2	+ 26	- 11
6.9	+2		+2		+2	- 26	+ 11
13.6			+2			+ 25	
27.0	-1		+2		+1	+ 19	- 10
32.0	-1			+2	+1	+ 14	- 7
31.7	+1			-2	+1	- 13	+ 7
9.5	-1		+2	+2	+1	- 9	+ 5
34.8	+1	+1		-2		- 7	
13.2		+1	+2		+2	+ 7	- 3
9,6	+1			+2		+ 6	
14.8				+2	+1	- 6	+ 3
14.2		-1	+2		+2	- 6	+ 3
5.6	+1		+2	+2	+2	- 6	+ 3
12.8	+2		+2	-2	+2	+ 6	- 2
14.7				-2	+1	- 5	+ 3
7.1			+2	+2	+1	- 5	+ 3
23.9	+1		+2	-2	+1	+ 5	- 3
29.5				+1		- 4	
15.4		+1		-2		- 4	
29.8	+1	-1				+ 4	
26.9	+1		-2			+ 4	
6.9	+2		+2		+1	- 4	+ 2
9.1	+1		+2			+ 3	
25.6	+1	+1				- 3	
9.4	+1	-1	+2		+2	- 3	
13.7	-2				+1	- 2	
32.6	-1		+2	-2	+1	- 2	
13.8	+2				+1	+ 2	
9.8	-1	-1	+2	+2	+2	- 2	
7.2		-1	+2	+2	+2	- 2	
27.8	+1				+2	- 2	
8.9	+1	+1	+2		+2	+ 2	
5.5	+3		+2		+2	- 2	

(b) A 69 by 5 matrix is formed from the elements of the argument portion of Table 1-1.

(c) The vector in (a) above is then multiplied by the matrix in (b) which results in a new vector with 69 elements.

(d) The sine and cosine is taken of each of the elements in (c) and then multiplied by their corresponding coefficient (Table 1-1) evaluated at T.

(e) The sine and cosine terms are separately summed to obtain values of $\Delta\psi$ and $\Delta\varepsilon$ respectively.

1.32 Polar Motion Determination

The coordinates x and y of the instantaneous pole to be used in Equation 1-8 are published annually by the central bureau of the International Polar Motion Service [Yumi, 1965]. The values for x and y at the epoch of observation are obtained by a second difference interpolation using Bessel's formula.

1.4 Optical Data

Agencies involved in the reduction of optical satellite observations have their own specifications; use formulas of varying accuracy; and use different methods and techniques. Therefore, the observations sent to the NASA data bank may be inconsistent according to:

(1) The coordinate system used,

(2) The corrections applied,

(3) The time system employed.

These inconsistencies are removed according to the procedure given in [Hotter, 1967].

In order to exemplify the observational data used in the adjustment of simultaneous optical observations, three simultaneous events are tabulated in Table 1-2. Optical data is assumed to be in the form of topocentric right ascensions and declinations in the true celestial coordinate system and UT1 time system.

1.5 Range Data

Range data is treated as if only containing random errors. This means that no error models are present in the adjustment which would absorb any systematic errors. Systematic errors are assumed to be removed by the observing and/or processing agencies. It is recognized that at present this is not necessarily the case, thus it is likely that the range adjustment program will need to be revised to accommodate suitable error models [Gross, in prep.].

In order to exemplify the information used in the case of simultaneous range observations, three events are tabulated in Table 1-3.

Table 1-2. Simultaneous Optical Data

Simultaneous Event	Ground Station	Universal Time (UT 1)			Date	Right Ascension (α)			Declination (δ)			Standard Dev. R. A. (arc sec.)	Standard Dev. Dec. (arc sec.)	Covariance (arc sec.)^2
						h	m	s	°	'	"			
1	1	21	11	0.0000	4 Aug 63	12	6	24.5269	+ 5	26	11.7732	3.09	2.09	-0.002
1	2	21	11	0.0000	4 Aug 63	10	37	59.2645	- 0	12	20.7854	3.01	2.06	0.006
1	5	21	11	0.0000	4 Aug 63	10	3	35.9231	+33	22	18.3210	3.86	2.50	-0.018
1	6	21	11	0.0000	4 Aug 63	9	24	2.3857	+12	32	47.6900	3.50	2.73	0.022
2	2	21	12	0.0000	4 Aug 63	11	20	41.2215	+ 6	18	40.0611	3.02	2.61	0.030
2	3	21	12	0.0000	4 Aug 63	13	37	31.8662	+34	44	0.9713	3.01	2.03	0.034
3	4	21	13	0.0000	4 Aug 63	13	5	55.3844	+31	7	12.2932	3.73	2.50	0.062
3	5	21	13	0.0000	4 Aug 63	13	8	59.3949	+57	58	26.2258	3.85	2.71	-0.066
3	6	21	13	0.0000	4 Aug 63	10	44	30.4435	+33	57	8.0444	3.01	2.03	0.070

Table 1-3. Simultaneous Range Data

Simultaneous Event	Ground Station	Universal Time (UT 1)	Date	Topocentric Range (r) (meters)	Std. Dev. (meters)
1	1	21 11 0.0000	4 Aug 63	1473865.0485	10.0
1	2	21 11 0.0000	4 Aug 63	1754116.2758	10.0
1	3	21 11 0.0000	4 Aug 63	1196451.8892	10.0
1	4	21 11 0.0000	4 Aug 63	1393289.7925	10.0
1	5	21 11 0.0000	4 Aug 63	1261431.6142	10.0
2	3	21 12 0.0000	4 Aug 63	1234760.4490	10.0
2	4	21 12 0.0000	4 Aug 63	1245522.1957	10.0
2	5	21 12 0.0000	4 Aug 63	1175798.3971	10.0
2	6	21 12 0.0000	4 Aug 63	1443743.8483	10.0
3	1	21 13 0.0000	4Aug 63	1454435.2321	10.0
3	3	21 13 0.0000	4 Aug 63	1382387.7661	10.0
3	4	21 13 0.0000	4 Aug 63	1206081.1251	10.0

2. THE OPTICAL ADJUSTMENT

2.1 The Mathematical Structure

The adjustment method is by least squares, where the parameters are the three dimensional rectangular coordinates of the ground stations and satellite positions*, while the observables are the topocentric range*, and topocentric declination and right ascension of the satellite.

The mathematical structure relating the parameters and the observables is a function of three vectors. The three vectors as depicted in Figure 2-1 are (the arrow over the symbol will be reserved for those vectors which have a finite magnitude as opposed to say vectors containing differential corrections):

(1) $\vec{X}_i$, the coordinate system origin to ground station vector,

(2) $\vec{X}_j$, the coordinate system origin to satellite position vector.

(3) $\vec{X}_{ij}$, the ground station i to satellite position j vector.

Thus

$$\vec{X}_j - \vec{X}_i = \vec{X}_{ij} \qquad 2\text{-}1$$

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* Needed in the algebraic derivation but in fact, in the numerical computation, they are either not needed, or obtained to a sufficient accuracy from the observed quantities.

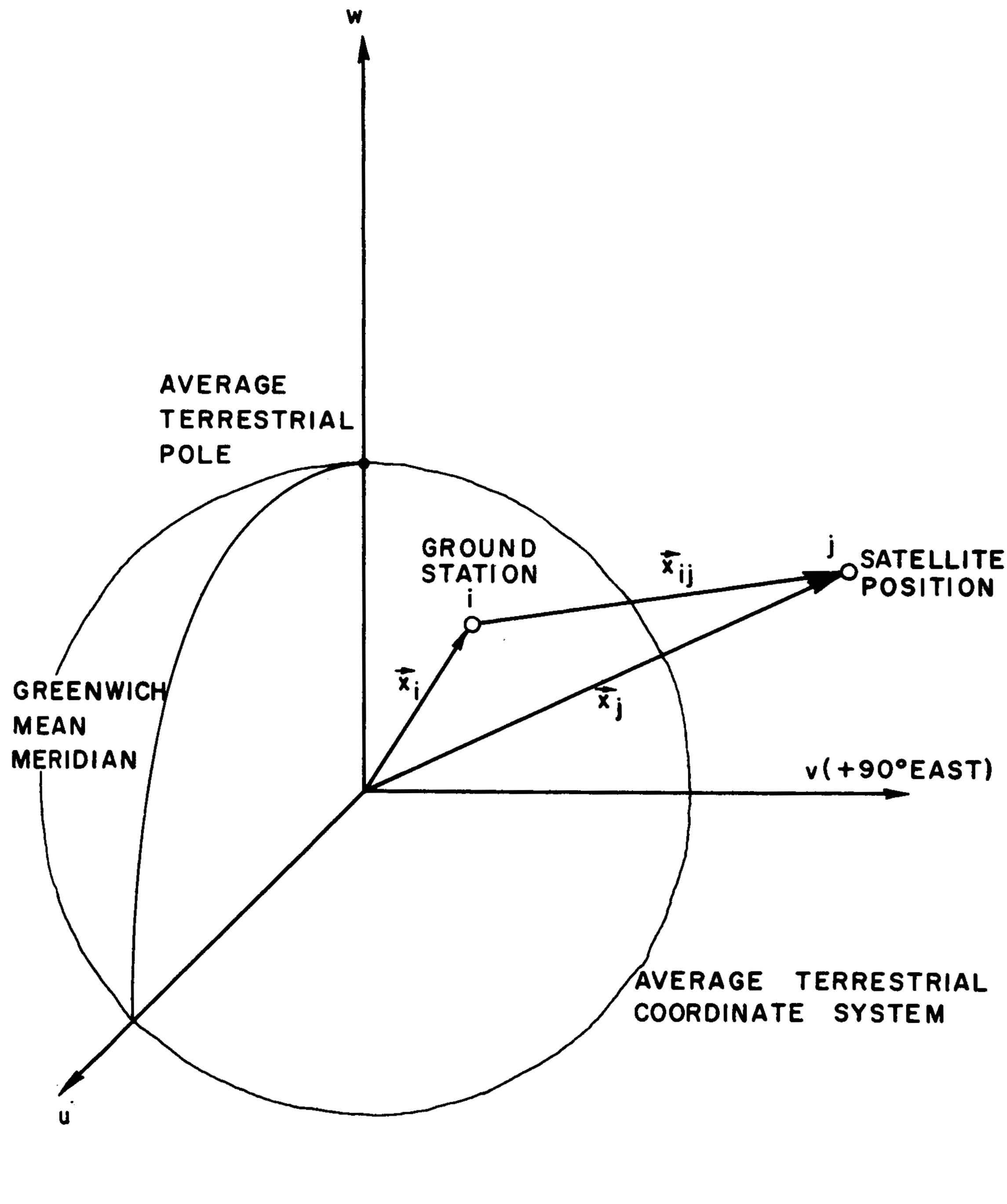

FIGURE 2-1. THE ADJUSTMENT COORDINATE SYSTEM.

or

$$F_{ij} = \vec{X}_j - \vec{X}_i - \vec{X}_{ij} = 0 \qquad 2\text{-}2$$

where

$$\vec{X}_j = \begin{vmatrix} u_j \\ v_j \\ w_j \end{vmatrix} \qquad 2\text{-}3$$

is a vector composed of the rectangular coordinates of an arbitrary satellite position;

$$\vec{X}_i = \begin{vmatrix} u_i \\ v_i \\ w_i \end{vmatrix} \qquad 2\text{-}3(a)$$

is a vector composed of the rectangular coordinates of an arbitrary ground station;

$$\vec{X}_{ij} = S \begin{vmatrix} r_{ij} & \cos\delta_{ij} & \cos\alpha_{ij} \\ r_{ij} & \cos\delta_{ij} & \sin\alpha_{ij} \\ r_{ij} & \sin\delta_{ij} & \end{vmatrix} , \qquad 2\text{-}4$$

r_{ij}, δ_{ij}, α_{ij} being the topocentric range, declination and right ascension from i to j, respectively, while S is the matrix which transforms the vector from the true celestial to the average terrestrial coordinate system (Section 1.3).

The point by point build-up of the network can be visualized in the following way. Given the components of the vectors $\vec{X}_i$ and $\vec{X}_{ij}$, $\vec{X}_j$ is computed. Then with this position j as known and a known vector from an unknown station to j, the coordinates of the unknown station are computed. This is extended to include many unknown and known stations, along with many redundant observations thereby necessitating an adjustment.

Strictly speaking, pure optical or range data does not permit such a procedure to be literally followed, however the adjustment framework (a form of colinearity) remains applicable.

2.2 The Linearized Form of the Mathematical Structure

The mathematical structure (Equation 2-2) is linearized by a Taylor series expansion about the preliminary values of the ground stations and satellite positions, and the observed topocentric values of the range, declination and right ascension. The result is (i and j designate ground and satellite points concerned and not dimensions of arrays)

$$A_{ij} X_{ij} + B_{ij} V_{ij} + W_{ij} = 0. \qquad 2\text{-}5$$

$$A_{ij} = \frac{\partial F_{ij}}{\partial \vec{X}_j, \vec{X}_i} = \begin{vmatrix} +1 & 0 & 0 & -1 & 0 & 0 \\ 0 & +1 & 0 & 0 & -1 & 0 \\ 0 & 0 & +1 & 0 & 0 & -1 \end{vmatrix} = [+I \mid -I] \; ; \qquad 2\text{-}6$$

$$X_{ij} = \begin{vmatrix} \vec{X}_j \\ \vec{X}_i \end{vmatrix} \quad , \qquad 2\text{-}7$$

where

$$X_j = \begin{vmatrix} du_j \\ dv_j \\ dw_j \end{vmatrix} \qquad 2\text{-}8$$

and

$$X_i = \begin{vmatrix} du_i \\ dv_i \\ dw_i \end{vmatrix} \qquad 2\text{-}9$$

are corrections to the preliminary values of the ground station and satellite position, respectively.

$$B_{ij} = \frac{\partial F_{ij}}{\partial \delta, \partial \alpha, \delta r} = S \; R_3 \; (-\alpha) \; R_2 \; (-90^{\circ} + \delta) \; C, \qquad 2\text{-}10$$

where S is defined by Equation 1-8, R_3 and R_2 are rotation matrices, and

$$C = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -\cos\delta & 0 \\ 0 & 0 & -1 \end{vmatrix} . \qquad 2\text{-}11$$

The matrix

$$\begin{vmatrix} r_{ij} & 0 & 0 \\ 0 & r_{ij} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

is omitted from the expression for B_{ij} since it is multiplied into

$$V_{ij} = \begin{vmatrix} \delta\delta_{ij} & \\ \delta\alpha_{ij} & \cos\,\delta_{ij} \\ \delta r_{ij} & \end{vmatrix} ,$$

namely,

$$V_{ij} = \begin{vmatrix} r_{ij}\delta\delta_{ij} & \\ r_{ij}\delta\alpha_{ij} & \cos\,\delta_{ij} \\ \delta r_{ij} & \end{vmatrix} ; \qquad 2\text{-}12$$

these are the residuals of the adjustment in units of meters ($\delta\delta$ and $\delta\alpha$ are in radians).

$$W_{ij} = \vec{X}^o_j - \vec{X}^o_i - \vec{X}^b_{ij} \qquad 2\text{-}13$$

is the evaluated mathematical structure where "o" designates "evaluated at preliminary values" and "b" designates "evaluated at observed values."

Up to now consideration has been given to the i^{th} and j^{th} ground and satellite positions. Extending this basic idea to include a redundant number of satellite positions observed from a multitude of known and unknown ground stations, the following matrix equation is built-up:

$$AX + BV + W = 0 \ , \qquad 2\text{-}14$$

where the original quantities A_{ij}, B_{ij}, etc. are simply submatrices of their corresponding unsubscripted counterparts.

2.3 Weighting the Declinations and Right Ascensions

The observed quantities in the optical case are considered as the topocentric declinations (δ) and right ascensions (α). The corresponding precision estimates resulting from a photographic plate adjustment or some other apriori estimate are m^2_δ and m^2_α, the variances, while $m_{\alpha\delta} \equiv m_{\delta\alpha}$ is the covariance. All units are arc seconds squared.

It is important to note that the weighting of the declinations and right ascensions is made on the basis of the estimates of variances of δ and α obtained from the plate adjustments and that it is assumed that the variance of δ and α do not vary according to the distance of the satellite from the particular observing ground station.

On the other hand, the weighted sum of squares of the residuals (Section 6.11) is conveniently chosen to have units of arc seconds squared, thus, the weights are to have units of (arc sec.)^2 m^{-2} since the units of the residuals have been stipulated (Equation 2-12) to be meters. Therefore, it is necessary to transform m_δ^2, m_α^2, and $m_{\delta\alpha}$ into linear units (meters) by the following formulas:

$$(m_\delta)^2 = \left| r \frac{m_\delta''}{\rho''} \right|^2 , \qquad 2\text{-}16$$

$$(m_\alpha)^2 = \left| r \frac{m_\alpha''}{\rho''} \right|^2 , \qquad 2\text{-}17$$

$$m_{\delta\alpha} = r^2 \frac{m_{\delta\alpha}^{('')2}}{(\rho'')^2} , \qquad 2\text{-}18$$

where r is the approximate topocentric range and

$$\rho'' = \frac{1}{\sin 1''} .$$

With the precision estimates in linear units the following variance-covariance matrix is formulated:

$$\Sigma_{\delta,\alpha,r} = \begin{vmatrix} \underline{m_\delta{}^2} & m_{\delta\alpha} & m_{\delta r} \\ & \underline{m_\alpha{}^2} & m_{\alpha r} \\ & & m_r{}^2 \end{vmatrix} \quad ,$$

where the new quantities m_r^2, $m_{\delta r}$, and $m_{\alpha r}$ are the variance of the range, covariance between the declination and range, and the covariance between the right ascension and range, respectively. If the correlation coefficients

$$\rho_{\delta r} = \frac{m_{\delta r}}{m_\delta m_r} = 0 \quad ,$$

$$\rho_{\alpha r} = \frac{m_{\alpha r}}{m_\alpha m_r} = 0 \quad ,$$

and

$$m_r \rightarrow \infty \quad ,$$

the weight matrix for a single direction is

$$P_{ij} = m_o^2 \begin{vmatrix} \begin{vmatrix} m_\delta^2 & m_{\delta\alpha} \\ m_{\alpha\delta} & m_\alpha^2 \end{vmatrix}^{-1} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 0 \end{vmatrix} \quad , \qquad 2\text{-}19$$

where m_0^2 is the apriori variance of unit weight in units of arc seconds squared.

Corresponding to P_{ij}, P denotes the weight matrix for the observed topocentric directions of the adjustment. P has the characteristic of containing non-zero 3 x 3 matrices only along the diagonal since the individual directions are assumed to be statistically independent.

The topocentric range is needed in Equations 2-16 to 2-18 to convert the precision of the directions from arc units into linear (meters) units. Four significant figures are required in the topocentric range. Equation 2-16 shows that the range need have no more significant figures than m''_δ or m''_α.

The topocentric range from an arbitrary ground station i in a given simultaneous event j is computed from

$$r_{ij} = [(u_j^o - u_i^o)^2 + (v_j^o - v_i^o)^2 + (w_j^o - w_i^o)]^{\frac{1}{2}} , \qquad 2\text{-}20$$

i = 1, 2, ..., m (number of stations in the event). u_i^o, v_i^o, w_i^o are the preliminary rectangular coordinates of the i^{th} ground station and are computed from

$$\vec{X}_i^o = \begin{vmatrix} u_i^o \\ v_i^o \\ w_i^o \end{vmatrix} = \begin{vmatrix} (N+H)\ \cos\phi\ \ \cos\lambda \\ (N+H)\ \cos\phi\ \ \sin\lambda \\ [N(1-e^2)+H]\ \sin\phi \end{vmatrix} , \qquad 2\text{-}21$$

ϕ, λ, H, N being the geodetic latitude and longitude, the ellipsoidal height, and prime vertical radius of curvature at point i, respectively, while e is the eccentricity of the reference ellipsoid. u_j^o, v_j^o, w_j^o are the preliminary rectangular coordinates of the j^{th} satellite position and are computed (note that these are needed only for the purpose of getting the approximate topocentric range) as follows:

(1) The ground vector, $\vec{X}_{12}$, between the first two stations listed in the particular simultaneous event, (see Table 1-2 and Figure 2-2) is computed according to

$$\vec{X}_{12} = \begin{vmatrix} u_2 - u_1 \\ v_2 - v_1 \\ w_2 - w_1 \end{vmatrix} . \qquad 2\text{-}22$$

(2) The ground station 1 to satellite position j unit vector $\vec{X}_{1j}$, is computed from

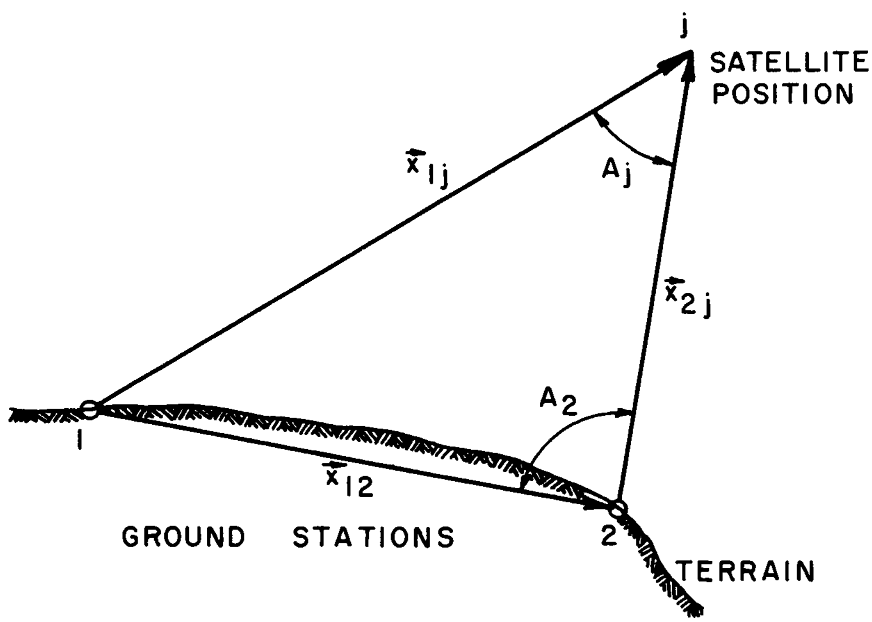

FIGURE 2-2. THE APPROXIMATE SATELLITE VECTOR.

$$\vec{X}_{1_j} = S \begin{vmatrix} \cos\delta \ \cos\alpha \\ \cos\delta \ \sin\alpha \\ \sin\delta \end{vmatrix} \qquad 2\text{-}24$$

where S is the transformation matrix of the true celestial to the average terrestrial coordinate systems (Section 1.3).

(3) The ground station 2 to satellite position j unit vector, $\vec{X}_{2_j}$, is computed as in (2).

(4) The angle, A_2, at ground station 2 is computed according to the following dot product:

$$\cos A_2 = \frac{\vec{X}_{21} \cdot \vec{X}_{2_j}}{|\vec{X}_{21}|\ (1)} \ . \qquad 2\text{-}25$$

(5) The angle, A_j, at the satellite position is computed from the following dot product:

$$\cos A_j = \frac{\vec{X}_{1_j} \cdot \vec{X}_{2_j}}{(1)\ (1)} \ . \qquad 2\text{-}26$$

(6) Finally the satellite position vector, $\vec{X}^o_j$, to be used in Equation 2-20 is computed from the following vector equation:

$$\vec{X}^o_j = \vec{X}^o_1 + r_{1_j} \vec{X}_{1_j} = \begin{vmatrix} u^o_j \\ v^o_j \\ w^o_j \end{vmatrix} , \qquad 2\text{-}27$$

where

$$r_{1_j} = |\vec{X}_{21}| \frac{\sin A_2}{\sin A_j} \quad . \qquad 2\text{-}28$$

2.4 Ground Stations as Constrained Quantities

In performing the adjustment, ground stations are fixed by one of the following two procedures. The first is simply to delete those rows and columns of the normal equations which belong to the ground stations in question; the second procedure is to over weight those particular ground stations.

Specifically, the quantities to be weighted are the rectangular coordinates u, v, w of the observing ground stations. The 3 x 3 weight matrix associated with the three dimensional rectangular coordinates of a particular ground station is denoted by P_k. The fixing of any ground station is achieved by specifying numerically large diagonal elements in P_k, thereby holding the coordinates of the ground station at its preliminary values.

P_1 is used to denote the matrix of weights, P_k, of all the weighted ground stations. Note, for the derivation to follow (Section 2.6) consider the satellite positions as also weighted with a weight matrix P_2, and further consider the matrix P_x as containing both P_1 and

P , namely

$$P_x = \begin{vmatrix} P_2 & 0 \\ 0 & P_1 \end{vmatrix} .$$

2.5 Spatial Chord Lengths

For the purpose of introducing scale, the spatial chord length between any two observing ground stations may be constrained at the value computed from their preliminary coordinates. This is most conveniently achieved by introducing the spatial chord length as a fictitious observation with a large weight [Uotila, 1967]. This procedure allows spatial chords to be treated as either observed or fixed by simply varying the weight.

The mathematical structure is

$$G_{kl} = [(u_l-u_k)^2 + (v_l-v_k)^2 + (w_l-w_k)^2]^{\frac{1}{2}} - L_{kl}, \qquad \text{2-29}$$

where the subscripts k and l refer to the two particular ground stations and not dimensions of arrays; L_{kl} is the numerical value at which the chord length is to be fixed.

The above mathematical structure is linearized by a Taylor's series expansion about the preliminary values of the ground station coordinates. The result is

$$C_{kl} X_{kl} - V_{kl} + D_{kl} = 0 , \qquad \text{2-30}$$

where

$$C_{kl} = \frac{\partial G_{kl}}{\partial u,\ \text{or}\ \partial v,\ \text{or}\ \partial w} , \qquad 2\text{-}31$$

$$= \left| \frac{u_l^o-u_k^o}{L_{kl}^o} , \frac{v_l^o-v_k^o}{L_{kl}^o} , \frac{w_l^o-w_k^o}{L_{kl}^o} , \;\middle|\; -\frac{u_l^o-u_k^o}{L_{kl}^o} , -\frac{v_l^o-v_k^o}{L_{kl}^o} , -\frac{w_l^o-w_k^o}{L_{kl}^o} \right| ,$$

2-32

$$= [\ T_l \mid T_k\ ] ; \qquad 2\text{-}33$$

$$X_{kl} = \begin{vmatrix} du_l \\ dv_l \\ dw_l \\ \hline du_k \\ dv_k \\ dw_k \end{vmatrix} \qquad 2\text{-}34$$

while

$$V_{kl} \simeq 0 \qquad 2\text{-}35$$

due to the large weight (see Equation 2-37);

$$D_{kl} = [\ (u_l^o-u_k^o)^2 + (v_l^o-v_k^o)^2 + (w_l^o-w_k^o)^2]^{\frac{1}{2}} - L_{kl}^o = 0 \qquad 2\text{-}36$$

according to the first equation of the section.

The weight applied to the constrained chord is

$$p_{kl} = \frac{m_o^2}{m_{kl}^2} \qquad 2\text{-}37$$

where m_o^2 is the variance of unit weight and m_{kl}^2 is the variance of the chord. A large weight results by stipulating a small enough value for m_{kl}^2.

Thus far consideration has been given to one spatial chord. The equation

$$CX - V_C + D = 0 \qquad 2\text{-}38$$

applies when more than one chord is constrained, thus the original quantities C_{kl}, D_{kl}, etc., are submatrices of their unsubscripted counterparts. P_C is used to represent the diagonal weight matrix of all constrained chords.

It will be convenient to defer a discussion of the contribution of the spatial chord constraint to the normal equations and to the sum of squares of weighted residuals to Sections 2.63 and 6.11, respectively.

2.6 The Normal Equations

2.61 Outline of Derivation

The normal equations are derived by minimizing the quadratic form

$$V'PV + X'P_xX$$

subject to the relation (Equation 2-14)

$$AX + BV + W = 0 \ .$$

Upon introduction of Lagrange multipliers K, the variation function is

$$\Phi = V'PV + X'P_xX - 2K'(AX + BV + W) \ , \qquad 2\text{-}39$$

where

V is the vector of residuals corresponding to the α's and δ's;

X is the vector of corrections to the preliminary ground and satellite positions;

P is the weight matrix for the α's and δ's;

P_x is the weight matrix for the ground and satellite positions;

A and B, and W are coefficient matricies and a constant vector, respectively, which were described in detail in Section 2.2.

Upon the differentiation of Equation 2-39 for the minimum condition [Uotila, 1967, p. 81], the expanded form of the normal equations becomes

$$\begin{vmatrix} -P_x & 0 & A' \\ 0 & -P & B' \\ A & B & 0 \end{vmatrix} \begin{vmatrix} X \\ V \\ K \end{vmatrix} + \begin{vmatrix} 0 \\ 0 \\ W \end{vmatrix} = 0 \; . \qquad 2\text{-}40$$

By a row and column transformation, the residual vector V is eliminated and the normal equations become

$$\begin{vmatrix} BP^{-1}B' & A \\ A' & -P_x \end{vmatrix} \begin{vmatrix} K \\ X \end{vmatrix} + \begin{vmatrix} W \\ 0 \end{vmatrix} = 0 \; . \qquad 2\text{-}41$$

Next the correlates are eliminated, thus resulting in

$$[A' \; (BP^{-1}B')^{-1} \; A + P_x] \; X + A' \; (BP^{-1}B')^{-1} \; W = 0 \; . \qquad 2\text{-}42$$

The following summation form of the non-zero 3 x 3 submatrices of the above equation is found by replacing the A, B, and P matrices with their expanded forms in terms of 3 x 3 submatrices (Equations 2-6, 2-10, and 2-19):

$$\left|\begin{array}{c|c} \sum_i (B_{ij}P_{ij}^{-1}B_{ij}')^{-1} + P_j & -(B_{ij}P_{ij}^{-1}B_{ij}')^{-1} \\ \hline -(B_{ij}P_{ij}^{-1}B_{ij}')^{-1} & \sum_j (B_{ij}P_{ij}^{-1}B_{ij}')^{-1} + P_i \end{array}\right| \left|\begin{array}{c} X_j \\ \hline X_i \end{array}\right| +$$

$$+ \left| \begin{array}{c} U_j = \\ \sum_i (B_{ij} P_{ij}^{-1} B'_{ij})^{-1} W_{ij} \\ \hline U_i = \\ -\sum_j (B_{ij} P_{ij}^{-1} B'_{ij})^{-1} W_{ij} \end{array} \right| = 0 \ , \qquad 2\text{-}43$$

where the non-zero 3 x 3 submatrices occur only on the diagonal and those ij 3 x 3 positions corresponding to a ground to satellite observation; $\sum_i$ indicates a summation over all ground stations observing satellite position j; $\sum_j$ indicates a summation over all satellite positions observed from ground station i. All summations contain only 3 x 3 and/or 3 x 1 matrices.

Elimination of X_s, the corrections to the satellite positions, from the above yields the following reduced normal equations:

$$N X_g + U_g = 0 \ , \qquad 2\text{-}44$$

where the X_g vector represents the unknown corrections to the preliminary rectangular coordinates of the ground stations; U_g is the constant vector; N is the coefficient matrix.

The coefficient matrix N is made up of 3 x 3 matrices. By letting

$$M_{ij}^{-1} = (B_{ij} P_{ij}^{-1} B_{ij}')^{-1} \qquad 2\text{-}45$$

$$= (B_{ij}^{-1})' \; P_{ij} B_{ij}^{-1} \qquad 2\text{-}46$$

in Equation 2-43, the expression for the 3 x 3 diagonal matrix corresponding to the k^{th} ground station is given by

$$N_{kk} = \sum_j M_{kj}^{-1} - \sum_j \{M_{kj}^{-1} \; (\sum_i M_{ij}^{-1})^{-1} M_{kj}^{-1}\} + P_k \; ; \qquad 2\text{-}47$$

Note the weight, P_j, for the j^{th} satellite position has been dropped in the second term of the above equation. The expression for the off diagonal 3 x 3 matrix corresponding to the k^{th} and the l^{th} ground stations is

$$N_{kl} = -\sum_j \{M_{kj}^{-1} \; (\sum_i M_{ij}^{-1})^{-1} \; M_{lj}^{-1}\} \quad , \qquad 2\text{-}48$$

where the summation $\sum_j$ is performed over all satellite events observed simultaneously from both ground stations k and l.

The constant vector of the normal equations (Equation 2-44) is made up of 3 x 1 vectors corresponding to each ground station. The vector, U_k, for the k^{th} ground station is given by

$$U_k = - \; (\sum_j M_{kj}^{-1} W_{kj}) + \sum_j \{M_{kj}^{-1} \; (\sum_i M_{ij}^{-1})^{-1} \; (\sum_i M_{ij}^{-1} W_{ij})\} \; , \qquad 2\text{-}49$$

where, according to Equation 2-13,

$$W_{ij} = \vec{X}_j^o - \vec{X}_i^o - \vec{X}_{ij}^b \ , \qquad 2\text{-}50$$

or

$$W_{ki} = \vec{X}_j^o - \vec{X}_k^o - \vec{X}_{kj}^b \ . \qquad 2\text{-}51$$

At first sight it seems that the preliminary coordinates of each satellite position are required, however substitution of Equations 2-50 and 2-51 into Equation 2-49 results in the cancelation or dropping out of terms containing $\vec{X}_j^o$ and the observed vector $\vec{X}_{ij}^b$ or $\vec{X}_{kj}^b$. Specifically,

$$U_k = -\sum_j \{M_{kj}^{-1} \ (\vec{X}_j^o - \vec{X}_k^o - \vec{X}_{kj}^b)\} +$$

$$+ \sum_j \{M_{kj}^{-1} \ (\sum_i M_{ij}^{-1})^{-1} \ [\sum_i M_{ij}^{-1}(\vec{X}_j^o - \vec{X}_i^o - \vec{X}_{ij}^b)\} \qquad 2\text{-}52$$

$$= -\sum_j \{M_{kj}^{-1}\vec{X}_j^o\} + (\sum_j M_{kj}^{-1}) \ \vec{X}_k^o + \sum_j \{M_{kj}^{-1}X_{kj}^b\} +$$

$$+ \sum_j \{M_{kj}^{-1} \ (\sum_i M_{ij}^{-1})^{-1} \ (\sum_i M_{ij}^{-1}\vec{X}_j^o)\} -$$

$$- \sum_j \{M_{kj}^{-1} \ (\sum_i M_{ij}^{-1})^{-1} (\sum_i M_{ij}^{-1}\vec{X}_i^o)\} -$$

$$- \sum_j \{M_{kj}^{-1} \ (\sum_i M_{ij}^{-1})^{-1} \ (\sum_i \ M_{ij}^{-1}\vec{X}_{ij}^b)\} \quad . \qquad 2\text{-}53$$

Terms 1 and 4 in the above cancel (i.e., $\vec{X}_j^o$, satellite coordinates drop out) because $\vec{X}_j^o$ can be factored out of $\sum_i$ in term 4 i.e.,

$$\sum_j \{M_{kj}^{-1} \left(\sum_i M_{ij}^{-1}\right)^{-1} \left(\sum_i M_{ij}^{-1}\right) \vec{X}_j^o\} = \left(\sum_j M_{kj}^{-1} \vec{X}_j^o\right) \qquad 2\text{-}54$$

which has an opposite sign to that of term 1. Terms 3 and 6 drop out because they are identically zero. This is because both terms contain products like

$$B_{ij}^{-1} \vec{X}_{ij}^b \text{ or } B_{kj}^{-1} \vec{X}_{kj}^b$$

where

$$B_{ij}^{-1} = C^{-1} R_2(90^o - \delta) R_3(\alpha) S'$$

and

$$S' \vec{X}_{ij}^b \equiv r_3'$$

where r_3 is the last row of the orthogonal matrix $R_2(90^o - \delta)\ R_3(\alpha)$; thus, taking into account the presence of C^{-1};

$$B_{ij}^{-1} \vec{X}_{ij}^b = \begin{vmatrix} 0 \\ 0 \\ -1 \end{vmatrix} ,$$

and since, in the optical adjustment P_{ij} has the form;

$$P_{ij} = \begin{vmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{vmatrix} ,$$

and using 2-46;

$$M_{ij}^{-1} \vec{X}_{ij}^{b} \equiv 0 \quad . \qquad 2\text{-}54a$$

The expression for the constant column becomes

$$U_k = \sum_j M_{kj}^{-1} \{\vec{X}_k^o - (\sum_i M_{ij}^{-1})^{-1} (\sum_i M_{ij}^{-1} \vec{X}_i^o)\} \quad . \qquad 2\text{-}55$$

In summary, the normal equations in the optical adjustment are formed by Equations 2-47, 2-48, and 2-55.

2.62 Weighted Ground Stations Contribution to the Normal Equations

In Section 2.4 the matter of fixing ground stations by weighting was discussed. Further, in Section 2.61 the normal equations pertaining to the ground stations were given. The weighting of ground stations is accomplished by the addition of P_k to Equation 2-47.

It is not until the other summations are completed that any consideration need be given as to how the ground stations are to be treated, namely:

(1) as parameters i.e.,

$$P_k = \begin{vmatrix} 0 & & \\ & 0 & \\ & & 0 \end{vmatrix} ,$$

(2) as fixed i.e., P_k having three numerically large diagonal elements. P_k is added to the 3 x 3 diagonal matrix of the k^{th} ground station in the reduced normal equation at the time of solution.

2.63 Spatial Chord Length Contribution to the Normal Equations

The normal equations (Equation 2-44) pertaining to the ground stations were derived in Section 2.61. The linearized spatial chord length equation (Equation 2-30) was derived in Section 2.5. The contribution of the latter to the normal equations may be found by first bordering the normal equations [Uotila, 1967, p. 74, Equation 194], thusly

$$\begin{vmatrix} N & C' \\ C & -P_c^{-1} \end{vmatrix} \begin{vmatrix} X \\ -K_c \end{vmatrix} + \begin{vmatrix} U \\ D \end{vmatrix} = 0 . \qquad 2\text{-}56$$

By transformation, the above is written as

$$\begin{vmatrix} -P_c^{-1} & C \\ C' & N \end{vmatrix} \begin{vmatrix} -K_c \\ X \end{vmatrix} + \begin{vmatrix} D \\ U \end{vmatrix} = 0 . \qquad 2\text{-}57$$

The elimination of K_c from the above yields

$$[N + C' P_c C] X + U + C' P_c D = 0 , \qquad 2\text{-}58$$

which is the most convenient formula from a programming standpoint. Specifically, a constrained spatial chord length between any two stations say k and l results in the following expressions:

$$T_k P_{kl} T_k' , \qquad 2\text{-}59$$

$$T_l P_{kl} T_l' , \qquad 2\text{-}60$$

$$T_k P_{kl} T_l' , \qquad 2\text{-}61$$

$$T_k' P_{kl} D_{kl} , \qquad 2\text{-}62$$

$$T' P_{kl} D_{kl} , \qquad 2\text{-}63$$

where all matrices in the above are defined in Section 2.5. The first three expressions in the above are 3 x 3 matrices and are added respectively to N_{kk}, N_{ll} (Equation 2-47) and N_{kl} (Equation 2-48); the last two expressions are added respectively to the constant columns U_k and U_l (Equation 2-55). Since $D_{kl} = 0$, according to Equation 2-36, there is no contribution to the constant column of the normal equations if the spatial chord is being constrained at the value computed from the preliminary values of the ground station coordinates.

2.7 Detection of Blunders in the Declinations and Right Ascensions, and/or Ground Station Coordinates

Blunders in the observed declinations and right ascensions and/or observing ground station coordinates are detected during the formation of the normal equations. The procedure used is to test the variance of unit weight that would result from a preliminary least square adjustment of each simultaneous event. In this adjustment the ground stations are held fixed. The residuals on the ijth observed α, δ pair from such a preliminary adjustment are the first two elements of the 3 x 1 vector

$$B_{ij}^{-1}\ (\vec{X}_i - \vec{X}_j^o)\quad \vec{X}_j^o = \{\sum_i M_{ij}^{-1}\}^{-1}\{\sum_i M_{ij}^{-1}\vec{X}_i\}$$

(the third element is the range to the preliminary adjusted satellite position), and therefore;

$$\sum_i V_{ij}' P_{ij}^{-1} V_{ij} = \sum_i (\vec{X}_i - \vec{X}_j^o)'\ M_{ij}^{-1}\ (\vec{X}_i - \vec{X}_j^o)$$

since the third element is dispensed within the product

$$P_{ij} B_{ij}^{-1}\ (\vec{X}_i - \vec{X}_j^o)$$

(see Equation 2-19).

Therefore;

$$m_o^2 = \frac{\sum_{\text{event}} (\vec{X}_i^o - \vec{X}_j^o)' \; M_{ij}^{-1} \; (\vec{X}_i^o - \vec{X}_j^o)}{2m-3} \quad , \qquad 2\text{-}64$$

where the numerator can be shown to be the sum square of the weighted residuals (arc seconds squared) of all the observed declinations and right ascensions in the event; m is the number of ground stations in the event.

If a number of rejected simultaneous events repeatedly contain a particular ground station, it is probably due to a blunder in the coordinates of the particular ground station rather than in the observed quantities. In this case, the preliminary coordinates of that ground station should be verified.

3. THE RANGING CASE ADJUSTMENT

3.1 The Mathematical Structure

The mathematical structure is

$$r_{ij} = [(u_j-u_i)^2 + (v_j-v_i)^2 + (w_j-w_i)^2]^{\frac{1}{2}} , \qquad 3\text{-}1$$

$$F_{ij} = [(u_j-u_i)^2 + (v_j-v_i)^2 + (w_j-w_i)^2]^{\frac{1}{2}}-r_{ij}= 0, \qquad 3\text{-}2$$

where the observable r_{ij} is the topocentric range from ground station i to satellite positon j; the parameters u_i, v_i, w_i and u_j, v_j, w_j are the three dimensional rectangular coordinates of the ground station i and satellite position j, respectively.

The basic mathematical structure above is extended to include simultaneous ranges from three or more ground stations. By increasing the number of simultaneous events along with the number of known and unknown ground stations, an adjustment is necessary.

3.2 The Linearized Form of the Mathematical Structure

The mathematical structure (Equation 3-2) is linearized by a Taylor series expansion about the preliminary values of the ground stations and satellites positions, and the observed value of the topocentric range. The result is (i and j refer to ground and satellite points and not to dimensions of arrays)

$$A_{ij}\, X_{ij} - v_{ij} + L_{ij} = 0 \;, \qquad 3\text{-}3$$

where

$$A_{ij} = \frac{\partial F_{ij}}{\partial u, \partial v, \partial w} = \left| \frac{u_j^o - u_i^o}{r_{ij}^b} \;,\; \frac{v_j^o - v_i^o}{r_{ij}^b} \;,\; \frac{w_j^o - w_i^o}{r_{ij}^b} \;\middle\vdots\; -\frac{u_j^o - u_i^o}{r_{ij}^b} \;,\; -\frac{v_j^o - v_i^o}{r_{ij}^b} \;,\; -\frac{w_j^o - w_i^o}{r_{ij}^b} \right| \;, \qquad 3\text{-}4$$

$$= [\, a_{ij} \mid -a_{ij} \,] \;, \qquad 3\text{-}5$$

where the superscripts "o" and "b" indicate preliminary quantities and observed values, respectively;

$$X_{ij} = \begin{vmatrix} X_j \\ X_i \end{vmatrix} \;,$$

where

$$X_j = \begin{vmatrix} du_j \\ dv_j \\ dw_j \end{vmatrix} \;,$$

and

$$X_i = \begin{vmatrix} du_i \\ dv_i \\ dw_i \end{vmatrix} \quad ;$$

v_{ij} is the residual of the adjustment in meters corresponding to the observed range r^b_{ij};

$$L_{ij} = r^o_{ij} - r^b_{ij} \qquad 3\text{-}6$$

is the difference between the preliminary range and the observed range.

Up to now consideration has been given to the i^{th} and j^{th} ground and satellite positions. Extending this to many positions, the following matrix equation is built-up:

$$AX - V + L = 0 \ , \qquad 3\text{-}7$$

where the original quantities A_{ij}, v_{ij}, etc., are subsets of their unsubscripted counterparts.

3.3 Weighting the Observed Ranges

The weighting of the observed topocentric range from ground station i to satellite position j is achieved by the following:

$$p_{ij} = \frac{m_o^2}{m_{ij}^2} \qquad 3\text{-}8$$

where m_o^2 is the variance of unit weight in units of meters squared and similarly m_{ij}^2 is the variance of the observed range in meters squared.

P denotes the diagonal weight matrix containing all the independent weights p_{ij} to be considered in the adjustment.

3.4 Weighting the Ground Stations

Weighting allows the ground stations to be treated as pure parameters or as fixed quantities. These aspects were treated in the optical case adjustment (Section 2.4) and can equally as well be used in the ranging adjustment.

3.5 Spatial Chord Lengths as Constrained Quantities

Spatial chord lengths between ground stations in the case of range adjustments are treated in the same way as in the optical case adjustment (Section 2.5). In fact, note that in the range adjustment, the spatial chord contribution is identical with that of a range observation except that two ground stations are involved rather than one ground and one satellite point.

3.6 The Normal Equations

3.61 Outline of Derivation

The variation function for the range adjustment is similar to the optical case, namely,

$$\Phi = V'PV + X'P_xX - 2K'(AX - V + L) \quad , \qquad 3\text{-}9$$

where

V is the vector of residuals corresponding to the range observations;

X is the vector of corrections to the preliminary ground and satellite positions;

P is the weight matrix for the ranges;

P_x is the weight matrix for the ground and satellite positions*;

K' is the vector of correlates;

The coefficient matrix A and the constant vector L were described in Section 3.2.

The differentiation of Equation 3-9 for the minimum condition results in the following expanded form of the normal equations:

$$\begin{vmatrix} -P_x & 0 & A' \\ 0 & -P & -I \\ A & -I & 0 \end{vmatrix} \begin{vmatrix} X \\ V \\ K \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ L \end{vmatrix} = 0 \ . \qquad 3\text{-}10$$

* As in the case of the optical adjustment, satellite positions and their corresponding weights were included for derivational purposes only.

After the elimination of the correlates and residuals,

$$A'PAX + A'PL = 0 \ , \qquad 3\text{-}11$$

which is nothing else but the normal equations corresponding to the variation of parameters method of least squares adjustment [Uotila, 1967]. As a result of replacing the A and P matrices with their expanded forms in terms of 1 x 3 vectors and 1 x 1 terms, respectively (Equations 3-5 and 3-8), the following summation form results:

$$\left|\begin{array}{c|c} \sum_i a'_{ij} p_{ij} a_{ij} + P_j & a'_{ij} p_{ij} a_{ij} \\ \hline a'_{ij} p_{ij} a_{ij} & \sum_j a'_{ij} p_{ij} a_{ij} + P_i \end{array}\right| \left|\begin{array}{c} X_j \\ \hline X_i \end{array}\right| + \left|\begin{array}{c} U_j = \\ \sum_i a'_{ij} p_{ij} L_{ij} \\ \hline U_j = \\ -\sum_j a'_{ij} p_{ij} L_{ij} \end{array}\right| = 0 \ ,$$

3-13

where $\sum_i$ indicates a summation over all ground stations observing satellite position j; $\sum_j$ indicates a summation over all satellite positions observed from ground station i. All summations are a function of scalars and three dimensional vectors; recall

$$a_{ij} = \left[\frac{u_j^o - u_i^o}{r_{ij}^b} \quad \frac{v_j^o - v_i^o}{r_{ij}^b} \quad \frac{w_j^o - w_i^o}{r_{ij}^b} \right] \ . \qquad 3\text{-}5$$

Elimination of the corrections to the preliminary coordinates of the satellite position, namely X_j from

Equation 3-13, results in the following three expressions: the 3 x 3 diagonal matrix corresponding to the k^{th} ground station is given by

$$N_{kk} = (\sum_j a'_{kj} p_{kj} a_{kj}) - \sum_j \{a'_{kj} p_{kj} a_{kj} (\sum_i a'_{ij} p_{ij} a_{ij})$$

$$a'_{kj} p_{kj} a_{kj}\} + P_k \quad . \qquad 3\text{-}14$$

the 3 x 3 off diagonal matrix corresponding to the k^{th} and the l^{th} ground stations is given by

$$N_{kl} = -\sum_j \{a'_{kj} p_{kj} a_{kj} \; (\sum_i a'_{ij} p_{ij} a_{ij})^{-1} a'_{lj} p_{lj} \, a_{lj}\} \; , \qquad 3\text{-}15$$

where the main summation $\sum_j$ is performed over all satellite positions observed simultaneously from both ground stations k and l; the constant vector of the k^{th} ground station is

$$U_k = -(\sum_j a'_{kj} p_{kj} L_{kj}) - \sum_j \{a'_{kj} p_{kj} a_{kj} (\sum_i a'_{ij} p_{ij} a_{ij})^{-1} \sum_i a'_{ij} p_{ij} L_{ij}\} \; . \qquad 3\text{-}16$$

In the above expressions, the weight matrix P_j of each satellite position was set equal to zero as there is no independent external source from which to get apriori variance estimates which could be used to derive weights.

The equivalent expression for the constant column U_k can be shown to have the following form:

$$U_k = -\sum_j a'_{kj} p_{kj} \bar{v}_{kj} \quad , \qquad 3\text{-}17$$

where $\bar{v}_{kj}$ is the residual of the particular observed range r_{kj} arising from a least squares adjustment of one simultaneous event with ground stations held fixed (see Section 3.7).

Computation of the components in Equations 3-14, 3-15, and 3-17 are discussed in Section 3.71.

3.62 Weighted Ground Station Contribution to the Normal Equations

The weighting of the ground stations in the ranging case is analogous to that of the optical case (Section 2.62). The weighting of the ground stations is accomplished by the addition of P_k to Equation 3-14.

3.63 Spatial Chord Length Contribution to the Normal Equations

As for the derivation in the optical adjustment, the range normal equations pertaining to the ground stations are bordered with the chord condition and then algeraically eliminated from the augmented system, thus determining their contribution to the normals (Section 2.63).

3.7 Detection of Blunders in the Observed Ranges and/or Ground Station Coordinates

Blunders in the observed topocentric ranges and/or ground station coordinates are detected during the formation of the normal equations. The procedure used is to test the variance of unit weight (Equation 3-26) arising from a preliminary least squares adjustment of each simultaneous event.

The preliminary adjustment is basically an iterative adjustment for the u_j, v_j, w_j rectangular coordinates of the satellite position by fixing the ground stations and applying the residuals of the adjustment to the observed ranges. The approximation to the parameters u_j, v_j, w_j is obtained by converting the so-called approximate geodetic coordinates of the satellite into rectangular coordinates by use of Equation 2-21. The approximate geodetic coordinates of the satellite are obtained by meaning the latitudes and longitudes of the ground stations involved in the simultaneous event and assuming a value of 1.6 megameters for the ellipsoidal height of the satellite. The idea that the above is crude is immediately rejected upon the knowledge that at most four iterations (to a tolerance of 1 cm in u_j, v_j, w_j) are required and that the electronic computer IBM 7094 performs these iterations somewhat more quickly than the time necessary to solve the corresponding simultaneous, exact, second order equations.

The equation giving the mathematical structure of this preliminary adjustment is identical to Equation 3-1, the mathematical structure for the main range adjustment. Since only three parameters are involved, the linearized form of the mathematical structure for m ground stations in one simultaneous event becomes

$$A\,X - \overline{V} + L = 0 \;, \qquad 3\text{-}18$$

where the coefficient matrix

$$A = \begin{vmatrix} u_j^o - u_1^o & v_j^o - v_1^o & w_j^o - w_1^o \\ u_j^o - u_2^o & v_j^o - v_2^o & w_j^o - w_2^o \\ \vdots & \vdots & \vdots \\ u_j^o - u_k^o & v_j^o - v_k^o & w_j^o - w_k^o \\ \vdots & \vdots & \vdots \\ u_j^o - u_m^o & v_j^o - v_m^o & w_j^o - w_m^o \end{vmatrix} \;; \qquad 3\text{-}19$$

the correction vector for the satellite coordinates

$$X = \begin{vmatrix} du_j \\ dv_j \\ dw_j \end{vmatrix} \;; \qquad 3\text{-}20$$

the residual vector for the ranges

$$\bar{v} = \begin{vmatrix} \bar{v}_{1j} \\ \bar{v}_{2j} \\ \vdots \\ \bar{v}_{kj} \\ \vdots \\ \bar{v}_{mj} \end{vmatrix} , \qquad 3\text{-}21$$

and the constant vector

$$L = \begin{vmatrix} r^o_{1j} - r^b_{1j} \\ r^o_{2j} - r^b_{2j} \\ \vdots \quad \vdots \\ r^o_{mj} - r^b_{mj} \end{vmatrix} , \qquad 3\text{-}22$$

where r^o_{1j} and r^b_{1j} are preliminary and observed ranges, respectively.

The normal equations

$$NX + U = 0 , \qquad 3\text{-}23$$

where

$$N = A'PA \qquad 3\text{-}24$$

and

$$U = A'PL \qquad 3\text{-}25$$

are solved for X by iteration until the elements of the vector X are less than 1 cm. At this point, X is entered into Equation 3-18 and the vector of residuals $\bar{V}$ is determined; the variance of unit weight is then computed according to

$$m_o^2 = \frac{\bar{V}'P\bar{V}}{m-3} \quad . \tag{3-26}$$

The complete set of data for the simultaneous event is printed out for evaluation in the case that the particular m_o^2 is greater than a chosen input value. At the same time, no contribution is made to the normal equations by the rejected event.

3.71 Additional Benefits of the Preliminary Simultaneous Event Adjustment

The quantities a_{kj} and $\bar{v}_{kj}$ needed in the formation of the normal equations (Equations 3-14, 3-15, and 3-17) are a side product of the preliminary adjustment of each simultaneous event. Specifically, a_{kj} is contained in the A matrix given by Equation 3-19, and $\bar{v}_{kj}$ is an element of the $\bar{V}$ vector of Equation 3-21.

4. ADDITION OF NORMAL EQUATIONS

Independent sets of normal equations formed from two or more batches of optical data can be added together. The basic idea of the combination of the normal equations is simply the algebraic addition of their corresponding terms. Letting n sets of normal equations be represented by

$$N_1 X + U_1 = 0 \;, \qquad 4\text{-}1$$

$$N_2 X + U_2 = 0 \;, \qquad 4\text{-}2$$

$$\vdots$$

$$N_n X + U_n = 0 \;,$$

and their corresponding variances of unit weight as m_1^2, $m_2^2, \cdots, m_n^2$; the addition is

$$(N_1 + p_{12} N_2 + \cdots + p_{1n} N_n) X + (U_1 + p_{12} U_2 + \cdots p_{1n} U_n) = 0. \qquad 4\text{-}3$$

In the above, the weights may be obtained as follows:

$$p_{12} = \frac{m_1^2}{m_2^2} \qquad 4\text{-}4$$

$$\vdots$$

$$p_n = \frac{m_1^2}{m_n^2}$$

where m_1^2, m_2^2, $\cdots$ m_n^2 must have the same apriori variance (see Sections 2.3 and 3.3) of unit weight.

The advantage of the above is obvious, namely, batches of observed data may be adjusted separately or as a part of a combined adjustment.

The same holds for the addition of two or more independent sets of range normal equations. The possibility for the addition of optical and range normal equations to each other is also possible.

5. SOLUTION OF NORMAL EQUATIONS AND FORMATION OF THE INVERSE WEIGHT MATRIX

5.1 Introduction

The normal equations for the optical and range adjustments are given in Sections 2 and 3, respectively. The general form of the normal equations is

$$NX + U = 0 \ , \qquad 5\text{-}1$$

where N is the coefficient matrix, X is the vector of unknowns, and U is the constant vector.

The adjusted values of the three dimensional rectangular coordinates of the observing ground stations are obtained by adding the corrections, X, to the preliminary values, X^o, namely

$$X^a = X^o + X \ . \qquad 5\text{-}2$$

The precision estimate of X^a is obtained in the usual manner (Section 6) i.e., through the inverse weight matrix, N^{-1}. For this reason the method of formation of N^{-1} will be shown in this section along with the method of solving for X.

The procedure used here [Uotila, 1967, pp. 22-23] to accomplish the above is a Gauss reduction (Section 5.2) and back solution (Section 5.3), and computation of the inverse by the method established by Banachiewich (Section 5.3).

Two features which are peculiar to the specific procedure used here are:

(1) The coefficient matrix, N, is broken down into 3 x 3 submatrices, and similarly the U vector is treated as composed of 3 x 1 vectors.

(2) The coefficient matrix, N, is compacted so that 3 x 3 zero submatrices are neither stored nor used in the computation.

The first feature is achieved rather naturally; it is because of the form of expressions N_{kk}, N_{kl}, and U_k (Equations 2-47, 2-48, 2-55, 3-14, 3-15, 3-17) which are used to build-up N and U. On the other hand, the second feature is achieved through programming logic. Specifically, a first matrix, L, is used to tag each 3 x 3 non-zero submatrix of N with a row and column number. A second matrix, F, with a one-to-one correspondence to the first, is then employed to tag the storage assigned to the particular 3 x 3 submatrix. The individual elements of the 3 x 3 submatrices are all stored in one large linear array, E.

For example consider

$$L = \begin{array}{c|ccccc|} (1) & 2 & 3 & & & \\ (2) & 3 & 5 & 7 & 9 & \\ (3) & 4 & 5 & 6 & 7 & 8 \\ (4) & 7 & 8 & & & \\ (5) & 5 & 7 & 8 & & \\ (6) & 7 & 8 & & & \\ (7) & 8 & & & & \\ (8) & & & & & \end{array} \qquad 5\text{-}3$$

as depicting 8 ground stations (listed along the left-hand side of the matrix) involved in a series of simultaneous events. The information is read as follows: Ground station (1) has at some time been involved in simultaneous event(s) with ground stations 2 and 3; Ground station (2) has been involved with 3, 5, 7, and 9; and so on. So for L(3,5) = 8, the 9 elements beginning with cell E(F (3,5)) are the elements of N_{38}, the 3 x 3 non-zero submatrix on row 3 column 8 of the coefficient matrix, N (Equation 5-1).

The reduced elements of N are stored in the locations previously created for elements in N. During reduction additional 3 x 3 matrices arise in locations where there were none originally in N, thus "drag storage" must be assigned. In doing so, the guide matrix L, and the storage tagging matrix F are updated to account for these additional matrices. Similar drag storage is also determined during the formation of the inverse N^{-1}.

Once the drag storage is determined, the reduction, back solution and inverse determinations are guided by L, the storage located by F, and the elements to be used in the computation found in E.

5.2 Reduction

The coefficient matrix of the normal equations is written as

$$N = SR \,, \qquad 5\text{-}4$$

where S is a lower triangular matrix with 3 x 3 identity matrices along the diagonal, and R is an upper triangular matrix. All matrices and vectors presented in this discussion are stipulated to be composed of 3 x 3 submatrices and 3 x 1 submatrices, respectively.

The reduction is accomplished by computing

$$S = I - T \qquad 5\text{-}5$$

from

$$N = R - TR \,, \qquad 5\text{-}6$$

or

$$R = N + TR \,, \qquad 5\text{-}7$$

where R and T (thus S) are built-up simultaneously.

The augmented matrix

$$[N,U] = \begin{vmatrix} n_{11} & n_{12} & n_{13} & \cdots & n_{1n} & u_1 \\ n'_{12} & n_{22} & n_{23} & & n_{2n} & u_2 \\ n'_{13} & n'_{23} & n_{33} & \cdots & n_{3n} & u_3 \\ n'_{14} & & \cdot & & & u_4 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & u_{n-1} \\ n'_{1n} & & & & n_{nn} & u_n \end{vmatrix} \qquad 5\text{-}8$$

is first reduced according to the algorithms

$$\bar{n}_{ij} = \bar{n}_{ij} - \bar{n}'_{ki}\,\bar{n}^{-1}_{kk}\,\bar{n}_{kj}\,, \qquad 5\text{-}9$$

$$k = 1,2,\cdots,n$$

$$i = k+1,\ k+2,\cdots,n$$

$$j = i,\ i+1,\cdots,n$$

defining

$$R = \begin{vmatrix} n_{11} & n_{12} & \cdot & \cdot & \cdot & n_{1n} \\ & \bar{n}_{22} & \bar{n}_{23} & \cdot & \cdot & \bar{n}_{2n} \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & & \cdot \\ & & & & & \bar{n}_{nn} \end{vmatrix}$$

and

$$\bar{u}_i = \bar{u}_i - \bar{n}_{ki}^{-1}\ \bar{n}_{kk}^{-1}\ \bar{u}_k, \qquad k = 1,2,\cdots,n$$
$$i = k+1,\cdots,n$$

defining

$$C = \begin{vmatrix} u_1 \\ \bar{u}_2 \\ \bar{u}_3 \\ \cdot \\ \cdot \\ \cdot \\ \bar{u}_n \end{vmatrix} . \qquad 5\text{-}10$$

A second algorithim (performed as part of Equation 5-9) namely

$$\bar{\bar{n}}_{ij} = I \qquad j = i, \qquad 5\text{-}11$$

$$\bar{\bar{n}}_{ij} = \bar{n}_{ii}^{-1}\ \bar{n}_{ij} \qquad j = i+1,\ i+2,\cdots,n, \qquad 5\text{-}12$$

$$\bar{\bar{u}}_i = \bar{n}_{ii}^{-1}\ \bar{u}_i \qquad i = 1,2,\cdots,n, \qquad 5\text{-}13$$

results in the following reduced matrices:

$$S' = \begin{vmatrix} I & \bar{\bar{n}}_{12} & \bar{\bar{n}}_{13} & \cdot & \cdot & \cdot & \bar{\bar{n}}_{1n} \\ 0 & I & \bar{\bar{n}}_{23} & & & & \bar{\bar{n}}_{2n} \\ 0 & 0 & I & \cdot & & & \\ \cdot & \cdot & \cdot & & \cdot & & \\ \cdot & \cdot & \cdot & & & & \\ 0 & 0 & 0 & & & 0 & I \end{vmatrix} \quad ; \qquad 5\text{-}14$$

$$-D = \begin{vmatrix} \bar{\bar{u}}_1 \\ \bar{\bar{u}}_2 \\ \bar{\bar{u}}_3 \\ \cdot \\ \cdot \\ \cdot \\ \bar{\bar{u}}_n \end{vmatrix} , \qquad 5\text{-}15$$

(S' and D used to obtain solution vector, X, - Section 5.3);

$$R^{-1} = \begin{vmatrix} \bar{n}_{11}^{-1} & & & \text{elements above} \\ & \bar{n}_{22}^{-1} & & \text{diagonal} \\ & & \bar{n}_{33}^{-1} & \\ \text{zeros} & & & \\ & \text{below} & & \\ & & \text{diagonal} & \\ & & & \bar{n}_{nn}^{-1} \end{vmatrix} \qquad 5\text{-}16$$

(used to obtain inverse - Section 5.4).

5.3 Back Solution

The back solution involves the determination of the unknown vector X from elements of the reduced matrices S' and D. Without derivation [Uotila, 1967, p. 28],

$$X = T'X - D , \qquad 5\text{-}17$$

recall

$$T = I - S' ,$$

or in summation form

$$x_i = \sum_{k = i+1}^{n} \bar{\bar{n}}_{ik} x_k + \bar{\bar{u}}_i . \qquad 5\text{-}18$$

5.4 Formation of Inverse

The inverse weight matrix, N^{-1}, will be computed by the method associated with the name of Banachiewich [Uotila, 1967, p. 31]. According to Equation 5-4, N^{-1}, can be computed from

$$N^{-1} = R^{-1} S^{-1} , \qquad 5\text{-}19$$

however, it turns out that N^{-1} can be formed without the aid of S^{-1} and further only the diagonal elements of R^{-1} are needed. The diagonal elements of R^{-1} are readily available since the inverse of an upper triangular matrix has as its diagonal elements the reciprocal of the diagonal elements of the triangular matrix itself and an exactly similar result holds if "elements is taken to mean 3 x 3." The diagonal elements of R^{-1} are computed by inverting the 3 x 3 diagonal matrices of R, and for computer space saving reasons are stored along the diagonal of S' (Equation 5-14).

From Equation 5-19

$$R^{-1} = N^{-1} S \ , \qquad 5\text{-}20$$

and further substituting in for S from Equation 5-5,

$$R^{-1} = N^{-1} (I - T) \ , \qquad 5\text{-}21$$

$$= N^{-1} - N^{-1} T \ , \qquad 5\text{-}22$$

and finally

$$N^{-1} = R^{-1} + N^{-1} T \ . \qquad 5\text{-}23$$

The corresponding summation equation for computing any 3 x 3 matrix of N^{-1} is

$$n^{ij} = \sum_{k = i+1}^{n} \bar{\bar{n}}_{ik} n^{kj} + \delta_{ij} \bar{n}_{ii}^{-1} \ , \qquad 5\text{-}24$$

taking into account that

$$\delta_{ij} = 0 \qquad \text{for } i \neq j \text{ ,} \qquad \text{5-25}$$

$$\delta_{ij} = 1 \qquad \text{for } i = j \text{ ,} \qquad \text{5-26}$$

and

$$n^{ij} = (n^{ji})' \text{ .} \qquad \text{5-27}$$

6. PRECISION OF GROUND STATIONS AFTER ADJUSTMENT

6.1 Variance of Unit Weight

The variance of unit weight for the total adjustment is given by the following expression:

$$m_o^2 = \frac{V' \; PV}{df} \;, \qquad 6\text{-}1$$

where V' PV is the sum of the squares of the weighted residuals of all observed quantities and df is the number of degrees of freedom in the least squares adjustment.

6.11 Optical Adjustment

Equation 6-1 will now be considered for the optical adjustment. The linearized mathematical structure according to Section 2.2, was shown to be of the form

$$AX + BV + W = 0 \;, \qquad 6\text{-}2$$

and

$$CX - V_C + D = 0 \;, \qquad 6\text{-}3$$

resulted from the spatial chord constraint of Section 2.5.

The corresponding expression for the computation of V' PV with a change in notation and the deletion of condition equations is [Uotila, 1967, p. 75, Equation 200]

$$V' \ PV = -W'K - D'K_C \ , \qquad 6\text{-}4$$

where the first term is the contribution from Equation 6-2, and the second term is the contribution from Equation 6-3. Since D = 0 (Equation 2-36),

$$V' \ PV = -W'K \ , \qquad 6\text{-}5$$

and by obtaining an expression for K from Equation 2-41,

$$V' \ PV = W' \ (BP^{-1}B')^{-1} \ (AX + W) \ , \qquad 6\text{-}6$$

or denoting

$$M = BP^{-1} \ B' \ ;$$

$$V' \ PV = W' \ M^{-1} \ AX + W' \ M^{-1} \ W \ , \qquad 6\text{-}7$$

and since;

$$X = - \{A' \ M^{-1} \ A + P_X\}^{-1} A' M^{-1} W$$

from Equation 2-42

$$V' \ PV = W'M^{-1}W - W'M^{-1}A \ \{A'M^{-1}A+P_X\}^{-1}A'M^{-1}W \ . \qquad 6\text{-}8$$

Note that $A'M^{-1}A+P_X$ and $A'M^{-1}W$ are the coefficient matrix and constant column of Equations 2-42 and 2-43.

Let the partitioning of Equation 2-43 be denoted as

$$\begin{vmatrix} A & B \\ B' & C \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} + \begin{vmatrix} U_1 \\ U_2 \end{vmatrix} = 0 \quad ,$$

then, using;

$$\begin{vmatrix} A & B \\ B' & C \end{vmatrix}^{-1} = \begin{vmatrix} A^{-1}+A^{-1}BEB'A^{-1} & -A^{-1}BE \\ -EB'A^{-1} & E \end{vmatrix}$$

$$E = (C - B'A^{-1}B)^{-1}$$

the second term of Equation 6-8 becomes;

$$Q_1 = W'M^{-1}A\{A'M^{-1}A+P_x\}^{-1}A'M^{-1}W = U_1'A^{-1}U_1+(U_2-B'A^{-1}U_1)' *$$

$$* \quad E(U_2-B'A^{-1}U_1)$$

but by the elimination that led to Equation 2-44 we see that $E = N^{-1}$ and $U_g = U_2-B'A^{-1}U_1$

therefore;

$$Q = U_1'A^{-1}U_1+U_g'N^{-1}U_g$$

or

$$Q = U_1'A^{-1}U_1-U_g'X_g$$

and

$$V' PV = W'M^{-1}W - U_1'A^{-1}U_1+U_g'X_g \quad .$$

Denote $Q_2 = W'M^{-1}W - U_1'A^{-1}U_1$

or, by considering Equation 2-43, this becomes;

$$Q_2 = \sum_{ij} W_{ij}M_{ij}^{-1}W_{ij} - \sum_j\{\sum_i M_{ij}^{-1}W_{ij}\}'\{\sum_i M_{ij}^{-1}\}^{-1}\{\sum_i M_{ij}^{-1}W_{ij}\} \quad .$$

Now using Equations 2-50, 2-54, and factorization and cancelation analogous to that in Equations 2-53 to 2-54, this becomes;

$$Q_2 = \sum_{ij} \vec{X}_i' M_{ij}^{-1} \vec{X}_i - \sum_j \{\sum_i M_{ij}^{-1} \vec{X}_i\}' \{\sum_i M_{ij}^{-1}\}^{-1} \{\sum_i M_{ij}^{-1} \vec{X}_i\}$$

which is easily shown to be identically equal to

$$Q_2 = \sum_{ij} (\vec{X}_i - \vec{X}_j^o)' \; M_{ij}^{-1} \; (\vec{X}_i - \vec{X}_j^o)$$

with $\vec{X}_j^o = \{\sum_i M_{ij}^{-1}\}^{-1} \; \{\sum_i M_{ij}^{-1} \vec{X}_i\}$.

So that finally;

$$V' PV = \sum_{ij} (\vec{X}_i - \vec{X}_j^o)' \; M_{ij}^{-1} \; (\vec{X}_i - \vec{X}_j^o) + U_g' X_g \quad .$$

Note that the first term in the above is the quadratic form of all the residuals arising from all simultaneous event adjustments with ground stations held fixed, and is computed and summed for each event by means of Equation 2-64 for the purpose of blunder detection (Section 2.7); the second term is found from;

$$U_g' X_g = D'C \qquad 6\text{-}19$$

where the vectors D' and C are defined by Equations 5-15 and 5-10, respectively.

The total number of degrees of freedom, df, to be used in Equation 6-1 is

df = number of equations - number unknowns,

$$df = (\sum_j 2m + c) - (3s + 3g) , \qquad 6\text{-}20$$

where 2m is the number of equations resulting from one simultaneous event (m = number of ground stations in a particular event j) and the summation is performed over all simultaneous events; c is the number of spatial chord constraint equations; 3s is the number of unknowns due to s number of satellite positions; 3g is the number of unknowns due to g number of unknown ground stations.

In conclusion,

$$m_o^2 = \frac{V'PV}{df} \qquad 6\text{-}21$$

has units of $(\text{arc sec.})^2$ since V has linear units-meters (Equation 2-12) and P has units of $(\text{arc sec.})^2 \cdot m^{-2}$ i.e.,

$$(m) \frac{(\text{arc sec.})^2}{m^2} (m) = (\text{arc sec.})^2 . \qquad 6\text{-}22$$

6.12 Range Adjustment

Equations 6-1 will now be discussed in the light of the range adjustment. Firstly, the expression for computing V'PV is, by an analogous argument to the optical case,

$$V'PV = \bar{V}'P\bar{V} - X_g'U_g \qquad 6\text{-}23$$

where $\bar{V}'P\bar{V}$ is the quadratic form of the residuals arising from the adjustment of simultaneous events - holding the ground stations fixed; the second term

$$X'_g \, U_g = D'C \qquad 6\text{-}24$$

is computed according to Equations 5-15 and 5-10, respectively. The spatial chord constraint does not contribute to V'PV as shown for the optical case adjustment argument (Equation 6-5).

The degrees of freedom, df, in the range adjustment is as usual

$$df = \text{number equations} - \text{number of unknowns}, \qquad 6\text{-}25$$

$$= (\sum_j m + r) - (3s + 3g) \quad , \qquad 6\text{-}26$$

where m is the number of ground stations, thus observed ranges, in a particular simultaneous event and the summation is performed over all simultaneous events; r again is the number of spatial chord constraint equations; 3s and 3g are the number of unknowns due to s number of satellite positions and g number of unknown ground stations, respectively.

In summary,

$$m_o^2 = \frac{V'PV}{df} \qquad 6\text{-}27$$

has units of m^2 since V has linear units-meters (Below Equation 2-9) and P is unitless (Equation 3-8).

6.2 Variances and Covariances of Ground Stations

6.21 Rectangular Coordinates

The variance-covariance matrix giving the precision of the adjusted rectangular ground station coordinates is

$$\Sigma_{\substack{u\\v\\w}} = m_o^2 \, N^{-1} \, , \qquad 6\text{-}28$$

where m_o^2 is the variance of unit weight arising from the adjustment (Section 6.1) and N^{-1} is the weight coefficient matrix discussed in Section 5.4.

The logical and correct units for the variance-covariance matrix is meters^2. To confirm this for the optical case, simply examine the units of m_o^2 and N^{-1}. m_o^2, according to Equation 6-21, has units of (arc sec.)^2. On the other hand, the examination of Equation 2-42 yields units of m^2 per (arc sec.)^2 for N^{-1}. Therefore units of

$$\Sigma_{\substack{u\\v\\w}} = (\text{arc sec.})^2 \, \frac{m^2}{(\text{arc sec.})^2} = m^2 . \qquad 6\text{-}29$$

A similar analysis for the range case adjustment reveals the same units for the variance-covariance matrix of the adjusted rectangular ground station coordinates.

The square root of the diagonal elements of

$$\Sigma_{\begin{smallmatrix}u\\v\\w\end{smallmatrix}} \qquad 6\text{-}30$$

yields the corresponding standard deviations in meters.

6.22 Geodetic Curvilinear Coordinates

The propagation of variances and covariances from curvilinear coordinates geodetic latitude, ϕ, and longitude, λ, and ellipsoidal height, H, all in meters to three dimensional rectangular coordinates, u, v, w is achieved by the following matrix equation

$$\Sigma_{\left|\begin{smallmatrix}u\\v\\w\end{smallmatrix}\right|} = G\, \Sigma_{\begin{smallmatrix}\phi\\\lambda\\H\end{smallmatrix}}\, G' \quad , \qquad 6\text{-}31$$

where

$$G = \begin{vmatrix} -\sin\phi\ \cos\lambda & -\cos\phi\ \sin\lambda & \cos\phi\ \cos\lambda \\ -\sin\phi\ \sin\lambda & \cos\phi\ \cos\lambda & \cos\phi\ \sin\lambda \\ \cos\phi & 0 & \sin\phi \end{vmatrix} . \qquad 6\text{-}32$$

Reversing the transformation depicted by Equation 6-31, the 3 x 3 variance-covariance matrix corresponding to ϕ, λ, H is

$$\Sigma_{\substack{\phi\\ \lambda\\ H}} = G^{-1}\ \Sigma_{\substack{u\\ v\\ w}}\ (G')^{-1} \qquad 6\text{-}33$$

$$= \begin{vmatrix} m_\phi^2 & m_{\phi\lambda} & m_{\phi H} \\ m_{\lambda\phi} & m_\lambda^2 & m_{\lambda H} \\ m_{H\phi} & m_{H\lambda} & m_H{}^2 \end{vmatrix} \qquad 6\text{-}34$$

all in (meters) .

In order to obtain the units

$$\begin{array}{ll} m_\phi^2 & (\text{arc sec.})^2 \\ m_\lambda^2 & \quad '' \\ m_{\phi\lambda} \equiv m_{\lambda\phi} & \quad '' \\ m_H^2 & \quad m^2 \end{array} \qquad 6\text{-}35$$

$$M_{\phi H} \equiv M_{H\phi};\ M_{H\lambda} \equiv M_{\lambda H} \quad , \quad \text{arc sec. x meter}$$

the elements of Equation 6-34 require the following modifications:

$$m_\phi''^{\,2} = (\frac{\rho''}{R + H}\, m_\phi)^2 \quad , \qquad 6\text{-}36$$

$$m_\lambda''^{\,2} = (\frac{\rho''}{R + H}\, m_\lambda)^2 \quad , \qquad 6\text{-}37$$

$$m_{\phi\lambda} \equiv m_{\lambda\phi} = (\frac{\rho''}{R + H})^2\ m_{\phi\lambda} \qquad 6\text{-}38$$

$$m_{H\phi} \equiv m_{\phi H} = \frac{\rho''}{R + H}\, m_{H\phi} \quad , \qquad 6\text{-}39$$

$$m_{H\lambda} \equiv m_{\lambda H} = \frac{\rho''}{R + H} m_{H\lambda} \quad , \qquad 6\text{-}40$$

where

$$\rho'' = \frac{1}{\sin 1''} \quad , \qquad 6\text{-}41$$

$$R = 6{,}370{,}000 \text{ m.} \qquad 6\text{-}42$$

(Note, R replaces the radius of curvature, N, in the prime vertical plane in the rigorous case - justification for simplification is given by the fact that only three significant figures are meaningful in propagation of variances whose magnitudes in m^2 or (arc sec.)^2 are in the units place.)

6.3 Correlation Between Ground Stations

The amount of correlation between the adjusted ground station coordinates is described in terms of the correlation coefficient. The correlation coefficient is defined as [Hamilton, 1964, p. 31]

$$\rho_{ij} = \frac{m_{ij}}{m_i \; m_j} \quad , \qquad 6\text{-}43$$

where i and j represent any two quantities associated with a variance-covariance matrix such as that of Equation 6-28; m_{ij} is the covariance, namely the off diagonal term of Equation 6-28; m_i' and m_j' are the standard deviations or square root of the i^{th} and j^{th} variances (diagonal terms), respectively.

6.4 Error Ellipsoid Computation

Error ellipsoid computation is made for each observing ground station considered as an unknown in the adjustment. The eigenvalues and eigenvectors [Hamilton, 1964, pp. 57 to 60] are computed in a topocentric three dimensional rectangular coordinate system with its origin at the particular ground station and its axes parallel to the mean terrestrial coordinate system (Section 1.2). For each point, there corresponds one eigenvalue (λ_{ii}) for each of the three mutually perpendicular axes of the ellipsoid; the direction of these three axes is given by their corresponding eigenvector (T^i).

The actual computation is as follows. The particular 3 x 3 on diagonal variance-covariance matrix, Σ, of Equation 6-28 is subjected to an orthogonal transformation

$$T' \Sigma T = \Lambda , \qquad 6\text{-}44$$

where Λ is a diagonal matrix and T is the orthogonal transformation matrix to be found which diagonalizes Σ. The transformation results in three homogeneous linear equations, namely

$$[\Sigma - \lambda_{ii} I] T^i = 0 , \qquad 6\text{-}45$$

which has a solution only if the determinant of the coefficient vanishes i.e.,

$$[\Sigma - \lambda_{ii} I] = 0 \qquad 6\text{-}46$$

or

$$\begin{vmatrix} m_{11}^2-\lambda_{11} & m_{12} & m_{13} \\ m_{21} & m_{22}^2-\lambda_{22} & m_{23} \\ m_{31} & m_{32} & m_{33}^2-\lambda_{33} \end{vmatrix} = 0 \ . \qquad 6\text{-}47$$

Once the eigenvalues are obtained from Equation 6-47, their corresponding eigenvectors are obtained from Equation 6-45 after substitution of λ_{ii}.

The length of the axes of the error ellipsoid are the square-roots of the corresponding eigenvalues. The spherical coordinates (spherical latitude, θ, and longitude, λ) which give the direction of each ellipsoidal axis are obtained from the components of the eigenvector

$$T^i = \begin{vmatrix} t_1 \\ t_2 \\ t_3 \end{vmatrix} , \qquad 6\text{-}48$$

namely

$$\tan \theta = \frac{t_3}{\sqrt{t_1{}^2 + t_2{}^2 + t_3{}^2}} \qquad 6\text{-}49$$

and

$$\tan \lambda = \frac{t_2}{t_1} \ . \qquad 6\text{-}50$$

7. SUMMARY AND FUTURE CHANGES TO THE SYSTEM

The adjustment system, as presented in this report and programmed in [Krakiwsky et al; in press] has the following capabilities:

(1) Normal equations can be formed for separate batches of optical or range data.

(2) Independent sets of normal equations can be added together.

(3) Normal equations can be solved giving adjusted rectangular and geodetic curvilinear coordinates.

(4) The weight coefficient matrix is obtained thus allowing the computation of error ellipsoids for all ground stations; further all correlation coefficients are computed for all stations.

(5) Scale may be introduced by a spatial chord constraint.

(6) The covariance between the right ascension and declination of one direction as well as their standard deviations can be utilized for rigorous weighting.

(7) The standard deviations of each range can be utilized for weighting.

It is planned in the future to include the following features:

(1) The solution of the degrees of freedom problem when weighting the ground stations with values between zero and infinity.

(2) Incorporation of error models in the range adjustment to eliminate residual systematic errors.

(3) Sequential build-up of solution and precision estimates.

(4) Direction constraints.

(5) Statistical tests.

(6) Use of magnetic tape files to increase the maximum number of ground stations from 150 to several thousand.

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