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GUIDANCE, FLIGHT MECHANICS AND TRAJECTORY OPTIMIZATION

Volume XII - Relative Motion, Guidance Equations for Terminal Rendezvous

by D. F. Bender and A. L. Blackford

Prepared by
NORTH AMERICAN AVIATION, INC.
Downey, Calif.
for George C. Marshall Space Flight Center

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FOREWORD

This report was prepared under contract NAS 8-11495 and is one of a series intended to illustrate analytical methods used in the fields of Guidance, Flight Mechanics, and Trajectory Optimization. Derivations, mechanizations and recommended procedures are given. Below is a complete list of the reports in the series.

Volume I	Coordinate Systems and Time Measure
Volume II	Observation Theory and Sensors
Volume III	The Two Body Problem
Volume IV	The Calculus of Variations and Modern Applications
Volume V	State Determination and/or Estimation
Volume VI	The N-Body Problem and Special Perturbation Techniques
Volume VII	The Pontryagin Maximum Principle
Volume VIII	Boost Guidance Equations
Volume IX	General Perturbations Theory
Volume X	Dynamic Programming
Volume XI	Guidance Equations for Orbital Operations
Volume XII	Relative Motion, Guidance Equations for Terminal Rendezvous
Volume XIII	Numerical Optimization Methods
Volume XIV	Entry Guidance Equations
Volume XV	Application of Optimization Techniques
Volume XVI	Mission Constraints and Trajectory Interfaces
Volume XVII	Guidance System Performance Analysis

The work was conducted under the direction of C. D. Baker, J. W. Winch, and D. P. Chandler, Aero-Astro Dynamics Laboratory, George C. Marshall Space Flight Center. The North American program was conducted under the direction of H. A. McCarty and G. E. Townsend.

ABSTRACT

This Monograph is intended to present a discussion of the principles and techniques of accomplishing a rendezvous between two spacecraft. In the context here, rendezvous is considered as the interface between the midcourse corrections of an orbital transfer maneuver which establishes the two spacecraft on nearly identical orbits and the docking maneuver which results in the physical contact of the two spacecraft. First consideration in the discussion is given to the development of the equations of relative motion of the two vehicles. To facilitate the use in guidance scheme, these equations are developed in various coordinate systems, with several choices for the independent variables, and with several simplifying assumptions. Next, guidance schemes are developed based on these equations of motion. As each guidance scheme is presented, its existence is in some way justified and the relative advantages and disadvantages as compared to the other schemes discussed. With this discussion enough information is available so that the elements of a rendezvous guidance scheme can be constructed for a particular set of conditions in which a rendezvous maneuver is required.

NAA acknowledges the effort of the following persons in the preparation of this Monograph.

Mr. H. A. McCarty, Program Manager
Mr. G. E. Townsend, Project Engineer
Mr. D. F. Bender, Author
Mr. A. L. Blackford, Author
Mrs. D. Chandler, MSFC Program Manager

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LIST OF FREQUENTLY USED SYMBOLS

$\underline{a}$	acceleration vector
b	acceleration magnitude (of thrust)
a_T	semimajor axis of target orbit
e	eccentricity
E	eccentric anomaly
f	force or control vector
F	fundamental matrix
G	transition matrix
h	angular momentum
H	Hamiltonian
k	switching function
m	mass
M	mean anomaly
n	mean motion
p	semilatus rectum
P	co-state variable
$\underline{P},\underline{Q},\underline{W}$	unit vectors centered at target vehicle describing an inertial reference
$\underline{r}$	radius vector from attracting body to rendezvous vehicle
$\underline{r}_1$	radius vector from attracting body to target vehicle
R,Ø,Z	components of cylindrical coordinate system centered at target vehicle
t	time

T	time difference between a reference time, t_o, and current time
$\underline{u},\underline{v},\underline{w}$	unit vectors of rotating coordinate systems centered at the rendezvous vehicle
$\underline{u}_1,\underline{v}_1,\underline{w}_1$	unit vectors of a rotating coordinate system centered at the rendezvous vehicle
u	acceleration (control)
v	velocity
x,y,z	components along the $\underline{u},\underline{v},\underline{w}$ or $\underline{u}_1,\underline{v}_1,\underline{w}_1$ unit vectors
X,Y,Z	components along the $\underline{P},Q,\underline{W}$ unit vectors
θ	true anomaly
μ	gravitational constant of attracting body
ξ_1, ξ_2, ξ_3	components of the relative position vector (ρ)
ρ	position vector between target and rendezvous vehicle

SUPERSCRIPTS

$\cdot$	$\frac{d}{dt}$
$\circ$	$\frac{d}{d_M}$
$'$	$\frac{d}{d\theta}$
$*$	$\frac{d}{dE}$
-	denotes a vector quantity

SUBSCRIPTS

_	denotes a vector quantity
i	denotes the ith component of a vector

SATELLITE RENDEZVOUS

1.0 STATEMENT OF THE PROBLEM

This monograph is directed to that portion of the rendezvous problem in which the relative distance and velocity between an active spacecraft and a passive target satellite must be reduced from moderate values (say 50 km and .5 km/sec) to small values (say less than 5.0 meters and 1.5 meters/sec). On-board relative position and velocity sensing are assumed for the purpose of allowing precise manual or automatic steering. These observed quantities are to be utilized to drive the state of the system to zero in a reasonable time with as little fuel as possible. Neither the gross orbital changes which have been brought about previous to this closing (rendezvous) maneuver, nor the final phase, known as docking, will be considered. The target satellite will be assumed to be in a closed orbit; and since perturbative influences (such as a non central gravity field) will be nearly the same on both vehicles with the result that their effect will be very small, the orbit will be assumed to be a two-body orbit. Since this monograph represents an attempt to survey the known information regarding the rendezvous problem, it will be analytical in nature and will not refer to any particular spacecraft or its capabilities.

The problem of station keeping is similar to that of rendezvous in that it is assumed that a satellite is to be maintained in a specified orbit with a specified phase within tolerances similar to those mentioned for rendezvous. Thus, in a sense, the target is a point which moves along a desired path (this path may not correspond to the motion in the actual gravitational field). On the other hand, the chase vehicle moves along a path relative to this desired path which is defined by the perturbative influences acting on the vehicle and the differences in the positions and velocities. Accordingly, the position coordinates of the chase vehicle may consequently deviate from those of the target. After such a deviation has accumulated for a period of time the problem of returning the active craft to the nominal path in a substantially shorter time is the rendezvous problem as presented. Of course, it is assumed that the active craft possesses a mechanism by which the deviations from its nominal trajectory are determined as they may be needed.

The discussions begin with the presentation of the field free case; i.e., the case in which the same gravity acts on both satellites. This problem is of little physical importance; however, it serves to provide valuable insight into a more rigorously formulated system. It might be surmised, at first thought, that in this case no guidance technique would be necessary since an astronaut could effect the rendezvous by line-of-sight thrusting. This would approach would, however, cause the motion of the active craft to be one of constant angular momentum about the target. That is, if an angular momentum caused by an initial small velocity (v_{po}) perpendicular to the line of sight exists at the distance r_o, then, if the distance is reduced to 10^{-3} r_o and only line-of-sight thrusting is used, the velocity perpendicular to the line of sight will become $10^3 v_{po}$. Thus, unless v_{po} is zero (initially), rendezvous is not possible with line-of-sight thrusting. For the more accurate approximation of linear terms in the equations of motion, the same situation pertains except that some sets of initial conditions would reduce the effect and others would magnify it. In either case, it is clear that a technique for managing the relative velocity perpendicular to the line of sight and for reducing it to zero as range and range-rate are reduced to zero is essential for rendezvous. Further, though an astronaut could learn to make the necessary corrections by trial and error, techniques for optimal and for automatic control are needed.

The rendezvous operation may be described as the overall solution to the following set of interrelated problems:

a. The state determination problem.

b. The trajectory determination and prediction problem.

c. The trajectory control problem.

Each of these problems is discussed briefly here:

a. The state determination problem - Before making a course correction, the current conditions (i.e., the orbit of the target vehicle and the relative position and velocity of the active vehicle), must be determined. It is assumed in this monograph that this information, which is taken to include error estimation, is available at the start of the problem, as well as at later times as it may be needed.

b. The trajectory determination and prediction problem - the future separation of the two vehicles must be predictable in some fashion in order that changes of velocity can be determined which will cause the separation to be reduced to zero at the same time the velocity difference is nulled. The degree of sophistication required in the equations of motion will depend on the time to make the maneuver and the levels of thrust that may be used. For times which are short compared to the period of the motion and for thrust accelerations which are considerably larger than the differences in the gravitational or other perturbation accelerations between the two satellites, the motion of the two vehicles approximates completely the field free space problem. On the other hand, if the time to rendezvous is of the order of a quarter of a revolution or longer, the equations must contain periodic effects and secular effects produced by the dynamics of the two bodies. However, since the object of the maneuver is to effect a reduction of the relative motion to zero, it is to be expected that approximate representations will be satisfactory as long as errors in the rendezvous caused by poor representation at large values of the relative coordinates can be corrected by subsequent thrusting as the rendezvous is approached. In fact, this capability for error compensation is required since noise and measurement errors in the data sensed must be taken into consideration. Both the model errors and the measurement errors could add to the fuel cost but under these conditions would not hinder the eventually successful rendezvous. The motion analysis in this monograph (Section 2.1) will not go beyond that of linear terms in the relative coordinates, since these terms are believed to be adequate for the ranges of relative motions to be considered. A brief discussion of the possible effects of perturbations due to air drag, earth oblateness, and solar-lunar gravitation is included in the last portion of Section 2.1.

c. The trajectory control problem - Having determined the future course and set up the capability of determining the velocity requirements to effect rendezvous, a philosophy and a procedure to obtain it must be generated, described, and shown to be successful. This development of the guidance scheme is the heart of the problem. Thus, a series of techniques which have been suggested for this purpose are described in Section 2.2 (Guidance Equations). The rendezvous which is effected with any given guidance scheme, however, may not be as close as desired because of errors in the data and in the engine performance. Further, rendezvous will not be optimal unless allowance is made for the stochastic nature of the problem. Optimization techniques and data filtering procedures will thus be important phases of the problem, and these will be described as found in the literature in Sections 2.3 and 2.4, respectively.

For evaluation of the various schemes and for assistance in choosing the state determination process, error analyses are required. The work that is available in this area will be described in Section 2.4

Finally, in Section 3 suggestions for choosing the specific approaches for a number of types of systems and the definition of interface problems associated with mid-course orbital transfer or with the final docking will be discussed.

2.0 STATE-OF-THE-ART

The significant analytical results concerning the friendly rendezvous with a passive target in orbit around a single attracting center are presented in this section. The first portion (Section 2.1) deals with the equations of motion and their solutions for coasting arcs and for arbitrary powered arcs. Some of the equations are used frequently as the coast arc solutions (closed forms) and are given (or referenced). The powered arc solution, on the other hand, is reduced to a set of indefinite integrals containing the acceleration.

In Section 2.2 a series of guidance schemes is developed for the field free and constant gravity field cases; a scheme is then developed for the linear gravity gradient representations with their linear equations of motion.

In Section 2.3 the methods that have been used for optimization of the rendezvous maneuver are discussed. Included in this development is a discussion of optimal stepwise thrusting for time optimal, fuel optimal, and power limited fuel optimal rendezvous and a brief discussion of optimal impulsive rendezvous and its adaptation for finite thrust cases.

2.1 Equations of Motion

In order to formulate velocity requirements for rendezvous guidance it is necessary to know how the relative motion of the target vehicle during rendezvous is influenced by the application of corrective thrust (to the spacecraft) and by the passage of time for the case when no thrust is being applied. To this end the equations of motion of the target with respect to the spacecraft (as opposed to the absolute motion of the two vehicles in a central force field), and the solutions to these equations are developed in this section. It is noted that in a previous monograph of this series (Reference 1.1) the solutions to the equations of relative motion are presented; however, since they are to be examined in detail and since it is desirable to extend the material in the reference to include sets of equations in terms of a variety of independent variables, the equation will be re-developed here.

2.1.1 Coordinate Systems

The coordinate system for the relative motion is usually centered at the target satellite and rotates with it. However, a significant simplification of the circumferential component occurs if the unit vectors are determined by the position of the active satellite. In this derivation, therefore, the reference directions in the plane of motion of the target are chosen by the projection of the position of the active vehicle onto the plane: the first axis ($\underline{U}$) being radial, the second axis ($\underline{V}$) circumferential in the plane of the motion, and the third axis ($\underline{W}$) binormal to this motion. (At this point the oblateness of the earth is neglected so that the target satellite moves in a truly periodic orbit in a fixed plane.) The usual distance forms with the origin at the target will be presented below in Section 2.1.3.2.

The target satellite is taken to be at the location

$$\underline{r_1} = r_1 \, \underline{U_1} \text{ where } r_1 = p(1 + e \cos \theta)^{-1} = a_T(1 - e \cos E) = a_T q \qquad 1.2$$

while the active (or chasing) satellite is taken to be at

$$\underline{r} = r_1(1 + \xi_1)\,\underline{U} + r_1\ \xi_3\,\underline{W} \qquad 1.3$$

where U lies in the orbit plane at the angle ξ_2 ahead of U_1, and where ξ_1, ξ_3 are small angles while ξ_2 is unlimited (Figure 1.1). Thus for circular target orbits, the chase vehicle is allowed to be anywhere inside a torus of small lateral dimensions centered on the orbit of the target. For elliptical target orbits of high eccentricity, the values r_1 and r may be so different for large ξ_2 that ξ_1 could become large. Or to put it another way, if a torus of reasonably small cross section centered at the radius of the semi-major axis of the target orbit does not include the whole elliptical orbit in its interior, then ξ_2 will have to be limited to moderately small angles in order to keep ξ_1 small.

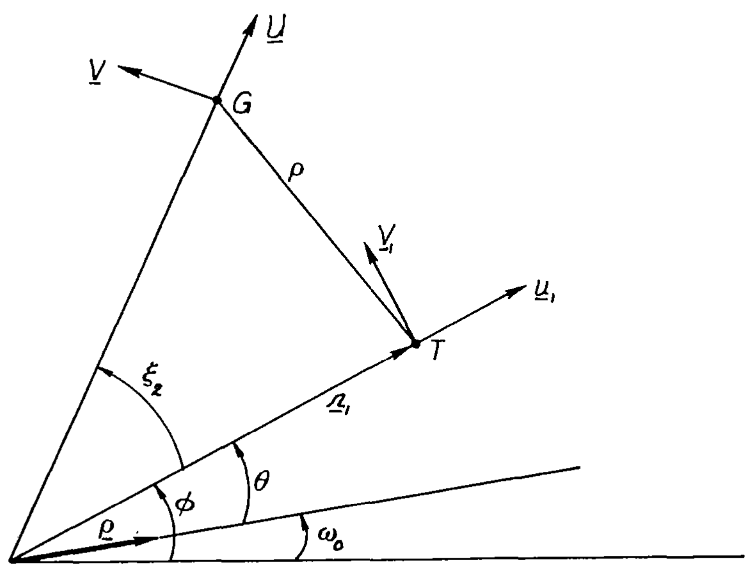


Figure 1.1 The Target Orbit Plane Vectors and Angles

Thus in cylindrical coordinates, the position of the target satellite is $(r_1, \phi, 0)$ and its motion satisfies the differential equation (the $\dot{}$ indicates d/dt)

$$\ddot{\underline{r}}_1 = -\mu r_1^{-2} \underline{U}_1 \qquad 1.4$$

while the active satellite is at $[r_1(1+\xi_1), \phi + \xi_2, r_1 \xi_3]$ and its motion satisfies

$$\ddot{\underline{r}} = -\mu r^{-3} \underline{r} + \underline{a} \qquad 1.5$$

where $\underline{a}$ is the instantaneous acceleration produced by the engines in affecting rendezvous.

The differential equation for the relative position $\underline{\rho} = \underline{r} - \underline{r}_1$ is then $\ddot{\underline{\rho}} = \Delta \underline{g} + \underline{a}$ where $\Delta \underline{g}$ is the difference in gravitational accelerations of the two satellites and is given by

$$\Delta \underline{g} = \mu (r_1^{-2} \underline{U}_1 - r^{-3} \underline{r}) \qquad 1.6$$

Equation 1.6 may have a wider application than indicated here, for Δg may represent the difference in all accelerations of the two satellites. Thus, the reference trajectory could be a simple orbit satisfying any portion of the total force equation. In fact, the idea of referring the motion of one satellite to that of another nearby has been used in lunar theory since the time of Euler in 1772; and it is doubtful if any of the sets of differential equations given below could be considered to be original in this century.

The signs of the coriolis terms in equations found in the literature are sometimes the opposites of those used in this monograph. The difference arises from a difference in the choice of coordinate axes, here X radial, Y circumferential ahead; whereas many authors use X circumferential back, Y radial.

2.1.2 Field Free Case

For the field free case, the vector difference $\Delta \underline{g}$ is assumed to be negligibly small and one obtains simply $\ddot{\underline{\rho}} = \underline{a}$. The solutions are immediately available as

$$\dot{\underline{\rho}} = \dot{\underline{\rho}}_0 + \int_0^t \underline{a}(\tau)\, d\tau \qquad 1.7$$

$$\underline{\rho} = \underline{\rho}_0 + \int_0^t \dot{\underline{\rho}}(\tau)\,d\tau \qquad 1.8$$

2.1.3 Approximate Equations

On the other hand, one may choose to obtain a set of linear differential equations for the components of ξ which may be chosen in order to develop a more accurate representation. This procedure will be adopted; but at the outset it is desirable to point out that solutions to the homogenous part of equations (i.e., for the case with no force, $\underline{a}$) that will be obtained are, in fact, already essentially available. These are the state transition matrices which have been discussed in the State Determination and/or Estimation Monograph (Reference 1.1). Of course, independent variable changes and simple coordinate changes will be necessary in order to obtain all the various forms that may be used.

2.1.3.1 Angular Forms of the Equations

The expression for $\ddot{\underline{\rho}} = \ddot{\underline{r}} - \ddot{\underline{r}}_1$ is developed in terms of r_1, θ, ξ_1, ξ_2, ξ_3 and their derivatives by making use of

$$\dot{\underline{U}}_1 = \dot{\theta}\underline{V}_1 , \quad \dot{\underline{U}} = (\dot{\theta} + \dot{\xi}_2)\underline{V} , \quad \dot{\underline{V}}_1 = -\dot{\theta}\underline{U}_1 , \text{ and } \dot{\underline{V}} = -(\dot{\theta} + \dot{\xi}_2)\underline{U}. \qquad 1.9$$

The gravitational terms, expressed in components along $\underline{U}$, $\underline{V}$, and $\underline{W}$, are

$$\Delta \underline{g} = -\mu r_1^{-2}\left\{\left[(1+\xi_1)K^{-3/2} - \cos\xi_2\right]\underline{U} - \sin\xi_2\,\underline{V} - \xi_3 K^{-3/2}\underline{W}\right\} \qquad 1.10$$

$$\text{where} \quad K = \frac{r^2}{r_1^2} = (1+\xi_1)^2 + \xi_3^2$$

The motion of the target satellite can be shown to satisfy

$$\ddot{\xi}_1 + \frac{2\dot{r}_1}{r_1}\dot{\xi}_1 - \dot{\xi}_2\left(2\dot{\theta} + \dot{\xi}_2\right)\left(1+\xi_1\right) - \frac{3\mu}{r_1^3}\left(\xi_1 - \xi_1^2 + \frac{\xi_3^2}{2} + \dots\right) = \frac{a_1}{r_1} \qquad 1.11a$$

$$\ddot{\xi}_2\left(1+\xi_1\right) + \frac{2\dot{r}_1}{r_1}\dot{\xi}_2\left(1+\xi_1\right) + 2\dot{\xi}_1\left(\theta + \dot{\xi}_2\right) = \frac{a_2}{r_1} \qquad 1.11b$$

$$\ddot{\xi}_3 + \frac{2\dot{r}_1}{r}\dot{\xi}_3 + \xi_3\left[\frac{\ddot{r}_1}{r_1} + \frac{\mu}{r_1^3}\left(1 - 3\xi_1 + \dots\right)\right] = \frac{a_3}{r_1} \qquad 1.11c$$

In this form the equations are valid to the second order in the ξ . It is seen that the truncation of an infinite series (K) is required in only the radial and the binormal component equations.

As already indicated, however, a linear theory is usually adequate for rendezvous discussions; therefore, these equations reduce to the following set of linear equations:

$$\begin{aligned} &\ddot{\xi}_1 + \frac{2he\sin\theta}{pr}\dot{\xi}_1 - 3n^2\left(\frac{a}{r}\right)^3\xi_1 - 2n\sqrt{1-e^2}\left(\frac{a}{r}\right)^2\dot{\xi}_2 = \frac{a_1}{r_1} \\ &\ddot{\xi}_2 + \frac{2he\sin\theta}{pr}\dot{\xi}_2 + 2n\sqrt{1-e^2}\left(\frac{a}{r}\right)^2\dot{\xi}_1 = \frac{a_2}{r_1} \\ &\ddot{\xi}_3 + \frac{2he\sin\theta}{pr}\dot{\xi}_3 + n^2\left(\frac{a}{r}\right)^4\left(1-e^2\right)\xi_3 = \frac{a_3}{r_1} \end{aligned} \qquad 1.12$$

where $n^2 = \mu a^{-3}$. 1.13

Here the independent variable is time; and as is customary, the dot indicates differentiation with respect to time.

The first important point to note is that the out-of-plane motion is decoupled from the in-plane motion, a feature that is characteristic of all linear sets. These equations can be changed so that the independent variable is the mean anomaly, M, since dM = n dt. This step is equivalent to making the unit of time equal to the time required for a change of one radian in mean anomaly. The resulting equations are (where the open dot $^\circ$ is used to indicate d/dM):

$$\overset{\circ\circ}{\xi}_1 + \frac{2ae\sin\theta}{r\sqrt{1-e^2}}\overset{\circ}{\xi}_1 - 3\frac{a}{r}\xi_1 - 2\frac{a}{r}\sqrt{1-e^2}\,\overset{\circ}{\xi}_2 = \frac{a_1}{\mu^2 r_1} \qquad 1.14a$$

$$\overset{\circ\circ}{\xi}_2 + \frac{2ae\sin\theta}{r\sqrt{1-e^2}}\overset{\circ}{\xi}_2 + 2\frac{a}{r}\sqrt{1-e^2}\,\overset{\circ}{\xi}_1 = \frac{a_2}{\mu^2 r_1} \qquad 1.14b$$

$$\overset{\circ\circ}{\xi}_3 + \frac{2ae\sin}{r\sqrt{1-e^2}}\overset{\circ}{\xi}_3 + \left(\frac{a}{r}\right)^4\left(1-e^2\right)\xi_3 = \frac{a_3}{\mu^2 r_1} \qquad 1.14c$$

A major simplification occurs if the independent variable is changed to the true anomaly, (θ), or to the argument of latitude ($\phi = \theta + \omega_o$). The transformation makes use of the conservation of angular momentum

$$h\,dt = r_1^2\,d\theta = r_1^2\,d\phi \qquad 1.15$$

Denoting $d/d\theta$ by the prime, ', there finally results:

$$\xi_1'' - 3\frac{r_1}{p}\xi - 2\xi_2' = r_1^3 h^{-2} a_1 \qquad 1.16a$$

$$\xi_2'' + 2\xi_1' = r_1^3 h^{-2} a_2 \qquad 1.16b$$

$$\xi_3'' + \xi_3 = r_1^3 h^{-2} a_3 \qquad 1.16c$$

For the case of coast arcs when $a_1 = a_2 = a_3 = 0$, it is to be noted that the out-of-plane motion is simple harmonic in terms of true anomaly and that the first integral for the circumferential equation can be written down at once.

Another form of the linear equations of motion which will be considered makes use of the eccentric anomaly, E, as the independent variable. To accomplish this transformation, the substitution

$$r_1 d\theta = a_T\sqrt{1-e^2}\, dE$$

or

$$q\, df = \sigma\, dE \qquad 1.17$$

is made where $q = 1 - e\cos E = r_1/a_T$ and $\sigma = \sqrt{1 - e^2}$.
Denoting $\left(\frac{d}{dE}\right)$ with the asterisk, *, there results:

$$\left(q\xi_1^*\right)^* - 3\xi_1 - 2\sigma\xi_2^* = \sigma q^2\frac{a_1}{g_a} \qquad 1.18a$$

$$\left(q\xi_2^*\right)^* + 2\sigma\xi_1^* = \sigma q^2\frac{a_2}{g_a} \qquad 1.18b$$

$$\left(q\xi_3^*\right)^* + \frac{\sigma^2\xi_3}{q} = \sigma q^2\frac{a_3}{g_a} \qquad 1.18c$$

where

$$g_a = \mu a_T^{-2} \qquad 1.19$$

equals the acceleration of gravity at the distance of the semi-major axis. For coast arcs the third equation is easily integrable as will be shown below, and the second equations possess an immediate integral as for Figure 1.16.

For all the equations given to this point, the reference orbit can be elliptical. If the path is circular, a further simplification occurs for each form presented. The set with time as the independent variable (Eq. 1.12) becomes ($r_1 = r_o$ = constant):

$$\ddot{\xi}_1 - 3n^2 \xi_1 - 2n \dot{\xi}_2 = \frac{a_1}{r_o} \qquad 1.20a$$

$$\ddot{\xi}_2 + 2n \dot{\xi}_1 = \frac{a_2}{r_o} \qquad 1.20b$$

$$\ddot{\xi}_3 + n^2 \xi_3 = \frac{a_3}{r_o} \qquad 1.20c$$

The remaining sets (Eqs. 1.14, 1.16, and 1.18) reduce to a single set because of the equality of the three anomalistic variables ($M = \theta = E$) for circular orbits. This set is

$$\xi_1'' - 3\xi_1 - 2\xi_2' = \frac{a_1}{g_a} = u_1 \qquad 1.21a$$

$$\xi_2'' + 2\xi_1' = \frac{a_2}{g_a} = u_2 \qquad 1.21b$$

$$\xi_3'' + \xi_3 = \frac{a_3}{g_a} = u_3 \qquad 1.21c$$

It is to be noticed that for linear systems and a circular reference orbit the set of equations has constant coefficients and is, therefore, easily integrated for the case of coast arcs ($\underline{U} = \underline{a} = \underline{0}$).

As already mentioned for the sets of equations in terms of true anomaly or eccentric anomaly, the second of the three equations possesses an immediate first integral for the no-thrust situation. This integral is a representation of the constant difference in the angular momentum per unit mass for the two vehicles. Thus, for the no-thrust case, there must exist three more independent integrals consisting of simple combinations of ξ_1, ξ_2, ξ'_1, ξ'_2 representing constant differences in other elliptical orbit elements (e.g., semi-major axes, arguments of perigee, times of perigee passage). This concept, in fact, yields a method for obtaining the integrals to the sets.

2.1.3.2 Distance Forms of the Equations

In the first place, let

$$\underline{\rho} = x\underline{U}_1 + y\underline{V}_1 + z\underline{W} \qquad 1.22$$

and assume an elliptical reference orbit. Note that the reference system is centered at the target. Now, using the derivative of $\underline{U}_1$ and $\underline{V}_1$ as in Section 2.1.3.1 the equations are found to be:

$$\left.\begin{aligned} \ddot{x} - y\ddot{\theta} + x\dot{\theta}^2 - 2y\dot{\theta} &= -\frac{\mu}{r_1^3}\left[(r_1 + x)K^{-3/2} - r_1\right] + a_x \\ \ddot{y} - x\ddot{\theta} + y\dot{\theta}^2 + 2x\dot{\theta} &= -\frac{\mu}{r_1^3}\left[y K^{-3/2}\right] + a_y \\ \ddot{z} \qquad\qquad\qquad\qquad &= -\frac{\mu}{r_1^3}\left[z K^{-3/2}\right] + a_z \end{aligned}\right\} \quad 1.23$$

Note that moving the origin to the target vehicle has the effect of causing a K term to occur in all three equations. The second-order form of these equations as used by Anthony and Sasaki (Reference 1.2) is obtained by introducing changes of scale for both distance and time. In this reference, the semi-major axis, a_T, of the elliptical reference is used as a normalizing variable; thus, $x = a_T x_1$, $y = a_T y_1$, $z = a_T z_1$, and $r_1 = a_T q$. The time is then changed to mean anomaly, M, and d/dM is represented with the open dot, °, as before. Including the second-order terms on the right, the equations become:

$$\left.\begin{aligned} \overset{\circ\circ}{x}_1 - y_1\overset{\circ\circ}{\theta} - x_1\overset{\circ}{\theta}{}^2 - 2y_1\overset{\circ}{\theta} &= \frac{2x_1}{q^3} + \frac{3}{2q^4}(2x_1^2 - y_1^2 - z_1^2) + \frac{a_x}{q_a} \\ \overset{\circ\circ}{y}_1 + x_1\overset{\circ\circ}{\theta} - y_1\overset{\circ}{\theta}{}^2 + 2x_1\overset{\circ}{\theta} &= -\frac{y_1}{q^3} + \frac{3x_1y_1}{q^4} + \frac{a_y}{q_a} \\ \overset{\circ\circ}{z}_1 \qquad\qquad\qquad\qquad &= -\frac{z_1}{q^3} + \frac{3x_1z_1}{q^4} + \frac{a_z}{q_a} \end{aligned}\right\} \quad 1.24$$

For circular orbits the equations simplify to equations of exactly the same form as Equations 1.20 thus indicating the equivalence of the two origins (either active or target) for rendezvous. Thus, one finds

$$\begin{aligned} \overset{\circ\circ}{x} - 3n^2x - 2n\overset{\circ}{y} &\simeq a_x \\ \overset{\circ\circ}{y} \qquad\quad + 2nx &\simeq a_y \\ \overset{\circ\circ}{z} + n^2 z \qquad\qquad &\simeq a_z \end{aligned} \quad 1.25$$

For the final two forms of the equations, consider that $\underline{\rho}$ is expressed in a set centered at the target in an elliptical orbit and oriented in a fixed set of inertial directions which are taken to be those of perigee ($\underline{P}$), $\underline{Q} = \underline{W}\times\underline{P}$ and $\underline{W}$ (binormal). First choose a rectangular set (Figure 1.2) where

$$\underline{\rho} = X\underline{P} + Y\underline{Q} + \bar{Z}\,\underline{W}$$

The result is:

$$\begin{aligned}
\ddot{\underline{\bar{X}}} + \frac{\mu}{r_1^3}(\bar{\underline{X}} - 3\bar{\underline{X}}\cos^2\theta - 3\bar{\underline{Y}}\cos\theta\sin\theta) &\simeq a_{\bar{X}} \\
\ddot{\underline{\bar{Y}}} + \frac{\mu}{r_1^3}(Y - 3\bar{\underline{X}}\sin\theta\cos\theta - 3\bar{\underline{Y}}\sin^2\theta) &\simeq a_{\bar{Y}} \qquad 1.26\\
\ddot{Z} + \frac{\mu}{r_1^3}Z &\simeq a_Z
\end{aligned}$$

Here, of course, the terms involving μ are the first linear terms in the series expansion of Δg.

Finally, using a cylindrical set of coordinates (R, γ, Z) where $\underline{\rho} = R\,\underline{L}$, the equations are seen to be

$$\begin{aligned}
\ddot{R} - R\dot{\gamma}^2 &\simeq -\frac{\mu R}{r_1^3}\left[1 - 3\cos^2(\gamma-\theta)\right] + a_L \\
R\ddot{\gamma} + 2\dot{R}\dot{\gamma} &\simeq -\frac{3\mu R}{r_1^3}\sin(\gamma-\theta)\cos(\gamma-\theta) + a_M \qquad 1.27\\
\ddot{Z} &\simeq -\frac{\mu}{r_1^3}Z + a_Z
\end{aligned}$$

From sets 1.26 and 1.27 the usual expression for circular reference orbits is obtained by substituting $n^2 = \mu/r_1^3$

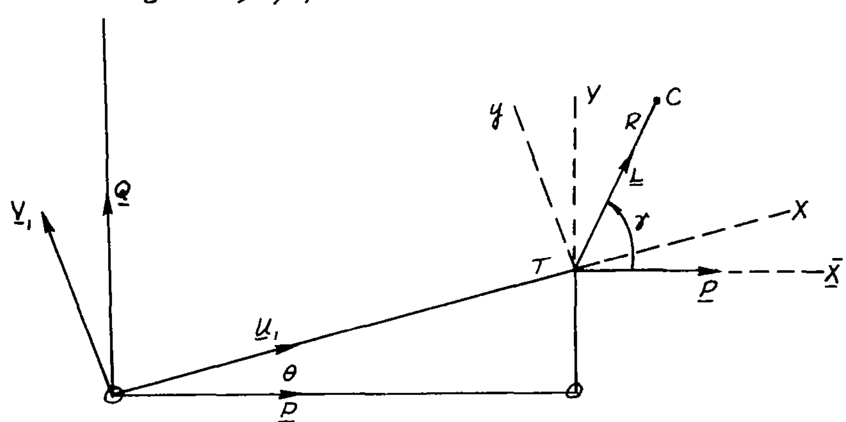


Figure 1.2 Target Centered Coordinate Systems

2.1.4 Solutions to the Equations of Motion

As has been mentioned in the discussion of the state transition matrices of the State Determination Monograph (Reference 1.1), this matrix represents the solutions to the homogeneous parts of various sets of the equations. However, rather than refer to them directly, solutions will be developed making use of matrix methods in this section; especially since it is desired to include the effects of the thrusting acceleration, a_3. The sets of equations will be split into the out-of-plane motion and the in-plane motion. Furthermore, the techniques of the matrix method will be illustrated in the solution for the out-of-plane motion; since this motion is seen to be simple harmonic motion with a forcing function (except when the reference orbit is elliptical and the independent variable is eccentric anomaly). Matrix methods and results are given by Leach (Reference 1.3), Tschauner and Hempel (Reference 1.4) for circular reference orbits and by Tschauner and Hampel (Reference 1.5) and Tschauner (Reference 1.6) for elliptic reference orbits.

2.1.4.1 The Out-of-Plane Motion

The out-of-plane motion can be represented by the differential equation

$$\ddot{\xi}_3 + n^2 \xi_3 = a_3/r_0 \qquad 1.28$$

for all cases except the set Equation 1.18, which will be considered later. The matrix methods require that the equations be expressed as linear first-order equations. This is accomplished by defining the two vectors ξ as

$$\zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix} = \begin{pmatrix} \xi_3 \\ \dot{\xi}_3 \end{pmatrix} \qquad 1.29$$

Thus, Equation 1.28 becomes

$$\dot{\zeta} = A\zeta + Ba_3/r_0 \qquad 1.30$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -n^2 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

To proceed, the fundamental matrix (F) for A must be found; that is, a set of independent solutions to

$$\dot{\zeta} = A\zeta \qquad 1.31$$

which form the columns of F must be found. This process is difficult in general; thus, sometimes it is preferable to make a transformation of variables to simplify the matrix A before attempting to find the fundamental matrix F. For the present equations, however, the solutions are known; and the fundamental matrix can be written in terms of real functions as

$$F(t) = \begin{pmatrix} \sin nt & \cos nt \\ n \cos nt & -n \sin nt \end{pmatrix} \qquad 1.32$$

The inverse matrix is

$$F^{-1}(t) = \begin{pmatrix} \sin nt & \frac{1}{n} \cos nt \\ \cos nt & -\frac{1}{n} \sin nt \end{pmatrix} \qquad 1.33$$

Now, since $\frac{d(F^{-1}F)}{dt} = 0$, it is easy to show that

$$\frac{d}{dt}(F^{-1}) = -F^{-1}A \qquad 1.34$$

for all problems of this kind. In order to obtain the solution, it is convenient to obtain a set of constants for equations with no forcing ($a_3 = 0$). This is accomplished by the substitution

$$z = F^{-1}(t)\zeta \qquad 1.35$$

which is seen to satisfy the differential equation

$$\dot{z} = F^{-1}(t)\, B\, a_3(t)/\Omega_0 \qquad 1.36$$

This equation is integrated to give $Z - Z_0$ and is then multiplied by $F(t)$ on the left to obtain the original variable, ζ . The result is

$$\zeta(t) = F(t)F^{-1}(t_0)\zeta_0 + F(t)\int_{t_0}^{t} F^{-1}(t)Ba_3(t)dt/\Omega_0 \qquad 1.37$$

The state transition matrix for these two variables is $F(t)F^{-1}(t_0)$. When the coefficients in the matrix A are constant, it is possible to write

$$F(t)F^{-1}(t_0) = G(t - t_0)$$

For this case $(t - t_o = T)$

$$G(T) = \begin{pmatrix} \cos nT & \frac{1}{n}\sin nT \\ -n\sin nT & \cos nT \end{pmatrix}$$

This solution, valid for circular reference orbits, is also applicable to elliptic reference orbits by changing nt to θ which is then true anomaly.

The solution to the out-of-plane problem for the case of elliptical target orbits and eccentric anomaly is given by Tschauner (Reference 1.5). Let

$$\zeta = \begin{pmatrix} \xi_3 \\ q\xi_3^* \end{pmatrix} \qquad 1.39$$

Thus, the set becomes

$$\zeta^* = A\zeta + Ba_3 \cdot \frac{\sigma}{g_a} \qquad 1.40$$

where

$$A = \begin{pmatrix} 0 & \frac{1}{q} \\ -\frac{\sigma^2}{q} & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 \\ q^2 \end{pmatrix}$$

The fundamental matrix is now

$$F(E) = \begin{pmatrix} \frac{\sin E}{q} & \frac{-\cos E + e}{q} \\ \frac{\cos E - e}{q} & \frac{\sigma^2 \sin E}{q} \end{pmatrix} \qquad 1.41$$

and

$$F^{-1}(E) = \begin{pmatrix} \frac{\sigma^2 \sin E}{q} & \frac{\cos E - e}{q} \\ \frac{-\cos E + e}{q} & \frac{\sin E}{q} \end{pmatrix} \qquad 1.42$$

In this case, the substitution

$$z = F^{-1}(E)\zeta \qquad 1.43$$

gives the equation $(q = r/a = 1 - e \cos E)$

$$z^* = \left(\frac{\cos E - e}{\sin E}\right) g\, \sigma a_3 / g_a \qquad 1.44$$

Thus, the solution is written

$$\zeta(E) = F(E)F^{-1}(E_0)\zeta_0 + F(E)\int_{E_0}^{E} \left(\frac{\cos E - e}{\sin E}\right) g \; \frac{\sigma a_3}{g_a}\, dE \qquad 1.45$$

The transition matrix then becomes

$$F(E)F^{-1}(E) = \frac{1}{g g_0} \begin{pmatrix} a & b \\ -\sigma^2 b & a \end{pmatrix} \qquad 1.46$$

where a and b are defined by the equations below:

$$a = \cos(E - E_0) - e(\cos E + \cos E_0) + e^2(1 - \sin E \sin E_0) \qquad 1.47a$$

$$b = \sin(E - E_0) - e(\sin E - \sin E_0) \qquad 1.47b$$

2.1.4.2 The In-Plane Motion

2.1.4.2.1 In-Plane Motion for Circular Target Orbit

For the case of the in-plane motion and the circular reference orbit, the first two equations of Equations 1.21 are used. To express these as a set of linear equations, let

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_1' \\ \xi_2' \end{pmatrix} \qquad 1.48$$

The set becomes

$$\eta' = A\eta + B\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad 1.49$$

where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The characteristic equation for A is $\lambda^4 + \lambda^2 = 0$ from which it is seen that the four independent functions out of which solutions are formed are 1, θ, sin θ, and cos θ. The fundamental matrix may be taken to be:

$$F(\theta) = \begin{pmatrix} 0 & 2 & \text{Sin}\ \theta & \text{Cos}\ \theta \\ 1 & -3\theta & 2\ \text{Cos}\ \theta & -2\ \text{Sin}\ \theta \\ 0 & 0 & \text{Cos}\ \theta & \text{Sin}\ \theta \\ 0 & -3 & -2\ \text{Sin}\ \theta & -2\ \text{Cos}\ \theta \end{pmatrix} \qquad 1.50$$

The determinant of F is equal to unity, and the inverse is

$$F^{-1}(\theta) = \begin{pmatrix} 6\ \theta & 1 & -2 & 3\ \theta \\ 2 & 0 & 0 & 1 \\ -3\ \text{Sin}\theta & 0 & \text{Cos}\theta & -2\ \text{Sin}\ \theta \\ -3\ \text{Cos}\theta & 0 & -\text{Sin}\theta & -2\ \text{Cos}\ \theta \end{pmatrix} \qquad 1.51$$

Thus, making the substitution

$$z = F^{-1}(\theta)\zeta \qquad 1.52$$

allows the differential equation for Z to be written as

$$z' = F^{-1}(\theta)\, B \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad 1.53$$

or

$$Z' = \begin{pmatrix} 2 & 3\ \theta \\ 0 & 1 \\ \text{Cos}\ \theta & -2\ \text{Sin}\ \theta \\ -\text{Sin}\ \theta & -2\ \text{Cos}\ \theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad 1.54$$

The state transition matrix for the variables η is thus

$$F(\theta) F^{-1}(\theta_0) = G(\theta - \theta_0)$$

This matrix is simply written as $G(\theta)$ (i.e., this notation is used to avoid writing)

$$G(\theta) = \begin{pmatrix} 4-3 \text{ Cos } \theta & 0 & \text{Sin } \theta & 2(1 - \text{Cos } \theta) \\ 6(\text{Sin } \theta - \theta) & 1 & 2(\text{Cos } \theta - 1) & 4 \text{ Sin } \theta - 3\theta \\ 3 \text{ Sin } \theta & 0 & \text{Cos } \theta & 2 \text{ Sin } \theta \\ 6(\text{Cos } \theta - 1) & 0 & -2 \text{ Sin } \theta & 4 \text{ Cos } \theta - 3 \end{pmatrix} \quad 1.55$$

This matrix, when combined with that of Equation 1-38 and both expressed in terms of $\theta = nT$ (and n added as needed to give dimensions correctly), are seen to be exactly that of page 145 of SID 65-1200-5 (Reference 1.1).

In the case of the representation of state transition matrix for locally level inertial systems (Table 2.4.2, page 147 of Reference 1.1), the coordinate transformation required is only that between inertial and rotating systems at the moment they are aligned. Reverting to $\theta = nT$, the rotation rate is one of the angular velocity, n, about the z or third axis and the transformation matrix, T, for $\xi = T X$ is

$$T = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I} & 0 \\ \hline 0 & -n & \\ n & 0 & I \end{array}\right] \quad 1.56$$

with

$$T^{-1} = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{I} & 0 \\ \hline 0 & n & \\ -n & 0 & I \end{array}\right] \quad 1.57$$

The state transition matrix for inertial locally level (at both times) may thus be obtained from $G(nT)$ and it is $\Phi_{i\ell} = T^{-1}G(nT)T =$

$$\begin{pmatrix} 2-\text{Cos } nT & \sin nT & 1/n \text{ Sin } nT & 2/n(1-\text{Cos } nT) \\ 2 \text{ Sin } nT-3nT & 2 \text{ Cos } nT-1 & 2/n(\text{Cos } nT-1) & 1/n(4 \text{ Sin } nT-3nT) \\ n(3nT \text{ Sin } nT) & n(1-\text{Cos } nT) & 2-\text{Cos } nT & 3nT-2 \text{ Sin } nT \\ n(\text{Cos } MT-1) & -n(\text{Sin } nT) & -\text{Sin } nT & 2 \text{ Cos } nT - 1 \end{pmatrix} \quad 1.58$$

The development of the system by Tschauner and Hampel (Reference 1.4) involves a substitution to simplify the matrix of coefficients. In addition, the out-of-plane motion will be included and the set of six equations solved with matrix notation for later reference. It is simplest to add the out-of-plane coordinates to the set of four in-plane variables of Equation 1.48 as

$$(w_5, w_6) = (\eta_5, \eta_6) = (\xi_3, \xi_3')$$

The set of differential equations is now written as

$$w' = Aw + f$$

where the transformation from the position and velocity differences (η) to w is given by

$$w = T\eta = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1/3 & 2/3 & 0 & 0 & 0 \\ 3/2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_1' \\ \xi_2' \\ \xi_3 \\ \xi_3' \end{bmatrix} \qquad 1.59$$

where

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

and where the forcing function, f, is

$$f^{T} = (u_2, \tfrac{2}{3}u_1, u_2, \tfrac{1}{2}u_1, 0, u_3)$$

The fundamental matrix for A can be written at once as

$$F(\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \theta & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \text{Cos } \theta & \text{Sin } \theta & 0 & 0 \\ 0 & 0 & -\text{Sin } \theta & \text{Cos } \theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{Cos } \theta & \text{Sin } \theta \\ 0 & 0 & 0 & 0 & -\text{Sin } \theta & \text{Cos } \theta \end{bmatrix} \qquad 1.60$$

for which the inverse is

$$F^{-1}(\theta) = F(-\theta)$$

The solutions to the equations can now be written in the form

$$w(\theta) = F(\theta)\, F^{-1}(\theta)\, w_0 - F(\theta) \int_{\theta_0}^{\theta} F^{-1}(\tau) f(\tau)\, d\tau \qquad 1.61$$

In this case, the matrix $G(\theta - \theta_o) = F(\theta)\ F^{-1}(\theta_o)$ is easily seen to be

$$G(\theta - \theta_0) = F(\theta - \theta_0) \qquad 1.62$$

Note that $F(0) = F^{-1}(\theta) = I$. In fact, since no loss of generality occurs by choosing $\theta_o = 0$, this value will be assumed for the remainder of this section. The six integrals in the solution for $w(\theta)$ (called "Z") will be:

$$\begin{aligned}
Z_1 &= \int_0^\theta u_2\, d\tau \\
Z_2 &= \int_0^\theta (\tau u_2 + \tfrac{2}{3} u_1)\, d\tau \\
Z_3 &= \int_0^\theta (u_2 \cos \tau - \tfrac{1}{2} u_1 \sin \tau)\, d\tau \\
Z_4 &= \int_0^\theta (u_2 \sin \tau + \tfrac{1}{2} u_1 \cos \tau)\, d\tau \\
Z_5 &= -\int_0^\theta u_3 \sin \tau\, d\tau \\
Z_6 &= \int_0^\theta u_3 \cos \tau\, d\tau
\end{aligned} \qquad 1.63$$

Finally, the solutions are expressed as

$$w(\theta) = F(\theta)\,(w_o - z(\theta)) \tag{1.64}$$

To repeat, the boundary conditions at $\theta = 0$ are $w = w_o$ and are seen to be satisfied. The final boundary or rendezvous conditions at $\theta = \theta_f$ are $\eta^T = w^T = (0, 0, 0, 0, 0, 0)$.

2.1.4.2.2 In-Plane Motion for Elliptical Target Orbits

A technique for obtaining the solutions to the in-plane motion has been mentioned (Section 2.1.3) and references made to two papers by J. Tschauner (References 1.5 and 1.6). It is suggested that these papers be reviewed as required; the ability to obtain these solutions should allow a completely satisfactory representation of the problem of rendezvous with targets in elliptic orbits.

2.1.5 Approximate Second Order Solutions

A solution to the equations of relative motion which includes both linear and quadratic terms in the gravity expansion and which is applicable to target orbits of small eccentricity is developed by Anthony and Sasaki in Reference 1.2. This work is essentially a combination of the work of London (Reference 1.7) who examined the effect of including the quadratic term for circular target orbits, and that of deVries (Reference 1.8) who considered target orbits of small eccentricity, but included only linear terms in the gravity model.

The equations of motion are given in terms of a rotating coordinate system centered at the target as in Figure 1.2. The equations of motion, before any approximations are made, were developed previously in time of non-dimensional variables as Equation (1.22). For convenience , this set is reproduced below-along with the definition of the non-dimensional variables.

$$\ddot{x} - y\ddot{\theta} - 2\dot{y}\dot{\theta} - \dot{\theta}^2 x - \left(\frac{1}{\rho^2}\right) + \frac{(x+\rho)}{(y^2+(x+\rho)^2+z^2)^{3/2}} = 0$$

$$\ddot{y} + x\ddot{\theta} + 2\dot{y}\dot{\theta} - \dot{\theta}^2 y + \frac{x}{(y^2+(x+\rho)^2+z^2)^{3/2}} = 0$$

$$\ddot{z} + \frac{z}{(y^2+(y+\rho)^2+z^2)^{3/2}} = 0$$

where

$$x = \frac{x}{a_T} \qquad y = \frac{y}{a_T} \qquad z = \frac{z}{a_T} \qquad \rho = \frac{r_1}{a_T} \qquad M = \left(\frac{\mu}{a_T^3}\right)^{1/2} T$$

and where the open dot superscript (e.g., $\mathring{x}$) refers to differentiation with respect to M. Expanding the nonlinear terms of the differential equations in powers of the coordinates and retaining linear and quadratic terms results in the set.

$$\begin{aligned} &\ddot{x} - y\ddot{\theta} - 2\dot{x}\dot{\theta} - \left(\frac{2}{\rho^3} + \dot{\theta}^2\right) x - \frac{3}{2\rho^4(y^2-2x^2+z^2)} = 0 \\ &\ddot{y} + x\ddot{\theta} + 2\dot{x}\dot{\theta} + \left(\frac{1}{\rho^3} - \dot{\theta}^2\right) y - \frac{3xy}{\rho^4} = 0 \\ &\ddot{z} + \frac{z}{\rho^3} - \frac{3zx}{\rho^4} = 0 \end{aligned} \tag{1.66}$$

Now, for orbits of small eccentricity, the variation of θ and ρ with time (and ultimately with M) can be written as a series expansion in the eccentricity, e.g.

$$\begin{aligned} &\mathring{\theta} = 1 + 2e\cos(\tau-\tau_p) + \frac{5}{2}e^2\cos 2(\tau-\tau_p) + \ldots \\ &\rho = 1 - e\cos(\tau-\tau_p) + \left(\frac{e^2}{2}\right)\left[1-\cos 2(\tau-\tau_p)\right] + \ldots \end{aligned} \tag{1.67}$$

where the subscript p refers to the condition at the time of periapsis passage.

If the nonlinear terms are omitted and the target orbit is circular, Equation (1.66) is identical to Equation (1.49) and a solution for x, y, z can be found in terms of the initial conditions x_0, y_0, z_0, $\mathring{x}_0$, $\mathring{y}_0$, $\mathring{z}_0$ by the use of the transition matrix Equation (1.58). If this solution is denoted by the subscript "c" then

$$x_c = \mathring{x}_0 \sin\tau - 2(2\mathring{y}_0 + 3\mathring{x}_0)\cos\tau + 2(2x_0 + \mathring{x}_0)$$

$$y_c = 2(2\mathring{y}_0 + 3x_0)\sin\tau + 2\mathring{x}_0\cos\tau - 3(2x_0 + \mathring{y}_0)\tau - (2\mathring{x}_0 - y_0)$$

$$z_c = \mathring{z}_0 \sin\tau + z_0\cos\tau$$

The solution to the nonlinear equation will be defined in terms of this solution and small corrections; that is, the solutions to the set (1.66) have the form

$$x = x_c + \delta x$$

$$y = y_c + \delta y$$

$$z = z_c + \delta z$$

If this solution set is substituted in (1.66), a set of differential equations for the variables δx , δy , and δz is produced. This set is then simplified by neglecting the smaller terms such as $x_c\,\delta x$, $e\,\delta x$, $e\,x_c^2$, $e^2 y_c$ etc. The resulting differential equations are

$$\delta\overset{\circ\circ}{x} - 2\delta\mathring{y} - 3\delta x = \frac{3}{2}(z_c^2 - y_c^2 - 2x_c^2) + e\big[(10x_c + 4\mathring{y}_c)\cos(\tau-\tau_p) - 2y_c\sin(\tau-\tau_p)\big]$$

$$\delta\overset{\circ\circ}{y} + 2\delta\mathring{x} = 3x_c y_c - e\big[(4\mathring{x}_c - y_c)\cos(\tau-\tau_p) - 2x_c\sin(\tau-\tau_p)\big]$$

$$\delta\overset{\circ\circ}{z} + \delta z = 3x_c z_c - 3e z_c\cos(\tau-\tau_p)$$

This set of equations is linear in terms of the known forcing functions; therefore, the solution is straightforward. For convenience the solution is given in two parts indicated by

$$\delta x = \delta x^o + \delta x^e\,,\quad \delta y = \delta y^o + \delta y^e\,,\quad \delta z = \delta z^o + \delta z^e$$

where the superscript 0 denotes the solution when the target orbit is circular, and the superscript e denotes the effect of small eccentricity on the solution. These solutions are given by

$$\delta x^p = A_0^p + A_1^p\sin\tau + A_2^p\cos\tau + A_3^p\sin 2\tau + A_4^p\cos 2\tau + A_5^p\tau + A_6^p\tau\sin\tau + A_7^p\tau\cos\tau + A_8^p\tau^2$$

$$\delta y^p = B_0^p + B_1^p \sin\tau + B_2^p \cos\tau + B_3^p \sin 2\tau + B_4^p \cos 2\tau$$
$$+ B_5^p \tau + B_6^p \tau \sin\tau + B_7^p \tau \cos\tau$$
$$\delta z^p = C_0^p + C_1^p \sin\tau + C_2^p \cos\tau + C_3^p \sin 2\tau + C_4^p \cos 2\tau$$
$$+ C_5^p \tau \sin\tau + C_6^p \tau \cos\tau$$

where the superscript p can be either o or e. The constants A_i^j , B_i^j , C_i^j are then given by

$$A_1^o = -12x_o\dot{x}_o - 7\dot{x}_o\dot{y}_o + 3y_o\dot{y}_o + 6x_oy_o - z_o\dot{z}_o$$

$$A_2^o = -2\dot{x}_o^2 + 18\dot{y}_ox_o + 5(\dot{y}_o)^2 + 15x_o^2 + \frac{3}{2}y_o^2 + \frac{1}{2}z_o^2 + \dot{z}_o^2$$

$$A_3^o = 2\dot{x}_o\dot{y}_o + 3x_o\dot{x}_o + \frac{1}{2}z_o\dot{z}_o$$

$$A_4^o = \frac{1}{2}\dot{x}_o^2 - 2\dot{y}_o^2 - 6\dot{y}_ox_o - \frac{9}{2}x_o^2 - \frac{1}{4}(\dot{z}_o)^2 + \frac{1}{4}z_o^2$$

$$A_5^o = -6x_oy_o - 3y_o\dot{y}_o + 6\dot{x}_o\dot{y}_o + 12x_o\dot{x}_o$$

$$A_6^o = -21\dot{y}_ox_o - 6\dot{y}_o^2 - 18x_o^2$$

$$A_7^o = -6x_o\dot{x}_o - 3\dot{x}_o\dot{y}_o$$

$$A_8^o = 18x_o^2 + 18\dot{y}_ox_o + \frac{9}{2}\dot{y}_o^2$$

$$B_1^o = 3\left(\frac{5}{2}x_o\dot{x}_o + \dot{x}_o\dot{y}_o - x_oy_o + \frac{1}{2}z_o\dot{z}_o\right)$$

$$B_1^o = 3\dot{x}_oy_o - 36x_o\dot{y}_o - 10\dot{y}_o^2 - 2\dot{x}_o^2 - 30x_o^2 - 3y_o^2 - z_o^2 - 2\dot{z}^2$$

$$B_2^o = -6x_o\dot{x}_o - 2\dot{x}_o\dot{y}_o + 3x_oy_o - 2z_o\dot{z}_o$$

$$B_3^o = -\dot{y}_o^2 + \frac{1}{4}\dot{x}_o^2 - 3x_o\dot{y}_o - \frac{9}{4}\dot{x}_o + \frac{1}{4}\dot{z}_o^2 - \frac{1}{4}z_o^2$$

$$B_4^o = -\dot{x}_o\dot{y}_o - \frac{3}{2}x_o\dot{x}_o + \frac{1}{2}z_o\dot{z}_o$$

$$B_5^o = 3\left(y_o^2 + \frac{11}{2}x_o^2 + 2\dot{y}_o^2 + \frac{1}{2}\dot{x}_o^2 + \frac{1}{2}\dot{z}^2 + \frac{1}{2}z_o^2 + 7x_o\dot{y}_o - \frac{2}{3}\dot{x}_oy_o\right)$$

$$B_6^o = -6x_o\dot{x}_o - 3\dot{x}_oy_o$$

$$B_7^o = 21x_o\dot{y}_o + 6\dot{y}_o^2 + 18x_o^2$$

$$C_0^o = \frac{3}{2}(\dot{x}_o\dot{z}_o - 2\dot{y}_oz_o - 3x_oz_o)$$

$$C_1^o = \dot{x}_oz_o + \dot{y}_o\dot{z}_o + 3x_o\dot{z}_o$$

$$C_2^o = -2\dot{x}_o\dot{z}_o + 2\dot{y}_oz_o + 3x_oz_o$$

$$C_3^o = -\frac{1}{2}(\dot{x}_oz_o - 2\dot{x}_o\dot{z}_o - 3x_o\dot{z}_o)$$

$$C_4^o = \frac{1}{2}(2\dot{y}_oz_o + 3x_oz_o + \dot{x}_o\dot{z}_o)$$

$$C_5^o = \frac{3}{2}(4x_oz_o + 2y_oz_o)$$

$$C_6^o = \frac{3}{2}(2\dot{y}_o\dot{z}_o - 4x_o\dot{z}_o)$$

$$A_0^e = (3\dot{x}_0 - 2y_0)\sin\tau_p - (13x_0 + 4\dot{y}_0)\cos\tau_p$$

$$A_1^e = \dot{y}_0 \sin\tau_p + 2x_0' \cos\tau_p$$

$$A_2^e = -2(2\dot{x}_0 - y_0)\sin\tau_p + (2\dot{y}_0 + 10x_0)\cos\tau_p$$

$$A_3^e = (2\dot{y}_0 + 3x_0)\sin\tau_p - \dot{x}_0\cos\tau_p$$

$$A_4^e = \dot{x}_0\sin\tau_p + (2\dot{y}_0 + 3x_0)\cos\tau_p$$

$$A_5^e = 0$$

$$A_6^e = (3\dot{y}_0 + 6x_0)\cos\tau_p$$

$$A_7^e = -(3\dot{y}_0 + 6x_0)\sin\tau_p$$

$$A_8^e = 0$$

$$B_0^e = (3\dot{y}_0 + \tfrac{7}{2}x_0)\sin\tau_p - (y_0 + \tfrac{1}{2}\dot{x}_0)\cos\tau_p$$

$$B_1^e = 3(2\dot{x}_0 - y_0)\sin\tau_p - 12x_0\cos\tau_p$$

$$B_2^e = -(6\dot{y}_0 + 8x_0)\sin\tau_p + (y_0 + 2\dot{x}_0)\cos\tau_p$$

$$B_3^e = -\tfrac{3}{2}\dot{x}_0\sin\tau_p - \tfrac{3}{2}(2\dot{y}_0 + 3x_0)\cos\tau_p$$

$$B_4^e = \tfrac{3}{2}(2\dot{y}_0 + 3x_0)\sin\tau_p - \tfrac{3}{2}\dot{x}_0\cos\tau_p$$

$$B_5^e = (3y_0 - 3\dot{x}_0)\sin\tau_p + (3\dot{y}_0 + 15x_0)\cos\tau_p$$

$$B_6^e = 3(\dot{y}_0 + 2x_0)\sin\tau_p$$

$$B_7^e = 3(\dot{y}_0 + 2x_0)\cos\tau_p$$

$$C_0^e = -\tfrac{3}{2}\dot{z}_0\sin\tau_p - \tfrac{3}{2}z_0\cos\tau_p$$

$$C_1^e = -z_0\sin\tau_p - \dot{z}_0\cos\tau_p$$

$$C_2^e = -2\dot{z}_0\sin\tau_p + z_0\cos\tau_p$$

$$C_3^e = \tfrac{1}{2}z_0\sin\tau_p + \tfrac{1}{2}\dot{z}_0\cos\tau_p$$

$$C_4^e = -\tfrac{1}{2}\dot{z}_0\sin\tau_p + \tfrac{1}{2}z_0\cos\tau_p$$

$$C_5^e = 0$$

$$C_6^e = 0$$

A comparison of the linear and quadratic solutions is given in Figure 1.3, 1.4, and 1.5, (from Reference 1.2) for a target orbit with apogee and perigee altitude of 400 miles and 200 miles. The figures were generated by assuming the relative position was zero at time zero, but that a non-zero relative velocity existed as indicated on each figure.

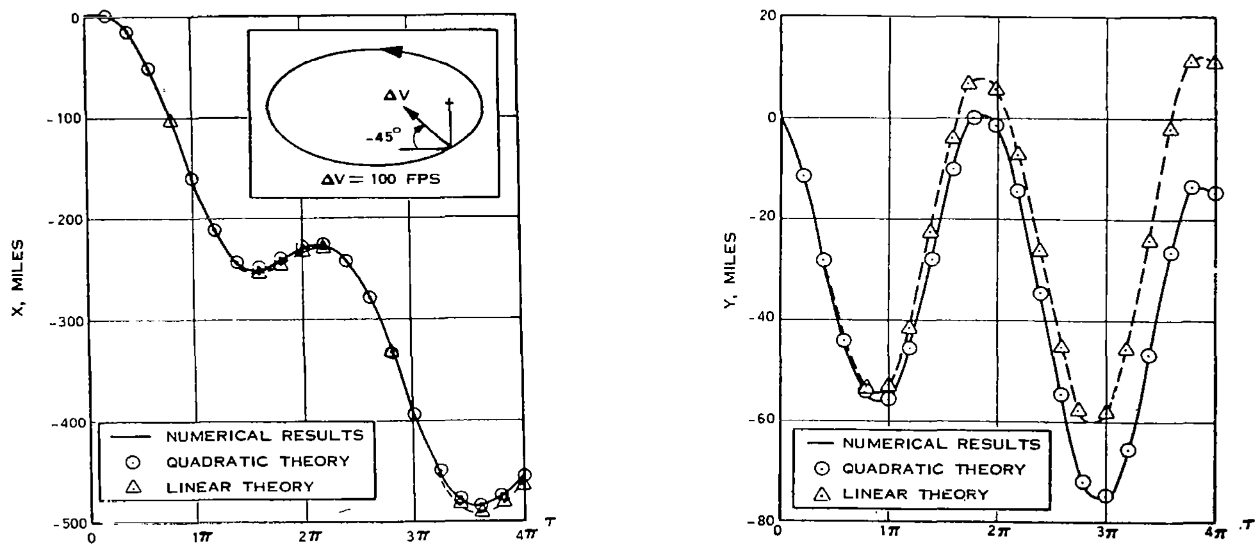


Figure 1.3
Comparison of Linear, Quadratic, & Exact (Numerical) Theory for 100 FPS Velocity Increment

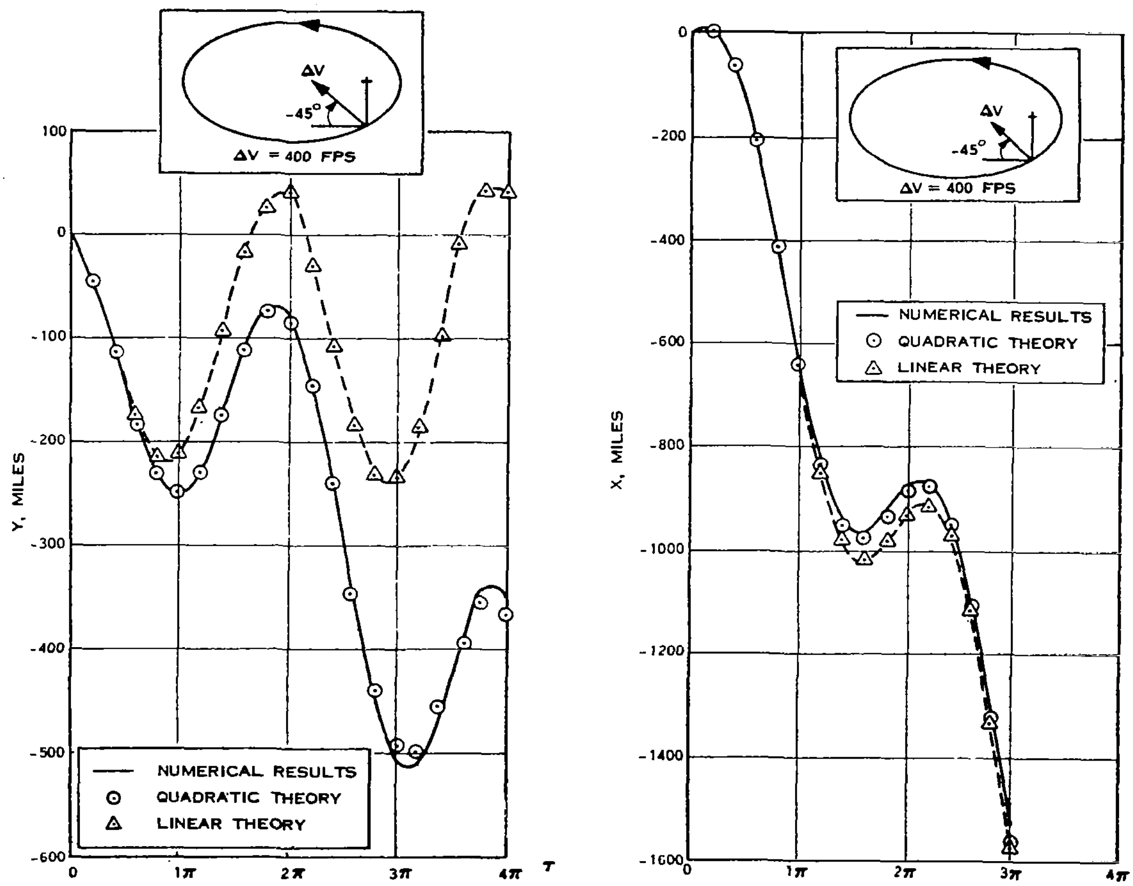


Figure 1.4
Comparison for Velocity Increment of 400 FB

As expected, the linear solutions are accurate for a short time until the distance involved becomes large. This divergence occurred in approximately 1/2 revolution for the trial cases. Thereafter, the linear results differ substantially from the actual (numerical) solution partically in the vertical component. The quadratic analysis yields accurate results for approximately two revolutions of the target.

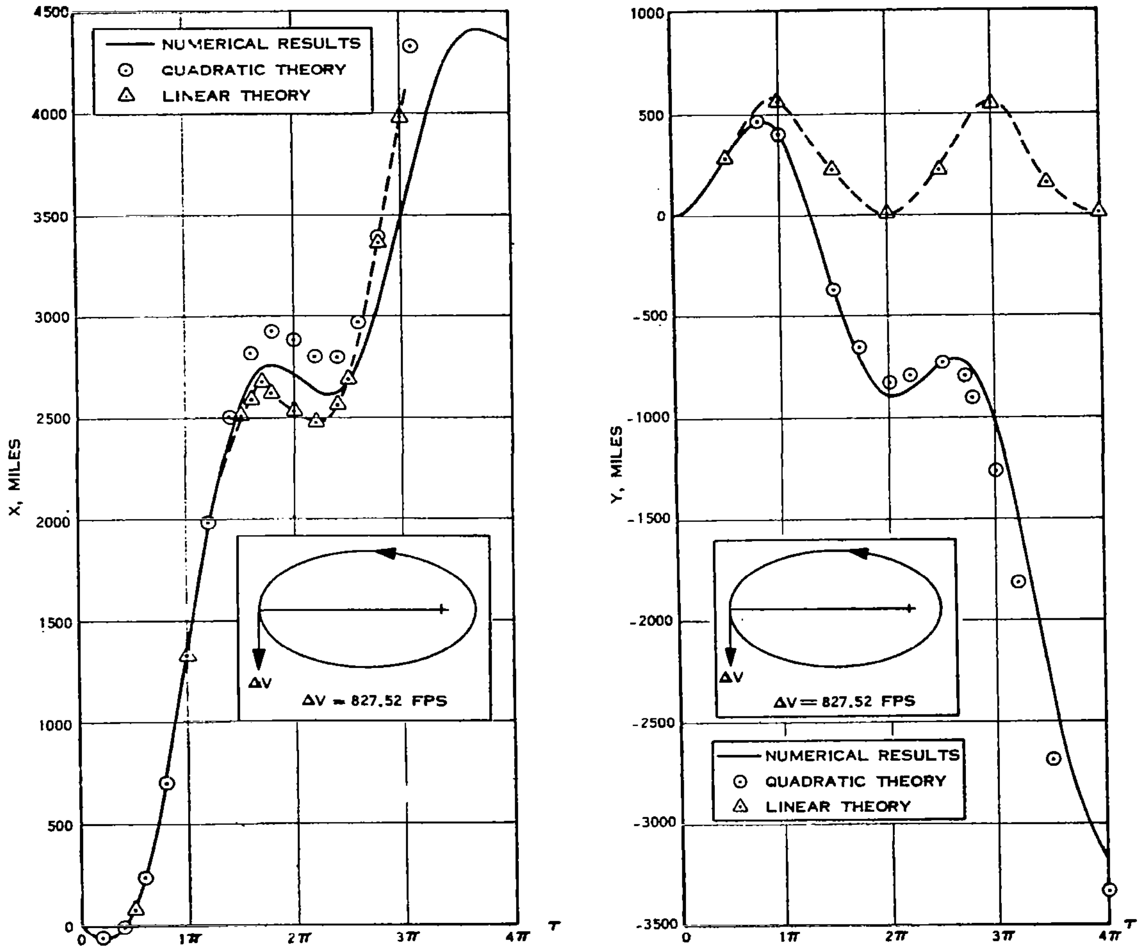


Figure 1.5
Extreme Case Comparison

Figures 1.5 illustrate a rather extreme case of velocity difference in which even the quadratic theory breaks down rapidly. However, even if this case for relative distance is less than 2500 miles, the predictions are quite accurate.

2.1.6 The Effects of Perturbations on Rendezvous

Satellite orbits are perturbed from pure conic sections by forces due to the earth's atmosphere, the non-sphericity of the earth, the attraction of the moon or sun, and the radiation pressure of solar radiation. As was indicated earlier, the term Δg really represents the difference between the accelerations of the two objects due to all forces. The earth's atmosphere could cause a significant effect for rendezvous maneuver sufficiently low, but the assumption in this study is that the maneuver is as a sufficiently high altitude (>150 Km) that air drag is negligible. The largest of the remaining forces in the region near the earth is that due to the oblateness of the earth. The magnitude of this force is about 10^{-3} times that of the inverse square force and while its gradient coefficient is twice that of the main term, it is clear that the magnitude of the contribution to Δg is not more than 1/500 that of the main term which is itself expressed only approximately. Consequently, all perturbative differences can be safely neglected in studying rendezvous in orbits near single attracting centers that are no more oblate than the earth. Rendezvous in earth-moon space is thus considered in the monograph only if near enough to one of the attracting centers, that the perturbative force due to the other is substantially smaller than the main gravity turn.

2.2 GUIDANCE EQUATIONS

In this section the methods that have been proposed for mechanizing the rendezvous are presented. In several cases, guidance schemes are chosen without regard to the degree of optimization; however, when the approach is taken, the considerations will be detailed [see Section (2.4)].

2.2.1 Fixed Inertial Line of Sight (LOS) - Free Space

It has been shown in a previous section that the "natural" maneuver of thrusting in the direction of the LOS will not, in general, produce rendezvous. However, for the special case with the relative velocity vector aligned parallel to the LOS, then thrust thrusting along the LOS is the optimum method of achieving rendezvous (in free space). This fact is demonstrated in Section 2.4 where optimization is discussed. This solution suggests that a maneuver which first orients the relative velocity vector along the LOS could be advantageous. Aligning the velocity vector in this way is equivalent to nulling the angular rate of the LOS in inertial space and can be accomplished by applying thrust normal to the LOS. The overall maneuver, thus, retains a degree of naturalness in that an astronaut performing a manual rendezvous can easily determine the directions in which the thrust is to be applied.

The discussion presented here is limited to its application as a manual back up guidance technique for Gemini as presented in papers by Chamberline and Rose, and Burton and Hayes (References 2.2 and 2.3), and to an extension of the technique by Steffan (Reference 2.4) which separates the guidance and navigation tasks. Because of the approximation that there is no relative acceleration due to gravity, the range of initial conditions for which this technique has acceptable accuracy is limited. A larger set of initial conditions can be handled if a number of "midcourse" corrections are made. However, from an efficiency standpoint, these midcourse corrections are undesirable. The principal advantage of this method lies in its use as a natural basis for manual guidance.

2.2.1.1 Manual Rendezvous Guidance

The technique of fixing the direction of the line of sight in inertial space effectively uncouples the linear and angular motion and reduces the problem to one dimension. This feature is a particularly useful in the case where a pilot is manually performing the rendezvous. Such a system is considered as a back-up for the Gemini missions (Reference 2.3); for the Gemini scheme, the pilot visually observes the relative motion between the spacecraft and the target vehicle with respect to a star background. Range and range-rate information are provided by radar or optical means. When the two vehicles are within a preselected distance, the pilot initiates a thrust maneuver normal to the line of sight until he observes that the relative (angular) motion has been eliminated. This process is continued throughout the rendezvous whenever relative motion is again noticed. The range and rate range are monitored so that the time to begin the braking maneuver can be determined.

2.2.1.2 Separation of Guidance-Navigation Tasks

In the previous section, the astronaut performing the rendezvous maneuver was required to navigate (i.e., determine when the relative motion has ceased) during periods of thrust application. A technique developed by Steffan (Reference 2.4) determines the time duration of the thrusting from data taken before thrust initiation. The angular rate of the LOS is allowed to oscillate between limits with the period of oscillation determined by thrusting in the LOS direction. This technique requires the application of several velocity increments normal to the LOS, the times of these applications are related to the period of the limit cycle and are controlled by controlling the range rate. The desired period of the limit cycle is then chosen so that the time between corrections is sufficient to allow for data taking and processing. This time will vary depending on how the data is being taken and processed, e.g., a range radar feeding information directly to a computer vs. optical measurements and hand calculations by an astronaut.

By the use of the rocket motor normal to the LOS, the rendezvous vehicle is established on a collision course with the target vehicle such that the direction of the LOS is stabilized, to within some limits, in inertial space. The approximate behavior of this limit cycle can be determined analyzing the expressions for the angular rate of the LOS as a function of time for (1) termination of normal thrust and (2) time for the initiation of the normal thrust. First the case of no thrust is considered.

A polar coordinate system will be used to describe the motion. In this system, the range (ρ) is defined as the distance from the rendezvous vehicle to the target vehicle; γ is measured from an inertial reference direction; and the origin of the coordinate system is at the target vehicle.

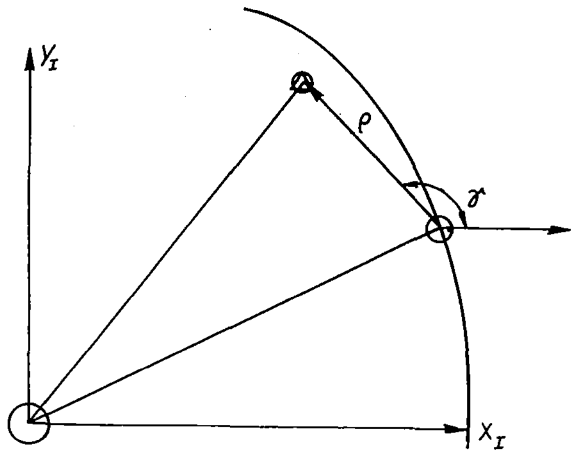


Figure 2.1
Polar Coordinate System

(Since the out-of-plane motion is uncoupled, only motion in two dimensions is considered.) Now, since for the case under consideration, there is no relative acceleration between the two vehicles due to the gravity gradient; thus the angular momentum of the system will remain constant (for period of no thrust). i.e.,

$$\rho^2 \dot{\gamma} = \rho_0^2 \dot{\gamma}_0 \tag{2.1}$$

The subscript, o, refers to some initial time. Therefore, if the normal control has been operating, the velocity vector will lie along the line of sight (to the first order) and the range as a function of time will be

$$\rho \approx \rho_0 + \dot{\rho}_0 t \tag{2.2}$$

With the use of this expression for range, the angular momentum equation can be written as

$$\dot{\gamma} \approx \frac{\dot{\gamma}_0}{\left(1 + \frac{\dot{\rho}}{\rho_0} t\right)^2} \tag{2.3}$$

Equation (2.3) is the desired relation for the angular rate ($\dot{\gamma}$) of the LOS for periods of free motion.

The kinetic energy of the system is

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left[\dot{\rho}^2 + (\rho \dot{\gamma})^2\right]$$

Thus, Lagranges equations of motion for periods of normal thrusting are seen to be

$$\rho \dot{\gamma}^2 + \ddot{\rho} = 0$$
$$\rho \ddot{\gamma} + 2\dot{\rho}\dot{\gamma} = a_n$$

where a_n is the acceleration due to the normal thrust. Now substituting relation for ρ as a function of time from Equation (2.2) in the second equation above gives

$$(\rho_0 + \dot{\rho}_0 t)\ddot{\gamma} + 2\dot{\rho}_0 \dot{\gamma} = a_n$$

or

$$\ddot{\gamma} + \left(\frac{2\dot{\rho}_0}{\rho_0 + \dot{\rho}_0 t}\right)\dot{\gamma} = \frac{a_n}{\rho_0 + \dot{\rho}_0 t}$$

This equation can be considered a first order differential equation in the variable $\dot{\theta}$, i.e.,

$$\frac{d\dot{\gamma}}{dt} + \left(\frac{2\dot{\rho}_0}{\rho_0 + \dot{\rho}_0 t}\right)\theta = \frac{a_n}{\rho_0 + \dot{\rho}_0 t}$$

But this equation has as a general solution

$$\dot{\gamma} = \frac{1}{(\rho_0 + \dot{\rho}_0 t)^2}\left[a_n t \left(\rho_0 + \frac{\dot{\rho}_0 t}{2}\right) + \rho_0^2 \dot{\gamma}_0\right] \qquad (2.4)$$

Equation (2.4) is the solution for the angular rate of the LOS ($\dot{\theta}$) during periods of normal thrusting. This equation, however, is not the one employed to determine the length of time that the normal thrust is to be applied, but rather the simpler equation which assumes $\dot{\rho} = 0$ is used. The time of normal thrusting is calculated as

$$t_n = \frac{\rho \dot{\gamma}_0}{a_n} \qquad (2.5)$$

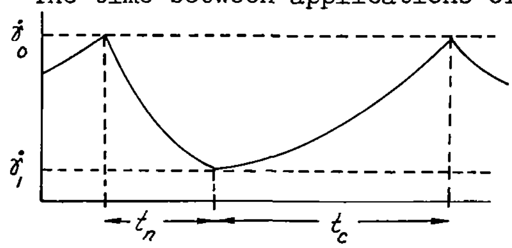
That is, if $\dot{\gamma}_0$ is the threshold value of $\dot{\gamma}$ and it is desired to drive $\dot{\gamma}$ to zero the length of time that normal thrusting should be applied is calculated from (2.5). If the motor used for normal control is actually operated for this time, $\dot{\gamma}$ will not be driven to zero. Rather, the value that it will attain, $\dot{\gamma}_1$, can be calculated by substituting the value of t_n from Equation (2.5) into Equation (2.4). This substitution gives

$$\dot{\gamma}_1 = \frac{a_n \dot{\rho}_0 \dot{\gamma}_0^2}{2(a_n - \dot{\rho}_0 \dot{\gamma}_0)^2}$$

Thus, assuming $|\dot{\gamma}_0 \dot{\rho}_0| << |a_n|$ (this approximation becomes better as $r \rightarrow 0$ as will be seen when motion along the LOS is discussed) the expression becomes

$$\dot{\gamma}_1 = \frac{\dot{\rho}_0}{2a_n} \dot{\gamma}_0^2 \tag{2.6}$$

The overall behavior of $\dot{\theta}$ can thus be determined from this equation and from the one given earlier for a no-thrust condition [Equation (2.3)]. This overall behavior will be a limit cycle between the values $\dot{\gamma}_0$ and $\dot{\gamma}_1$ as characterized in Figure (2.4). The time between applications of normal thrust (t_c) and $\emptyset$ can be calculated from Equations (2.3) and (2.6). Letting $\tau = -\frac{\rho_0}{\dot{\rho}_0}$ (the time until rendezvous occurs assuming no radial thrust is applied) quation (2.3) results in



$\dot{\gamma}$ Limit Cycle
Figure 2.2

$$1 - \frac{t_c}{\tau} = \sqrt{\frac{\dot{\gamma}_1}{\dot{\gamma}_0}}$$

or

$$t_c = \tau\left(1 - \sqrt{\frac{\dot{\gamma}_1}{\dot{\gamma}_0}}\right)$$

Finally, substituting for $\dot{\gamma}_1$ from Equation (2.6) gives

$$t_c = \tau\left(1 - \sqrt{\frac{\dot{\rho}_0 \dot{\gamma}_0}{2a_n}}\right) \tag{2.7}$$

A certain minimum time will be necessary to take and process the data to determine the current position and velocity. Therefore, it will be necessary to require that the time between corrections, t_c, be longer than the minimum data taking time. Since $\dot{\rho}$ is to be driven to zero and $\dot{\gamma}_0$ is small, it can be seen from Equation (2.7) that a minimum t_c will be maintained if a minimum τ is maintained. In turn, a constant τ (or more practically a constant range of values for τ) is obtained by a proper choice of thrusting periods for the LOS thruster. A discussion of motion along the LOS is now called for so that the behavior of $\dot{\gamma}$ during periods of LOS thrusting can be determined.

If the angular-rate control system is operating, the approximate range-rate and range equations for periods of LOS thrusting are

$$\dot{\rho} = \dot{\rho}_0 + a_\ell t$$
$$\rho = \rho_0 + \dot{\rho}_0 t + \frac{1}{2} a_\ell t^2$$

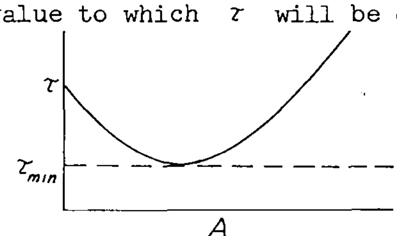
where a_ℓ is the acceleration along the line of sight. The corresponding expression for τ is thus

$$\tau = -\frac{\dot{\rho}}{\rho} = -\frac{\rho_0 + \dot{\rho}_0 t + \frac{1}{2} a_\ell t^2}{\dot{\rho}_0 + a_\ell t}$$

and

$$\frac{d\tau}{dt} = -\left[1 - a_\ell \frac{(\rho_0 + \dot{\rho}_0 t + \frac{a_\ell}{2} t^2)}{(\dot{\rho}_0 + a_\ell t)^2}\right] \qquad (2.8)$$

If the desired limits of τ are $\tau_{min} \leq t \leq \tau_{max}$ and if the slope $\frac{d\tau}{dt}$ is positive at t = 0, then the LOS thrusting is initiated whenever $\tau \leq \tau_{min}$ and terminated whenever $\tau \geq \tau_{max}$. If the slope of $\frac{d\tau}{dt}$ is negative at t=0, then τ will continue to decrease up to the point at which $\frac{d\tau}{dt}$ changes sign. (Figure 2.3) In this case, the minimum value to which τ will be driven must be predicted so that thrusting can begin sufficiently early to prevent $\tau \leq \tau_{min}$. The point at which reaches its minimum value is determined by solving Equation (2.8) with $\frac{d\tau}{dt} = 0$. The result is



Motion of τ

Figure 2.3

$$t_{min} = \frac{1}{a_\ell}\left[-\dot{\rho}_0 - \sqrt{2 a_\ell \rho_0 - \dot{\rho}_0^{\,2}}\right]$$

and the corresponding value of τ_{min} obtained by substituting this time is the equation for τ is:

$$\tau_m = \frac{2\rho_0 - \frac{\dot{\rho}^2}{a_\ell}}{\sqrt{2 a_\ell \rho_0 - \dot{\rho}_0^{\,2}}}$$

This equation may now be solved for ρ_o and replacing τ_m by τ_{min} (the desired lower limit)

$$\rho_o = \frac{\dot{\rho}_o^2}{2a_\ell} + \frac{a_\ell}{2}\tau_{min}^2$$

Thus, for the case when the initial slope $\frac{d\tau}{dt}$ is negative the motor is fired whenever this equation is satisfied and firing is terminated whenever $\tau \geq \tau_{max}$. If this scheme is used to control the velocity along the line of sight, the time until rendezvous will remain between the limits τ_{min} and τ_{max} and the range will be forced to decrease approximately exponentially with time. As the range becomes sufficiently small, the range rate control loop is opened and a docking maneuver initiated.

2.2.2 Coriolis Balance

The two previous sections considered guidance schemes based on nulling the angular rate of the line of sight. In this section, a technique where the line of sight is allowed to rotate at a constant rate is considered. It will be seen that this restriction is equivalent to requiring that the force normal to the line of sight be equal to the coriolis acceleration. Hence, the descriptive title "Coriolis Balance". (Reference 2.5).

Using a polar coordinate system centered at the target vehicle and assuming no relative gravitational acceleration, the equations of motion are (as in Section 2.1)

$$\begin{aligned} \ddot{\rho} - \rho\dot{\gamma}^2 &= a_R \\ \rho\ddot{\gamma} + 2\dot{\rho}\dot{\gamma} &= a_n \end{aligned} \tag{2.9}$$

Now, if the LOS acceleration, a_R, equal to zero and the normal acceleration is equal to the coriolis acceleration ($a_n = 2\dot{\rho}\dot{\gamma}$) these equations become

$$\ddot{\rho} - \rho\dot{\theta}^2 = 0$$

$$\ddot{\gamma} = 0$$

But, the second equation shows that the angular rate is constant and that the range equation can be written as

$$\ddot{\rho} - k^2\rho = 0\,, \quad k = \dot{\gamma} = \text{CONSTANT} \tag{2.10}$$

Equation (2.10) now shows that the coriolis balance technique also uncouples the angular motion from the range motion. The problem is now to show that a collision will result if this acceleration is applied. Consider the solution to Equation (2.10) at the boundary $\rho = 0$.

$$\rho = 0 = \frac{\dot{\rho}_0}{k} \sinh kt + \rho_0 \cosh kt = 0$$

This equation will have a solution for positive time, if $\dot{\rho}_0 < 0$ and $|\dot{\rho}_0 k| > \rho_0$. If these conditions are not met, or if the time to rendezvous is unsuitable, a velocity impulse along the line of sight can be added to remedy the situation.

The solution to the Equations (2.9) for an acceleration is applied along the line of sight (u_R) is now

$$\rho = \frac{\dot{\rho}_0}{k} \sinh kt + \left(\rho_0 + \frac{a_R}{\rho^2}\right) \cosh kt - \frac{a_R}{k^2}$$

$$\dot{\rho} = \left(k \rho_0 + \frac{a_R}{k}\right) \sinh kt + \dot{\rho}_0 \cosh kt$$

If the conditions for rendezvous (i.e., $\rho = 0$, $\dot{\rho} = 0$) are inserted in these equations, it can be seen that a solution can be obtained for a_r equal to a constant. Thus, the rocket motor furnishing thrust along the line of sight need not be throttleable. However, the normal thrust motor must furnish thrust according to

$$a_n = 2 \dot{\rho} \dot{\gamma}$$

and is therefore required to be throttleable.

The author or Reference (2.5) compares this scheme to a scheme which nulls the angular rate similar to that described in the last section. The comparison indicated that the coriolis balance technique requires a lower thrust level for both the range and normal rocket motors; however, the total impulse and flight time are greater.

2.2.3 Improved Model - The Inclusion of a Gravity Gradient

An improvement in the description of the gravity model can be made which will increase the range of initial conditions over which the preceding methods are valid, yet still retain the simple expressions for determining the duration of thrusting. The improved gravity model consists of approximating the difference in the gravitational acceleration of the two vehicles (Δg) by

$$\Delta g = \frac{\partial g}{\partial r} \Delta \bar{r}$$

If the matrix $\left[\frac{\partial g}{\partial r}\right]$ is taken as a function of time, then the linear gravity model discussed in Section 2.1.3 can be applied. However, the matrix can be approximated by a constant for short time periods and a less than precise gravity model suitable for improving the free space model is obtained. The value of the constant matrix can be obtained as the average value of $(\partial g/\partial r)$ during the thrusting period, i.e., if K denotes the constant value which approximates $(\partial g/\partial r)$ then

$$K = \frac{\int_0^T \left(\frac{\partial g}{\partial r}\right) dt}{\int_0^T dt}$$

Using this approximation, the equations of motion are

$$\ddot{\underline{\rho}} - K\underline{\rho} = \underline{a}$$

A solution to this equation is easily obtained if it is rewritten as set of two first-order equations

$$\underline{y}_1 \triangleq \underline{\rho}$$

$$\underline{y}_2 \triangleq \dot{\underline{\rho}}$$

then

$$\dot{\underline{y}}_1 = \underline{y}_2$$

$$\dot{\underline{y}}_2 = K\underline{y}_1 + \underline{a}$$

or

$$\dot{\underline{y}} = A\underline{y} + B\underline{a}$$

where

$$A = \begin{pmatrix} 0 & I \\ K & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ I \end{pmatrix}$$

The solution for Y is

$$y = e^{At} y(0) + \int_0^t e^{t-\tau} B\underline{a} \, d\tau$$

In the previous sections, only the integral term on the right hand side of the above equation was obtained. Thus, the only modification to the equations developed in those sections is the addition of the expression involving the initial conditions. Mechanization is, thus, similar.

2.2.4 Impulsive Rendezvous Techniques

In this section, the development of guidance equations in a linear gravity field will be restricted to determining an impulsive velocity correction such that a rendezvous will take place at some time in the future. The techniques of this section can be thought of as defining a reference trajectory along which the rendezvous vehicle is to travel. Once this reference trajectory has been established, the actual steering equations (i.e., orientation of the thrust vector as a function of time) can be determined by the methods discussed in a previous monograph of this series on boost-guidance equations (Reference 2.16). The section below gives a discussion of two simple techniques for utilizing calculated velocity impulses.

Impulsive velocity changes have been considered in detail in the monograph on midcourse guidance (Reference 2.15) and this reference is recommended for a more complete formulation including the effects of errors in data and optimization of multiple impulse schemes.

2.2.4.1 Approximating Velocity Impulses

The use of a model having a linear variation in gravity generates the necessity of neglecting the effects of finite burning time in order that closed form solutions be obtainable. Although, the use of impulsive velocity changes is essential to the simplification of the equations, their physical realization can be only approximate. If a true impulse could be achieved, then the guidance mechanization would simply be to orient the thrustor along the direction of the required velocity change and apply an impulse equal to the magnitude of the required change. Since the rocket motor must burn for a finite time (the length of time depends upon the thrust capability in relation to the magnitude of the velocity increment) the orientation of the required velocity increment will, generally, not be the same at the initiation and termination of the thrusting period. Thus, if the rocket

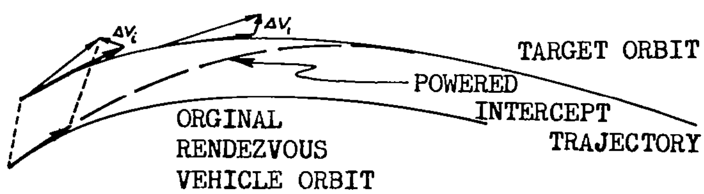


Figure 2.4
Direction of Velocity Correction

Rocket motors were oriented along the required ΔV vector and maintained in that orientation throughout the thrusting period, the resulting velocity increment would have the correct magnitude, but it would be directly incorrectly.

One method which can be employed (and has been employed in the Gemini missions (Reference 2.3)) to compensate for the finite thrusting time is to center the thrusting period about the time at which the velocity impulse is to be applied. For example, if it is calculated, that the velocity change to be applied at t = to to achieve rendezvous is ΔV, and the acceleration produced by the rocket motor is T/M then the length of time the motor is required to burn is

$$\Delta t = \Delta V \cdot \frac{M}{T}$$

The thrust period is then centered about time t_o by initiating the thrust at $t = t_o - \frac{\Delta t}{2}$.

Another method of employing impulsive computations which can be used when the computation time is negligible compared to the thrust period is afforded via definition of required velocity. Required velocity is that velocity necessary to achieve rendezvous from the present position in the absence of external forces (except gravity). In this method, the thrust vector is realigned along a new velocity to be gained vector (i.e., the difference between the required velocity and the actual velocity) at each computation cycle and thrust is terminated when the velocity to be gained is driven to zero. In order to insure that the computation of required velocity can be accomplished rapidly, it is written as an expansion about its initial value (the calculation of which may take considerably more time than the subsequent calculations).

This method, discussed by Gunckel in Reference (2.8), requires that a multiplication by a constant matrix and an addition be performed to determine the required velocity at the next time point. That is, if an impulsive velocity has been determined at t = 0 and thrust initiated, then at time Δt, the actual velocity is compared with the velocity calculated from the following equation to determine if thrusting should continue.

$$\underline{V}_R(\Delta t) = \underline{V}_R(0) + \int_0^{\Delta t} g \, dt + \int_0^{\Delta t} \left[\frac{\partial V_R}{\partial f(0)}\right] \int_0 (\bar{V} - \bar{V}_R)_{t=0} \, dt$$

V_R	=	required velocity
$\underline{V}$	=	actual velocity
g	=	acceleration due to gravity
$\left[\frac{\partial V_R}{\partial \rho(0)}\right]$	=	matrix of partial derivatives of required velocity with respect to position evaluated along the desired trajectory as a function of time

This calculation is made as rapidly as necessary to provide the required accuracy. In this process, the matrix $[\partial V_R / \partial \rho(0)]$ and a are considered as constants over several computation periods. If the thrust period is relatively long, the matrix of partial derivatives and the acceleration vector will have to be updated to account for the inaccuracy in the model.

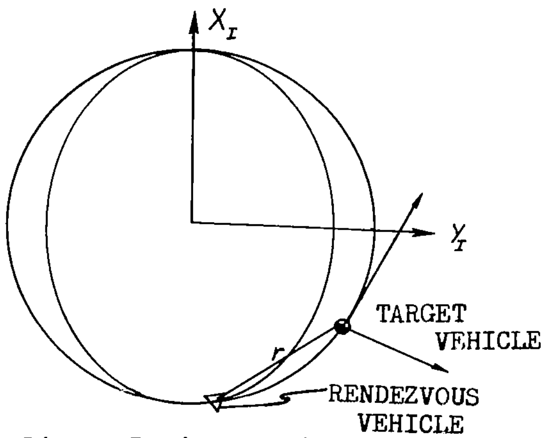
This updating can be accomplished, however, at a much slower rate than is necessary for steering and cut-off calculations. In the reference, Gunckel proposes to update one element of the matrix each computation cycle; thus, updating the complete matrix in nine cycles. Since the quantities under the integral sign are taken to be constant during the time period, the above equation can be written as

$$\underline{V}_R(\Delta t) = \underline{V}_R(0) + \underline{g}\Delta t + \frac{\partial V_R}{\partial \rho(0)} \left[(V(0)\right] - V_R(0))\Delta t$$

2.2.4.2 Two Impulse Rendezvous

The first linear impulsive guidance scheme to be considered was prepared by Clohessy and Wiltshire (Reference 2.1) and is the basis of the Gemini automatic rendezvous guidance (Reference 2.2 and 2.3). Two major velocity impulses are used through several smaller mid-course corrections calculated from the same equations and based on intermediate measurements of the state attained by the first change in velocity may also be required.

The first velocity impulse is applied at the beginning of the rendezvous maneuver and establishes the target and rendezvous vehicle on a collision course. The second impulse is required to null the range rate at the time of closure. Consider a target vehicle in a circular orbit and a coordinate system fixed to the target such that the Y axis is always along the radius from the center of attraction, the X axis is circumferential in the direction of the motion of the target, and the Z axis completes the right handed XYZ set. If the rendezvous vehicle is in an orbit which closely approximates that of the target vehicle, then the transition matrix of Equation (1.58) can be used to describe the relative motion. In this scheme, the only concern is the reduction of the range to zero; thus, the only part of the matrix which is needed is rewritten below.



Linear Rendezvous Coordinate System

Figure 2.5

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}_{t=\tau} = \begin{pmatrix} 1 & 6(\sin n\tau - n\tau) & 0 & \frac{1}{n}(4\sin n\tau - 3n\tau) & \frac{2}{n}(\cos n\tau - 1) & 0 \\ 0 & 4 - 3\cos n\tau & 0 & \frac{2}{n}(1 - \cos n\tau) & \frac{1}{n}(\sin n\tau) & 0 \\ 0 & 0 & \cos n\tau & 0 & 0 & \frac{1}{n}\sin n\tau \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ \dot{X} \\ \dot{Y} \\ \dot{Z} \end{pmatrix}_{t=0}$$

If a collision is to occur at time τ then the components on the left of this equation must be zero. If this substitution is made and the matrix multiplication indicated on the right performed, the three resulting equations are

$$0 = x_0 + 6(\sin n\tau)y_0 + \frac{1}{n}(4\sin n\tau - 3n\tau)\dot{x}_0 + \frac{2}{n}(\cos n\tau - 1)\dot{y}_0$$

$$0 = 4y_0 - 3\cos n\tau\, y_0 + \frac{2}{n}(1-\cos n\tau)\dot{x}_0 + \frac{1}{n}\sin n\tau\, \dot{y}_0$$

$$0 = \cos n\tau\, z_0 + \frac{\sin n\tau}{n}\dot{z}_0$$

Assuming that the position at t = 0 is fixed, the velocity necessary to achieve a collision at time τ can be determined by solving the above sets of equations for $\dot{X}_0$, $\dot{Y}_0$, and $\dot{Z}_0$.

$$\left.\begin{aligned}
\dot{x}_0 &= n\left\{\frac{x_0\sin n\tau + y_0[-6n\tau\sin n\tau + 14(1-\cos n\tau)]}{3n\tau\sin n\tau - 8(1-\cos n\tau)}\right\} \\
\dot{y}_0 &= n\left\{\frac{x_0\, 2(\cos n\tau - 1) + y_0[4\sin n\tau - 3n\tau\cos n\tau]}{3n\tau\sin n\tau - 8(1-\cos n\tau)}\right\} \\
\dot{z}_0 &= n z_0 \cot n\tau
\end{aligned}\right\} \qquad (2.11)$$

Thus, if the rendezvous vehicle is at a position (X_0, Y_0, Z_0) with velocity (V_{X_0}, V_{Y_0}, V_{Z_0}) and the desired time to rendezvous is τ then the impulsive velocity increment necessary is given by.

$$\Delta V_x = \dot{x}_0 - V_{x_0}$$

$$\Delta V_y = \dot{y}_0 = V_{y_0}$$

$$\Delta V_z = \dot{z}_0 = V_{z_0}$$

The velocity which must be nulled by the braking impulse can now be determined by again applying the transition matrix to the initial conditions X_0, Y_0, Z_0, $\dot{X}_0$, $\dot{Y}_0$, $\dot{Z}_0$.

$$\dot{x}_\tau = 3n\sin n\tau\, x_0 + \cos n\tau\, \dot{x}_0 + 2\sin n\tau\, \dot{y}_0$$

$$\dot{y}_\tau = 6n(\cos n\tau - 1)x_0 - 2\sin n\tau\, \dot{x}_0 + 4(\cos n\tau - 3)\dot{y}_0$$

$$\dot{z}_\tau = -n\sin n\tau\, z_0 + \cos n\tau\, \dot{z}_0$$

Note that this method requires knowledge of the time at which rendezvous is to occur. However, there are several factors which will indicate a suitable choice for τ. For example, in the Gemini rendezvous, it was found that the velocity requirements for a rendezvous maneuver which caused interceptions in three-fourths of an orbit were less than for other fractions, therefore, τ is chosen to cause such an interception (Reference 2.2). Another criteria would be the adjustment of the closing rate to be compatible with the rocket motor which is to supply the braking thrust. Data taking and processing requirements could also place a particular requirement on . For a generalization to finite thrusts, see below Section 2.3.1 and the work of Tschauner and Hempel (Reference 1.4).

2.2.4.3 Extension to Non-Circular Orbits

The state transition matrix used in the derivation of the required velocity in the last section was limited to circular orbit . If the transition matrix of Equation (1.58) is replaced by the symbolic notation

$$G(t,t_o)=\left[\begin{array}{c|c} \dfrac{\partial \underline{r}_t}{\partial \underline{r}_o} & \dfrac{\partial \underline{r}_t}{\partial \underline{V}_o} \\ \hline \dfrac{\partial \underline{V}_t}{\partial \underline{r}_o} & \dfrac{\partial \underline{V}_t}{\partial \underline{V}_o} \end{array}\right]$$

the technique of the previous section can be applied to any orbit. In the case of a circular orbit, the transition matrix has been given (Equation 1.58 or Figure 1.58). For other orbits, the matrix is available in an analytic form (Reference 1.1, 1.5, 1.6, 1.9). References 1.1 and 1.9 gives expressions for transition matrices for elliptic and hyperbolic orbits. For extension to finite thrust, see the work of Tschanner (Reference 1.5 and 1.6)

Very little use appears to have been made of the results for elliptic target orbits and it is suggested here that they should be of significant use. For example, possibly the use of the long set of approximate equations for orbits of small eccentricity developed by Anthony and Sasaki (Reference 1.2 and Section 2.2.4.5 below) could be replaced entirely and with no limitation on eccentricity.

2.2.4.4 Multiple Impulse Rendezvous

A modification of the previous scheme developed by Shapiro (Reference 2.11) employs several velocity corrections and drives the range rate to zero by the successive application of velocity pulses rather than by a single impulse at zero range. This technique achieves a degree of flexibility over the previous scheme in that it is also applicable to targets in an elliptical orbit.

The first velocity correction is computed from Equation (2.11) based on a ficticious time to rendezvous τ . Subsequent velocity corrections are computed on the basis of a time to rendezvous equal to (τ - kt) where t is the time elapsed since the initiation of the rendezvous maneuver and k is a constant. The desired velocity magnitude decreases at each computation period because of the dilated time scale and approaches zero as t $\rightarrow \tau/k$. To see that this is so, consider the solution (2.11) as t $\rightarrow \tau/k$. In this case, the approximations $\sin n(\tau - kt) \approx n(\tau - kt)$ and $\cos n(\tau - kt) \approx \frac{n^2(\tau-kt)^2}{2}$ are valid and Equation (2.2.1) reduces to

$$\dot{x} = -\frac{x}{\tau - kt} - yn \tag{2.12}$$

$$\dot{y} = -\frac{y}{\tau - kt} - x_n$$

The solution to these equations are

$$\left.\begin{aligned} x &= \left[n(\tau - kt)\right]^{1/2}\left\{A \sin\left[\frac{n(\tau-kt)}{k}\right] + B\cos\left[\frac{n(\tau-kt)}{k}\right]\right\} \\ y &= \left[n(\tau - kt)\right]^{1/2}\left\{-B \sin\left[\frac{n(\tau-kt)}{k}\right] + A\cos\left[\frac{n(\tau-kt)}{k}\right]\right\} \end{aligned}\right\} \tag{2.13}$$

From these equations, it is seen that the position and velocity will simultaneously go to zero if $k < 1$. However, if the acceleration is examined, it will be found that for $\frac{1}{2} \leq k < 1$ infinite acceleration is required as $t \rightarrow \tau/k$. Therefore, for practical rendezvous, k must be restricted to the range $0 < k < \frac{1}{2}$.

Flight control is accomplished by generating an error signal, V_e, from the continuous monitoring of the actual and desired velocity

$$V_\epsilon = \sqrt{(\dot{x}_A - \dot{x}_D)^2 + (\dot{y}_A^2 - \dot{y}_D)^2}$$

and igniting the rocket motor whenever this value reaches a specified threshold. The thrust angle (i.e., the angle between the thrust vector and the x axis) is

$$\theta = \arccos\left(\frac{\dot{X}_A - \dot{X}_D}{V_\epsilon}\right)$$

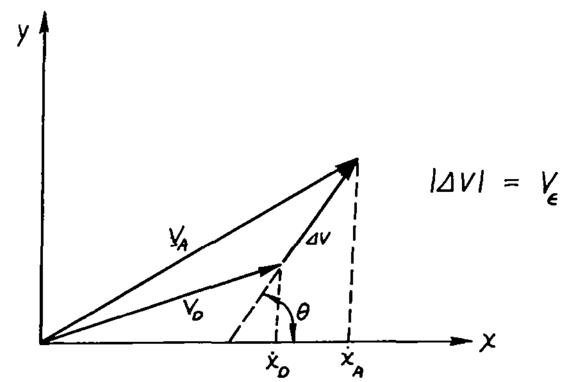


Figure 2.6
In-Plane Steering Angle

The relative out of plane motion ($\dot{Z}_D$) is sinusoidal and uncoupled from the planar motion. Thus, the out of plane motion could be nulled separately; alternately, the desired $\dot{Z}$ can be selected as some function of the Z error, e.g.,

$$\dot{z}_D = -kz$$

The total velocity error in this case is

$$V'_\epsilon = \sqrt{(\dot{x}_A - \dot{x}_D)^2 + (\dot{y}_A - \dot{y}_D)^2 + (\dot{z}_A - \dot{z}_D)^2}$$

and the out of plane thrust angle is

$$\phi = \arctan\left(\frac{z_D - z_A}{V_\epsilon}\right)$$

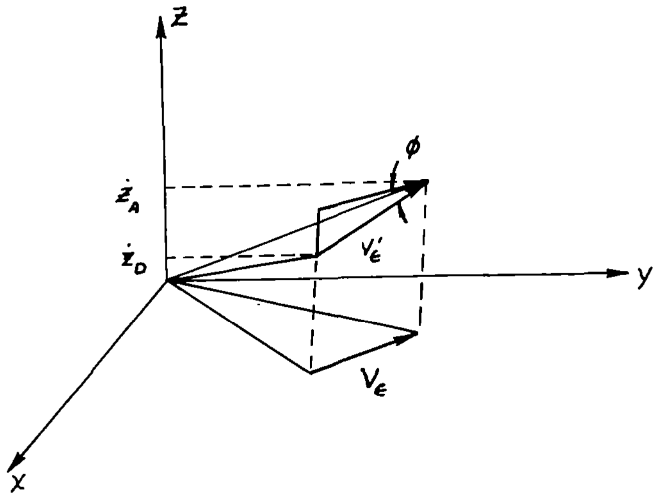


Figure 2.7
Out of Plane Steering Angle

If an elliptic target orbit is considered, each successive calculation of the desired velocity from Equation (2.11) can be viewed as a problem involving a new set of initial conditions and a new circular orbit with a period slightly different than at the last calculation. Since the limits as $t \to \tau/k$ of Equations (2.12) and (2.13) remains, the same, even if n is a function of time, the implication is that a rendezvous will occur for an elliptic orbit. Shapiro [Reference (2.11)] has shown that a rendezvous will not occur for elliptic target orbits by the use of a single application of Equation (2.11) as in the method discussed in Section (2.2.2.1).

The general idea of this method can be used with other schemes by changing the time to rendezvous at each correction point so as to gradually reduce the range rate. If a range range-rate schedule is used, the time to rendezvous can be calculated from the present position and desired range-rate. For example, a fixed range range-rate schedule being considered for an Apollo rendezvous is given in Table 2.1 (Reference 2.9)

Fixed Range: Range-Rate Schedule

Range (N. Mi.)	Desired Range Rate (fps)
5	-100
1.5	-20
.25	-5

Sears and Filleman, in Reference (2.10), chose a desired range rate proportional to the square root of the range. Many other range range-rate schedules could be used and the choice of a particular one could be based on factors such as (1) the maximum relative velocity expected at the beginning of the rendezvous maneuvers (2) the type and capabilities of the propulsion system to be used in the maneuver (3) data acquisition and processing requirements and (4) back-up guidance requirements.

2.2.4.5 Second Order Improvement

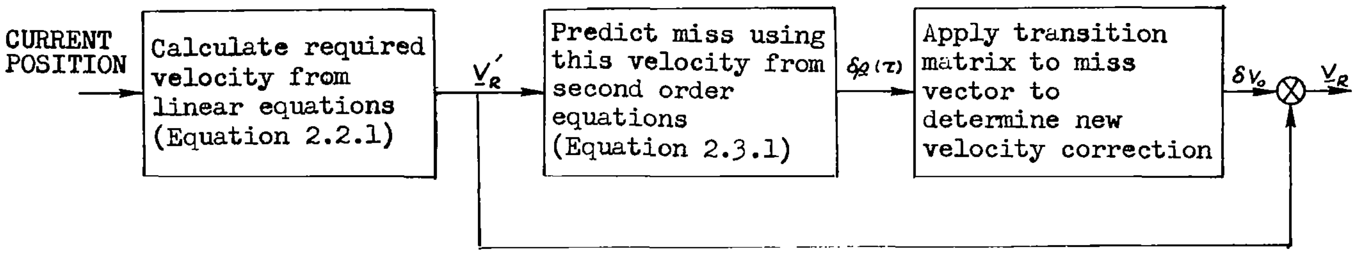
If the required rendezvous velocity is calculated from Equation (2.11), then the miss at the target (i.e., the difference between the target and rendezvous vehicle) at the time rendezvous is to occur will be zero to first order. However, if the initial separation distance is large or if the selected rendezvous time is large, the actual target miss may be significantly greater than zero, just due to model errors. In these cases, an on-board system based exclusively on the linear equation must include midcourse corrections to achieve reasonable miss distances. An alternative is to use equations of motion which have increased accuracy because of the inclusion of second order terms in the expansion of the coordinates. Such equations have been developed in Section 2.1.5 and are of the form

$$\underline{\rho}(t) = \underline{\alpha}_0 + \underline{\alpha}_1 t + \underline{\alpha}_2 \sin t + \underline{\alpha}_3 \cos t + \underline{\alpha}_4 \sin 2t + \underline{\alpha}_5 \cos 2t + \underline{\alpha}_6\, t \sin t + \underline{\alpha}_7\, t \cos t \tag{2.14}$$

where $\underline{\rho}(\tau)$ is the relative position vector (to second order in the coordinates) for target orbits of small eccentricity. The vector $\underline{\alpha}_i$ has components A_i, B_i, C_i which are in general non-linear functions of the initial position and velocity. (See Section 2.1.5 for the exact expressions for A_i, B_i, and C_{i_e}). One way to improve the miss at the target would be to calculate the required initial velocities from Equation (2.14). This step cannot be accomplished directly, however, because of the non-linear nature of the α_i; thus, appeal to numerical methods must be made. An alternative to a numerical solution is developed by Anthony and Sasaki (Reference 1.2) which assumes a solution as a sum of the linear solution plus an error term.

$$\dot{\underline{\rho}}(0) = \dot{\underline{\rho}}_l(0) + \underline{\epsilon} \tag{2.15}$$

where ρ_1 is the required velocity computed from linearized equations such as Equation (2.11). Substitution of Equation (2.15) in Equation (2.11) yields equations for the ϵ_i's. A second technique which yields essentially the same result as that of Anthony and Sasaki method was developed by Bonomo and Schlegel [Reference (2.12)]. This second method seems to be more formalized and compact for on-board implementation and is, therefore, chosen for detailed discussion here. The method consists of first determining the velocity required for rendezvous by the use of linear equations; next the miss resulting from the use of this velocity is calculated using Equation (2.14); and finally, the transition matrix is applied to this miss to determine a new velocity correction.



Block Diagram of Second Order Velocity Correction
Figure 2.8

If $\delta\rho(\tau)$ denotes the miss distance to the second order resulting from the use of the first order velocity correction at t = 0, then the relation between the miss and the required correction to the linear velocity increment is given in terms of the transition matrix $G(\tau, 0)$ as

$$\begin{bmatrix} \delta\rho(\tau) \\ \delta\dot{\rho}(\tau) \end{bmatrix} = G(\tau, 0) \begin{bmatrix} \delta\rho(0) \\ \delta \underline{V}(0) \end{bmatrix}$$

The transition matrix can be partitioned as

$$G(\tau, 0) = \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right]$$

Therefore, the desired relation between the second order miss and the required velocity correction is given by (since it is assumed that $\delta\rho(0) = 0$)

$$\delta\rho(\tau) = G_{12}\, \delta\underline{V}(0)$$

or

$$\delta V(0) = G_{12}^{-1}\, \delta\rho(\tau)$$

The velocity correction, $\delta V(0)$, represent the corrections which must be added to the "first order" required velocity so that the miss distance at rendezvous will be reduced to zero.

Figures 2.9 and 2.10 (taken from Reference 2.12) illustrate a comparison of the first and second order scheme described above.

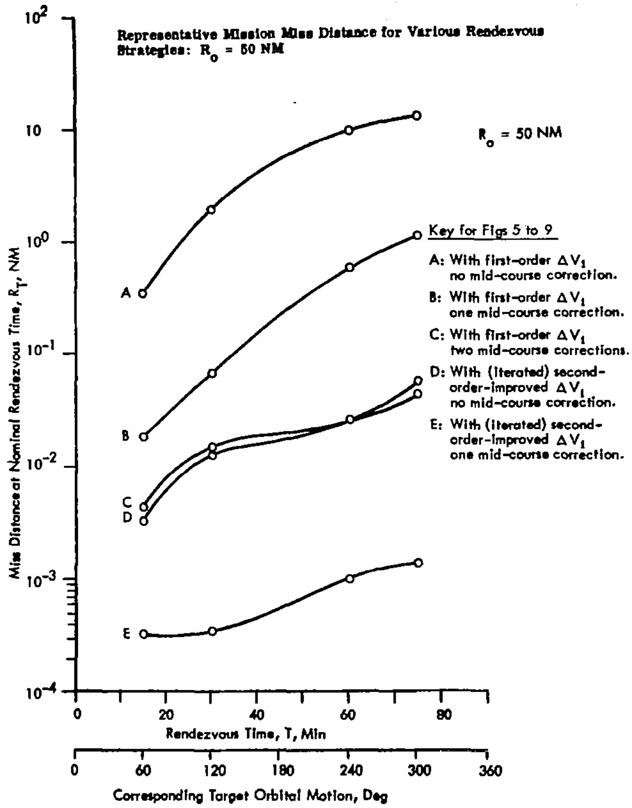


Figure 2.9

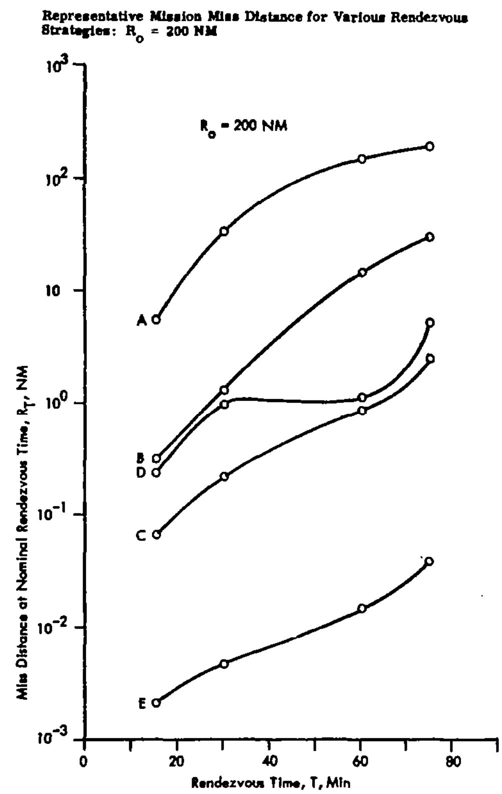


Figure 2.10

As can be seen in these figures, the effectiveness of the improvement depend upon the initial separation distance and rendezvous time τ . As a consequence of reducing the number of midcourse corrections, the second order technique requires a smaller total Δ V to perform the rendezvous maneuver. This difference becomes more significant for large initial separations and times.

2.2.4.6 Direct Calculation using Two-Body Orbits

Perhaps the most straightforward method of determining the impulsive velocity required for rendezvous is obtained by using the equations of absolute motion of the two vehicles. The relative motion can then be determined as the difference of the respective absolute motions. However, since the relative positions and velocities will be several orders of magnitude smaller than their absolute counterparts, it is seen that great computational precision is required. Furthermore, the errors in the knowledge of the orbits of the two vehicles must be small in order to maintain acceptably small errors in the relative motion. The method has the advantage, however, of providing rendezvous velocity requirements from a much wider range of initial conditions than any of the methods discussed previously.

In order to use the two-body equations of motion to determine the required velocity a time at which rendezvous is to occur must be selected. Some criteria for selecting the rendezvous time, τ, have been given in Sections 2.2.3.2 and 2.2.3.4. For the purpose of illustrating the method, it is assumed that the time till rendezvous is determined from some range range-rate function, i.e.,

$$\dot{r}_d = f(r) \tag{2.16}$$

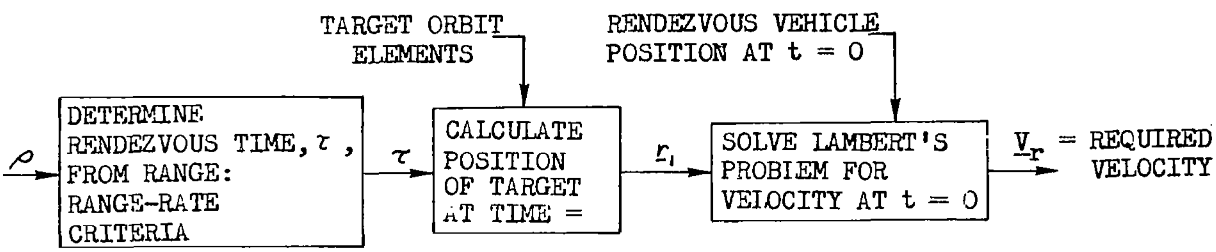
as in Section 2.2.3.4. The procedure is (1) measure the relative range; (2) determine the desired range-rate ($\dot{r}_d$) from Equation (2.16), (3) calculate the time of rendezvous, τ, as

$$\tau = \frac{\rho_m}{\dot{r}_d}$$

where ρ_m = measured range

$\dot{r}_d$ = desired range rate

(4) determine the position of the target at τ using two-body equations and knowledge the target orbit (5) compute the required velocity by solving Lambert's problem using the position of the target at τ, the position of the rendezvous vehicle at t = 0 and the desired flight time τ.



Block Diagram of Conic Rendezvous Calculation
Figure 2.11

If the position and velocity of the target vehicle are expressed in an inertial coordinate system whose origin is at the center of attraction, then the position at time τ can be calculated from the following (sequential) set of equations

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\dot{s}^2 \triangleq \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$r\dot{r} = x\dot{x} + y\dot{y} + z\dot{z}$$

$$\frac{1}{a} = \frac{2}{r} - \frac{\dot{s}^2}{\mu}$$

$$e \cos E_0 = 1 - \frac{r}{a}$$

$$e \sin E_0 = \frac{r\dot{r}}{\sqrt{\mu a}}$$

$$e = \sqrt{(e \cos E_0)^2 + (e \sin E_0)^2}$$

$$n = \frac{\sqrt{\mu a}}{a^2}$$

$$E_0 = \arctan \frac{(e \sin E_0)}{(e \cos E_0)}$$

$$M_\tau = n\tau + E_0 - e \sin E_0$$

$$E_\tau - e \sin E_\tau = M_\tau \tag{2.17}$$

$$\underline{r}(\tau) = \frac{a}{r}\left[(\cos E_\tau - e) \cos E_0 + \sin E_0 \sin E_\tau\right] \underline{r}\,(t=0)$$
$$+ \frac{1}{n}\left[(\cos E_0 - e) \sin E_\tau - (\cos E_\tau - e) \cos E_\tau\right] \underline{\dot{r}}\,(t=0)$$

The solutions of all of these equations are straightforward except that for (2.17). Kepler's equation requires an iterative procedure to obtain E_τ .

Once the position of the target vehicle at time τ is known, the velocity required of the rendezvous vehicle at t = 0 can be found. First, the set of equations (2.18), which constitute Lambert's theorem (see Reference 2.13) solved iteratively for a, α , and β .

$$\left.\begin{aligned} r(0) + r_1(\tau) + C &= 4a \sin^2 \frac{\alpha}{2} \\ r(0) + r_1(\tau) - C &= 4a \sin^2 \frac{\beta}{2} \\ \tau &= \sqrt{\frac{a^3}{\mu}} \left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right] \end{aligned}\right\} \tag{2.18}$$

$$c = |\underline{r}(0) - r_1(\tau)|$$

Next, the eccentricity e is determined from the set

$$\Delta E = \alpha - \beta$$

$$r(0) = a\,(1 - e \cos E_1)$$

$$r_1(\tau) = a\,(1 - e \cos(E_1 + \Delta E))$$

Now, the magnitude of the required velocity, V, can be determined from the energy equation

$$V^2 = \frac{2\mu}{r(0)} - \frac{\mu}{a}$$

This magnitude can be associated with a velocity component normal to the radius (V_M) which is found from the conservation of angular momentum

$$V_n = \sqrt{\frac{\mu a (1-e^2)}{r^2(0)}}$$

and a radial component (V) found from

$$V_r = \sqrt{V^2 - V_n^2}$$

Finally, the required velocity vector can be written as

$$\underline{V}_R = V_r \frac{\underline{r}(0)}{r(0)} + V_n \frac{(\underline{r}_1 \times \underline{r}_2) \times \underline{r}_1}{|(\underline{r}_1 \times \underline{r}_2) \times r_1|}$$

A variation of this procedure called 'miss distance guidance' and is described by Gunckel in Reference 2.8. In this method, both the target and rendezvous vehicle orbits are integrated forward to the rendezvous time. The difference in their positions, at this time, is the miss distance. The velocity required for rendezvous is then found by operating on the miss distance with the transition matrix.

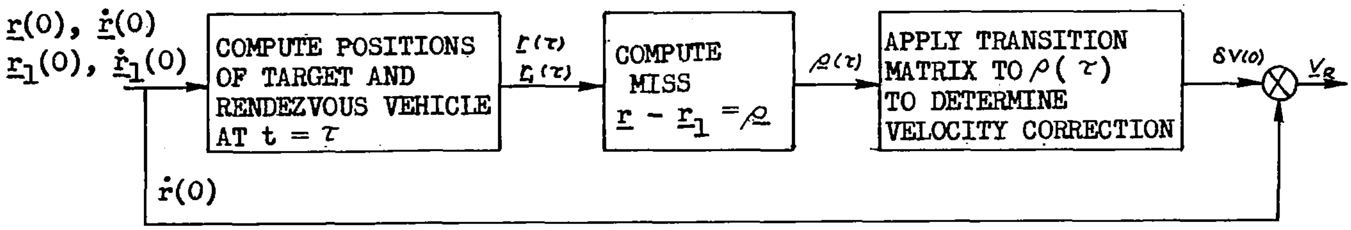
$$\delta \underline{V} = \left[\frac{\partial \underline{V}(0)}{\partial \underline{r}(\tau)}\right] (\underline{r}(\tau)) - r_1(\tau))$$

where $$\underline{V}_R = \underline{V}(0) + \delta \underline{V}$$

$\underline{V}_R$ = velocity vector required of the rendezvous vehicle at t = 0

$\underline{V}(0)$ = velocity of the rendezvous vehicle at t = 0

$\left[\frac{\partial \underline{V}(0)}{\partial \underline{r}(\tau)}\right]$ = transition matrix relative variations in position at $t = \tau$ to velocity at t = 0



Block Diagram of Miss Distance Guidance
Figure 2.12

Because of the fact that the solution to Lambert's problem, in the first method, is replaced by a repeated application of the equations determining the position followed by a matrix operation, this second method holds considerable computational advantage over the first. This simplification is purchased, however, at the expense of generality in the method.

Some of the difficulties mentioned in the first paragraph of this section can be overcome by the incorporation of direct measurement of the relative range into the equation defining the velocity. Such an incorporation is discussec by Gedeon (Reference 2.14). In this reference, Gedeon compares the direct measurement of the relative range with that calculated by the conic equations for application to a process employing differential corrections of the transfer orbit parameters. Only an outline of this procedure is presented here, since the reference uses Herrick's universal parameters rather than the orbital elements discussed to this point. The coordinate system is such that the Z axis is normal to the "transfer" orbit and x is along the radius $\underline{r}$. Let the subscripts m and c denote the measured and calculated values of the relative position (Δx , Δy and Δz); then the residuals (δx , δy) are defined by

$$\delta x = \Delta x_c - \Delta x_m$$

$$\delta \Delta y = \frac{\Delta y_c - \Delta y_m}{\cos i}$$

where

$$i = \arctan \left(\frac{\Delta z m}{y(\tau) + \Delta y_m} \right)$$

A relation between these residuals and variations in the elements of the transfer orbit can be written as

$$\begin{pmatrix} \delta a \\ \delta D_0 \end{pmatrix} = A \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

when $\quad a \quad = \quad$ semi major axis

$\quad D_0 \quad = \quad$ one of Herrick's universal variables

$\quad A \quad = \quad$ matrix

A similar relation can be written for the velocity and variation in the orbit parameters

$$\begin{pmatrix} \delta \dot{x} \\ \delta \dot{y} \end{pmatrix} = B \begin{pmatrix} \delta a \\ \delta D_0 \end{pmatrix} = BA \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

The "correct" velocity required for rendezvous is then given by

$$\dot{x}_r = \dot{x} + \delta \dot{x}$$

$$\dot{y}_r = (\dot{y} + \delta \dot{y}) \cos i'$$

$$\dot{z}_r = (\dot{y}_i + \delta \dot{y}_i) \sin i'$$

2.3 PROPORTIONAL GUIDANCE

Proportional guidance refers to a class of guidance laws which exhibits smooth thrust functions chosen to be proportional to some function of range and range rate. The mechanization of these laws requires throttable motors, this requirement may be undesirable when compared with the relative simplicity and reliability of constant thrust motors. Also, these methods are expected to be less optimum than those using constant-thrust motors since optimization of maneuvers indicates that thrusting should occur in periods of either full or zero thrust (see Section 2.4). The computation of the thrust vector, however, is quite simple since the gravity model is not introduced into the equations.

An example of proportional guidance is presented by Sears and Fellman (Reference 2.10). In this reference, the rocket thrust is determined from the equations

$$T_r = S_1[\dot{\rho} - \sqrt{\rho}] - S_2(\rho \dot{\gamma}_{LS})$$

$$T_c = S_3[\dot{\rho} - \sqrt{\rho}] + S_4(\rho \dot{\gamma}_{LS})$$

$$T_z = S_5[\dot{\rho} - \sqrt{\rho}] + S_6(\rho \dot{\gamma}_{LS})$$

where S_1, S_2, S_3, S_4, S_5, and S_6 are constants and $_{LS}$ is the angular rate of the line of sight. (The subscripts r, c and z refer to the radial, circumferential and out of plane direction.) If the constants S_1 and S_3 are made equal and S_2 and S_4 are set to zero, the thrust direction will be along the line of sight and the thrust magnitude will be determined by the difference between the range rate and the square root of the range. For this case, the trajectory as seen by an observer on the target satellite will appear as in Figure 2.13.

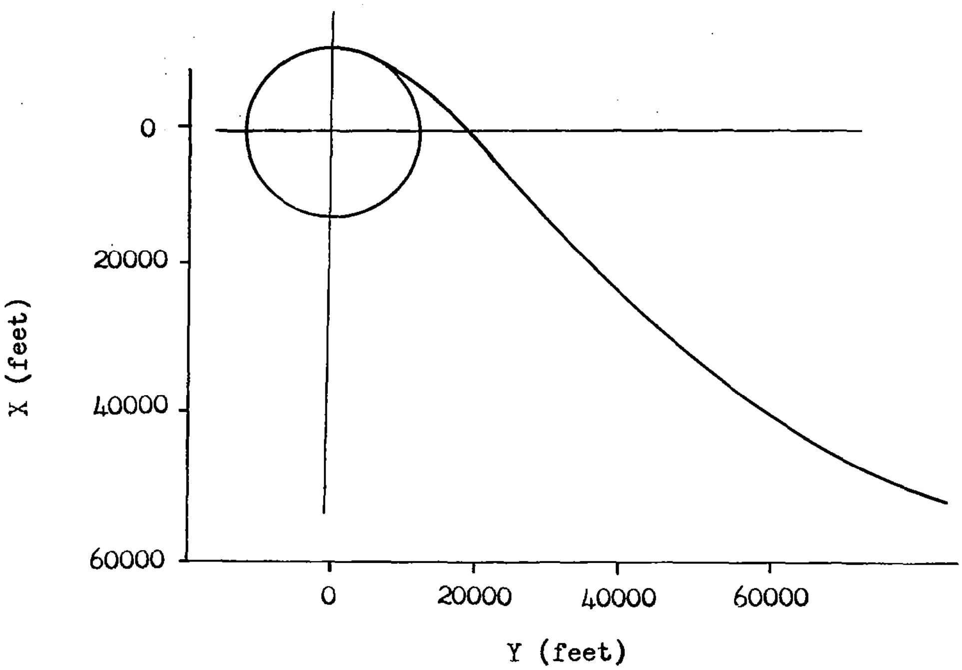


Figure 2.13
Rendezvous Trajectory with Thrust Along the Line of Sight

If all four of the guidance constants S_1, S_2, S_3 and S_4 are made equal, a rendezvous will occur and the trajectory will appear as in Figure 2.14

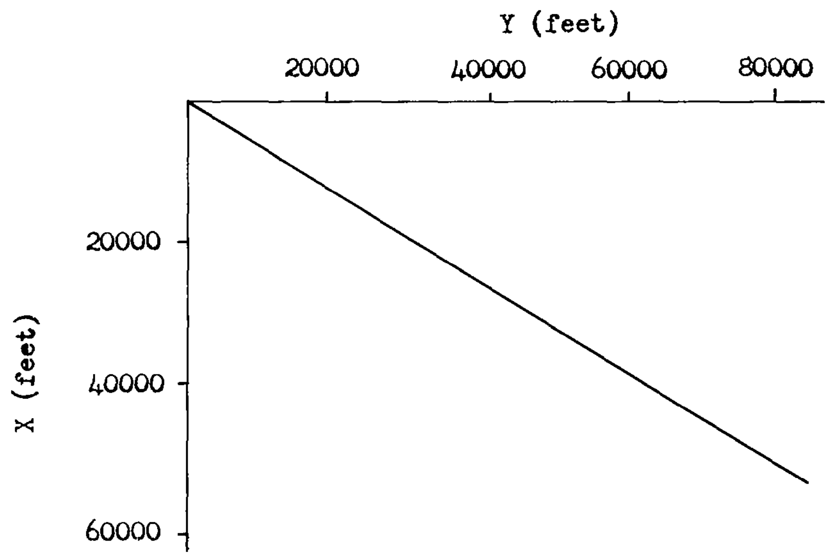


Figure 2.14
Rendezvous Trajectory with All Sensitivities Made Equal

Another example of proportional guidance is given by Green (Reference 2.17). A discussion of this method, call logarithmic guidance, is presented here since a comparison between this type of guidance and an optimum guidance technique is made in Section 2.3. In this method, the vehicle control system attempts to adjust the thrust so that the equations below are satisfied

$$\frac{\rho}{\dot{\rho}} = K_1 \frac{\dot{\rho}}{\rho}$$

$$\frac{\ddot{\gamma}}{\dot{\gamma}} = K_2 \frac{\dot{\rho}}{\rho}$$

where K_1 and K_2 are constants. These equations may be integrated to illustrate the logarithmic behavior

$$\frac{\dot{\rho}}{\dot{\rho}_0} = \left(\frac{\rho}{\rho_0}\right)^{K_1} \quad \text{or} \quad \ln \frac{\dot{\rho}}{\dot{\rho}_0} = K_1 \ln \frac{\rho}{\rho_0}$$

$$\frac{\dot{\gamma}}{\dot{\gamma}_0} = \left(\frac{\rho}{\rho_0}\right)^{K_2} \quad \text{or} \quad \ln \frac{\dot{\gamma}}{\dot{\gamma}_0} = K_2 \ln \frac{\rho}{\rho_0}$$

The first equation may be integrated to give

$$\rho = \rho_0 \left(1 + \frac{\dot{\rho}_0}{\rho_0}(1-K_1)t\right) \qquad K_1 \neq 1$$

$$\rho = \rho_0 \exp\left(-\frac{\dot{\rho}_0}{\rho_0} t\right) \qquad K_1 = 1$$

and differential yielding

$$\ddot{\rho} = K_1 \frac{\dot{\rho}_0^2}{\rho_0^{2K}} \rho^{2K-1}$$

Thus, the time to achieve rendezvous can be determined as

$$\tau = -\frac{r_0}{\dot{r}_0}\left(\frac{1}{1-K}\right)$$

From these equations, the following bounds on K_1 can be determined

$K_1 < 1$ (to achieve rendezvous in a finite time)

$K_1 \geq \frac{1}{2}$ (to keep $\ddot{\rho} < \infty$ as ρ approaches zero)

Manipulation of the equations involving γ results in

$$\dot{\gamma} = \dot{\gamma}_0 \left[1 + \frac{\rho_0}{\rho_0} \left(\frac{1}{1-K} \right) t \right]^{\frac{K_2}{1-K}}$$

$$\gamma = \gamma_0 + \left(\frac{1}{K_2 - K_1 + 1} \right) \frac{\dot{\gamma}_0 \rho_0}{\dot{\rho}_0} \left[\left(\frac{\rho}{\rho_0} \right)^{(K_2 - K_1 + 1)} - 1 \right]$$

$$\ddot{\gamma} = \frac{K_2 \dot{\rho}_0 \dot{\gamma}_0}{\rho_0^{(K_2 + K_1)}} \rho^{(K_2 + K_1 - 1)}$$

From this equation, it is seen that the condition

$$K_2 + K \; -1 > 0$$

must be met if $\ddot{\gamma}$ is to remain finite as ρ approaches zero.

2.4 OPTIMIZATION OF THE RENDEZVOUS MANEUVER

The cost of transporting fuel to space is so great that it is essential in the design of the maneuvers to determine those which require the least fuel and to choose the guidance mechanizations which approximate the optimum. Eventhough the considerations of this monograph deal with small orbital changes, the problem can be studied from the point of view of orbital transfers. Thus, it is known from the work of several authors (Reference 3.1 through 3.5) that impulsive transfer of either one or two impulses will be the optimal transfer (minimum Δ V requirement) between orbits such as are involved here. Further, a phasing technique similar to that discussed by Strahly (3.6) which involves splitting one of the two impulses into two portions which are used as an integral number of revolutions apart can yield rendezvous with the impulse of optimum two-impulse transfer. Some possibilities of this technique were demonstrated by Bender (3.7). Consequently, there exists a determinable lower bound to the velocity increment to rendezvous for any given case. This bound can be used to evaluate the effectiveness of any chosen scheme for its nearness to being optimal.

There is a feature of optimal two-impulse orbital transfer which partially removes the need to optimize. This is the fact that for orbit pairs which do not intersect deeply and which are fairly near to one another all the way around, the effect of varying the departure point around the first orbit is not very significant (see Figures 5 and 6 of Reference 3.8). In addition to this feature of this class of optimum, impulsive transfers is the well known result due to Lawden (3.1) which states that optimal space maneuvers employing a rocket motor are constructed of zero and maximum thrust segments. Thus, rendezvous schemes universally utilize this principle in their design.

The first area of work to be discussed concerns the optimization of rendezvous using stepwise thrusting either as a series of impulsive thrusts or as a series of finite burns separated by coasting arcs. This technique actually results from a development of switching functions from the Pontryagin maximum principle. In this second and third portions of the section, rendezvous studies involving continuous thrust are described. These sections deal (for the most part) with the final phase of the rendezvous maneuver; that is to say, the two objects are on the verge of a very near collision and the engine is to be used with a single burn to alter this situation and produce rendezvous. In this area of study, field free equations are, in general, satisfactory and most work deals with such motion.

There is an area of low thrust rendezvous studies in which power limited engines are considered. The cost functions for the fuel for such engines is quadratic, thus the linear differential equations of motion yield a set of equations for the optimum situation which is linear and consequently solvable. Applications of this approach have been largely for interplanetary studies which anticipate electric propulsion systems. However, closed form solutions obtainable for this type of system have been suggested by Goldstein et. al., (3.9) and Bryson (3.10) as useful aids in studying the rendezvous maneuver. The method of Bryson will be described in Section 2.3.2 and finally the continuous low thrust studies (with linear cost function) will be presented in Section 2.3.3.

2.4.1 Optimal Stepwise Thrusting

Assuming that a spacecraft is propelled by means of a chemical rocket engine, optimization of space maneuvers on the basis of minimum fuel implies optimization on the velocity increment provided by the engine. That is

$$J = \int_0^T |a| \, dt = \quad \text{a minimum} \qquad 4.1$$

It is assumed that the engine, when it is turned on, operates at a fixed fuel rate and that this condition implies a fixed acceleration because the total change in velocity for the rendezvous is expected to be provided by less than 5% of the total weight in fuel. If an unlimited time is allowed for a rendezvous or transfer maneuver, the optimal thrust program solution will revert to the impulsive solution in which many very short duration pulses at each passage of certain points on the orbit (Breakwell Ref. 4.2) are applied. Another variation of the rendezvous problem formulation is the time optimal problem in which a fixed acceleration and an upper limit to the total amount of fuel (i.e. ΔV) available are assumed; the solution then seeks the program for rendezvous in the shortest time possible. Both procedures lead to a series of engine burns and both procedures require the solution of two point boundary value problem as will be shown.

Most of the results to be presented in this section refer to the linear problem with a circular reference orbit. The solutions to the equations of motion are found in section 2.1. (Equations 1 - 21 and 1 - 48 to 1 - 63). Before proceeding to the application of the Pontryagin Maximum Principle to these problems, some results obtained by Tschauner and Hempel (Ref. 1.4) will be indicated. (Tschauner has recently extended the results to the case of targets in elliptical orbits (Ref. 1.5 and 1.6).) These authors consider the rendezvous maneuver on the basis of linear terms and a circular reference orbit and separate the motion into the in-plane and out-of-plane motions so that the quantity being minimized is

$$J = J_{12} + J_3 = \frac{1}{g_a} \int_0^T \left(|a_{INPLANE}| + |a_{\perp PLANE}| \right) dt$$

$$= \int_0^{\theta_f} \sqrt{u_1^2 + u_2^2} \, d\theta + \int_0^{\theta_f} |u_3| \, d\theta$$

where g_a = the acceleration of gravity at the target orbit.

The out-of-plane motion is simple harmonic and is driven to zero by a series of oppositely directed thrusts applied every half cycle, the last one of which yields $\xi_3 = 0$, $\xi'_3 = 0$. The duration (ϕ) of each pulse depends on the initial motion, the number of pulses desired, and the maximum thrust of the engine. The optimum cost is that of a single impulse applied at the time the vehicle crosses the desired plane. This cost is

$$J_3{}_{(opt)} = \sqrt{\xi_{30}^2 + \xi_{30}'^2} \qquad 4.3$$

For the finite thrust case of (m+1) intervals of acceleration (u_3) and duration ϕ, the cost is

$$J_3 = u_3 (m+1) \phi = \frac{\phi/2}{\sin \phi/2} J_3{}_{opt} \qquad 4.4$$

This technique is shown to satisfy the Pontryagin Maximum Principle by the development of the switching function.

For the in-plane motion Tschauner and Hempel point out that

$$J_n = \int_0^T \sqrt{u_1^2 + u_2^2}\, d\theta \geq \int |u_2|\, d\theta \geq \left|\int u_2\, d\theta\right| = |\eta_1(0)| \qquad 4.5$$

This equation shows that η_1 (o) is a lower limit for the velocity increment needed, and that this minimum can be reached only if there is no radial thrust and the circumferential thrust does not involve a change in direction. On the basis of this argument, u_1 is set = 0 and the problem is reduced to a one dimensional motion.

The condition that u_2 does not suffer a change in direction is known to occur when the two orbits do not intersect, since for this case optimum impulsive transfer is very nearly cotangential, and the angular momentum is increased by both burns (the active vehicle is assumed to be in the orbit closer to the force center). Tschauner and Hempel proceed to develop criteria and equations for a three burn maneuver of the form shown in Figure 4.1.

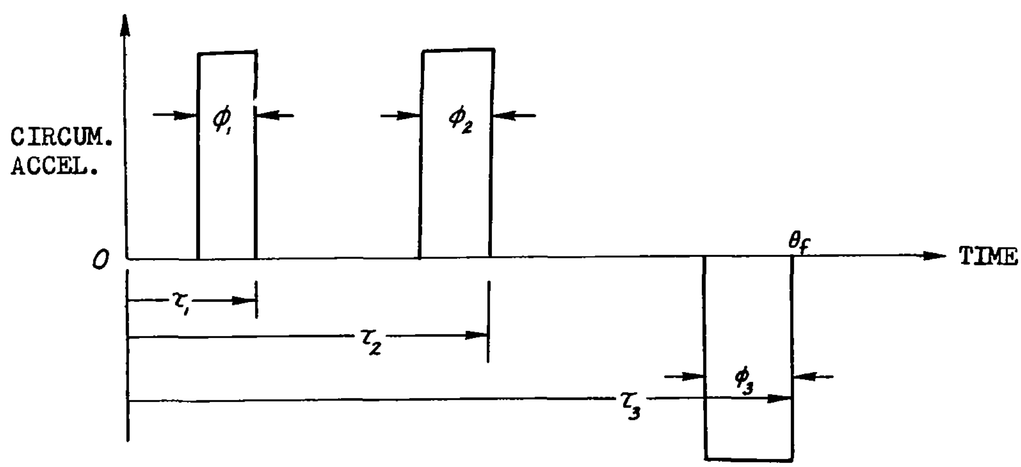


Figure 4.1. Three-Burn Maneuver to Rendezvous

Note that if it can occur, this scheme will include the optimal two-burn case when the thrust is not reversed.

The integrals z_1, z_2, z_3, and z_4 can be easily found and set equal to $w_1(0)$, $w_2(0)$, $w_3(0)$, $w_4(0)$. They are:

$$z_1 = w_1(0) = u_0(\phi_1 + \phi_2 - \phi_3)$$

$$z_2 = w_2(0) = u_0\left[\left(\tau_1 - \frac{\phi_1}{2}\right)\phi_1 + \left(\tau_2 + \frac{\phi_2}{2}\right)\phi_2 - \left(\tau_3 - \frac{\phi_3}{2}\right)\phi_3\right] \tag{4.6}$$

$$z_3 = w_3(0) = [\sin\tau - \sin(\tau_1 - \phi_1) + \sin\tau_2 - \sin(\tau_2 - \phi_2)$$
$$- \sin\tau_3 + \sin(\tau_3 - \phi_3)]$$

$$z_4 = w_4(0) = u_0[\cos(\tau_1 - \phi_1) - \cos\tau_1 + \cos(\tau_2 + \phi_2) - \cos\tau_2$$
$$+ \cos\tau_3 - \cos(\tau_3 - \phi_3)]$$

Assuming that $\theta_f = \tau_3$ is given, it is seen that there are four equations to determine the unknowns ϕ_1, ϕ_2, ϕ_3, τ_1, and τ_2. Since the fuel consumption is equal to $(\phi_1 + \phi_2 + \phi_3)$, this sum can be equated to some reasonable value as the extra equation.

The inversion of these equations to find the coast and burn intervals is not possible in a closed form, but by means of special assumptions Tschauner and Hempel are able to solve the special case of optimal rendezvous from an inner non-intersecting orbit. The results are a generalization from impulsive to finite thrust of the impulsive splitting technique suggested in Ref. 4.6 and 4.7.

Before leaving the work of Tschauner, it is to be noted that in Reference 1.6, a similar pair of thrusting programs for elliptic reference orbits for the out-of-plane and the in-plane portions of a rendezvous with a target in an elliptic orbit is developed. The independent variable for this analysis was eccentric anomaly (the analysis turns out to be more manageable). Again, no radial thrusting is assumed and again, the case of three impulses to rendezvous is developed. The analysis is somewhat involved and the reader is referred to the paper for the details and the equations.

Goldstein et al, (Reference 4.9) utilizes linear equations for the in-plane motion only and describes a procedure for obtaining the optimum thrusting and steering program. The problem is formulated as a Mayer problem in the calculus of variations and the switching function is developed. A sequence of thrusting programs is then constructed and developed in such a way as to approach that one which satisfies the maximizing conditions. Computational results for a series of cases are given and these will be summarized below.

The analysis to be presented is similar to that of Paiewonsky and Woodrow. In this case, the full set of equations for the circular reference orbit is used because the function to be minimized involves all three thrust directions, that is, Equation 4.1 is used. In addition, both the time optimal and fuel optimal rendezvous problems are developed. Much of the analysis is common to the two procedures; therefore, these discussions will be carried together until it finally becomes necessary to distinguish one problem from the other. These two problems are in fact very similar to the time and fuel optimal orbit transfer problems which are described by McIntyre (Ref. 4.12, Section 2.3.4, pages 61-68) as illustrations of the Pontryagin Maximum Principle. The differences between rendezvous and transfer problems are: first, that for the three dimensional transfer problem only five variables corresponding to some set of five distinct orbital elements have to be matched at the end, whereas for rendezvous six variables are needed so that the final position on the target orbit matches a given phase situation; and second, that the transfer problem is extremely non-linear while the differential equations for the present problem are linear with constant coefficients.

In order to attack either of these two problems in three dimensions, it is necessary to combine the differential equations of motion and one for a measure of fuel consumption into a set of seven linear differential equations. The equations of motion were integrated in section 2.1.4.1 as follows: to begin with, the differential equations of motion are: (Eq. 1.21).

$$\begin{aligned} \xi_1'' - 3\xi_1 - 2\xi_2' &= u_1 \\ \xi_2'' \qquad + 2\xi_1' &= u_2 \\ \xi_3'' + \xi_3 \qquad &= u_3 \end{aligned} \tag{4.10}$$

where the ξ's are distances expressed in units of the radius of the circular reference orbit; the independent variable, Θ, can be taken as the true anomaly or the mean anomaly in radians (time in units of the time for the reference particle to move one radian around its orbit); and u_i are the forces/unit mass in units of the acceleration of gravity at the reference orbit.

The variables $(\xi_1, \xi_2, \xi_1', \xi_2', \xi_3, \xi_3') = \eta^T$ were transformed to w^T according to Eq. 1.59 and the solution was obtained as (Eq. 1.64) as

$$w = F(\Theta)\ (w_o - z) \tag{4.11}$$

where F (Θ) is the matrix given by Eq. 1.60. Then, the six integrals (z) of the components of the forcing variables u_1, u_2, u_3 are given in Eq. 1.63.

Now the acceleration produced by the engine is described by three control variables: b, α, β where the magnitude of the acceleration will be specified by, b. (This variable will have only the values u_o or 0 as called for by the switching function) and the components are described by two angles, α, β, similar to latitude and longitude in the rotating coordinate system as shown in Figure 4,2. Thus, the forcing functions are specified by the controls according to

$$\begin{aligned} .u_1 &= b \cos\alpha \cos\beta \\ u_2 &= b \cos\alpha \sin\beta \\ u_3 &= b \sin\alpha \end{aligned} \qquad 4.12$$

where the permissible values are

$$-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2},$$

$$0 \leq \beta \leq 2\pi, \text{ AND}$$

$$b = 0 \text{ or } U_o \text{ ONLY}$$

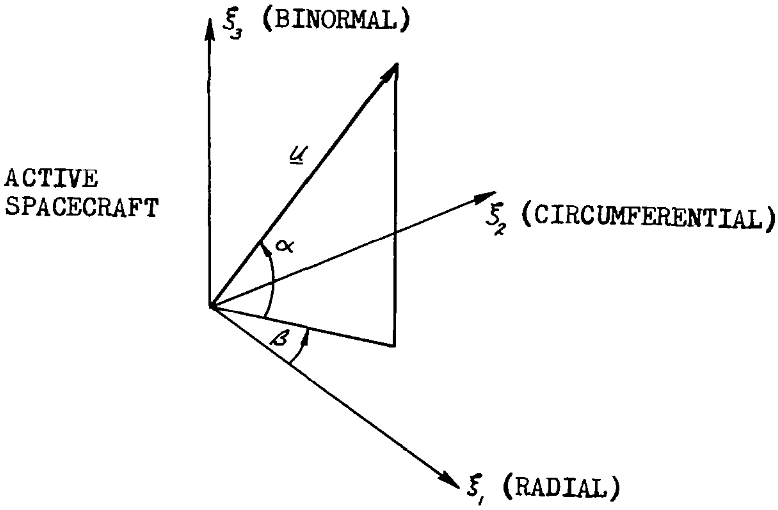


Figure 4, 2 Thrust Direction at Active Craft

For a measure of the fuel consumption (assumed to be a small fraction of the total mass of the spacecraft), the total velocity increment added will be used as the seventh variable w_7. This differential equation is

$$w'_7 = b \qquad \text{with} \quad w_7(0) = 0. \tag{4.13}$$

The integral is expressed simply as

$$w_7 = \int_0^{\theta_0} b \, d\tau \tag{4.14}$$

Many presentations of the Pontryagin Maximum Principle such as that given by Kopp (Ref. 4.13) require the introduction of an additional variable when the payoff function involves the time; here, the payoff function is the final time itself. The treatment by McIntyre (Ref. 4.12) on the other hand allows J to contain θ_f; this procedure will be followed, though it is noted that the adjoint variables and the Hamiltonian can take the same form for both problems. Thus, seven adjoint variables corresponding to the seven state variables w, w_7 are introduced as p, p_7 and the Hamiltonian is

$$H = p^T (A w + f) + p_7 b \tag{4.15}$$

where the forcing vector is $f^T = (u_2, \frac{2}{3} u_1, u_2, \frac{1}{2} u_1, 0, u_3)$

and where the differential equations for the seven variables p, p_7 are:

$$p' = -A^T p \tag{4.16}$$

$$p_7' = 0 \tag{4.17}$$

Since a fundamental motion for $-A^T$ is $F^T(-\theta)$, the solutions can be written as

$$p = F^T(-\theta) \; F(-\theta_0) \, p_o = F^T(-\theta) \, p_o \tag{4.18}$$

$$p_7 = P_7 \text{, a constant,} \tag{4.19}$$

where $\theta_o = 0$ and $F(0) = I$

The control variables in this problem are to be chosen so as to maximize the Hamiltonian along the path. In order to determine these control conditions, note that H can be written as:

$$H = p_1 b \cos\alpha \sin\beta + p_2 (w_1 + \tfrac{2}{3} b \cos\alpha \cos\beta)$$
$$+ p_3 (w_4 + b \cos\alpha \sin\beta) \qquad 4.20$$
$$+ p_4 (-w_3 + \tfrac{1}{2} b \cos\alpha \cos\beta)$$
$$+ p_5 w_6 + p_6 (-w_5 + b \sin\alpha) + p_7 b \qquad 4.21$$
$$= A b \cos\alpha \cos\beta + B b \cos\alpha \sin\beta + C b \sin\alpha$$
$$+ p_7 b + p_2 w_1 - p_4 w_3 + p_5 w_6 - p_6 w_5 + p_3 w_4$$

where $A = \frac{2}{3} p_2 + \frac{1}{2} p_4$ 4.22

$B = p_1 + p_2.$

$C = p_6$

Since α and β have continuous ranges, H is maximized by setting $\frac{\partial H}{\partial \alpha} =$ 0 and $\frac{\partial H}{\partial \beta} = 0$. These conditions yield

$$(-A \sin\alpha \cos\beta - B \sin\alpha \sin\beta + C \cos\alpha) b = 0 \qquad 4.23$$

$$(-A \cos\alpha \sin\beta + B \cos\alpha \cos\beta) b = 0$$

Eq. 4.23 shows that (when $b \neq 0$)

$$\tan\beta = \frac{B}{A}$$

and hence that

$$\sin\beta = \frac{\pm B}{\sqrt{A^2 + B^2}} \quad , \quad \cos\beta = \frac{\pm A}{\sqrt{A^2 + B^2}} \qquad 4.24$$

The sign ambiguity resulting from the square root is resolved by requiring that second derivative be negative. Thus, it is seen that only the positive sign is required. The quadrant in which β lies ($0 \leq \beta \leq 2\pi$) is thus determined by the signs of A and B.

In a similar manner, Eq. 4.23a now yields

$$\tan\alpha = \frac{C}{\sqrt{A^2 + B^2}} \qquad 4.25$$

Hence

$$\sin\alpha = \frac{C}{D} \qquad \cos\alpha = \frac{\sqrt{A^2 + B^2}}{D} \qquad 4.26$$

where

$$D = \sqrt{A^2 + B^2 + C^2}$$

Again, a test of the second derivatives will indicate that only the plus sign for D is allowed. The quadrant for α ($-\pi/2 \le \alpha \le \pi/2$) is thus determined by the sign of C.

The portion of the Hamiltonian containing the control is

$$H_c = b\ (D + p_7) \quad = \quad b\ (D + P_7) \tag{4.27}$$

Thus, it is clear that to maximize the Hamiltonian when $D + P_7 > 0$, the value of b should be u_o, whereas when $D + P_7 < 0$, b should be 0. The case when $D + P_7 = 0$ over an appreciable time is called a singular arc and is discussed by McIntyre (p 67 of Reference 4.12). Such arcs are not excluded and have not been found to occur for these problems. Note that since D is either positive or zero, the value of P_7 must be negative for coasting arcs to occur. The function $k = D + P_7$ is called the switching function since it controls the value of the thrust. Thus, to summarize

$$\begin{array}{ll} \text{If } k < 0, \text{ then } b = 0 & \text{a coasting arc, or} \\ \text{if } k > 0, \text{ then } b = u_o & \text{a full power thrusting arc} \end{array} \tag{4.28}$$

and the thrust components are

$$\begin{aligned} u_1 &= b \quad A/D \\ u_2 &= b \quad B/D \\ u_3 &= b \quad D/C. \end{aligned} \tag{4.29}$$

The distinction between the time optimal problem and the fuel optimal problem lies in the function to be optimized, the constraints, the boundary conditions on the adjoint variables, and the Hamiltonians. The time optimal problem is the determination of the controls such that

$$J = \theta_f \quad \text{is a minimum} \tag{4.30}$$

under the constraint that $w_7(\theta_f) \le \Delta V$, the total allowable velocity increment. It is necessary to include the constraint $w_7(\theta_f) \le \Delta V$ in some way since otherwise $P_7 = 0$ and the engine will be turned on all the time. (That is, the time to rendezvous would be minimized if there were no coast arc; that is, the vehicle should accelerate thus increasing the closing rate until it is necessary to reverse the thrust to be able to stop at the target). If the fuel is limited, this result suggests that the optimal time will be attained by using all of the fuel and hence, $w_7(\theta_f) = \Delta V$ is the boundary condition. Thus, the seventh constraint is taken to be $\psi_7 = w_7(\theta_f) - \Delta V = 0$ for the time optimal problem.

In contrast, the fuel optimal problem is the determination of the controls such that

$$J = w_7(\theta_f) \text{ is a minimum} \tag{4.31}$$

under the constraint that $\theta_f \leq T$. Suppose that a local optimum has been found which produces rendezvous as the result of a one, two, or three burn maneuver. This solution is the program sought and represents a lower limit to the possible value of T which might be imposed in view of the thrusting limitations. Hence, the final time θ will be left open for the fuel optimal case; that is, only locally optimal solutions will be sought in the neighborhood of one, two, or three burn maneuvers. The boundary conditions on the adjoint variables and the Hamiltonian are given by (Equation 2.3.36, P. 55 Reference 4, 12)

$$\left. p_i + \sum_j \mu_j \frac{\partial \psi_j}{\partial w_i} + \frac{\partial J}{\partial w_i} = 0 \right]^{\theta_f} \qquad i = 1, \ldots, 7 \tag{4.32}$$

$$\left. H = \frac{\partial J}{\partial \theta} + \sum_j \mu_j \frac{\partial \psi_j}{\partial \theta} \right]^{\theta_f} \tag{4.33}$$

where $\psi_j(\theta_f) = 0$ are the boundary conditions for the variables $w_1, \ldots, w_7$.

In summary, for the time optimal problem, the results are (at $\theta = \theta_f$)

$$\begin{aligned} p &+ \mu = 0 \qquad \text{(for the first six)} \\ p_7 &+ \mu_7 = 0 \\ H &= 1 \end{aligned} \tag{4.34}$$

and for the fuel optimal problem (time open)

$$\begin{aligned} p &+ \mu = 0 \qquad \text{(for the first six)} \\ p_7 &+ 1 = 0 \qquad (p_7 = -1) \\ H &= 0 \end{aligned} \tag{4.35}$$

The computational problem involved in finding the optimal maneuver is one of determining the six components of μ (or of p_o) from the six conditions $w(\theta_f) = 0$ (and μ_7 from $w_7(\theta_f) = \Delta V$ for the time optimal problem). Thus, the problem has been reduced to a two point boundary value problem (as may be expected, this problem is one of considerable sensitivity). Paiewonsky and

Woodrow have shown that the Neustadt method of solution which changes such problems to maximum-minimum problems can be used in this case. They illustrate the convergence properties for three methods of solving the maximum-minimum problem. However, regardless of the approach taken in solving the two-point boundary value problem, the most important feature of the success or lack of it is the starting point assumed. Without a reasonably good guess for the first attempt, it has been found that optimum transfer and rendezvous problems may not converge. Fortunately, there are at least two schemes for obtaining valid initial estimates for the solutions. One method uses the solution to the quadratic payoff function problem which can be found explicitly for rendezvous (Section 2.4.2 below). Another scheme bases the initial guess for the problem solution of the impulsive solution.

Still another approach has been suggested by McCue (Reference 3.14). In this reference, the application of quasilinearization to solve the two-point boundary value problem along with the use of impulsive transfers as the scheme for generating the first guess of the solution has been shown to be capable of yielding optimum finite thrust coplanar orbital transfers between arbitrary elliptic orbits. This particular scheme has not been applied to the linear rendezvous problem as formulated above to the author's knowledge but it should be a rapidly converging device for the problem. Another demonstration of the use of the impulsive case as a starting point is given Handelsman (Ref. 4.15).

Actually, the computations involved in locating optimal rendezvous are required only in the case of low thrust systems. The pratical solution for optimal rendezvous to any problem where the thrust capability is of the order of 1 m/sec^2 (3 ft/sec^2) can be obtained from the optimal impulsive solution by replacing the impulsive thrusts by finite burns as is shown by Robbins (Ref. 4.24). In this analysis, it was assumed that the thrust was of sufficiently large magnitude that thrusting arcs would not be excessively long. This is a very modest requirement because, as Robbins points out, an error of not more than .2% in the velocity requirement can be assured if the central angle traversed during the burn does not exceed .22 radians. As an example, consider a circular orbit at 300 KM above the earth; this angle corresponds to a burn of nearly 200 seconds (this time, generally increases as the altitude increases).

As an example of the application of this material, consider Reference 4.11. In this reference Paiewonsky and Woodrow analyze the problem of time optimal rendezvous in three dimension from two separate conditions at a series of available ΔV's. For the case illustrated, the situation is somewhat like an abort problem in that the two craft are separating at the initial time and they are to be rejoined. Thus, an immediate first burn is required which must at least change the sign of the relative velocity. Several tables of error sensitivities were shown which illustrated the capability of the procedure and the interesting feature that three burns are optimal in certain cases; some results are reproduced in Tables 4.1 and 4.2. When the amount of fuel available is large, a single burn time-optimal trajectory occurs as is expected. For a moderate amount of fuel (the case of Table 4.1), two burns occur; while for a reduced amount of fuel near the absolute minimum (the case of Table 4.2), an intermediate burn also develops. The thrust required during this intermediate burn is largely out-of-plane and has the effect of significantly reducing the out-of-plane velocity when the out-of-plane distance is small. The body of Tables 4.1 and 4.2 contain measurement error sensitivity coefficients $\partial \eta_i(q)/\partial \eta_i(0)$ where the six variables are $\eta^T(\xi_1, \xi_2, \xi_1', \xi_2', \xi_3, \xi_3')$. It is seen that the quantities in the matrix depend only on the unit of time (here it is the second). The results represent the effects of errors in the control due to the measurement errors of the initial conditions, the vehicle being at its nominal position. (They are not the usually obtained errors due to errors in actual position with nominal control.) The very high dependence of rendezvous errors on initial velocity errors indicates the need for updating the relative velocity information and for making the required alterations in the thrusting program.

As another example, Goldstein et. al. (Reference 4.9) compare numerical results of four methods of determining the velocity increment to rendezvous for a series of planar rendezvous problems for a circular target orbit at 300 n. mi = 557 KM altitude. Two of their figures are shown as Figure 4.3 and 4.4. These figures show the velocity increment to rendezvous as a function of the time from rendezvous. The four methods of determining the velocity increment and the labels for the figures are: (1) the power limited optimal theory of Section 2.4.2 (a curve labeled Phase I); (2) the fuel optimal (fixed time) theory above (a few points labeled P or L depending on the initial guess from Method 1 or 4), (3) two impulse computation for a series of times (a curve labeled Two Impulse), and (4) logarithmic or proportional guidance for the case where the relative separation is initially decreasing (a series of points labeled with crosses and the value of K $(=(\ddot{\rho}/\dot{\rho})/(\dot{\rho}/\rho))$. These transfers were optimized on $[(\ddot{\gamma}/\gamma)/(\dot{\rho}/\rho)]$. The thrust magnitudes vary between the methods used, but range below .6 ft/sec^2 = .2m/sec^2 for the cases in Figure 4.3 and below 1.5 ft/sec^2 = .5 m/sec^2 for the cases in Figure 4.4. The most important conclusion to be observed from these results is that the two-impulse rendezvous is an extremely good approximation to the optimum if the relative phase is such that the rendezvous transfer is nearly an optimal transfer. Furthermore, as has already been noted, the impulse function for for optimal two impulse transfers for such orbits in the rendezvous problem has a very broad minima.

Table 4.1
Measurement Error Sensitivity Coefficients

	1	2	3	4	5	6
1	4.08	3.56	.0025	.0148	.01	-.0007
2	.07	- .96	-.0039	-.0047	-.01	+.0044
3	-825.06	1753.25	3.4236	6.1922	34.38	-3.8879
4	-1753.25	653.17	-2.1082	2.9518	0	.1307
5	.05	+.02	.0017	+1.7092	.06	-.0008
6	34.38	-8.59	1.2890	+ .0020	858.84	-1.4426

Initial Conditions $\eta_1 = -\eta_2 = \eta_5 = 60{,}000$ ft = 18.3 km

$\eta_3 = -\eta_4 = \eta_6 = 100$ ft/sec = 30.5 meters/sec

Engine Accel. = 1.0 ft/sec^2 = .305 m/sec^2

Total velocity increment = 450 ft/sec = 137 m/sec

Burn 5.0 min; Coast 15.0 min; Burn 2.5 min

Table 4.2
Measurement Error Sensitivity Coefficients

1	-4.02	34.16	.0194	.0055	-.05	.0129
2	.0	-1.04	-.0022	.0002	+.02	-.0017
3	885.22	1667.31	9.2189	-3.0149	42.97	7.4381
4	-1727.47	15,581.59	6.4954	2.6177	-17.19	3.6556
5	.04	-.05	.0082	-.0009	.01	.0054
5	-8.59	-8.59	6.6294	-.5799	842.25	5.4075

Initial conditions $\eta_1 = -\eta_2 = \eta_5 = 60{,}000$ ft = 18.3 km

$\eta_3 = -\eta_4 = \eta_6 = 100$ ft/sec - 30.5 m/sec

Engine Accel = 1.0 ft/sec^2 = .305 m/sec^2

Total ΔV = 250 ft/sec = 137 m/sec

Burn 2.7 min; Coast 27 min; Burn 1.6 min; Coast 22 min; Burn .8 min

Figure 4.3
Velocity Increments to Rendezvous (Ref. 4a)

Initial Conditions, $_1$ = 139,000 ft = 42.3 km,
$_2$ = 72,300 ft = 22.0 km,
$_3$ = 335. ft/sec= 104. m/sec
$_4$ = 170 ft/sec = 51.7 m/sec

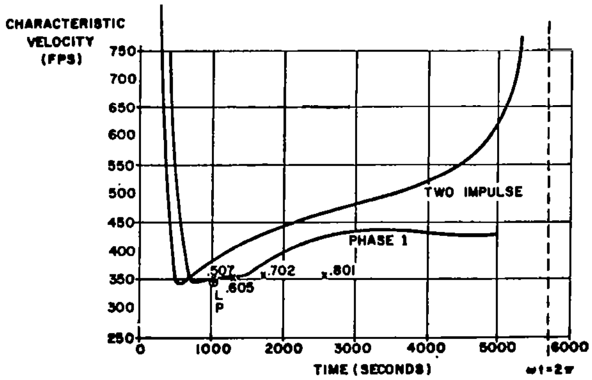
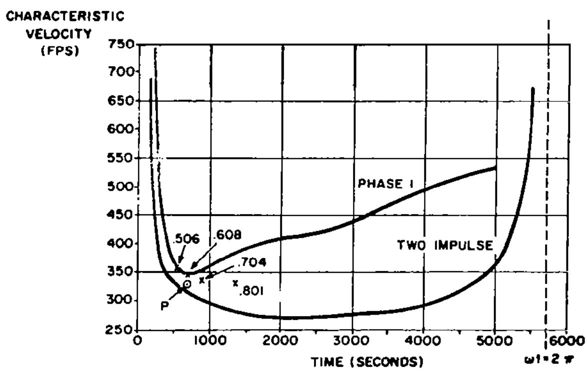


Figure 4.4
Velocity Increments to Rendezvous (Ref 4a)

Initial Conditions, $\eta_1 = \eta_2 = 60{,}000$ ft = 18.3 km

$\eta_3 = -100$ ft/sec = -305 m/sec

$\eta_4 = -350$ ft/sec = 107 m/sec



2.4.2 Optimum Power Limited Rendezvous

This analysis is concerned with a class of vehicles which contain a rocket engine which is power limited and which has variable thrust by its ability to vary the exhaust velocity of the fuel. When operated at full (constant) power, the engine pay-off function for fuel consumption of such an engine is

$$J = \int_0^T a^2 dt$$

This pay-off function and the linearized equations of motion have been used to study low thrust flights to Mars and Venus by Melbourne and Sauer (4-15) and by Gobetz (4-16). However, the analysis presented is due to Bryson (4-10).

As is indicated from the title of Bryson's paper, (Reference 4-10) includes interception and soft landing. This capability is occasioned by his use of the cost function

$$2J = c_v \underline{v} \cdot \underline{v} + c_r \underline{\rho} \cdot \underline{\rho} + \int_{t_0}^{T} \underline{u} \cdot \underline{u} \, dt$$

Where

$\underline{\rho}$ = relative separation

$\underline{V}$ = relative velocity

$\underline{u}$ = acceleration due to power limited engine

t_0, T = initial terminal time

C_r, C_v are scalar weighting factors

The linear differential equations for this problem are

$$\dot{\underline{\rho}} = \underline{v}$$

$$\dot{\underline{v}} = \Delta \underline{g} + \underline{u}$$

where $\Delta \underline{g}$ = difference in acceleration due to gravity between vehicle and target ($\Delta \underline{g}$ = 0 for rendezvous)

Thus, the Hamiltonian of the problem is

$$H = \frac{1}{2} \underline{u} \cdot \underline{u} + \underline{\lambda}_v \cdot (\Delta \underline{g} + \underline{u}) + \underline{\lambda}_r \cdot \underline{v}$$

If it is assumed that $\underline{\Delta g}$ is constant, the differential equations for the multipliers become

$$\dot{\underline{\lambda}}_v = -\frac{\partial H}{\partial \underline{v}} = -\underline{\lambda}_r$$

$$\dot{\underline{\lambda}}_r = -\frac{\partial H}{\partial \underline{r}} = 0$$

and the optimality condition yields

$$\frac{\partial H}{\partial \underline{u}} = \underline{u} + \underline{\lambda}_v = 0$$

These equations are readily integrated to obtain (the initial position and velocity are $\underline{\rho}_0$ and $\underline{v}_0$ and the initial time is taken to be $t_0 = 0$.)

$$\underline{\lambda}_r = \underline{A}$$

$$\underline{\lambda}_v = \underline{B} - \underline{A}(t - t_0)$$

$$\underline{v} = \underline{\Delta g}\, t + \underline{B}t - \tfrac{1}{2}\underline{A}t^2 + \underline{v}_0$$

$$\underline{\rho} = \tfrac{1}{2}\underline{\Delta g}\, t^2 + \tfrac{1}{2}\underline{B}t^2 - \tfrac{1}{6}\underline{A}t^3 + \underline{v}_0 t + \underline{\rho}_0$$

Where the boundary conditions at t = T for $\underline{\lambda}_r$ and $\underline{\lambda}_v$ are

$$\underline{\lambda}_r = c_r \underline{v}(T)$$

$$\underline{\lambda}_v = c_v \underline{r}(T)$$

The problem is to find $\underline{A}$ and $\underline{B}$ in terms of $\underline{r}_0$ and $\underline{\rho}_0$. This inversion is easily carried out and the result is:

$$\underline{A} = \frac{C_r(1 + C_v T)(\tfrac{1}{2}\underline{\Delta g}\, T^2 + \underline{v}_0 T + \underline{\rho}_0) - C_v C_r \frac{T^2}{2}(\underline{\Delta g}\, T + \underline{v}_0)}{D}$$

$$\underline{B} = \frac{C_r T(1 + C_v \frac{T}{2})(\tfrac{1}{2}\underline{\Delta g}\, T^2 + \underline{v}_0 T + \underline{\rho}_0) + C_v(1 - C_r \frac{T^3}{6})(\underline{\Delta g}\, T + \underline{v}_0)}{D}$$

where $D = (1 + C_v T)(1 - C_r \frac{T^3}{6}) + C_r \frac{T^3}{2}(1 + C_v \frac{T}{2})$

In order to apply these results to the rendezvous problem in free space, $\underline{\Delta g}$ is equated to zero and $\underline{A}$ and $\underline{B}$ are obtained for $C_v, C_r >> 1$. The value of the initial time is referred to t_0. The result of these manipulations is:

$$\underline{A} = \frac{6}{(T - t_0)^3}\left[2\underline{\rho}_0 + \underline{v}_0(T - t_0)\right]$$

$$\underline{B} = \frac{2}{(T-t_0)^2}\left[3\underline{\rho}_0 + 2\underline{v}_0 (T-t_0)\right]$$

When these expressions are substituted into the control law, the result may be interpreted as a sampled data feedback law where the last measurement occurred at t_o. Thus,

$$\underline{u} = \frac{-2}{(T-t_0)}\left[3\underline{\rho}_0 + 2\underline{v}_0 (T-t_0)\right] + \frac{6(t-t_0)}{(T-t_0)^3}\left[2\underline{\rho}_0 + \underline{v}_0 (T-t_0)\right]$$

Another result may be obtained by imagining that the position and velocity are continuously sensed and immediately used to correct the thrust. In this event, a $\underline{\rho}_0$, $\underline{v}_0$, and t_o become $\underline{\rho}$, $\underline{v}$, and t and a continuous feedback guidance law is obtained, which is

$$\underline{u} = -\frac{6\underline{\rho}}{(T-t)^2} - \frac{4\underline{v}}{(T-t)}$$

To express this law in terms of $\mathbf{u}_R$ (toward the target) and $\mathbf{u}_\perp$ (normal to the line of sight), note that $\underline{\rho} = -R\underline{L}$, $\underline{v} = -R\underline{L} - R\ \underline{M}$. Next adopting the notation of expressing a small displacement from the nominal path as a small angle,

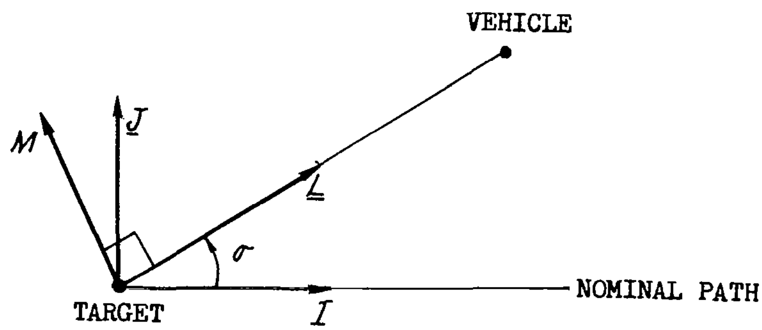


Figure 4.5 - Geometry of Rendezvous

Yields

$$u_\perp = \underline{u}\cdot\underline{J} = 4V_c\dot{\sigma} + \frac{2V_c\sigma}{T-t} \quad \text{where} \quad V_c = \frac{R}{T-t} \simeq -\dot{R}$$

$$u_R = \underline{u}\cdot\underline{I} = \frac{6V_c + 4\dot{R}}{T-t} - 4V_c\sigma\dot{\sigma}$$

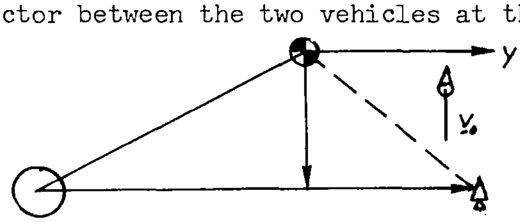
Examples of the effectiveness of this form of proportional guidance for chemical engines used in earth orbital rendezvous are given by Goldstein, et.al, (Reference 4.9) and an illustration of their results is given in Section 2.4.1.

2.4.3 Optimum Continuous Thrust Guidance for the Final Maneuvers

2.4.3.1 Free Space Model

In this section, the optimum steering program for the final thrust period of a rendezvous is developed. That is, it is assumed that the two vehicles have been established on nearly identical trajectories such that the range rate is negative and the range is large enough that rendezvous can be accomplished without overshooting the target. A single thrust period will accomplish the maneuver, but depending on when the maneuver is initiated, a coast period may be required before thrusting begins. A switching function, however, is not developed in the course of the optimization as was done in the last section. Rather, the time to initiate the thrust maneuver is determined after the optimum steering program for the thrust period has been determined. Although the simplifications necessary for this derivation are much more restrictive than those of the last section, the result is a closed form expression for the steering function in terms of the initial conditions of the relative motion. In contrast, the thrust components of the previous section were given in terms of a two-point boundary value problem whose solution is analytically intractable. Because of the closed form of the solution, it is applicable for use in an on-board guidance system whereas the previous method is probably not because of the two-point boundary value problem whose solution must be obtained by iterative techniques which are often slow to converge.

The coordinate system in which the problem is considered is two-dimensional, non-rotating, and fixed to the target vehicle. The X axis is in the direction of the relative velocity vector between the two vehicles at the start of the rendezvous maneuver. The rendezvous vehicle is assumed to have a constant thrust motor which provides a constant acceleration a_o. The equations of motion are:



$$\ddot{x} = a_o \cos\theta$$
$$\ddot{y} = a_o \sin\theta \qquad (4.36)$$

where θ is the angle between the thrust vector and the X axis. The problem is to find θ as a function of time so that the fuel required for the rendezvous maneuver is minimum. Since the constant thrust motor is operating continuously, the amount of fuel required is proportional to the time that the motor is operating. Therefore, minimizing the fuel is equivalent to minimizing the time of burning, i.e., the problem is

$$\text{find } \theta\ (t) \qquad\qquad 0 \le t \le t_f$$

such that t_f is minimum and $\underline{x}\ (t_f) = \underline{\dot{x}}\ (t_f) = 0$. The optimizations will be performed using the Pontryagin Maximum Principle as in the last section. The notation used here is the same as in the previous monograph on the Pontryagin Maximum Principle (Reference 1.8). In order to make use of the maximum principle in the form given in the reference, Equation (4.36) must be reduced to a system of first order differential equations. This is accomplished by making the change of variables.

$$x_1 = x \, , \quad x_2 = y \, , \quad x_3 = \dot{x} \, , \quad x_4 = \dot{y}$$

The new equations of motion are:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \cos\theta \\ \sin\theta \end{pmatrix}$$

or $\dot{\underline{X}} = A \, \underline{X} + Ba_o$. (4.37)

The target set in which the terminal state must lie is given by:

$$\Psi(x_f \, , t_f) \triangleq \left\{ \psi_i : \psi_i = x_i \, (t_f) = 0 \, , \quad i = 2, 3, 4 \right\}$$

and, the function to be minimized is:

$$\phi(x_f \, , t_f) \triangleq t_f$$

The differential constraints are given by Equation (4.37); (in component form) they are:

$$\dot{x}_i = \sum A_{ij} \, x_j + B_j \, a_o$$

where A_{ij} and B_j are elements of A and B, respectively.

Notice that the target set $\Psi(x_f , t_f)$ does not contain the condition $x_1 (t_f) = 0$. This condition is unnecessary since X_1 (t_f) can be nulled by advancing or delaying the time to start the rendezvous maneuver. If X_1 (t_f) were constrained to be zero, then the problem would have to be reformulated to allow for coast periods, i.e., the acceleration should be allowed two values $a_o \theta_o$, and 0 instead of just a_o . In addition, the final time of the maneuver would no longer be proportional to the fuel used.

The Hamiltonian for the problem is:

$$H = P^T \dot{x} = \underline{P}^T (Ax + Ba_0)$$

$$H = P_1 x_3 + P_2 x_4 + P_3 a_0 \cos\theta + P_4 a_0 \sin\theta$$

But, the Pontryagin Maximum Principle states that the function $\theta(t)$ which minimizes the coast, $\phi(t)$, is that function which causes the Hamiltonian to be a maximum at each instant of time. Therefore, this value of θ can be found by setting the derivative $dH/d\theta$ equal to zero:

$$\frac{dH}{d\theta} = -P_3 a_0 \sin\theta + P_4 a_0 \cos\theta = 0$$

or

$$\tan\theta = \frac{P_4}{P_3}$$

The P_i are the co-state variables and obey the differential equations:

$$\dot{P}_i = -\frac{\partial H}{\partial x_i}$$

or

$$\dot{P}_1 = 0\,,\quad \dot{P}_2 = 0\,,\quad \dot{P}_3 = -P_1\,,\quad \dot{P}_4 = -P_2$$

Since, the boundary condition on P_1 is:

$$P_1(t_f) + \sum \alpha_j \cancelto{0}{\frac{\partial \psi_j}{\partial x_1}} + \cancelto{0}{\frac{\partial \phi}{\partial x_i}} = 0$$

or

$$P_1(t_f) = 0$$

And since $\dot{P}_1 = 0$, the solution for $P_1(t)$ is:

$$P_1(t) = 0$$

Solving the differential equations for P_3 and P_4 with $P_1 = 0$ gives

$$P_3 = P_3(0)\,,\quad P_4 = P_4(0) + P_2(0)t$$

and the steering law becomes:

$$\tan\theta = \frac{P_4(0) + P_2(0)t}{P_3(0)}$$

or letting

$$C_1 = \frac{P_4(0)}{P_3(0)} \quad \text{and} \quad C_2 = \frac{P_2(0)}{P_3(0)} \tag{4.38}$$

$$\boxed{\tan\theta = C_1 + C_2 t}$$

This equation gives the optimum steering schedule as a function of time. It remains to evaluate C_1 and C_2 from the initial condition. This objective is realized by first writing Equation (4.38) in terms of $\sin\theta$ and $\cos\theta$.

$$\sin\theta = \frac{C_1 + C_2 t}{\sqrt{1+(C_1+C_2 t)^2}} \qquad \cos\theta = \frac{1}{\sqrt{1+(C_1+C_2 t)^2}}$$

These values can now be used in the expression for X and Y velocity.

$$\dot{x} = \int_0^{t_f} a_0 \cos\theta \, dt \quad , \quad \dot{y} = \int_0^{t_f} a_0 \sin\theta \, dt$$

Performing the indicated integration gives:

$$\left.\begin{aligned} \dot{x}(t_f) &= -V_0 + \frac{a_0}{C_2} \sinh^{-1}(C_1 + C_2 t) - \frac{a_0}{C_2} \sinh^{-1}(C_1) \\ \dot{y}(t_f) &= \frac{a_0}{C_2} \sqrt{1+(C_1+C_2 t)^2} - \frac{a_0}{C_2}\sqrt{1+C_1^2} \end{aligned}\right\} \tag{4.39}$$

(Recall that the coordinate system was chosen so that the initial velocity vector is along the X axis therefore $\dot{x}_0 = V_0$, $\dot{y}_0 = 0$.

Applying the terminal conditions reduces the y equation to:

$$t_f = -\frac{2C_1}{C_2}$$

Substitutions of this in the $\dot{x}$ equation gives

$$V_0 = -\frac{2a_0}{C_2} \tag{4.40}$$

Integrating the second equation in (4.39) to get $Y(t_f)$ and applying the boundary conditions gives

$$0 = y_0 + \frac{a_0 C_1}{C_2^2}\sqrt{1+C_1^2} - \frac{a_0}{C_2^2} \sinh^{-1} C_1$$

This equation and Equation (4.40) are adequate to determine the constants C_1 and C_2 from the initial condition. The complete solution is summarized below:

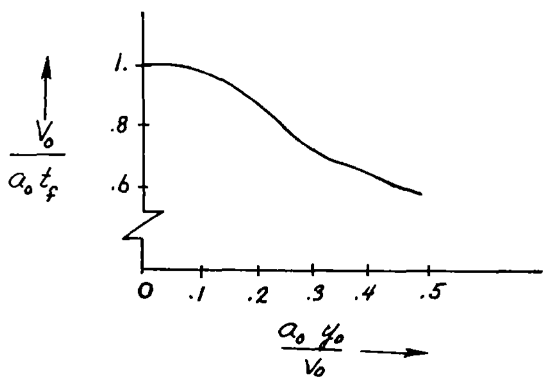
$$\boxed{\begin{aligned} \tan\theta &= C_1 + C_2 t \\ V_0 &= -\frac{2a}{C_2}\sinh^{-1}(C_1) \\ 0 &= y_0 + \frac{a_0 C_1}{C_2^2}\sqrt{1+C_1^2} - \frac{a_0}{C_2^2}\sinh^{-1} C_1 \end{aligned}}$$

If the equation for t_f is substituted in the relations for C_1 and C_2, a parametric relation for t_f in terms of the initial conditions results. This relation is informative since t_f is proportional to cost of the maneuver, and since it is required to determine the length of the initial coast period.

$$\frac{V_0}{a_0 t_f} = \frac{1}{C_1}\sinh^{-1} C_1$$

$$\frac{a_0 y_0}{V_0^2} = \frac{1}{4}\left[\frac{1}{\sinh^{-1} C_1} + \frac{C_2\sqrt{1+C_1^2}}{(\sinh^{-1} C_1)^2}\right]$$

The quantities $\frac{V_0}{a_0 t_f}$, and $\frac{a_0 y_0}{V_0^2}$ can be plotted with C_1 as a parameter obtaining a graphical solution for t_f.



Graphical Solution for t_f
Figure 4.7

Once t_f has been determined, the miss distance in the x direction can be determined by integrating the $\dot{x}$ equation

$$M_x = x_0 - \int_0^{t_f} \dot{x}(t)\, dt$$

where $\dot{x}(t)$ is given by the first equation of (4.38). This miss is to be reduced to zero by an initial coast period of t_0 seconds with

$$t_0 = \frac{M_x}{V}$$

An earlier paper by Kidd and Soule (Reference 4.18) also discusses optimum rendezvous in a gravity free space. The basis of the optimization is that the optimum initial conditions for a rendezvous maneuver have the relative velocity vector oriented along the line of sight. (This conclusion can be verified by examination of Figure 4.2 which shows that for constant values of the other parameters, a minimum t_f occurs when $\psi_0 = 0$.) If it is assumed that midcourse corrections have placed the rendezvous vehicle on a trajectory such that the relative velocity vector is nearly parallel to the line of sight, then the thrust vector will make only small angles with the LOS. In this paper, the cost function was selected to be the amount of energy expended normal to the line of sight (i.e., the integral of the impulse due to the non-zero angle between the LOS and the thrust vector) and the optimization was performed by a variational technique. The range of initial conditions over which the results are valid are limited by the same assumption as the first method of this section with the additional restriction that the initial relative velocity vector must be nearly parallel to the LOS. This method seems to have no advantage over Gunckel's method and, therefore, will not be discussed further here.

2.4.3.2 Linear Gravity Model

The analysis in Section 2.4.1 is limited as an on-board mechanization because of the necessity of using iterative methods to achieve a solution. However, a method discussed in Reference (4.18) approximates an optimum policy for the terminal maneuver; although an iterative solution is still required, the convergence was found, by the authors of (4.18), to be rapid for a wide range of initial conditions. The basis of the approximation is the observation that for a range of trajectories, there are only small variations in the thrust angle (γ). This observation suggests an expansion for γ of the form

$$\gamma = \gamma_0 + a + bt + cf(t) \tag{4.41}$$

where $\gamma_0 >> (a + bt + cf(t))$

The procedure assumes that this is the form of the optimal steering angle, then inserts this form in the equations of motion and finally determines the constants so that rendezvous actually occurs. Assuming a circular target orbit,(the equations of relative motion are derived in Section 2.1.3.2) and neglecting the out of plane motion, these equations are

$$\ddot{x} = \frac{f_T}{M} \cos \gamma + 2n\dot{y} + 3n^2 x$$

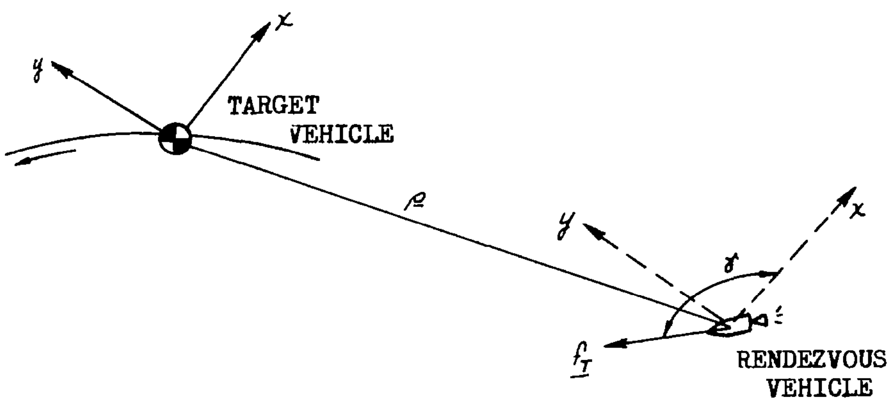
$$y = \frac{f_T}{M} \sin \gamma - 2n\dot{x}$$

where

f_T = constant rocket motor thrust

M = instantaneous mass = $M_o - M_t$

γ = direction of rocket thrust referenced to the axis



Coordinate Definition
Figure 4.8

The solution to this set of equations is

$$x(t) = x_c + \frac{f_T}{n}\left[\int_0^t \frac{\sin[\gamma(\tau)] \sin[(t-\tau)n] dt}{M_0 + \dot{M}\tau} + 2\int_0^t \frac{\sin[\gamma(\tau)][1-\cos[(t-\tau)n\, d\tau]}{M_0 + \dot{M}\tau}\right]$$

$$y(t) = y_c + \frac{f_T}{n}\Big[\int_0^t \frac{\cos\gamma(\tau)}{M_0+\dot M\tau}\sin(t-\tau)n \;-2\int_0^t \frac{\sin\gamma(\tau)}{M_0+\dot M\tau}\Big\{1-\cos[(t-\tau)n]\Big\}dt$$
$$-3\int_0^t \frac{\cos\gamma(\tau)}{M_0+\dot M\tau}\Big\{(t-\tau)n-\sin[(t-\tau)n]\Big\}\Big]$$

$$\dot x(t) = \dot x_c + f_T\Big[\int_0^t \frac{\sin\gamma(\tau)}{M_0+\dot M\tau}\cos[(t-\tau)n]\,d\tau + 2\int_0^t \frac{\sin\gamma(\tau)}{M_0+\dot M\tau}\sin[(t-\tau)n]\,d\tau\Big]$$

$$\dot y(t) = \dot y_c + f_T\Big[\int_0^t \frac{\cos\gamma(\tau)\cos[(t-\tau)n]\,d\tau}{M_0+\dot M\gamma} - 2\int_0^t \frac{\sin\gamma(t)\sin[(t-\tau)n]\,dt}{M_0+\dot M\tau}$$
$$+3\int \frac{\cos\theta(\tau)}{M_0+\dot M\tau}\Big\{\cos[(t-\tau)n]+1\Big\}d\tau\Big]$$

where $x_c, y_c, \dot x_c, \dot y_c$ are the solutions to the homogeneous equations which are found by applying the transition matrix for circular orbits to the initial conditions

$$\begin{bmatrix} x_c \\ y_c \\ \dot x_c \\ \dot y_c \end{bmatrix} = G(t,0)\begin{bmatrix} x(0) \\ y(0) \\ \dot x(0) \\ \dot y(0) \end{bmatrix}$$

with $G(t_1 0)$ given by Equation 1.58. When the approximation for γ (Equation 4.41) is substituted into the previous solutions, the resulting equations can be written in the form

$$x(t_f) = 0 = R_{10} + R_{11}a + R_{12}b + R_{13}c$$
$$y(t_f) = 0 = R_{20} + R_{21}a + R_{22}b + R_{23}c$$
$$\dot x(t_f) = 0 = R_{30} + R_{31}a + R_{32}b + R_{33}c$$
$$\dot y(t_f) = 0 = R_{40} + R_{41}a + R_{42}b + R_{43}c$$

where a, b, c, and t_f are constants and the R_{ij}'s are nonlinear functions of γ_0 and t_f. This set of equations must be solved for a, b, c and t_f; this objective is accomplished by first estimating t_f and θ_0 from the first two equations in the set with a, b and c set equal to zero. This guess for t_f and γ_0 is next substituted in the three equations:

$$x(t_f) = y(t_f) = \dot{x}(t_f) = 0$$

and this set is solved for a, b, and c. Substitution of these values of a, b, and c in the equation for $\dot{y}(t_f)$ will generally not result in it being zero, so further iteration (on time) is made. The time increment is chosen by seeking an increment which causes the remainder from $\dot{y}(t_f)$ to change sign. Half this interval is then used to recompute a, b, c and subsequently $\dot{y}(t_f)$. This process is then continued until the remainder from $\dot{y}(t_f)$ becomes acceptably small.

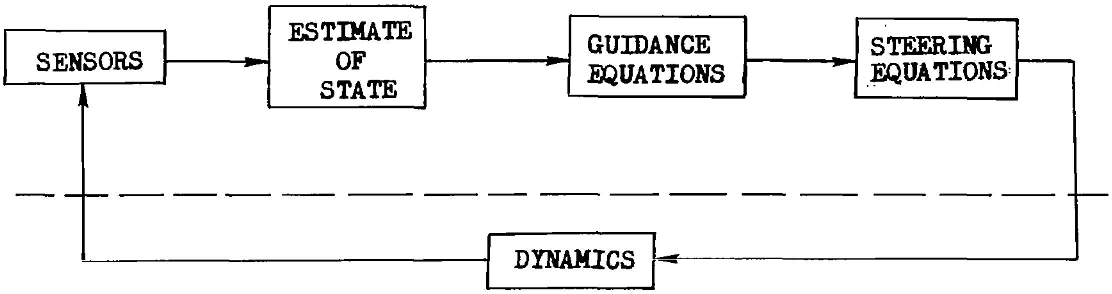
The authors of Reference (4.18) investigated three forms for f(t) in Equations (1.21). These forms were: (1) t^2, (2) $1/(t+1)$, and (3) $\sin t$. Of these three, only the quadratic one proved unacceptable because of the inability to maintain the small angle approximations. The other two forms were found to be stable at all points on the trajectory, and the trajectories obtained were very close to the optimum trajectories as determined in Section (2.4.1).

3.0 RECOMMENDED PROCEDURES

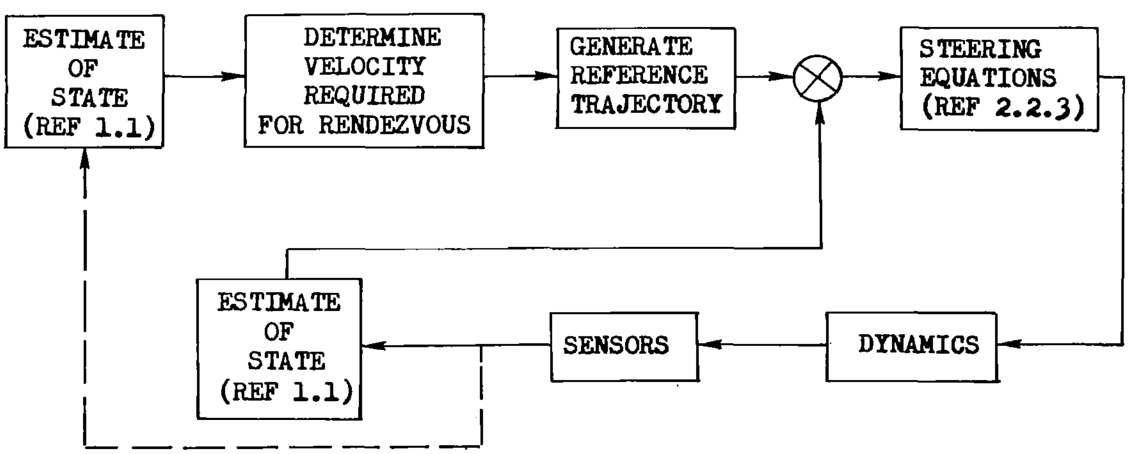
In the preceding sections of this Monograph, the equations of motion relevent to the rendezvous maneuver have been discussed and several methods of incorporating the various form of these equations into guidance concepts have been proffered. Of the guidance techniques discussed, none is clearly superior to all the others and to recommend a particular scheme for all applications would be folly. As with many engineering applications, the trade-offs between the various techniques are one of flexibility vs. complexity and efficiency. Thus, for example, the price that is paid for requiring a rendezvous guidance system capable of initiating the maneuver at extreme ranges is the additional computer capacity to mechanize equations such as those in Section 2.2.3.6 or to provide the additional fuel required for several applications of the simpler equations of Section 2.2.3.4. While it would at first appear that guidance schemes which are optimized to conserve fuel could be generally recommended, a closer look reveals that there are the same sort of criteria to be employed in this area. At present, computational methods for solving optimization problems where they would do the most good (i.e., early in the mission when the separation is large) are not suitable for a closed loop, on-board guidance system. The difficulty here is that large separations require the use of sophisticated gravity model, and the use of such models in the optimization problem leads to two-point boundary value problems which require iterative solutions as was seen in Section 2.3.1. On ther other hand, if the separation distance is small enough to allow a good description of the motion with gravational acceleration totally neglected, then the optimization problem has an analytic solution as in Section 2.4.2. However, the fuel savings in some cases is so small that it is not worth considering, and a guidance scheme which could accommodate a wider range of initial conditions is preferred. (This determination can be made only after examination of the particular problem of interest, and it is not intended to imply that this technique should never be used.)

Although no single method can be recommended for all applications, a general procedure for selecting an appropriate guidance scheme can be suggested. It is felt that first consideration should be given to one of the two-impulse schemes of Section 2.2.3. The choice of the degree of sophistication to be used in the gravity model will be determined by the desired range of initial conditions, the expected computational capabilities of the particular configuration, and the amount of fuel available for the maneuver. If the most severe constraints appear to be on weight or size of the vehicle, then the linear gravity model of Section 2.2.3.2 might be first investigated to see if it will produce acceptable rendezvous from the desired range of initial conditions and with the available fuel. Alternately, if the space and weight of the computing equipment is relatively unrestricted, as, for example, it might be if the computation were done on some other vehicle or on the ground, then the two body equations of Section 2.2.3.6 might be first investigated. These equations might also be the first choice for investigation if they were used for some other portion of the flight as with the Apollo case where basically the same equations serve for midcourse and rendezvous. Finally, a comparison between the amount of fuel used for the maneuver selected and the theoretical minimum amount as determined by an optimization process, such as discussed in Section 2.4.1 should be performed to determine if the fuel savings is significant enough to warrant investigation of an approximate optimization technique.

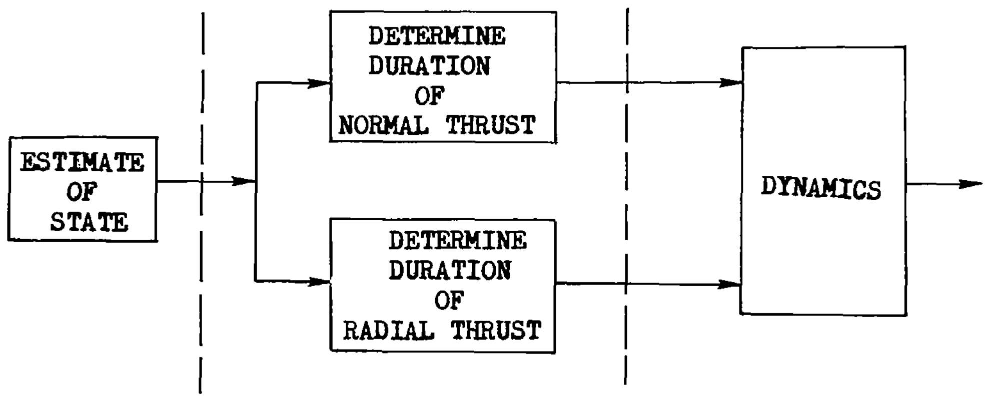
A general block diagram of the rendezvous guidance process is shown in the following sketch.



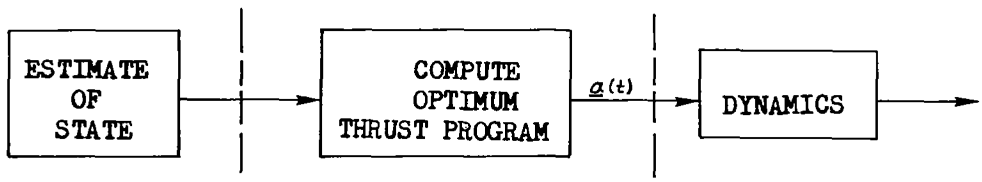
In some of the rendezvous guidance schemes discussed, as for example the impulsive technique in Section 2.2.3, the interface between the determination of the required velocity and the steering equations was not discussed. In these cases, the required velocity can be used to generate a reference trajectory and the methods of the Monograph on boost guidance Reference (2.23) used to define the steering commands. The guidance and steering equation blocks of the sketch can be represented in more detail as



For the guidance technique which nulls the angular rage of the LOS, the direction of thrust are along the range and normal and steering is achieved by determining the length of time each rocket motor is to operate. For this situation, the link between the state estimation and the dynamics is represented as



Still another representation is possible for the optimum guidance scheme of Section 2.3.2 since this scheme determines the steering angles directly. For this method, the two blocks of the first sketch can be combined as



Although the sensors necessary to measure the quantities necessary for a particular guidance scheme or the methods used to smooth the raw data have been discussed in this Monograph, these portions of the guidance and navigation process have been included in the above diagrams for completeness. Both of these subjects have been discussed in detail in previous monographs of this series (Reference 1.1 and 2.22).

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