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SENSITIVITY DESIGN TECHNIQUE FOR OPTIMAL CONTROL

Submitted on behalf of

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Introduction

This report describes a feasibility study conducted to investigate the possibility of using optimal sensitivity design techniques to design a feedback control system for a large, flexible booster. The vehicle to be controlled is described in NASA MSFC "Model Vehicle No. 2". The design objective is to maintain the vehicle in the immediate neighborhood of a predefined nominal trajectory in spite of disturbances acting on the vehicle. These disturbances are both external, such as random wind gusts, and internal such as changes in vehicle parameters. An additional design objective is to keep the bending of the vehicle within design limits, preferably as small as possible.

The nominal trajectory of the vehicle is described by a set of nonlinear differential equations. However, small perturbations of the vehicle's motion about the nominal can be adequately characterized by a set of linear incremental differential equations with time varying coefficients. The data needed to calculate these coefficients were obtained from the "Model Vehicle No. 2" data package.[1]

The Problem

The vehicle to be controlled is a large flexible booster of the Saturn type. Control is exerted by gimballing four of the eight propulsion engines. The control objectives can be divided into categories -- rigid body objectives and bending body objectives.

The object of the rigid body control is to maintain the vehicle as close as possible to its nominal trajectory in spite of random wind gusts. As a first approximation it is assumed that the necessary parameters of the ficticious rigid body can be measured exactly (cf. Fisher).[2]

Unfortunately, because the vehicle is flexible, this rigid body is nonexistent. Rather the vehicle behaves like an unsupported beam with a (controllable) torque applied to one end. Gimballing the engines to provide a control torque for the rigid body excites bending modes in the vehicle. If ignored, these oscillations could reach a point where the vehicle exceeds its elastic limits and is destroyed. A successful control strategy must prevent this. Because of the large size of the vehicle, however, full scale experimental determinations of the values of the parameters involved in the bending is difficult. Thus, the successful control strategy must limit the bending when the parameters are known only approximately and change with time. For simplicity only the first bending mode is considered and its equation is assumed to be that of a linear oscillator.

$$\ddot{\eta} + 2\zeta\omega\dot{\eta} + \omega^2\eta = \frac{R' Y(X_\beta)}{M_1}\beta$$

The right hand side of the equation is the engine excitation of the mode. Combining this first bending mode with the rigid body description gives the vehicle description in terms of five state variables.

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \dot{\phi}\\ \alpha-\alpha_w\\ \eta\\ \dot{\eta}\end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0\\ 0 & 0 & \frac{-N\ell_{cp}}{I_{xx}} & 0 & 0\\ -\frac{F-X}{MV} & 1 & \frac{-N'}{MV} & 0 & 0\\ 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & -\omega^2 & -2\zeta\omega \end{pmatrix}\begin{pmatrix}\phi\\ \dot{\phi}\\ \alpha-\alpha_w\\ \eta\\ \dot{\eta}\end{pmatrix} + \begin{pmatrix}0\\ -\frac{R'\ell_{cg}}{I_{xx}}\\ -\frac{R'}{MV}\\ 0\\ \frac{R' Y(X_\beta)}{M_1}\end{pmatrix}\beta$$

The numerical values of these time varying coefficients are obtained in tabular form from Ref. 1.

Sensitivity Design Method[3,4,5]

The trajectory of an optimal control system is sensitive to changes in the vehicle parameters. Thus, it is important to develop a control strategy which compensates for the effect of these changing parameters, preferably without having to track them. The control system should be designed so that the vehicle trajectory approximates the nominal trajectory corresponding to the nominal parameter values, in spite of variations of these parameters from their nominal values. Adjoining a sensitivity index and sensitivity equations to the standard optimal control problem makes this possible.

To consider the problem in more concrete terms, it is necessary to make the following assumptions, all of which are met by the flexible booster problems:

1. The process has a finite number of state equations of the form:

$$\dot{x} = f(x, \beta, q, t) \qquad x(t_o) = c$$

where

x: n-dimensional state vector

β: r-dimensional control vector

q: m-dimensional parameter vector with nominal value q_o.

2. The parameters appear as coefficients in the differential equation.
3. Any parameter variation does not change the number of state variables.

It is also assumed that f satisfies the continuity conditions which guarantee a unique solution $x(t)$ with $x(t_o) = c$ once the control β and the parameter q are specified.

Since closed loop control is desired, β can be written as

$$\beta = \beta(x,t)$$

For a given control strategy let the nominal trajectory corresponding to go be

$$x = x_o(t,\beta_o,q_o)$$

Then the trajectory deviation due to the parameter variation

$$q = q_o + \Delta q \quad \text{is given by}$$

$$\Delta x = x\ (t,\beta,q) - x\ (t,\beta_o,q_o)$$

The magnitude of the deviation can be defined in terms of $\|x\|$, the Euclidean norm of x. Expanding this deviation in a Taylor Series gives -

$$\Delta x = J\left(\frac{x}{q}\right)(q - q_o) + \text{higher order terms}$$

where

$$J\left(\frac{x}{q}\right) = \begin{pmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots \\ \vdots & \ddots & \\ \frac{\partial x_n}{\partial q_1} & \cdots & \frac{\partial x_n}{\partial q_m} \end{pmatrix}$$

To simplify the mathematics at this point, q is assumed to be a scalar, that is a single parameter. For small perturbations, the higher order terms in the Taylor Series can be neglected, giving -

$$\Delta x \approx \delta x = \left(\frac{\partial x}{\partial q}\right)\delta q$$

If an upper bound on the parameter variation $\delta q(t)$ is known or assumed, then a bound can be found for δx, the first order trajectory dispersion.

For suitably small δq, then, $\|\delta x\|$ can be limited by limiting

$$z(t) \triangleq \left\| \frac{\partial x}{\partial q} \delta q \right\| .$$

At a point on the perturbed trajectory $(x,q) = (x_0+\Delta x, q_0+\Delta q)$ the state equations become

$$\dot{x}_0 + \Delta\dot{x} = f\{x_0+\Delta x, \beta(x_0+\Delta x, t), q_0+\Delta q, t\}$$

Expanding this in a taylor Series about (x_o, q_o) -

$$\dot{x}_0 + \Delta\dot{x} = f + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial x}\right)\Delta x + \left(\frac{\partial f}{\partial q}\right)\Delta q$$

$+$ higher order terms

where f an its derivatives are evaluated at (x_o, q_o). Again, for small $\delta q = (q - q_o)$ the higher order terms are negligible, leaving -

$$\delta\dot{x} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial x}\right)\delta x + \left(\frac{\partial f}{\partial q}\right)\delta q$$

where δx is the first order approximation to Δx.

If δq is a (known) bound on the parameter variation which yields the maximum trajectory dispersion Δx, then, letting $z(t) \triangleq \|\delta x\|$ as above, the sensitivity equation can be written as

$$\dot{z} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial x}\right)z + \left(\frac{\partial f}{\partial q}\right)\delta q$$

This is a linear differential equation defined on the nominal trajectory. If the trajectory dispersion due to parameter variation is to be limited, the nominal trajectory and nominal control must be chosen to limit z(t). This is done by incorporating the sensitivity measure into the optimal control problem. To do this the optimal control problem is reformulated to minimize the sum of the original index of performance J and an index of sensitivity J subject to the constraints of the vehicle and the sensitivity equation.

The augmented optimal control problem thus becomes

determine $$\beta^*(x,t) = \left\{ \beta \,\middle|\, \min_{\beta} (J + \hat{J}) \right\}$$

where $$J = \int_{t_0}^{t_f} f_0(x, v, \beta, t)\, dt$$

$$\hat{J} = \int_{t_0}^{t_f} g_0(z,t)\, dt$$

subject to $$\dot{x} = f(x, \beta, q_0, v, t) \qquad x(t_0) = c$$

$$\dot{z} = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \beta} \frac{\partial \beta}{\partial x} \right) z + \frac{\partial f}{\partial q} \delta q \qquad z(t_0) = 0$$

where f_o and g_o are positive definite functions of x, v, β, and z.

Upon inspection it is found that this problem does not admit to a solution as there is no relation which yields the structural information necessary to construct $\beta^*(t)$. Specifically $\frac{\partial \beta}{\partial x}$ is unknown. The problem is underspecified. Therefore, the designer imposes this structural information in the form of a feedback control law. This completes the specification of the problem and allows it to be solved. The designer, however, must be judicious in his choice of feedback structure, since the resulting "optimal" closed loop system is then optimal only with respect to the specific type of feedback structure specified. Two rather general structures are considered below.

Cascade Gains

Consider the control system shown in Fig. 1. This consists of a set of cascade gains with unit feedback around the plant. The control law for this case is

$$\beta(x,t) = K(v - x)$$

where K is the vector of feedback gains and v is the desired state response. The gains are to be chosen to reduce the sensitivity of the nominal trajectory to parameter variations and to keep the vehicle as close to the nominal trajectory as possible.

Using this control, the plant equation becomes -

$$\dot{x} = f(x, K(v-x), q_0, t)$$

and the sensitivity equation is

$$\dot{z} = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial \beta} K\right) z + \frac{\partial f}{\partial q} \delta q$$

Thus, the optimal control problem becomes

$$\min_{K} (J + J) \qquad \text{where} \qquad J = \int_{t_0}^{t_f} f_0(x, K(v-x), t)\, dt, \qquad \hat{J} = \int_{t_0}^{t_f} q_0(z,t)\, dt$$

subject to the state and sensitivity equations.

The solution of this problem by standard parameter optimization methods gives a set of feedback gains K which is really a trade off between the two objectives of the problem. Two targets are being aimed at with only one arrow. The next section improves on this situation.

Feedback Gains with Autonomous Input

Consider the system of Fig. 2. This time control is exerted by a set of feedback gains and a pre-computed autonomous input (or prefilter).

To solve this problem it is necessary to combine control signal and parameter optimization.

First, parameter optimization is used to determine the feedback gains that offer the greatest protection against trajectory perturbations due to parameter changes. Then, constraining the feedback gains to be constant, the autonomous input is selected to keep the vehicle as near the desired trajectory as possible while further minimizing parameter sensitivity if possible. Thus, the control law is of the form:

$$\beta(x,t) = u(t) - Kx(t)$$

where K is the constant feedback matrix and u(t) the autonomous input - The nominal process equation is now

$$\dot{x} = f(x,\ u - Kx,\ q_o, t) \qquad\qquad x(t_o) = c$$

The sensitivity equation is

$$\dot{z} = \left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial \beta} K\right) z + \frac{\partial f}{\partial q}\,\delta q \qquad\qquad z(t_o) = 0$$

Assuming that the index of performance integrand can be written as

$$f_o\ (x,\ v,\ \beta,\ t) = f_1(x,v,t) + f_2(\beta,t)$$

i.e. that no cross-product terms appear, the problem can be stated in the following form:

$$\min_{u} \left\{ \int_{t_o}^{t_f} f_1\, dt + \min_{K} \left[\int_{t_o}^{t_f} (f_2 + g_o)\, dt \right] \right\}$$

subject to the state and sensitivity equations.

In order to adapt this problem to standard optimization techniques it can be restated as two related problems.

1. Feedback Gain Selection.

$$\min_{K} \int_{t_0}^{t_f} (f_2+g_0)\,dt$$

subject to the state and sensitivity equations.

2. Autonomous Input Selection:

$$\min_{u} \int_{t_0}^{t_f} (f_0+g_0)\,dt$$

subject to the state and sensitivity equations and $\dot{K} = 0$.

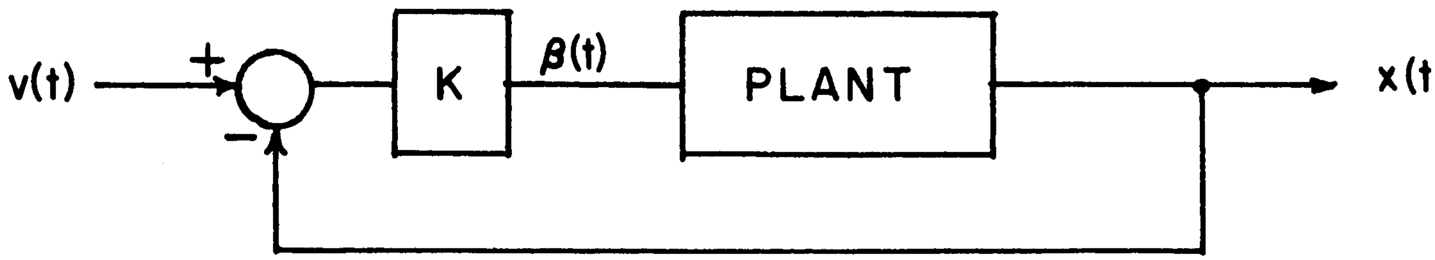
Results

The cascade gain configuration was solved on a digital computer. Since this was a feasibility study, weighting matrices were not selected to give the best possible results, rather the first weights tried that gave reasonable results were used. With more time invested in this selection results should be better.

Figure 3 shows pitch error and normalized bending for sensitivity weighting equal to zero. This corresponds to the usual parameter optimization solution. In both graphs the solid line indicates the response of the vehicle for nominal bending frequency, ω_o. The dotted line indicates the response for a -20% perturbation in bending frequency. The errors in this case are not only much greater, but they exhibit diverging oscillations.

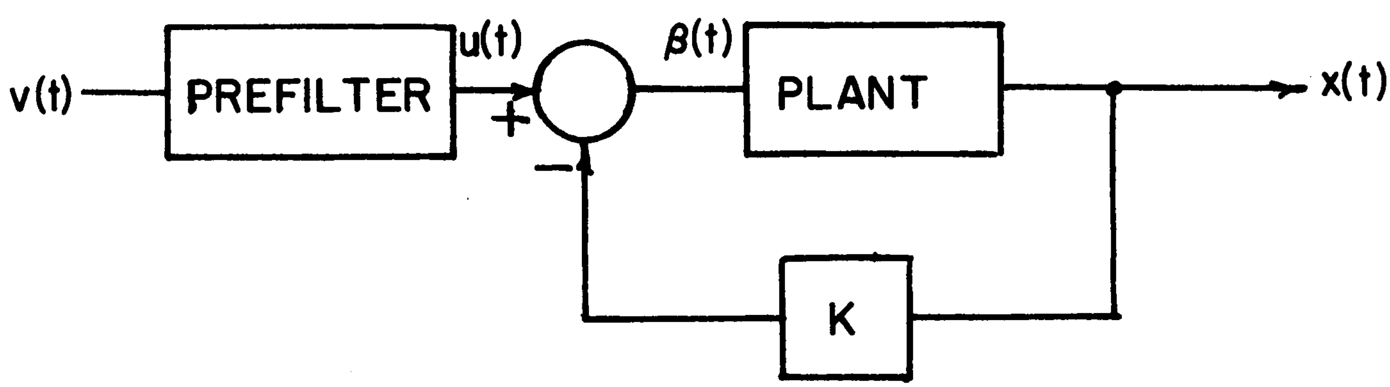
Figure 4 shows the same quantities with the sensitivity weighting increased to 10,000. Since the state and control weighting was not changed, the results for nominal bending frequency are slightly worse than those above. However for the -20% perturbation in ω the results are clearly greatly improved and in fact, quite acceptable.

This study has proven the feasibility of using optimal sensitivity design techniques to design a control system for a large, flexible booster. Work is currently underway to use the techniques on a more sophisticated vehicle model where, for instance, state information is not directly available. Further development of the Feedback Gain Autonomous Input Technique is also underway and this is expected to give even better results than the Cascade Gain Configuration.



CASCADE GAINS SYSTEM

FIGURE 1



FEEDBACK GAINS — PREFILTER SYSTEM

FIGURE 2

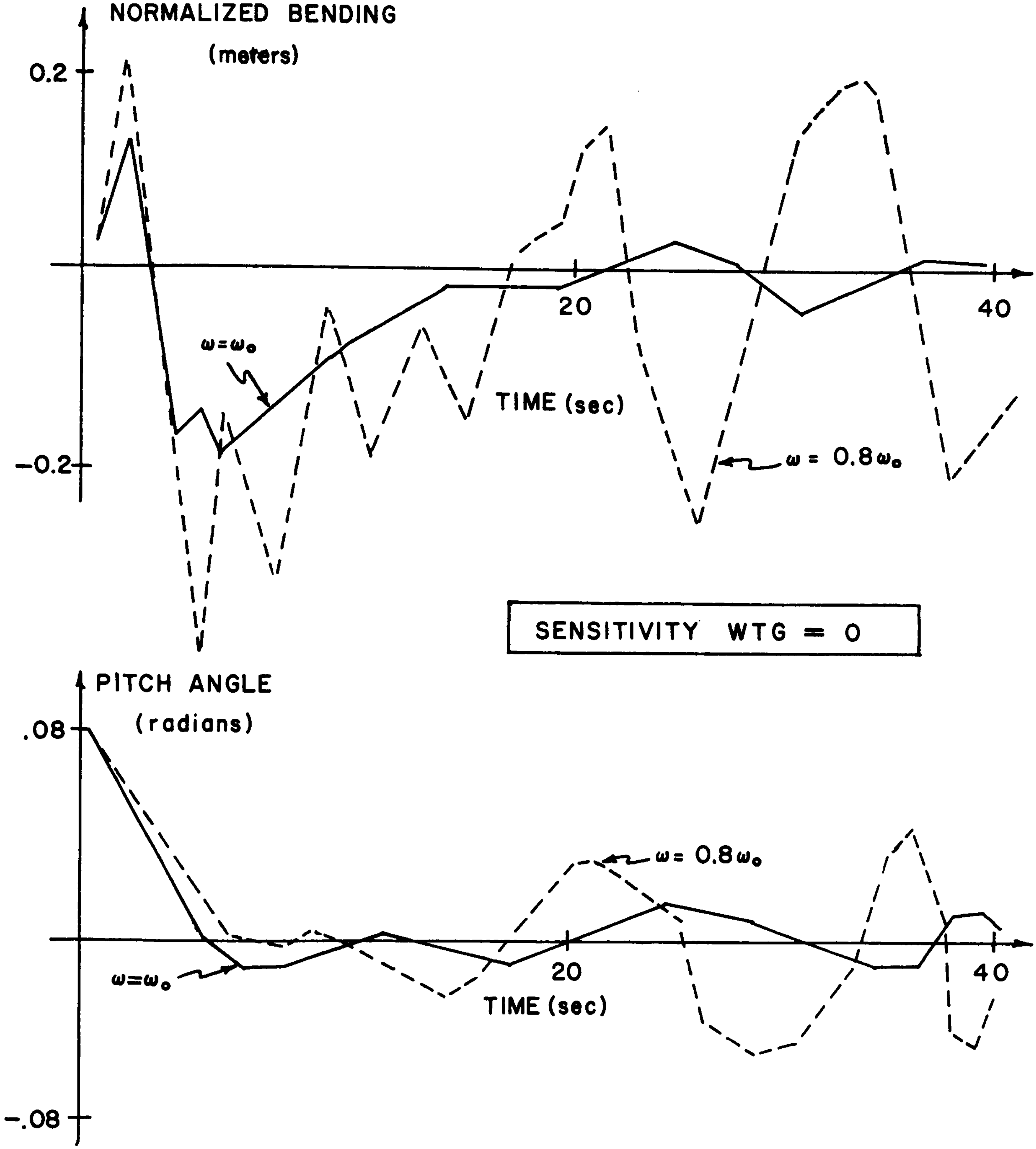
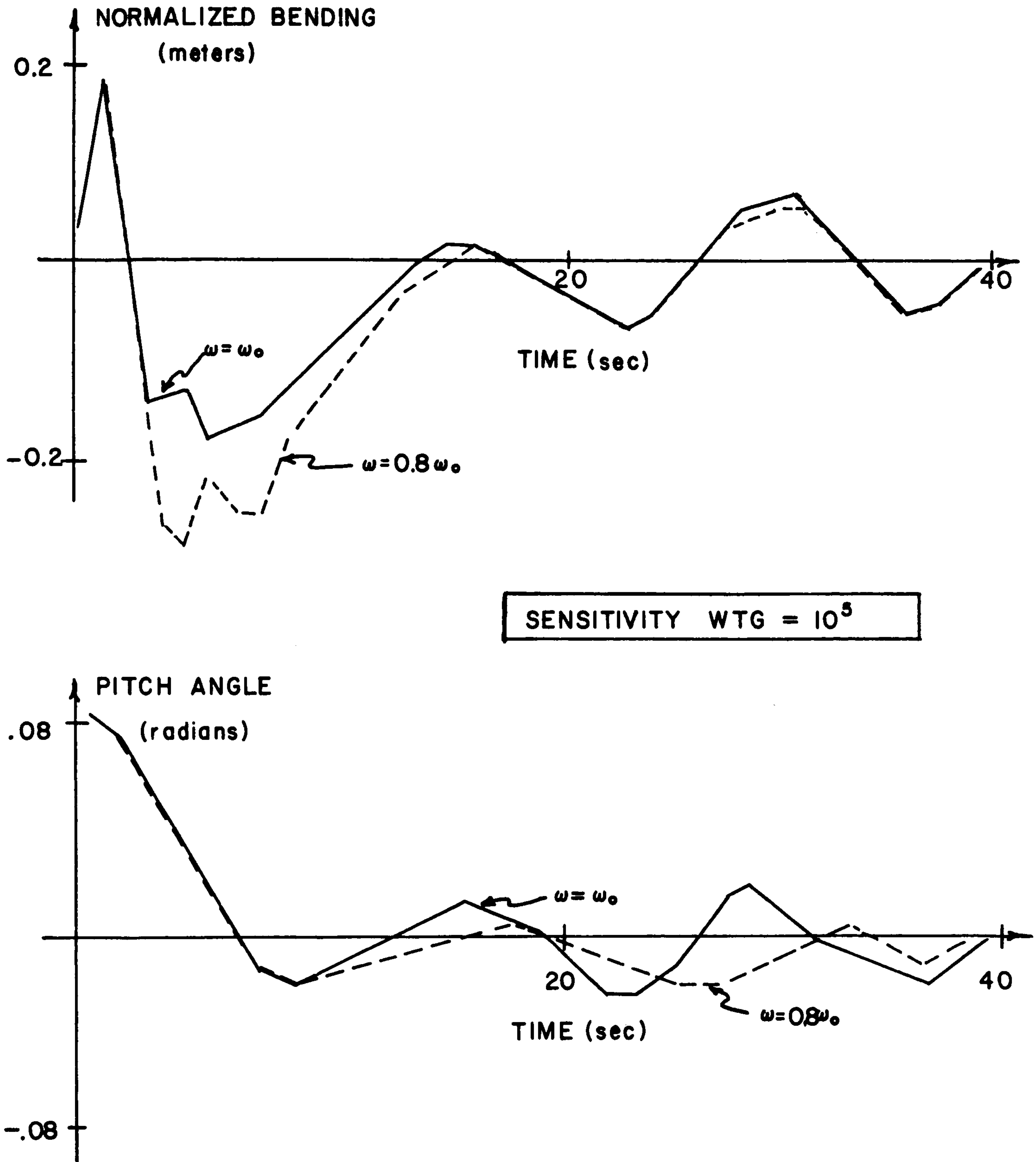

NORMALIZED BENDING
(meters)
0.2
-0.2
20
40
$\omega = \omega_o$
TIME (sec)
$\omega = 0.8\omega_o$
SENSITIVITY WTG = 0
PITCH ANGLE
(radians)
.08
-.08
$\omega = 0.8\omega_o$
$\omega = \omega_o$
20
40
TIME (sec)

FIGURE 3



NORMALIZED BENDING
(meters)
0.2
-0.2
20
40
TIME (sec)
ω=ω₀
ω=0.8ω₀
SENSITIVITY WTG = 10⁵
PITCH ANGLE
(radians)
.08
-.08
20
40
TIME (sec)
ω=ω₀
ω=0.8ω₀

FIGURE 4

DEFINITION OF SYMBOLS

F	Total thrust of Booster
K	Amplifier gain
lcg	Distance from engine gimbal to vehicle center of gravity
lcp	Distance from vehicle center of gravity to center of pressure
M	Total vehicle mass
M_1	Generalized mass of first bending mode
N'	Aerodynamic force
g	(unknown) parameter
R'	Control thrust
u	Autonomous input (output of prefilter)
v	Reference input
V	Vehicle velocity
x	State vector
X	Drag Force
$Y(X_\beta)$	Normalized displacement at engine gimbal
z	Sensitivity vector
α	Attack angle
α_w	Wind induced attack angle
β	Engine gimbal angle
ϕ	Attitude angle
ζ_1	First bending mode damping
η_1	First bending mode amplitude
ω_1	First bending mode frequency.

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