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ABSTRACT

The problem of radiative transfer in a medium with a strongly anisotropic phase function is considered. Traditional methods of solution of the transfer equation have not proved practicable. Recent calculations using the Neumann solution, Romanova's method, and the "doubling method" of van de Hulst are described. To facilitate the study of absorption features under conditions of multiple scattering, the probability distribution of photon optical paths is introduced. When appropriately normalized, this distribution satisfies a transfer equation.

I. INTRODUCTION

The processes of diffuse reflection, transmission, and absorption of radiation by clouds of aerosol layers are of basic importance in both planetary and terrestrial atmospheric physics. Problems that are formally very similar present themselves also in studies of multiple scattering of charged particles, (1) multiple scattering of light by colloidal solutions, (2) the diffusion of fast neutrons, (3) and certain fields of astrophysics, such as the investigation of the reflection nebulae and the diffuse light in the galaxy.(4)

In all these instances, the basic problem involves solving the equation of transfer in a turbid medium. A scattering medium may be called turbid when the scattering diagram of the individual particles of the medium (or more rigorously, the scattering diagram of a unit volume in the case of a polydisperse medium) is strongly asymmetric. This will be the case for the scattering of radiation, for example, when the scattering centers have dimensions comparable with or larger than the wavelength of the radiation. Such particles scatter an incident plane wave predominantly in the forward direction. (5)

[redacted]

II. THE PHASE FUNCTION

Using the nomenclature of Chandrasekhar, (6) we shall describe the angular scattering pattern of the individual scattering act by a phase function $\Phi(\alpha)$, where $\Phi(\alpha)/4\pi$ is the probability that a scattered photon will be deviated by an angle α. We shall consider Φ to be a scaler, and thus ignore polarization effects.

Since the difficulty of the multiple-scattering problem increases markedly with growing complexity of the phase function, most of the literature of radiative transfer theory has been concerned with very simple phase functions, either isotropic scattering, Rayleigh scattering, or scattering with a phase function $\Phi = a + b \cos \alpha$. Rayleigh scattering is, of course, appropriate for particles with dimensions much smaller than the wavelength of the radiation, while isotropic scattering happens to apply to a stellar atmosphere in local thermodynamic equilibrium in the gray case. The third phase function mentioned, $\Phi = a + b \cos \alpha$, has been used as a test case for asymmetric scattering. (7, 8, 9, 10)

The striking departures from the simple phase functions mentioned above, which regularly arise in the optics of planetary atmospheres, are illustrated in figure 1. The significant departures from Rayleigh scattering, which occur even under conditions of very large horizontal visibility, are apparent. The presence of cloud droplets increases the forward elongation of the phase function by an additional order of magnitude or more (fig. 2).

The present paper is thus concerned with the solution of the equation of radiative transfer when the phase function is strongly elongated, either in the forward or the backward direction. We shall confine the discussion to methods that give, in principle, rigorous solutions to the equation and shall not discuss approximate methods based on simpler physical models (for example, Eddington's approximation or the two-stream method, (9) diffusion-type methods, (13) or the method utilized for extensive computations by Feigel'son et al. (14)

III. TRADITIONAL METHODS

The problem posed here may, in principle, be solved with the aid of methods that are now familiar. For example, in the case of a plane-parallel homogeneous layer, the equation of transfer is an integro-differential equation in three variables (two angular variables plus a coordinate specifying depth in the layer). The dependence on azimuthal angle may be removed by expanding the phase function and the intensity (which is the unknown quantity) in Legendre polynominals. The remaining integration over the polar angle may be replaced by a sum using, for example, a gaussian quadrature formula. There results a system of linear first-order ordinary differential equations. (6) Difficulty arises in practice with this method because the expansion of an elongated phase function in Legendre polynominals requires a large number of terms N (for example, scattering by a spherical water droplet with a circumference-to-wavelength ratio of 20 requires approximately 45 terms). (15) Consistency than requires that the number n of simultaneous equations to be solved be chosen such that $2n - 1 \geq 2N$. In consequence, numerical problems attending the solution become very severe (e.g., the characteristic equation that must be solved is an algebraic equation of order n/ 2).

Similar practical difficulties arise in the application of the coupled nonlinear integral equations which arise in the alternate formulation of the transfer problem first given by Ambartsumyan (16) and Chandrasekhar (6) on the basis of principles of invariance (but which may also be derived directly from the equation of transfer). (17, 18) Moreover, the nonlinearity of the equations of this method gives rise to problems in the numerical calculations and interpretation of results. Thus, Mullikin has shown that the solutions to certain of the equations obtained from the principles of invariance are not in general unique unless additional constraints are specified, (19) in the absence of which an instability in the numerical calculations can result for optical depths greater than about 1.5. That similar problems occur in the formulation of this approach called by Bellman "invariant imbedding" (20) has also been shown by Mullikin. (21)

Mullikin's method, (22) which utilizes linear singular integral equations, would also seem to be impracticable when applied to very elongated phase functions.

IV. RECENT DEVELOPMENTS

Let us, for concreteness, consider the following problem. A homogeneous plane-parallel layer of total optical thickness τ^* is illuminated by monochromatic radiation. The sources of radiation may be either internal or external to the layer. The single scattering albedo is a. We designate the optical depth measured vertically from the top of the layer by τ, the angle between a given direction and the direction of increasing τ by θ, and the azimuthal angle by ϕ. We further set $\Omega = (\theta, \phi)$ and $\mu = \cos\theta$. We seek the specific intensity $I(\tau, \Omega)$ of radiation in the layer. The equation of transfer for this problem is then

$$\mu \frac{dI}{d\tau} = -I(\tau, \Omega) + S(\tau, \Omega) \tag{1a}$$

$$S(\tau, \Omega) = \frac{a}{4\pi} \int d\Omega' \, \Phi(\Omega, \Omega') \, I(\tau, \Omega') + S_1(\tau, \Omega) \quad . \tag{1b}$$

Most frequently we shall be interested in the case when the source of radiation is external and at a large distance from the layer. We may then consider the top of the layer to be illuminated uniformly by parallel radiation incident at an angle $\Omega = (\theta_0, \phi = 0)$. It is then convenient to define I as the diffuse radiation only (that is, radiation that has been scattered at least once). The inhomogeneous term in equation (1) then takes the form

$$S_1(\tau, \Omega) = \frac{a\,\Phi(\Omega, \Omega_0)\, e^{-\tau/\mu_0}}{4\pi\,\mu_0} \tag{2}$$

and the boundary conditions are

$$I(0, \Omega) = 0 \qquad \mu > 0$$

$$I(\tau^*, \Omega) = 0 \qquad \mu < 0 \quad . \tag{3}$$

A. Small-Angle Approximation

Because the difficulties in the numerical solution of equation (1) are associated with the forward peak in Φ and I , it is natural to try to separate off that portion of I that is strongly elongated in the forward direction. Romanova (23) has proposed a method for achieving this based on the small-angle approximation familiar from the theory of scattering of fast particles. The small-angle approximation $\overline{I}$ is the solution of the equation

$$\mu_0 \frac{d\overline{I}}{d\tau} = -\overline{I}(\tau, \Omega) + S(\tau, \Omega) \quad , \tag{4a}$$

where $S(T, \Omega)$ is given by equations (1b) and (2), and the boundary conditions are

$$\overline{I}(0, \Omega) = 0 \qquad (-1 \leq \mu \leq 1) \quad ; \tag{4b}$$

the variable μ in equation (1) has been replaced by the constant μ_0, and the boundary conditions assume that there is no reflection from the layer. The solution to this approximate problem is known. If we now set

$$I = \overline{I} + \tilde{I} \tag{5}$$

we easily find that $\tilde{I}$ satisfies

$$\mu \frac{d\tilde{I}}{d\tau} = -\tilde{I}(\tau, \Omega) + \frac{a}{4\pi} \int d\Omega' \, \Phi(\Omega, \Omega') \, \tilde{I}(\tau, \Omega') + (\mu_0 - \mu) \frac{d\bar{I}}{d\tau} \quad , \tag{6}$$

a transfer equation of the same form as our original problem. Hopefully, the angular dependence of $\tilde{I}$ is much less pronounced than that of I.

The new equation (6) may of course be solved by any of the traditional methods. Romanova (24) has found that by expressing $\tilde{I}$ through a Lagrangian interpolation formula with unknown coefficients, accurate ($\sim$ 5 percent) values for the intensity may be found by use of as few as five angular points in the interval $0 \leq \theta \leq \pi/2$ (the azimuthal components of $\tilde{I}$ separate in the usual way). In figure 3, Romanova's calculations ((25) and private communication) for a layer of optical thickness $\tau^* = 2.5$ are compared with the solution obtained from the Neumann series (see below). The results agree quite well except for the directly backscattered radiation where the small-angle method does not predict the backscattered peak. (Since the Neumann series computation did not use enough angular points to completely define the amplitude of the maxima and minima in the reflected light, Romanova's calculations may be slightly more accurate than would appear from the figure.) Romanova's calculations for thicker layers ($\tau^* = 12, \infty$) confirm a number of intuitive expectations:

1. $I(\mu)$ becomes more and more isotropic as we go to greater depths.
2. In the deep regime (angular dependence of I independent of depth, I independent of azimuth) I is more anisotropic as the absorption increases.
3. τ_0 (the optical depth at which the deep regime sets in) is greater, the greater the asymmetry of Φ and the larger the absorption.

B. Neumann Solution

It is well known that equation (1) may be rewritten as an integral equation for the source function S:

$$S(\tau, \Omega) = a\Lambda\{S\} + S_1 \quad , \tag{7}$$

where

$$\Lambda\{\ldots\} = \frac{1}{4\pi} \int d\Omega' \, \Phi(\Omega, \Omega') \int_0^{\tau^*} d\tau' k(\tau - \tau', \mu) \{\ldots\} \tag{8a}$$

$$k(t, \mu) = \begin{cases} \dfrac{e^{-t/\mu}}{|\mu|} & \dfrac{t}{\mu} > 0 \\ 0 & \dfrac{t}{\mu} < 0 \quad . \end{cases} \tag{8b}$$

Formal manipulation of equation (7) then gives the solution

$$S(\tau, \Omega) = (1\, a\Lambda)^{-1} \{S_1\} = \sum_{n=0}^{\infty} (a\Lambda)^n \{S_1\} \quad , \tag{9}$$

where $(1\, a\Lambda)^{-1}$ is the inverse of the operator $(1 - \Lambda)$. It may be shown that the sum in equation (9), known as the Neumann series, is rigorously the unique solution to the problem. (17) Physically, the Neumann series represents nothing more than the sum of successive orders of scattering. This solution has been known for some time, but has generally been regarded as being too slowly convergent for practical use. However, with the use of modern computing machines this is not a serious problem for moderate optical depths and the method has a number of advantages (27):

1. All operations are linear and the result at each stage has a direct physical interpretation.
2. There is no problem for transmission at an angle equal to the angle of incidence, where Chandrasekhar's X and Y equations have a singularity.

3. The effect of changing the absorption in the layer (if the total optical thickness τ^* remains the same) is accounted for very simply through the factor a^n in the nth-order scattering.
4. There are no special problems in the conservative case (a = 1).

Finally, there is a point that seems not to have been emphasized until rather recently. Examination of the eigenvalue equation

$$\lambda_i \psi_i = \Lambda\{\psi_i\}$$

shows that there is a maximum eigenvalue ($\lambda_1 = \eta$) that is nondegenerate and that dominates the modulus of the other eigenvalues. (27, 28) This means that after a finite number of terms n_0 of the series in equation (9) have been computed, the series may be truncated and the remainder replaced by a geometric series with ratio ($a\eta$). Such truncation is particularly important for asymmetric scattering, when the Λ operator involves a double or triple numerical integration, so that successive iterations can become time consuming.

An example of the approach to the eigenfunction condition is shown in figure 4 for the phase function used in (29):

$$\Phi(\cos \alpha) = b\Phi_{HG}(g_1) + (1 - b)\ \Phi_{HG}(g_2) \quad , \tag{10}$$

where

$$\Phi_{HG}(\cos \alpha, g) = \frac{1 - g^2}{(1 + g^2 - 2g \cos \alpha)^{3/2}} \quad .$$

Computations of diffuse reflection and transmission by thin layers for phase functions characteristic of terrestrial hazes and fogs have been made by Irvine. (29) Values of η as a function of optical depth for two of these phase functions and also for isotropic scattering are shown in figure 5. In figures 6a and b are shown examples of the diffuse reflection and transmission for such phase functions.

C. Doubling Method

The disadvantage of the Neumann solution described in the previous section is its slow convergence for thicker layers. As van de Hulst has pointed out, however, we can improve this convergence by using the following approach:

If we know the reflection $R(\Omega, \Omega_0) = I(0, \Omega)$ and the transmission $T(\Omega, \Omega_0) = I(\tau^*, \Omega)$ for a layer of optical thickness τ^*, then we know implicitly the reflection and transmission by a layer of optical thickness $2\tau^*$. The latter quantities, as may be seen from figure 7, will each be given by an infinite series, but these series will converge much more rapidly than the corresponding Neumann series. The mathematical formulation of this method has been described by van de Hulst. (27)

Just as with the Neumann series, the ratio of successive terms, R_{n+1}/R_n or T_{n+1}/T_n, approaches limit, say α, so that the series may be truncated and the remainder replaced by a geometric series.

Some computations of diffuse reflection and transmission by this method for an elongated phase function are shown in figures 8a and b. (30)

V. ABSORPTION BANDS

The entire discussion thus far has been implicitly confined to the case of radiation within a wavelength interval (or, in the case of neutron scattering, of particles within an energy interval) such that the scattering characteristics of the atmosphere (Φ, a, τ^*) are essentially constant. However, the study of absorption lines or bands, across which the absorption coefficient of the atmosphere will be a rapidly changing function of frequency, is of great importance to atmospheric and planetary physics. From the relative intensity and width of such absorption features one may hope to deduce the thermodynamic characteristics of the layer in which they were produced, as well as an estimate of the amount of absorbing constituent present.

There are a number of difficult problems associated with the interpretation of such absorption features:

1. If the band width $\Delta\nu$ of the detector being employed is such that the absorption coefficient of the atmosphere changes appreciably across $\Delta\nu$, then one will observe an appropriately weighted average of the monochromatic solutions I_ν to the equation of transfer. It is not immediately obvious how to construct such an average in a simple manner, since the frequency enters I_ν not only through the albedo a, but also through the optical depth τ and total optical thickness τ^*.

2. In studying the shape of an absorption feature it would be convenient if, instead of computing independent solutions to the equation of transfer I_ν at a sufficient number of points to define the line shape, one could solve the equation of transfer once at a frequency outside of the absorption feature, and then apply a suitable correction to obtain the intensity for any arbitrary absorption.

3. In order to estimate the abundance of any absorbing constituent above some given level in an atmosphere, it is necessary to know the mean path traveled by photons in that part of the atmosphere above such a height. Because of the occurrence of multiple scattering in problems of interest, the calculation of this path length is not a simple matter.

These problems are, in fact, closely related. Let us assume (as is frequently the case) that the scattering characteristics of the atmosphere are independent of frequency within the wavelength interval of the absorption feature. It is then natural to introduce the probability $p(\lambda;\ \tau,\ \Omega)\ d\lambda$ that a photon contributing to $I(\tau,\ \Omega)$ has traveled an optical path for scattering between λ and $\lambda + d\lambda$. (31) Here $I = I_\nu\ \Delta\nu$ is the intensity that would be observed within the interval $\Delta\nu$ if the absorption band were absent. If the absorption coefficient of the atmosphere is changing rapidly across $\Delta\nu$, the transmission function for the feature of width $\Delta\nu$ will no longer be exponential, but may have a more general form $\psi_{\Delta\nu}(\ell)$, where ℓ is geometric path length (for example, ψ might be the transmission on the Elsässer model or the Goody model, which are appropriate for the absorption bands of CO_2 and H_2O respectively). (32) The intensity integrated across $\Delta\nu$ that will be observed is then clearly

$$I_{\Delta\nu}(\Omega) = I(\Omega) \int_0^\infty d\lambda\ p(\lambda,\ \Omega)\ 4\left(\frac{\lambda}{\sigma}\right), \tag{11}$$

where σ is the scattering coefficient of the layer (in units of inverse length). We have thus separated the radiative-transfer aspects of the problem (which involve finding I and p) from the complications due to the introduction of absorption, which are accounted for by the simple quadrature.(11)

The first moment of $p(\lambda)$

$$\langle \lambda \rangle = \int_0^\infty d\lambda \; \lambda \; p(\lambda) \quad , \tag{12}$$

is the mean optical path traveled by radiation in the layer in the absence of absorption.

A. Mean Path Lengths

We may investigate $\langle \lambda \rangle$ without explicit knowledge of the form of $p(\lambda)$. Consider the form of equation (11) for monochromatic radiation so that $\psi = e^{-\beta\lambda}$, where $\beta = \sigma/\kappa$ and κ is the coefficient of true absorption. Differentiating both sides with respect to β, we find that

$$\langle \lambda \rangle = - \frac{1}{I(\beta = 0)} \frac{\partial I(\beta)}{\partial \beta} \Bigg|_{\beta = 0} \quad ; \tag{13}$$

or if we write I with its arguments in the form usual in astrophysics

$$\langle \lambda \rangle = \frac{1}{I(\tau, \tau^*, \Omega, a = 1)} \left[\frac{\partial I}{\partial a} - \frac{\tau \, \partial I}{\partial \tau} - \tau^* \frac{\partial I}{\partial \tau^*} \right]_{a = 1} \quad . \tag{14}$$

Let us denote the integral term on the right-hand side of equation (11) by $\langle \psi \rangle$. We may approximate this term through use of $\langle \lambda \rangle$ in certain limiting cases. For example, when the layer is thin $(\tau^*(s + s_0) \ll 1)$, most of the radiation will come from path lengths for which $\lambda \ll 1$. If we then expand ψ to first order in λ, we find that

$$\langle \psi \rangle = \psi(\langle \lambda \rangle) \qquad (\lambda \ll 1) \quad . \tag{15}$$

If the strength of the absorption enters ψ through a parameter b such that $\psi = \psi(b\lambda)$ (for example, in the strong-line limit the Elsässer function is $\psi = \mathrm{erfc}\sqrt{b\lambda}$) equation (15) will hold provided that $b\lambda \ll 1$.

In order to estimate the abundance of an absorbing constituent above a certain level in the atmosphere, one may wish to know λ_{eff} as a function of the depth of the level, where λ_{eff} is defined by $\psi(\lambda_{eff}) = \langle\psi\rangle$ (that is, λ_{eff} is the laboratory path length that would give the same absorption observed in the multiple-scattering case). We shall have $\lambda_{eff} = \langle\lambda\rangle$ under the same conditions for which equation (15) holds (that is, $\lambda \ll 1$, or $b\lambda \ll 1$). When these conditions do not hold, λ_{eff} will be different for different frequencies within an absorbtion feature, so that the profile of a given line or band will be different in the laboratory from that observed after multiple scattering under identical thermodynamic conditions.

B. Probability Distribution (33)

From the form of equation (11) we see that it may be useful to renormalize p and define the new quantity

$$\mathcal{J}(\lambda;\ \tau,\ \Omega) = I(\tau,\ \Omega)\ p(\lambda;\ \tau,\ \Omega)\ . \tag{16}$$

The quantity $\mathcal{J}$ satisfies an equation of transfer which, in the case of a plane-parallel layer, takes the form

$$\frac{\partial \mathcal{J}}{\partial \lambda} + \cos\theta\ \frac{\partial \mathcal{J}}{\partial \tau} = -\mathcal{J}(\lambda,\ \tau,\ \tau^*,\ \Omega) + \mathcal{S} \tag{17a}$$

$$\mathcal{S}(\lambda,\ \tau,\ \tau^*,\ \Omega) = \frac{1}{4\pi}\int_{4\pi} d\Omega'\ \Phi(\Omega\ ,\Omega')\mathcal{J}(\Omega') + \mathcal{S}_1 \tag{17b}$$

$$\mathcal{J}(\lambda = \infty) = \mathcal{J}(\lambda = 0) = 0 \quad , \tag{17c}$$

where $\mathcal{S}$ is the relevant source function; the boundary conditions with respect to τ are given by equation (3). Note that since $\mathcal{D}$ is defined with respect to a conservative layer, the albedo a does not enter equation (17).

One may also formulate principles of invariance satisfied by $\mathcal{D}$ and obtain corresponding nonlinear integral equations for the reflectivity $\mathcal{R}(\lambda)$ and transmissivity $\mathcal{T}(\lambda)$, from which it may be seen that

$$\mathcal{R}(\lambda, \Omega, \Omega_0) = \mathcal{R}(\lambda, \Omega_0, \Omega) \tag{18}$$

$$\mathcal{T}(\lambda, \Omega, \Omega_0) = \mathcal{T}(\lambda, \Omega_0, \Omega) \quad ,$$

which is a principle of reciprocity.

In analogy to equation (9), the Neumann solution for $\mathcal{D}$ takes the form

$$\mathcal{D}(\lambda, \tau, \Omega) = \sum_{n=1}^{\infty} \Xi^{n-1} \{\mathcal{D}_1(\lambda, \tau', \Omega')\} \tag{19a}$$

$$\Xi\{\ldots\} = \frac{1}{4\pi} \int_{4\pi} d\Omega \; \Phi(\Omega, \Omega) \int_0^{\tau^*} d\tau \; k(\tau - \tau, \mu) \, \Upsilon_{\tau\tau\mu} \{\ldots\}, \tag{19b}$$

where k is given by equation (8b) and we have introduced the shift operator

$$\Upsilon_{\tau\tau'\mu'}\{f(\lambda)\} = f(\lambda - \frac{\tau - \tau'}{\mu'}) \quad . \tag{20}$$

For thicker layers, use of equation (19) is not so convenient as noting that equation (17) is identical to the equation of transfer for a time-dependent radiation field. The latter problem is amenable to Laplace transfer techniques, (34) and it is easy to show that

$$\mathcal{J}(\lambda, \Omega, \tau) = L^{-1}_{\lambda\beta} \{I(\beta; \Omega, \tau)\} \quad , \tag{21}$$

where $L^{-1}_{\lambda\beta}$ is the inverse Laplace transform operator, and the notation $I(\beta; \Omega, \tau)$ means that the entire effect of true absorption on the intensity is expressed through the argument β (i.e., τ is the optical depth due to scattering only). In the case of reflection from a semi-infinite or from a very thick layer, $\mathcal{J}(\tau = 0) = \mathcal{R}$ may be obtained analytically in the limits of large and small absorption. By inverting certain expressions due to Rozenberg that are valid in the limiting case $\beta << 1$, using the Neumann solution when $\beta << 1$, and interpolating with the aid of the requirement of conservation of flux, Romanova (35) was able to obtain the results shown in figure 9 for

$$A(\lambda) = \int d\Omega' \mu' \mathcal{R}(\lambda; \Omega', \Omega_0) \quad . \tag{22}$$

Note that, as is intuitively obvious, the distribution of path lengths is much more concentrated toward small λ for isotropic scattering than for forward-directed scattering.

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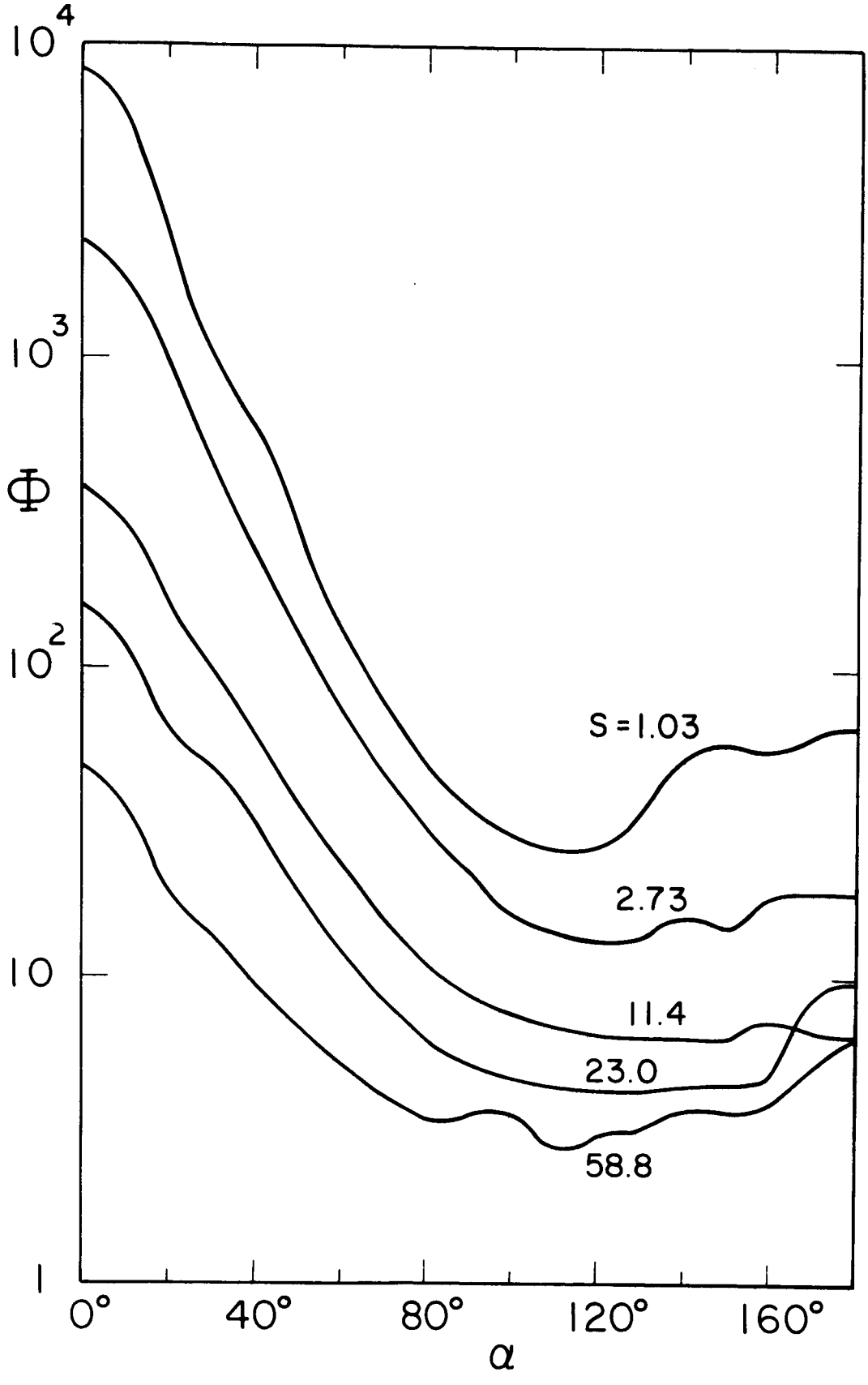

Figure 1. Phase functions (not normalized) for the lower terrestrial atmosphere under various conditions of horizontal visibility S = 3.91/z (z = volume extinction coefficient). (11)

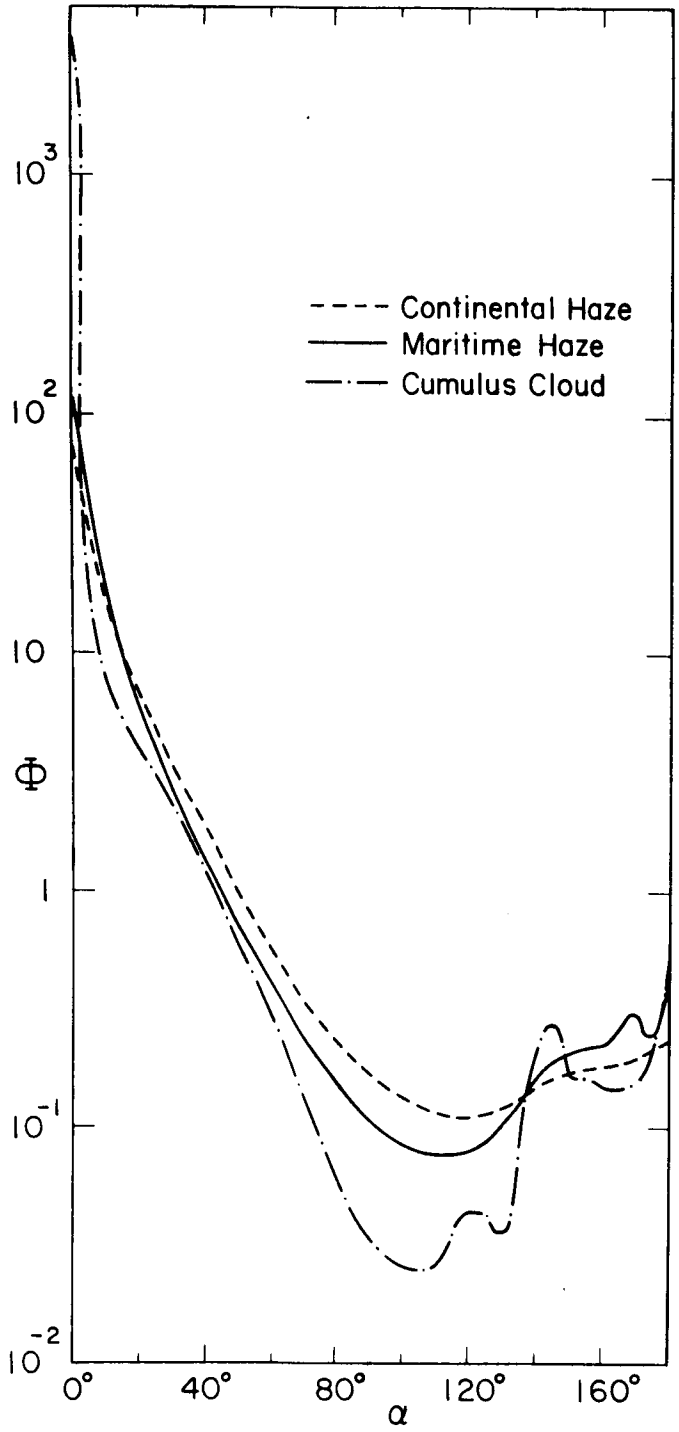

Figure 2. Phase functions for cloud and haze layers at $\lambda = 0.45\ \mu$. (12)

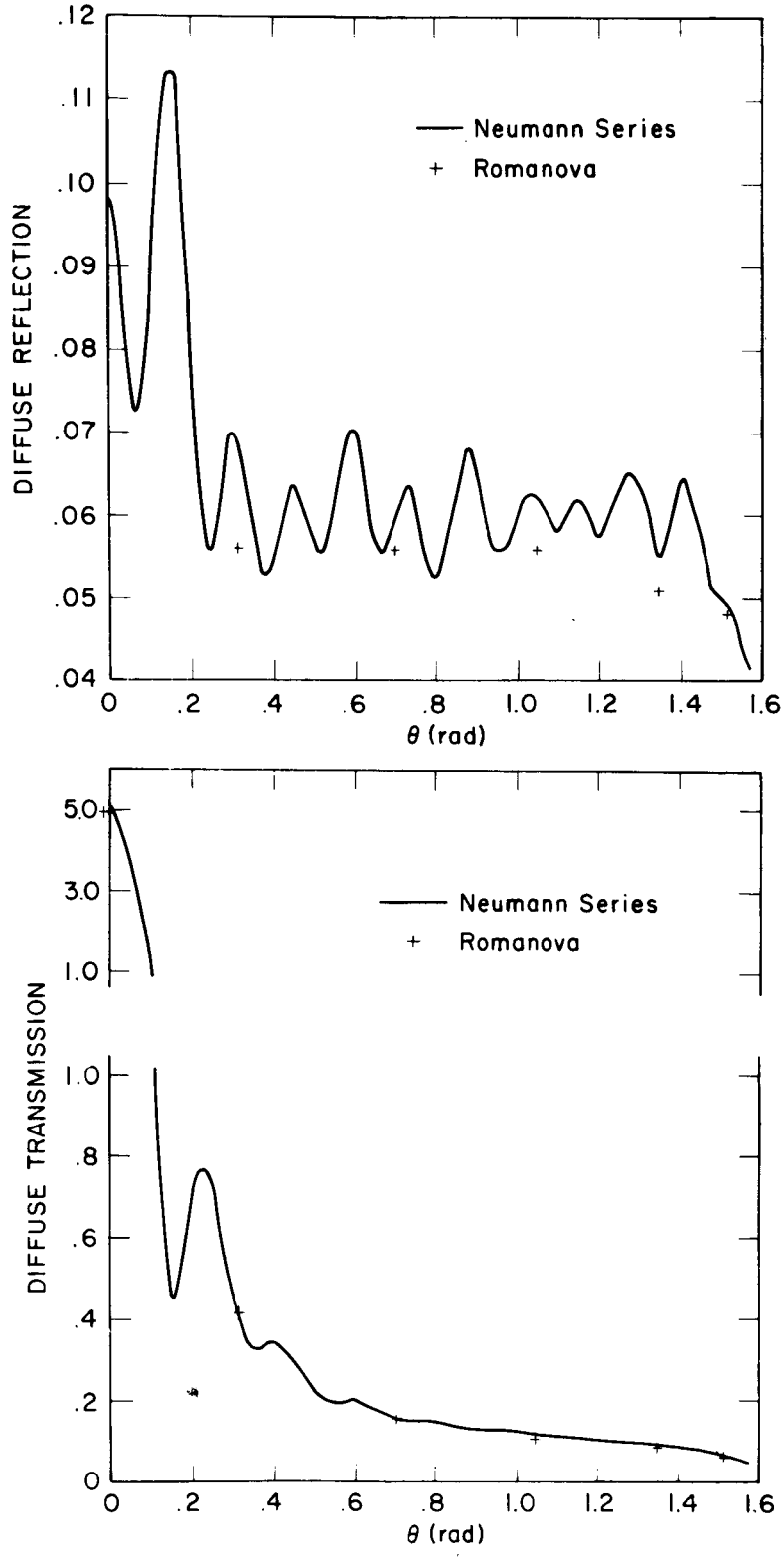

Figure 3. Diffuse reflection (a) and transmission (b) from a homogeneous, plane-parallel layer of optical thickness $\tau^* = 2.5$ for normal incidence and Mie scattering ($m = 1.33$, $2\pi r/\lambda = 20$). Angle θ measured from outward normal. (25, 26)

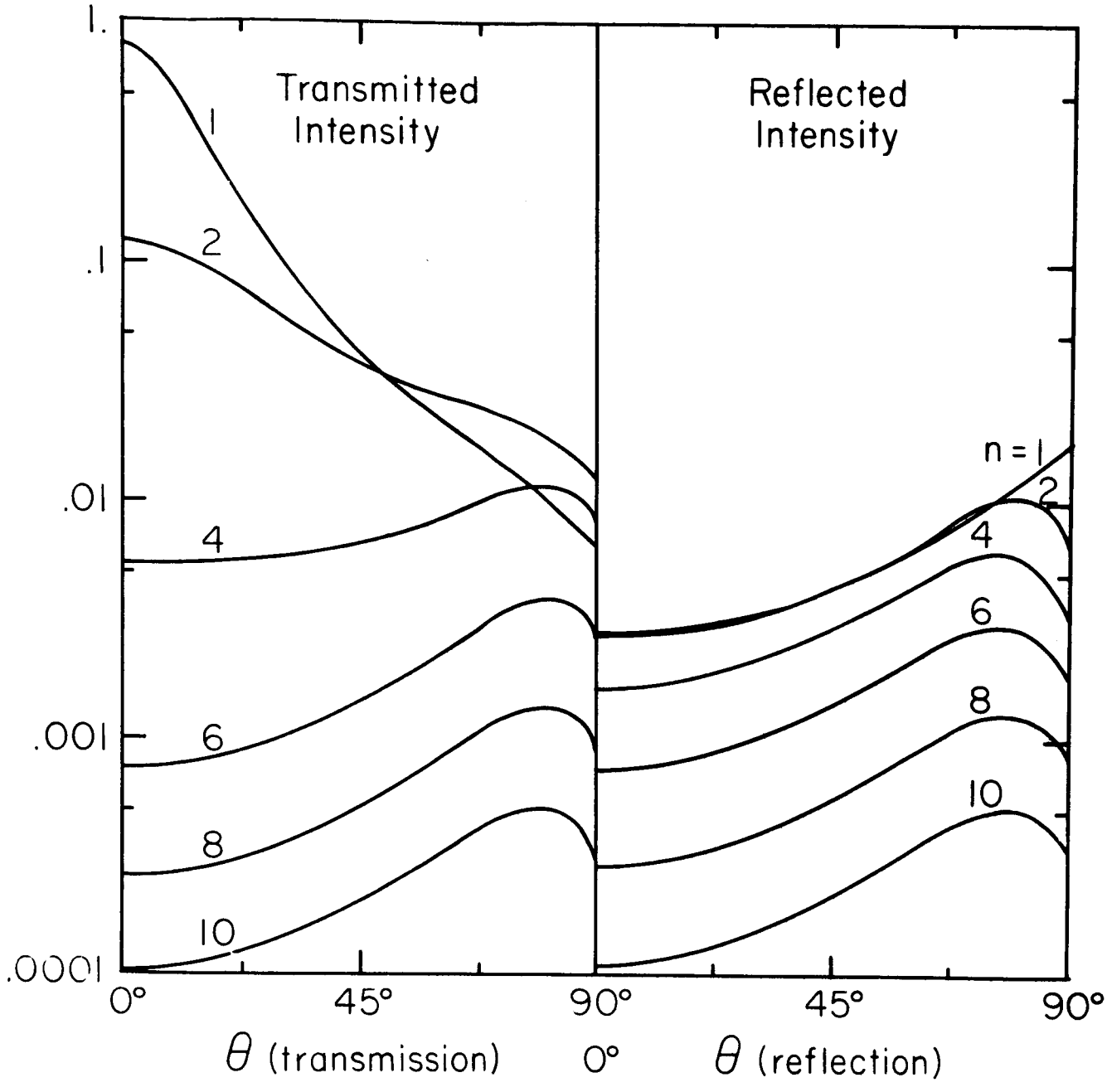

Figure 4. Diffuse reflection and transmission in successive orders of scattering n from a plane-parallel atmosphere of unit optical thickness for normal incidence and the phase function equation (10) with $g_1 = 0.75$ and $b = 1$. Angle θ measured from outward normal. (26)

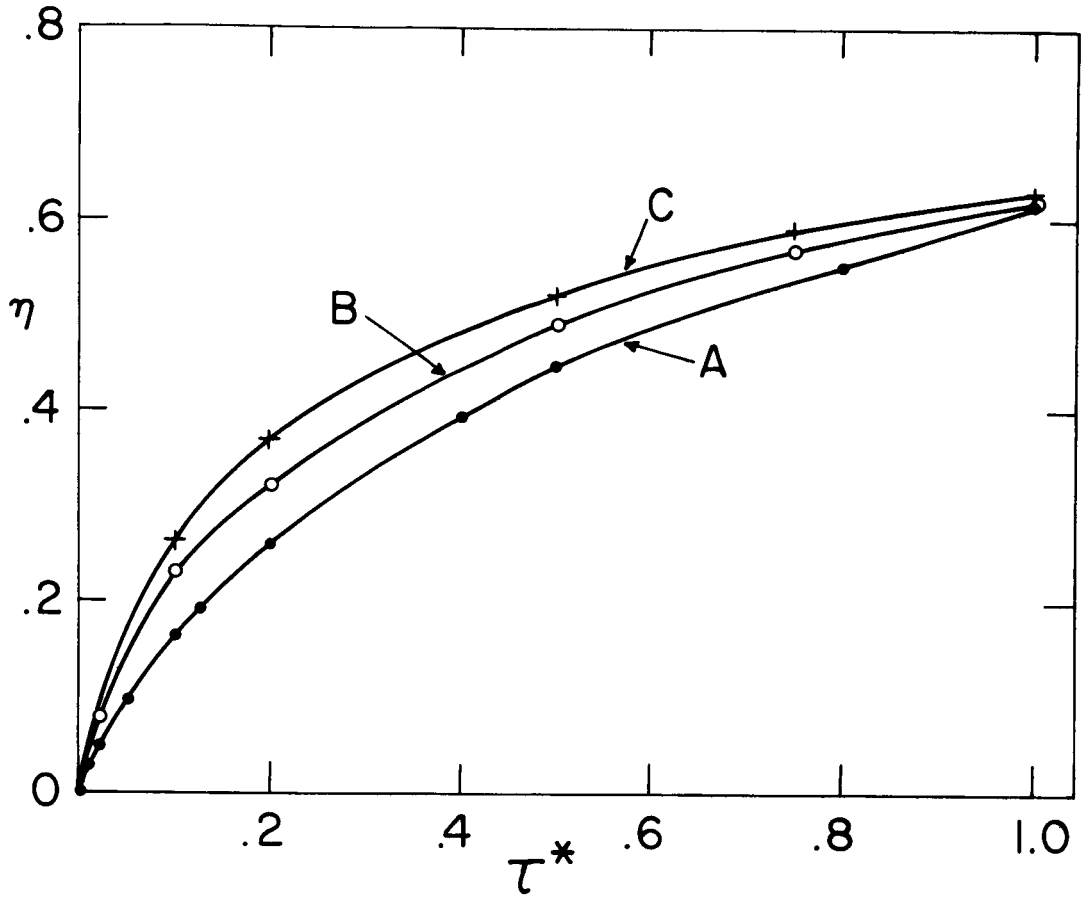

Figure 5. The eigenvalue η versus optical thickness for three phase functions of the form of equation (10) (26):

A — $g_1 = g_2 = 0$ (isotropic scattering),
B — $g_1 = 0.75$, $b = 1$
C — $g_1 = 0.824$, $g_2 = -0.55$, $b = 0.9724$.

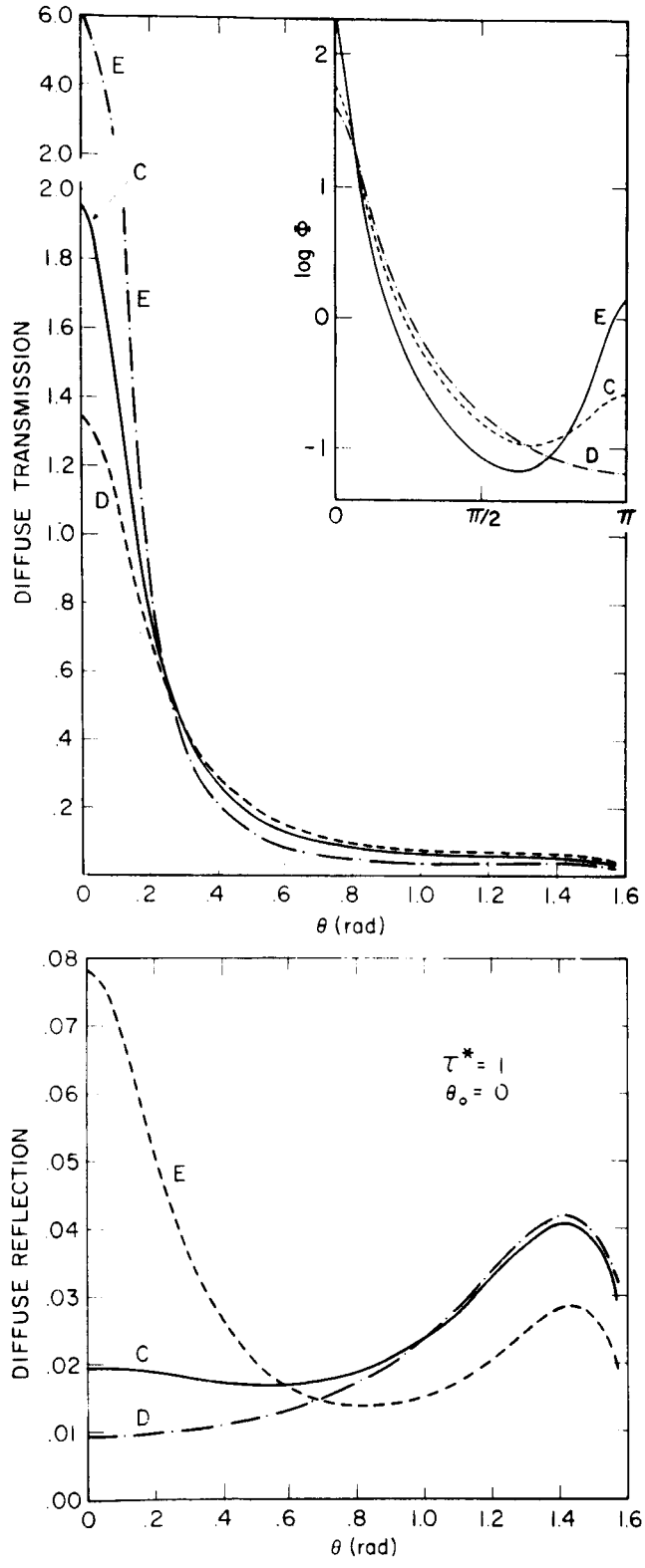

Figure 6. Diffuse transmission and reflection from a plane-parallel layer of optical-thickness unity for normal incidence and phase functions of the type equation (10) (26, 29):

C — $g_1 = 0.824$, $g_2 = -0.55$, $b = 0.9724$,
D — $g_1 = 0.7861$, $b = 1$ (same first moment as C),
E — $g_1 = 0.90$, $g_2 = -0.75$, $b = 0.95$.

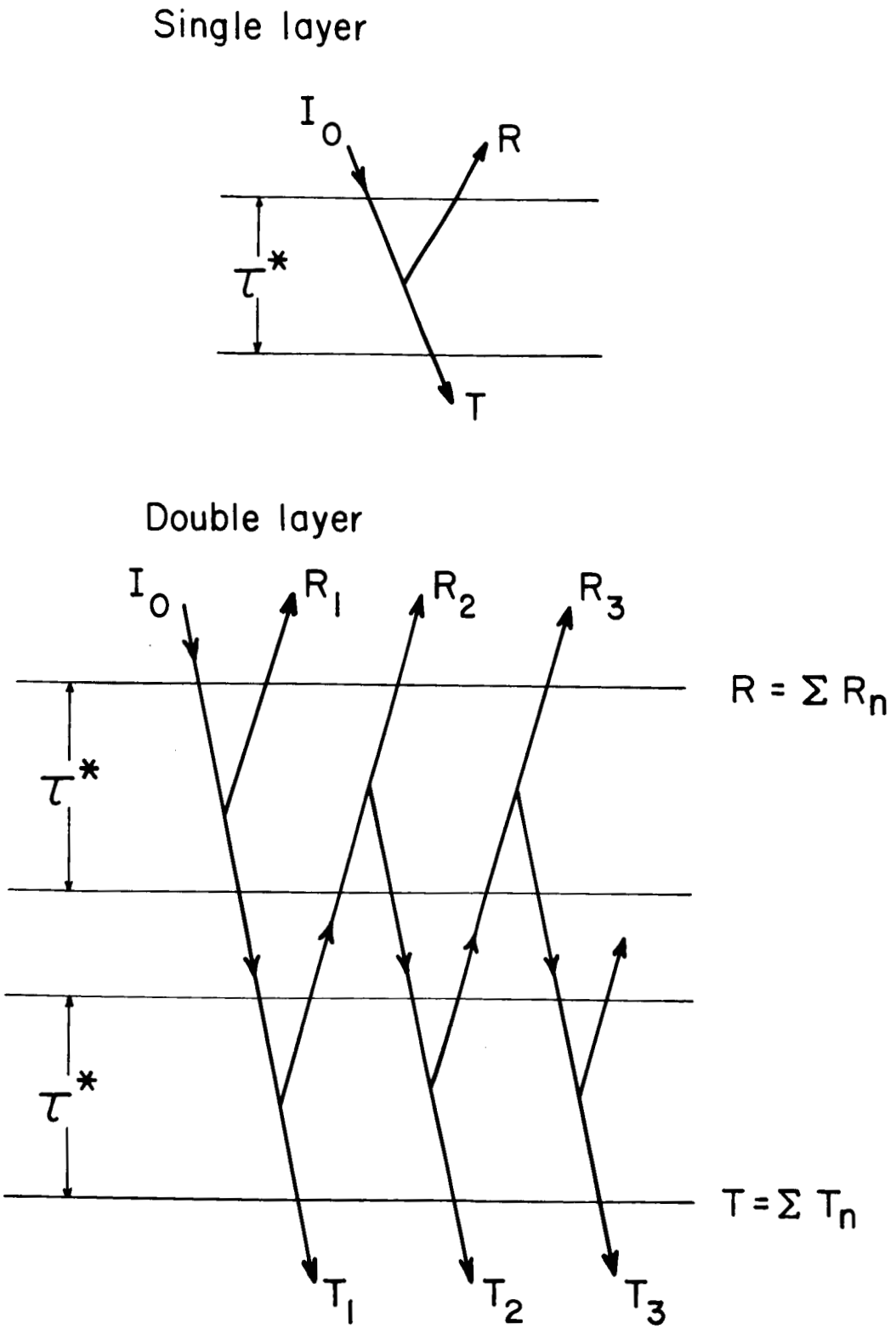

Figure 7. Geometry of the doubling method (after (27)).

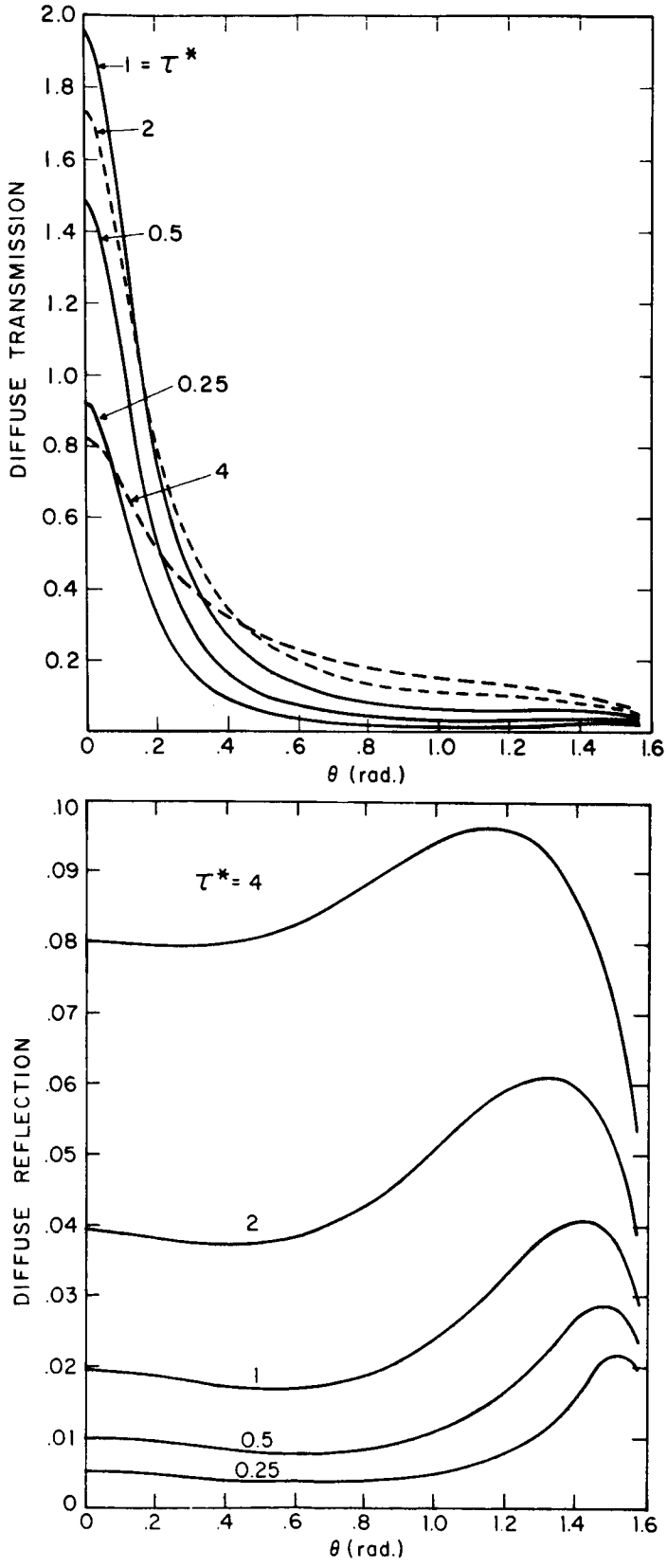

Figure 8. Diffuse transmission and reflection by a plane-parallel layer of optical thickness τ^* for normal incidence and phase function C (see fig. 6a). (30)

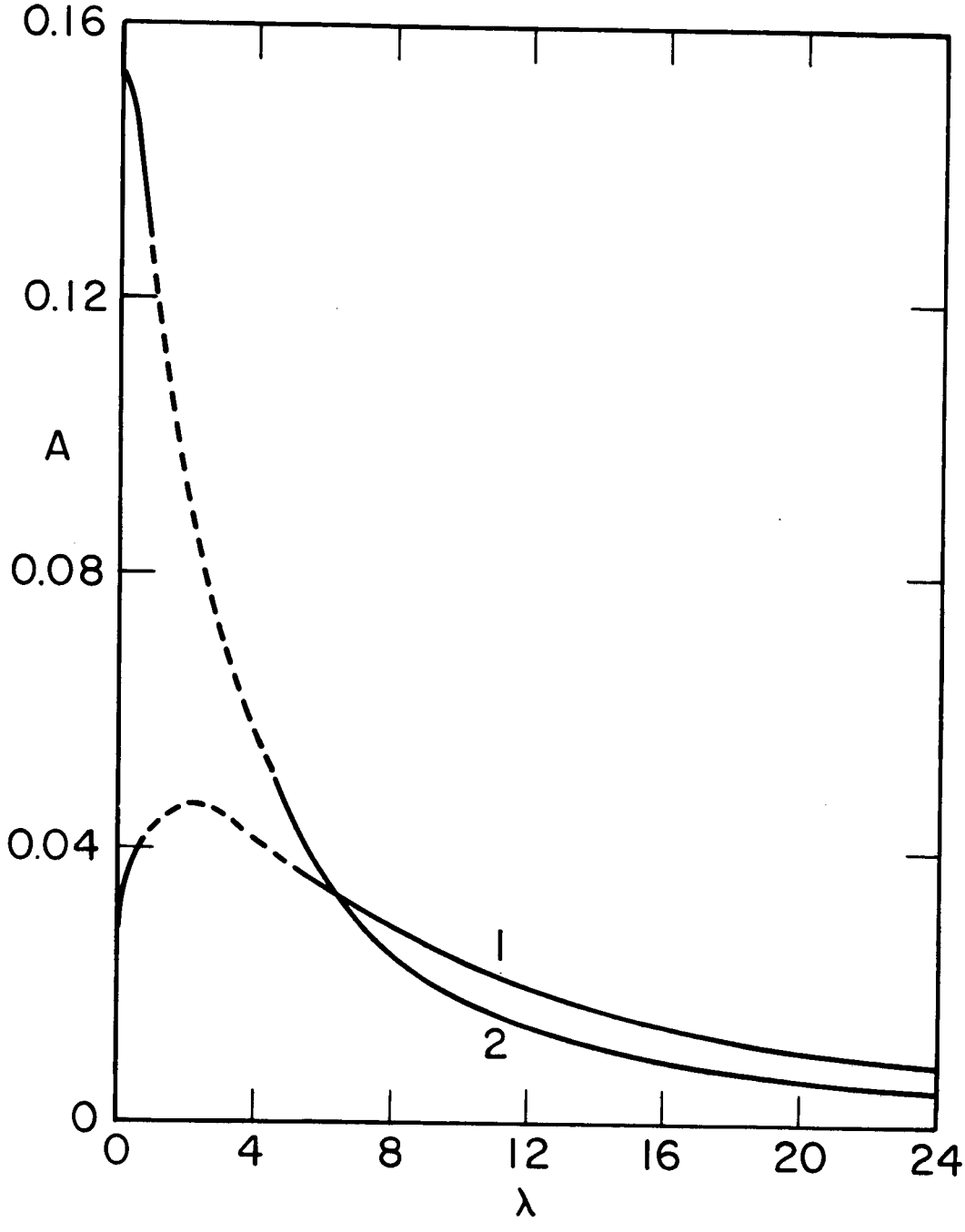

Figure 9. Probability distribution of photon optical paths in total flux reflected from a semi-infinite, conservative layer for normal incidence: (1) Mie scattering ($m = 1.33$, $2\pi r/\lambda = 20$), (2) isotropic scattering. (35)