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ANALYSIS OF GEODETIC SATELLITE TRACKING DATA TO DETERMINE TESSERAL HARMONICS OF THE EARTH'S GRAVITATIONAL FIELD[1]

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ABSTRACT

Determinations of 88 tesseral harmonic coefficients of the gravitational field were made from camera tracking of seven satellites plus Doppler tracking of five satellites. It was found that addition of Doppler tracking of satellites which also have appreciable camera tracking had relatively little effect on the results. It is felt that not more than 50 of the coefficients are adequately determined. The improvement primarily required is more tracking of high inclination satellites; refinement of the dynamical theory used may also help.

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The analyses described in this paper are in continuation of those reported 1½ years ago [Kaula, 1966a]. These investigations are distinguished from other determinations of the earth's gravitational field principally in using an entirely analytic dynamical theory. The principal changes from the previous solution were 1) the incorporation of Doppler tracking data, and 2) an increase in the number of gravitational harmonic coefficients in the solution.

Incorporation of Doppler Data. Tracking by the U.S. Navy "Transit" Network was received in the form of Doppler frequencies, scaled to a reference frequency of about 107 MHz, at intervals of 16 seconds. To utilize these data in the same computer programs as the camera data, and to economize computer time, the following conversion and compression was applied to the Doppler data: 1) the form was converted to range rate in "canonical" units: earth radii/(806.8137 secs.); 2) the time was converted from WWV emitted to A1; 3) observations within 15° of the horizon were omitted, and tropospheric refraction corrections applied; 4) 3 or 4 observations at equal intervals over each pass were selected; 5) for one day at a time, an orbit was fitted to these observations by iterated least squares, taking into account variations of the gravitational field up to ℓ, m = 4,4; 6) from this orbit, the range-rate was calculated for each of the original 16-second interval observations; 7) for each pass, a combination of a polynomial in time and a station position shift was fitted to the residuals of the observed with respect to the computed range rates; 8) at three times within each pass, a range rate was calculated as the sum of the range rate from the orbit fitted for the day plus the polynomial & station shift fitted to the pass. The final information written on a binary tape for use in the subsequent analysis included as one record for each pass: a type number identifying the data as range rate; the tracking station number; the number of observations in the pass; the GST and A1 time (in Modified Julian Days) of the start of the pass; the three aggregated range rates formed by the process described above; and the time after pass start for each of these range rates.

Selection of Spherical Harmonic Coefficients. The zonal harmonics were held fixed at the values given in Table 2 of Kaula [1966a]. The tesseral harmonics selected for solution were all those for which a normalized coefficient of magnitude $8 \times 10^{-6}/\ell^2$ caused a perturbation of at least 10 meters amplitude in one satellite or at least 5 meters amplitude in two satellites, as listed in Table 3 of Kaula [1966a]: all coefficients thru 6,6; 7,1 thru 7,5; 8,1 thru 8,6; 9,1 and 9,2; 10,1 and 10,2; 11,1; and and 12,1; plus the small-divisor or near-resonant, harmonics: 9,9; 12,12; 13,12; 14,12; 15,12 thru 15,14; and 17,14.

Thus there were a total of 88 unknowns common to all orbits. With 7 unknowns represented by the Keplerian elements plus an acceleration parameter for each arc, the computer storage capacity for the normal equations as currently dimensioned was equalled. An increase of capacity to at least 145 unknowns could be accomplished with very little difficulty. In the solutions described herein, the positions of 16 Baker-Nunn camera and 33 Transit Doppler tracking stations were held fixed at the values obtained by Gaposhkin [1966] and Anderle & Smith [1967] respectively. It is intended to modify the programs to increase the capacity for unknowns and to solve for station position shifts when warranted by the accuracy of the solution for gravitational coefficients. So far, this stage has not been reached.

Summary of Satellites. The satellites used are summarized in Table 1. For the five satellites which also were used in the 1966 solution the data are essentially the same (except for 5 more months of Transit 4A), because 1963 was the year of minimum disturbances of atmospheric density by solar activity. There are minor modifications in the arcs actually used, however, because of changes in acceptance criteria for arcs: as well as number of iterations and number of observations (32 for Transit 4A, 40 for Vanguard 2, 60 for the others), a chi-square test was applied.

The significant additions to the data are the tracking of Courier 1B (28.2°), GEOS 1 (59.5°), and Beacon Explorer B (79.7°). It was found that adding a satellite of different orbital inclination made much more difference in the solution than did adding Doppler tracking. Considerable testing was done using different weights of the Doppler tracking relative to the camera tracking of GEOS 1, in particular, with very little variation in the results. While this situation adds to our confidence that the Doppler portions of the program are correct and accurate, it means that the major benefit of adding the capability to analyze Doppler data will not come until it enables analysis of orbits of appreciably different inclination than the set in Table 1: in particular, a polar orbiter.

In addition to Doppler tracking of a polar satellite, it is desirable that the amount of tracking of Beacon Explorer B be increased appreciably and that tracking of all satellites from more overseas stations be added so as to give a better distribution of observations than indicated by Table 2. The poor distribution apparently arises in part from the unavailability for administrative reasons of tracking from some overseas stations. This maldistribution is more severe than that tested by Anderle [1966].

Supplemental Data. Because the station positions were held fixed, of the three types of supplemental equations used in the earlier analyses only the 24-hour satellite orbit accelerations were applied (see Table 4 of Kaula [1966a]). Carrying these equations at unit weight, they have a mild influence on the solutions for the 2,2; 3,1; and 3,3 coefficients. It is planned to add some of the more accurate recent accelerations derived by Wagner [1967].

Manner of Analysis. The method of partitioned normals as described by Kaula [1966a, Eq. (1)-(2)] was utilized, so that there was no limit on the number of orbital arcs which could be analyzed In addition, one reference frequency correction per pass was included as an additional optional unknown to be separated out of the normals in the same manner as the orbital elements. Exercise of this option, however, appeared to make little difference in the results for the gravitational coefficients.

The normal equation blocks generated from the Doppler data were kept separate from the blocks generated from the camera data, in order to facilitate the testing of different relative weights of Doppler vs. camera tracking. However, as mentioned previously, variety of tracking type seems to make much less difference than variety of orbital specifications.

Results. The best solution (by the criterion of minimum discrepancy from terrestrial gravimetry [Kaula, 1966b]) is given in Table 3. This solution utilized a priori standard derivations of $\pm 10^{-5}/\ell^2$ for non-resonating coefficients of degree $\ell \geq 7$. This limitation is disappointing; the variety of inclinations is such that more than a three-fold ambiguity in periodicity of perturbations by tesseral harmonics should be resolvable. Of the two inadequacies which are most likely to cause this result, insufficient amount of data and error in dynamical theory, the former is easier to rectify, and hence is being tested first.

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TABLE 1: SATELLITE SPECIFICATIONS

Satellite	a Earth Radii	e	I Deg.	Days/ Arc	No. Arcs	Total Obs.	Starting Date	Ending Date	Type Tracking
Courier 1B	1.171	0.02	28.2	17	3	193	'65 Jun 11	'65 Oct 9	O
Vanguard 2	1.302	0.16	32.9	18	12	696	'62 Dec 31	'63 Dec 25	O
Transit 4B	1.163	0.01	32.4	9	2	1350	'62 Apr 21	'62 Jun 23	D
Echo 1 Rocket	1.250	0.01	47.2	18	14	1380	'63 Jan 1	'63 Dec 26	O
Anna 1B	1.177	0.01	50.1	18	15	1322	'62 Dec 31	'63 Oct 22	O
				9	2	3930	'63 May 16	'63 Jun 4	D
Geos 1	1.266	0.07	59.5	18	7	1126	'65 Nov 4	'66 Jun 10	O
				10	6	4768	'66 Jul 1	'67 Feb 9	D
Transit 4A	1.147	0.01	66.8	18	14	536	'62 Apr 6	'63 Dec 26	O
				9	2	2556	'62 Jul 19	'62 Aug 7	D
Beacon Expl. B	1.154	0.01	79.7	9	2	2496	'65 Jan 30	'65 May 9	D
Midas 4	1.568	0.01	95.9	30	12	3021	'62 Aug 3	'63 Dec 25	O

TABLE 2: GEOGRAPHIC DISTRIBUTION OF DOPPLER TRACKING

Number of passes observed from stations within each octant

Latitude N \ Longitude E:	25–115	115–205	205–295	295–25
90 to 0	0	1109	3724	651
0 to -90	333	352	0	315

TABLE 3: GEOPOTENTIAL FULLY NORMALIZED SPHERICAL HARMONIC COEFFICIENTS x 10^6

Degree ℓ	Order m	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	Degree ℓ	Order m	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$	Degree ℓ	Order m	$\bar{C}_{\ell m}$	$\bar{S}_{\ell m}$
2	2	2.45	-1.37	6	3	0.14	0.23	9	1	0.07	-0.06
3	1	1.99	0.13	6	4	-0.16	-0.84	9	2	0.01	0.02
3	2	0.80	-0.71	6	5	-0.24	-0.54	9	9	-0.18	-0.14
3	3	0.47	1.27	6	6	-0.30	-0.80	10	1	0.00	0.00
4	1	-0.58	-0.39	7	1	0.17	0.05	10	2	-0.03	0.05
4	2	0.40	0.68	7	2	0.34	0.04	11	1	-0.03	-0.04
4	3	1.02	0.08	7	3	-0.01	-0.09	12	1	-0.05	-0.03
4	4	-0.36	-0.32	7	4	-0.11	0.06	12	12	-0.11	-0.01
5	1	-0.09	0.02	7	5	0.05	-0.03	13	12	-0.08	0.08
5	2	0.84	-0.14	8	1	-0.02	0.12	14	12	-0.05	-0.04
5	3	-0.50	-0.06	8	2	0.10	-0.10	15	12	-0.08	0.01
5	4	0.36	0.28	8	3	0.08	0.11	15	13	-0.03	-0.07
5	5	-0.22	-0.14	8	4	-0.05	0.02	15	14	-0.00	0.02
6	1	-0.13	0.05	8	5	-0.02	-0.01	17	14	-0.05	0.12
6	2	0.10	-0.40	8	6	-0.03	0.02				

TESSERAL HARMONICS OF THE EARTH'S GRAVITATIONAL FIELD FROM CAMERA TRACKING OF SATELLITES

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Tesseral Harmonics of the Earth's Gravitational Field from Camera Tracking of Satellites[1]

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A total of 7234 Baker-Nunn camera observations of 5 satellites were analyzed to determine simultaneously 44 tesseral harmonic coefficients of the gravitational field, 36 station coordinates, and 511 orbital elements. Supplementary observational data incorporated in the solution included accelerations of 24-hour satellites and directions between tracking stations from simultaneous observations; observation equations were also written for the differences between geometrical and gravitational geoid heights at tracking stations. Several variations in relative weighting of different observational data and a priori variances of parameters were tested. The previous independent solution most closely approached was that by Anderle based on Doppler data, from which the rms discrepancy was $\pm 0.18 \times 10^{-6}$ for 38 normalized harmonic coefficients, or ± 7 m in total geoid height. An equatorial radius of 6,378,160 $\pm$ 5 m was obtained.

INTRODUCTION

The analyses described in this paper are a continuation of those reported three years ago [*Kaula*, 1963*a*, *b*]. They are an appreciable improvement over the previous analyses because of better observations of more recent orbits, better methods of analysis, and better use of supplemental data. This investigation is one of four principal efforts in the determination of tesseral harmonics of the gravitational field. The complexity of such investigations makes it desirable that there be independent efforts which differ not only in the tracking data but also in the techniques of analysis.

CHANGES FROM PREVIOUS SOLUTIONS

The dynamical theory applied, formation of partial derivatives, use of observational and timing variances, formation of observational equations, and accumulation of normal equations are essentially the same as described by *Kaula* [1963*a*, *b*; see also *Kaula*, 1966*a*]. The most significant improvement is in the data, Baker-Nunn camera observations of the Smithsonian Astrophysical Observatory. The satellites used are somewhat better distributed in inclination, and, all being later than 1962 March 7, are appreciably less affected by drag than those used in the earlier analyses. The satellite data are summarized in Table 1. In determining the preliminary orbits, I rejected arcs for the final analysis not only if the number of observations was insufficient but also if excessive iterations were required to obtain a satisfactory fit. The greatest deficiency of camera tracking, using solar illumination, appears to be an inability to obtain a good distribution of observations of satellites that are low enough to be sensitive to the variations of the gravitational field (perigee below 1200 km) and are of inclination appreciably higher than the latitudes of the tracking stations (less than 37°). Thus the most sensitive satellite used in this study, $1961o_1$, is the poorest observed, whereas the best observed, $1961\alpha\delta_1$, is so high as to be useless for determining gravitational harmonics above the fourth degree.

To solve for a maximum number of tesseral harmonics, the geopotential central term GM was held fixed at 3.986009×10^{14} m^3/sec^2, the mean of values determined from Ranger lunar probes [*Sjogren and Trask*, 1965], and the zonal harmonics J_2 through J_7 were held fixed at the values [*Kozai*, 1964; *King-Hele and Cook*, 1965; *King-Hele et al.*, 1965] given in Table 2. Perturbations due to these zonal harmonics, as well as lunisolar perturbations of more than 10^{-8} amplitude, were calculated in analyses of both preliminary and final orbits. Arbitrary poly-

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TABLE 1. Specifications for Close Satellite Orbits

	Satellite				
	1959α1	1960ι2	1961ο1	1961αδ1	1962βμ1
Name	Vanguard 2	Echo 1 Rocket	Transit 4A	Midas 4	Anna 1B
Epoch	1963 Jan. 27.0	1963 Jan. 10.0	1962 May 21.0	1962 Aug. 18.0	1963 Jan. 9.0
Semimajor axis	1.301994	1.250052	1.146988	1.568120	1.177254
Eccentricity	0.16417	0.01139	0.00799	0.01209	0.00707
Inclination	0.57383	0.82437	1.16620	1.67302	0.87514
Argument of perigee	3.13491	1.93573	1.18658	1.67305	0.94214
Longitude of node	2.87158	0.79776	0.46898	6.27650	2.84671
Mean anomaly	1.76589	5.92654	3.92748	0.67818	0.80524
Min. acceleration*	-0.51×10^{-9}	-1.00×10^{-9}	0.02×10^{-9}	-0.25×10^{-9}	-0.44×10^{-9}
Max. acceleration*	7.01×10^{-9}	1.35×10^{-9}	0.90×10^{-9}	0.41×10^{-9}	0.27×10^{-9}
Perigee motion/day	0.09238	0.05200	−0.01210	−0.01708	0.04364
Node motion/day	−0.06141	−0.05415	−0.01438	0.00367	−0.04119
Periods/day	11.48	12.20	13.86	8.68	13.35
Max. A/m, cm²/g	0.21	0.27	0.12	0.08	0.07
Min. A/m, cm²/g	0.21	0.08	0.11	0.02	0.07
Perigee height, km	560	1500	880	3500	1080
Starting date	1963 Jan. 18	1963 Jan. 1	1962 May 12	1962 Aug. 3	1962 Dec. 31
Ending date	1963 Nov. 20	1963 Sep. 28	1963 Jul. 24	1963 Oct. 27	1963 Nov. 2
Number of arcs	13	15	15	15	15
Days/arc	18	18	18	30	18
Min. obs./arc	42	67	32	61	61
Total observations	790	1628	612	2882	1322
SAO Spec. Rept. Nos.	185	185	148,185	147,185	168

* Units for acceleration: dn/dt in radians/(806.8 sec)², where n is mean motion.

nomials were limited to a t^2 term in the mean anomaly, making seven orbital constants for each arc.

To solve, in effect, for an indefinite number of orbital constants simultaneously with tesseral harmonic coefficients and corrections to station coordinates, I used the technique of partitioned normals; i.e., writing the normal equations as [*Kaula*, 1966*a*, pp. 104–106]

$$\begin{pmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{s}_1 \\ \mathbf{s}_2 \end{pmatrix} \qquad (1)$$

where **N** is the matrix of normal equation coefficients, **z** is the vector of corrections of parameters, and **s** is the vector of normal equation constants, makes it possible to write a solution for $\mathbf{z}_1$ alone:

$$\mathbf{z}_1 = [\mathbf{N}_{11} - \mathbf{N}_{12}\mathbf{N}_{22}^{-1}\mathbf{N}_{21}]^{-1} \cdot [\mathbf{s}_1 - \mathbf{N}_{12}\mathbf{N}_{22}^{-1}\mathbf{s}_2] \qquad (2)$$

If $\mathbf{z}_2$ is the correction to orbital constants, which are peculiar to each arc, the nonzero elements in the matrix $\mathbf{N}_{22}$ will be in a series of square blocks down the main diagonal, one block per arc. Hence the inversion $\mathbf{N}_{22}^{-1}$ and the subtractions of $\mathbf{N}_{12}\mathbf{N}_{22}^{-1}\mathbf{N}_{21}$ and $\mathbf{N}_{12}\mathbf{N}_{22}^{-1}\mathbf{s}_2$ in (2) can be made separately for each arc. Therefore, at any time only those parts of the normal equations pertaining to the parameters common to all arcs—the corrections $\mathbf{z}_1$ to tesseral harmonic coefficients and stations coordinates—plus the parts peculiar to the one arc being analyzed need to be stored in the computer. This technique is also used by *Anderle* [1966] and *Guier and Newton* [1965] in analyzing Transit Doppler tracking data; it is probably the principal difference in method from the iterative technique used by *Izsak* [1966] and *Gaposhkin* [1966] in analyzing the Baker-Nunn camera tracking data.

The principal inaccuracies in the calculations, aside from neglect of drag, are believed to be the absence of short-period J_2^2 terms in the orbital theory of *Brouwer* [1959] and the failure

TABLE 2. Fixed Zonal Harmonics

l	J_l 10^{-6}	$\bar{C}_{l0}$ 10^{-6}
2	1082.70	−484.198
3	−2.55	0.965
4	−1.50	0.500
5	−0.15	0.045
6	0.50	−0.140
7	−0.37	0.090

TABLE 3. Subscripts lm of Geopotential Coefficients $\bar{C}_{lm}$, $\bar{S}_{lm}$ of Magnitude $\pm 8 \times 10^{-6}/l^2$ Causing Perturbations of Amplitude of More than 5 Meters

Zonal harmonics and tesseral harmonics which are of degree 4 or lower or which are near-resonant are omitted.

				Terms lm Causing Perturbations of Amplitude		
Satellite	a	e	I	More than 20 Meters	10 to 20 Meters	5 to 10 Meters
$1959\alpha_1$	1.302	0.16	32.9°	51,52,61,62, 63,71,81	53,72,83, 101,111	54,64,73,74,82,84,92, 93,102,104,122,141
$1960\iota_2$	1.250	0.01	47.2°	51,61	52,63,64, 82,101	53,54,62,65,71,72,81, 85
$1961o_1$	1.147	0.01	66.8°	51,61,62,63, 65	52,53,54, 55,64,66, 72,81,84, 101,121	71,73,74,75,76,86,87, 91,92,102,103,111
$1961\alpha\delta_1$	1.568	0.01	95.9°			61,62
$1962\beta\mu_1$	1.77	0.01	50.1°	51,52,61,63, 64	53,62,65, 71,81,82, 101	54,55,72,73,74,75,83, 85,86,92,102,111,121

to correct station positions to a common epoch for latitude variation [*Veis*, 1960, pp. 97–98]. Both these defects are of the order of ± 10 m or less in effect. The parameters to be determined were therefore selected as being of greater expected effect. Experience indicates that tracking stations as far apart as the Baker-Nunn cameras should to this level of accuracy be considered as moving separately. Hence 36 of the unknowns in $\mathbf{z}_1$ are corrections to station coordinates. To select the tesseral harmonic coefficients to be determined in addition to the low-degree terms up to degree and order l, m of 4, 4 and the small divisor terms for which m is approximately equal to the number of revolutions per day and l is odd, I calculated orbital perturbations under the assumption that the normalized coefficients $\bar{C}_{lm}$, $\bar{S}_{lm}$ are $\pm 8 \times 10^{-6}/l^2$ in magnitude, a rule-of-thumb which appears quite good up to about degree 15 [*Kaula*, 1966*b*]. The results of this calculation appear in Table 3. Twenty-two coefficients of degrees 5 through 8 were selected on the basis of perturbing at least two satellites more than ± 10 m; l, m = 10, 1 was omitted as being difficult to distinguish from 41, 61, and 81 using the number of satellites available.

The small-divisor, or near-resonant, harmonics [*Anderle*, 1965; *Yionoulis*, 1965] under the $\pm 8 \times 10^{-6}/l^2$ assumption were significant for satellites $1960\iota_2$ (twelfth order), $1961o_1$ (fourteenth order), and $1962\beta\mu_1$ (thirteenth order) but not for $1959\alpha_1$ or $1961\alpha\delta_1$. The particular degrees selected for solution were those which happened to have the largest partial derivatives. The procedure for evaluating these partial derivatives is exactly the same as for the lower-degree harmonics, with the important precaution that the rate for a perturbation of the mean anomaly through the perturbation of the semimajor axis is *not* assumed to be an integer multiple of the mean motion.

More specifically, for a disturbing function term of the form

$$R_{lmpq} = K_{lmpq}(a, e, I)\begin{Bmatrix}\sin\\\cos\end{Bmatrix}\{(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)\} \qquad (3)$$

where θ is Greenwich sidereal time, and (a, e, I, ω, M, Ω) are the Keplerian elements—semimajor axis, eccentricity, inclination, perigee argument, mean anomaly, and nodal longitude, respectively. The indirect perturbation of the mean anomaly is

$$\Delta_2 M_{lmpq} = \int \frac{\partial n}{\partial a} \int \frac{2}{na}\frac{\partial R_{lmpq}}{\partial M}\,dt\,dt$$

$$= \frac{-\dfrac{3}{a^2} K_{lmpq}(a, e, I)(l-2p+q)\begin{Bmatrix}-\cos\\\sin\end{Bmatrix}\{(l-2p)\omega + (l-2p+q)M + m(\Omega-\theta)\}}{[(l-2p)\dot{\omega} + (l-2p+q)\dot{M} + m(\dot{\Omega}-\dot{\theta})]} \qquad (4)$$

TABLE 4. Twenty-Four-Hour Satellite Orbits

Satellite		1963 31*A*					1964 47*A*	1965 28*A*
Name				Syncom 2			Syncom 3	Early Bird
Inclination				33°			0.1°	0.2°
Start longitude		305.1°	244.7°	174.0°	118.0°	81.0°	179.2°	330.7°
End longitude		302.4°	197.5°	161.5°	102.2°	52.0°	178.2°	330.7°
Observed acceleration $\times 10^9$		−1.962	1.888	0.435	−2.203	0.849	1.476	−1.291
Deviation		±28	±74	±44	±44	±54	±62	±9
Amplitude factors of partial derivatives	Q_{22}			0.7775×10^{-3}			0.9144×10^{-3}	
	Q_{31}			-0.0155×10^{-3}			-0.0582×10^{-3}	
	Q_{33}			0.1752×10^{-3}			0.2253×10^{-3}	
	Q_{42}			0.0008×10^{-3}			-0.0182×10^{-3}	
	Q_{44}			0.0344×10^{-3}			0.0482×10^{-3}	

Accelerations and partial derivatives in radians/(planetary time unit)^2, where planetary time unit = 806.8137 sec.

where n *is* the mean motion, $\mu^{1/2}a^{-3/2}$ [*Kaula*, 1966*a*, p. 49].

To strengthen the solution, two types of supplemental data were included: the accelerations of 24-hour synchronous satellites and the mutual directions of tracking stations obtained from simultaneous satellite observations, which are different from those used in the dynamical calculations.

The acceleration in longitude of a 24-hour satellite appears in an observation equation of the form

$$\sum_{(l-m)\,\text{even}} Q_{lm}[\bar{C}_{lm} \sin m\lambda - \bar{S}_{lm} \cos m\lambda] = \ddot{\lambda}_0 + \delta\ddot{\lambda}_0 \qquad (5)$$

where

$$Q_{lm} = \left[\frac{(l-m)!\,(2l+1)}{(l+m)!}\right]^{1/2} 3n^2 m\left(\frac{a_e}{a}\right)^l \cdot F_{lmp}(I)G_{lp0}(e) \qquad (6)$$

in which p is $(l - m)/2$, a_e is the equatorial semimajor axis, and $F_{lmp}(I)$ and $G_{lp0}(e)$ are polynomial functions of the inclination and eccentricity, respectively [*Kaula*, 1966*a*, p. 51]. The observed accelerations $\ddot{\lambda}_0$ (corrected for lunisolar perturbations) and their standard deviations $\sigma(\ddot{\lambda}_0)$ were taken from the work of *Wagner* [1966]. Five accelerations of satellite 1963 31*A* at a variety of longitudes and one acceleration each of 1964 47*A* and 1965 28*A* were used, as summarized in Table 4.

The direction of one tracking station from another as obtained by simultaneous observations of satellites appears in an observation equation of the form

$$\begin{Bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{Bmatrix} \mathbf{R}_{lu} \cdot [\mathbf{u}_j + \Delta\mathbf{u}_j - (\mathbf{u}_i + \Delta\mathbf{u}_i)]/|\mathbf{u}_j - \mathbf{u}_i| = \delta\mathbf{l} \qquad (7)$$

where $\mathbf{R}_{lu}$ is the rotation matrix from coordinates referred to the earth's pole and Greenwich meridian to coordinates with the 1 axis along the line from station i to station j and the 2 axis along the major axis of the error ellipse of the observed direction:

$$\mathbf{R}_{lu} = \mathbf{R}_1(\rho)\mathbf{R}_2(-\varphi)\mathbf{R}_3(\lambda) \qquad (8)$$

In equation 7, φ and λ constitute the observed direction of station j from station i in the form of latitude and longitude, and ρ is the angle between the normal to the meridian plane defined by λ and the major axis of the error ellipse.

The directions between 14 pairs of Baker-Nunn camera stations derived by *Aardom et al.* [1965] from 615 pairs of quasi-simultaneous observations of satellites of about 3700 km altitude are given in the form of direction cosines **c** with respect to polar-Greenwich axes of station j from station i. The standard deviations are given in the form of the semimajor and semiminor axes a and b of the error ellipse and the angle θ between the major axis and the normal to the plane defined by the stations and the earth's center. To apply these observations in (6) and (7), we have

$$\varphi = \sin^{-1} c_3$$
$$\lambda = \tan^{-1} c_2/c_1$$
$$\mathbf{n} = \mathbf{u}_2 \times \mathbf{u}_1$$
$$\mathbf{m} = \begin{cases} -\sin\varphi\cos\lambda \\ -\sin\varphi\sin\lambda \\ \cos\varphi \end{cases} \qquad (9)$$
$$\mathbf{k} = \begin{cases} -\sin\lambda \\ \cos\lambda \end{cases}$$
$$\rho = \tan^{-1}(\mathbf{n}\cdot\mathbf{m}/\mathbf{n}\cdot\mathbf{k}) + \theta - \pi/2$$

The semimajor axis of the error ellipse was always within 18° of the station-center plane. The number of observation pairs used for each position line varied from 5 to 90; the standard ellipse semimajor axis varied from ± 2.3 to $\pm 10.5 \times 10^{-6}$; and the semiminor axis varied from ± 0.9 to $\pm 3.9 \times 10^{-6}$. The stations appearing in these 14 pairs of equations are noted in the last column of Table 7.

We can also write as an observation equation the fact that the geometrical geoid height derived from the position of a tracking station should differ from the gravitational geoid height, calculated for the same point from the harmonic coefficients, only by the contribution δN_{GR} of variations in the gravitational field of higher degree than those represented by the coefficients

$$(1 \quad 0 \quad 0)\mathbf{R}_{lm}\,\Delta\mathbf{u} - a_e \sum_{l,m} \bar{P}_{lm}(\sin\varphi) \cdot [\bar{C}_{lm}\cos m\lambda + \bar{S}_{lm}\sin m\lambda] = \delta N_{GR} - N_{GE} \qquad (10)$$

where $\mathbf{R}_{l\mu}$ is defined by (8) (the rotation about the 1 axis being of no effect), using the position φ, λ of the station, $\bar{P}_{lm}$ is the normalized associated Legendre function, and N_{GE} is the geometrically calculated geoid height, obtained from station position **u**, the station height above sea level h, and the reference ellipsoid of semimajor axis 6,378,165 m and flattening of 1/298.25, corresponding to the potential coefficient J_2 in Table 1. Also applied as a fixed correction are the contributions of the fixed zonal harmonics to the gravitational geoid

TABLE 5. Datum Weights and a priori Standard Deviations of Parameters

Solution	Datum Weights			Parameter, a priori Standard Deviations	
	Close Satellites*	24-Hour Satellites	Mutual Directions and Geoid Heights†	Station Positions, m	Gravity Coefficients $\bar{C}_{lm}$, $\bar{S}_{lm}$ 10^{-6}
A	1	1	1	∞	∞
B	1	1	0	∞	∞
C	Varied	1	0	∞	∞
D	Varied	21.2	Moderate	∞	∞
E	Varied	21.2	High	∞	∞
F	1	1	1	10	∞
G	1	1	0	10	∞
H	Varied	1	0	10	∞
I	Varied	21.2	Moderate	10	∞
J	Varied	1	1	10	∞
K	Varied	1	1	10	Deg 2–4: ∞; 5–8: $8/l^2$
L	Varied	1	0	∞	Deg 2–4: ∞; 5–8: $8/l^2$
M	1	1	0	10	All $8/l^2$
N	Varied	1	1	∞	∞
O	Varied	1	1	a‡	∞
P	Varied	1	1	b	∞
Q	Varied	1	1	c	∞

* Varied satellite weighting: 1959α_1, 2.05; 1960ι_2, 1.00; 1961o_1, 2.70; 1961$\alpha\delta_1$, 0.55; 1962$\beta\mu_1$, 1.20.
† Moderate weighting: directions 10.5, heights 16.4. High weighting: directions 110, heights 270.
‡ Station weighting a–c: all stations ∞, except a. Station 1 fixed in all coordinates. b. Station 1 fixed in longitude and radius. c. Station 1 fixed in longitude only.

height. Since the semimajor axis a_e is used in calculating N_{GE} in (10), the mean radial shift of the tracking stations can be considered as a correction to the semimajor axis. The standard deviation of the 'observation' δN_{GR} in (10) was estimated to be ± 20 m as follows. The 49 coefficients fixed or being determined on the $\pm 8 \times 10^{-6}/l^2$ rule contribute a mean square of $(26 \text{ m})^2$ to the geoid height, which was subtracted from the $(33 \text{ m})^2$ mean square estimated from autocovariance analysis of gravimetry [*Kaula*, 1959, p. 2418].

In combining widely differing types of data, the relative weighting is necessarily somewhat arbitrary, particularly when the observational variances are derived in different ways. For the satellite observations, variances based on observational residuals of previous analyses were used: $(12.0'')^2$ direction and $(0.050 \text{ sec})^2$ time [*Kaula*, 1963*b*, p. 5184]. For the 24-hour satellite accelerations and the directions between stations, the variances produced by the least-squares analyses of *Wagner* [1966] and *Aardom et al.* [1965], respectively, were used.

Furthermore, when one type of data is represented by many more observations than another, as was the case for the close satellite data (14,468 equations) compared with the supplemental data (47 equations), the neglect of covariances in the former will be much more significant, and the use of the correct variances in simple least squares will result in an overweighting of the more numerous relative to the less numerous.

For the foregoing considerations the computer program was so modified that when the normal equations for a particular satellite had been generated, they were saved on tape to be read off and multiplied by the weighting factor before being added to the combined normal equations. In this manner, additional solutions with different combinations of weights could be made. A further capability which was included for these short-time additional solutions was change in preassigned variances and starting values for the parameters.

Some of the data weighting and preassigned standard deviations of parameters tried are given in Table 5. The varied satellite weights and the supplemental equation weights in excess of 100 were calculated on the basis of making each satellite and each block of supplemental data of equal weight; the square roots of these 'high' weights are the 'moderate' weights between 10 and 100 in Table 5. However, since the satellite variances are probably too large and the supplemental variances probably too small, the smaller weights for the supplemental data are probably more realistic. In any case, over quite a wide range of weights the influence in the solution will appear for any datum which differs significantly from the bulk of the data in its sensitivity to certain parameters.

As discussed by *Kaula* [1966*b*], solutions for a set of station coordinates from close satellite tracking are subject to systematic error in orientation. In the iterative solutions from camera data by *Izsak* [1966], *Veis* [1965], and *Gaposhkin* [1966], the over-all orientation is essentially fixed by correcting orbital longitudes and station longitudes at alternate stages. In the solutions from Doppler data by *Anderle* [1966] and *Guier and Newton* [1965], one station is held fixed to establish a longitude reference. In the analyses described in this paper, several solutions (A through E, L and N in Table 5) were made in which all stations were left free to move, in the hope that adequate orientation would be obtained from the inertially referred directions constituted by the camera observations. The opposite alternative of fixing one station in one or more coordinates was also tried (solutions O, P, and Q). However, there is no reason to give preference to one station over another, and it seems better to treat all stations equally and to allow some influence on the camera directions by preassigning variances to all station positions (solutions F through K and M). The use of such preassigned variances gives weight, in effect, to the solution on which the station coordinates were based.

Missing from Table 5 are some obvious alternatives: omitting or giving higher weight to the 24-hour satellite data, restraining the fifth- to eighth-degree gravitational coefficients completely, including or omitting mutual direction and geoid height equations separately, etc. Most of these alternatives were tested at an earlier stage, with a set of close satellite data differing in some respects from those used in the final analysis. In these tests the variations in the weighting of the 24-hour satellites had a considerable effect: their omission resulted in a wider scatter of results for the coefficients $\bar{C}_{22}$,

TABLE 6. Coefficients of the Gravitational Field

Degree l	Order m	Alternative Solutions: F $\bar{C}_{lm}$ 10^{-6}	F $\bar{S}_{lm}$	H $\bar{C}_{lm}$ 10^{-6}	H $\bar{S}_{lm}$	I $\bar{C}_{lm}$ 10^{-6}	I $\bar{S}_{lm}$	N $\bar{C}_{lm}$ 10^{-6}	N $\bar{S}_{lm}$	O $\bar{C}_{lm}$ 10^{-6}	O $\bar{S}_{lm}$	Q $\bar{C}_{lm}$ 10^{-6}	Q $\bar{S}_{lm}$	Preferred Solution: J $\bar{C}_{lm}$ 10^{-6}	J $\bar{S}_{lm}$	σ_{lm}
2	2	2.44	−1.38	2.43	−1.39	2.48	1.37	2.47	−1.39	2.43	−1.42	2.46	−1.44	2.43	−1.39	±0.02
3	1	1.97	0.11	1.94	0.15	1.91	0.03	1.97	0.19	1.90	0.21	1.89	0.21	1.94	0.15	0.05
3	2	0.71	−0.83	0.72	−0.76	0.84	−0.77	0.66	−0.65	0.81	−0.81	0.64	−0.68	0.72	−0.78	0.10
3	3	0.50	1.25	0.56	1.24	0.23	1.26	0.42	1.22	0.56	1.23	0.51	1.20	0.55	1.24	0.09
4	1	−0.65	−0.51	−0.61	−0.49	−0.60	−0.53	−0.62	−0.50	−0.57	−0.44	−0.60	−0.53	−0.61	−0.49	0.03
4	2	0.37	0.73	0.32	0.71	0.34	0.71	0.35	0.77	0.31	0.69	0.32	0.70	0.33	0.71	0.05
4	3	0.86	0.14	0.89	0.07	0.93	0.00	0.93	0.05	0.89	0.05	0.91	0.07	0.89	0.07	0.06
4	4	−0.45	−0.04	−0.31	−0.15	−0.07	−0.05	−0.34	−0.07	−0.06	0.05	−0.16	0.04	−0.31	0.11	0.11
5	1	−0.08	0.02	−0.05	0.02	−0.04	0.05	−0.08	0.01	−0.03	0.01	−0.03	0.01	−0.05	0.03	0.04
5	2	0.73	−0.18	0.76	−0.13	0.65	−0.28	0.76	−0.06	0.80	−0.14	0.80	−0.05	0.75	−0.17	0.09
5	3	−0.63	0.09	−0.54	0.12	−0.75	0.12	−0.73	0.27	−0.62	0.02	−0.59	0.17	−0.61	0.15	0.11
6	1	−0.18	0.16	−0.18	0.11	−0.20	0.16	−0.16	0.16	−0.17	0.13	−0.16	0.17	−0.18	0.12	0.04
6	2	0.11	−0.38	0.03	−0.38	0.04	−0.35	0.07	−0.33	0.05	−0.37	0.09	−0.37	0.04	−0.38	0.06
6	3	0.05	0.47	0.14	0.35	0.08	0.39	0.17	0.29	0.12	0.34	0.14	0.34	0.12	0.35	0.08
6	4	0.03	−0.53	0.12	−0.49	0.24	−0.43	0.24	−0.80	0.24	−0.55	0.28	−0.71	0.13	−0.50	0.10
6	5	−0.01	−0.35	−0.10	−0.37	−0.19	−0.43	−0.15	−0.54	−0.17	−0.47	−0.15	−0.58	−0.11	−0.37	0.11
7	1	0.21	0.09	0.21	0.11	0.19	0.09	0.23	0.14	0.19	0.13	0.20	0.14	0.21	0.11	0.05
8	1	−0.04	0.00	−0.04	0.05	−0.05	0.02	−0.08	0.03	0.00	0.09	−0.05	0.00	−0.05	0.05	0.05
8	2	0.17	−0.05	0.09	−0.07	0.09	−0.06	−0.07	−0.05	0.05	−0.05	−0.06	−0.06	0.09	−0.07	±0.08

TABLE 7. Station Positions
Rectangular coordinates **u** referred to the equator and Greenwich meridian.

				Coordinate Shifts								
				Alternative Solutions						Preferred		
No.	Station and Number of Observations		Starting Coordinates, m	F, m	H, m	I, m	N, m	O, m	Q, m	J, m	σ, m	In Direction Eq. with Sta. No.
1.	Organ Pass (926)	u_1	−1 535 753	−32	−31	−26	−121	0	19	−38	6	7,9,10,12
		u_2	−5 167 000	18	27	13	98	0	61	18	5	
		u_3	3 401 047	19	47	4	178	0	158	27	5	
2.	Olifantsfontein		5 056 133	7	17	18	7	26	−11	18	6	8,9,10
	(664)		2 716 489	−19	−32	−28	−114	−72	−54	−29	7	
			−2 775 832	−9	−9	−1	−50	−19	−39	−8	7	
3.	Woomera (719)		−3 983 738	10	10	−6	66	22	19	6	7	
			3 743 127	−40	−46	−42	−78	−71	−117	−45	6	
			−3 275 615	1	4	5	−32	4	−19	6	6	
4.	San Fernando		5 105 610	−22	−36	−11	−98	−39	−6	−23	5	8,9,10
	(790)		−0 555 226	−18	−25	+3	−73	−38	−17	−19	7	
			3 769 693	30	57	25	192	73	172	39	5	
5.	Tokyo (339)		−3 946 697	21	26	31	150	71	94	25	7	6
			3 366 293	−5	−6	−22	−26	−18	68	−8	7	
			3 698 858	16	13	24	176	56	150	14	7	
6.	Naini Tal (678)		1 018 206	13	24	6	61	29	2	15	7	5,8
			5 471 103	−9	−10	−6	−104	−32	−73	−11	5	
			3 109 620	34	26	47	218	91	194	40	5	
7.	Arequipa (518)		1 942 768	8	11	4	−40	−19	30	2	5	1,9,10,11
			−5 804 089	2	4	−4	38	−5	38	−1	5	
			−1 796 968	8	4	32	51	43	93	11	7	
8.	Shiraz (564)		3 376 887	1	8	−12	−11	−2	−47	−2	6	2,4,6
			4 403 994	−25	−31	−15	−113	−52	−79	−27	6	
			3 136 264	29	38	41	210	83	186	34	5	
9.	Curaçao (484)		2 251 822	11	−5	8	−38	−20	29	6	5	1,2,4,7,10,11
			−5 816 923	12	−15	13	62	7	58	10	5	
			1 327 171	3	−1	10	147	19	156	11	5	
10.	Jupiter (567)		0 976 281	−10	−17	−4	−76	−34	−2	−12	5	1,2,4,7,9
			−5 601 390	7	13	2	70	0	59	7	5	
			2 880 247	34	34	34	194	35	186	40	5	
11.	Villa Dolores		2 280 572	3	−20	8	−44	−26	24	−4	5	7,9
	(552)		−4 914 580	24	21	8	85	33	95	18	6	
			−3 355 464	−1	1	29	−31	1	−13	4	6	
12.	Maui (623)	u_1	−5 466 063	8	12	9	38	19	66	11	6	1
		u_2	−2 404 286	11	22	8	94	32	41	18	6	
		u_3	2 242 180	36	36	45	189	51	179	39	6	

$\bar{S}_{33}$ as well as some others, whereas weighting them heavily distorted $\bar{C}_{31}$, $\bar{S}_{31}$ from the values strongly indicated by the close satellite data. Varying the weights of the geometrical data and restraining the higher gravitational coefficients appeared to have little effect on the solution for the low-degree coefficients. Also tested was omission of each close satellite, one at a time, in a solution for the low-degree gravitational coefficients. As anticipated, omission of 1961$\alpha\delta_1$, the least sensitive satellite, had least effect and omission of 1961o_1 had greatest effect.

Results

The principal test of the value of different solutions was intended to be the χ^2 test: if the original estimates of weights, variances, and covariances are good (and if the formulation of the problem is correct), the corrected quadratic sum should be close to the degrees of freedom. In other words, the quantity

$$q = [\mathbf{f}^T\mathbf{W}^{-1}\mathbf{f} - \mathbf{z}^T\mathbf{s}]/(n - p) \qquad (11)$$

should be close to unity, where **f** is the vector of observation equation constants; **W** is the weighted covariance matrix; n is the number of observations; p is the number of parameters; and **z** and **s** are the solution and normal equation constant vectors, as in (1). The q's obtained varied from 1.18 (solution B) to 1.54 (solution E). However, much of this variation is due to the weights incorporated in the sums in the numerator, but not in the denominator, of (11). If the number of observations n is changed from $\sum_i n_i$ to $\sum_i \omega_i n_i$, where ω_i is the weight of data of type i, the q's vary from 1.01 (solution E) to 1.33 (solution F); A, D, F through K and M through Q are all between 1.25 and 1.33. Of those which are distinctly lower, B, C, and E all fail to utilize the mutual direction and geoid height data. On the other hand, E overutilizes these data; i.e., some of the geometrical geoid heights resulting from solution E agree with the gravitational geoid heights within a meter, which is not possible without distorting the lower-degree gravitational coefficients by forcing them to absorb much of the higher-degree contributions to the station geoid heights.

Hence the choice of preferred solution must be based on more selective indicators of the essential quality of sensitivity of data to parameters determined. The most obvious weakness is that of over-all orientation: when all 36 station coordinates are free to shift, erratic results are obtained, as shown by solution N in Table 7. Some constraint must be applied, as it has been in all previous analyses of close satellite tracking. Such constraint necessarily amounts to some weighting of previous solutions. The station positions obtained by the iterative satellite orbit analysis of *Izsak* [1966] and *Gaposhkin* [1966] now seem superior to starting values based on terrestrial data, as used by *Kaula* [1963*a*, *b*]—certainly so for stations not connected to continental datums. The next choice is between expressing this weighting by fixing one station (solutions O, P, Q) or by assigning a priori variances to all station positions (solutions F through K and M). As previously discussed, the latter seems better in principle, in that no preference is given to any one station; the results in Tables 6 and 7 do not appear to markedly contradict this choice.

The two solutions which assigned a priori variances to gravitational coefficients, K and M, differed negligibly in their results from solutions J and G, respectively, the maximum changes being decreases in absolute magnitude of 0.09 to 0.11 $\times 10^{-6}$ in two or three fifth- and sixth-degree coefficients. Of the remaining solutions, F through J, F, I, and J are preferable to G and H because they incorporate the supplemental data, and H, I, and J are preferable to F and G because they give relatively greater weight to the sensitive lower satellites 1961o_1 and 1959α_1 than to the insensitive high satellite 1961α^{δ}_1. The two preferred solutions, I and J, differ in the weight assigned to the supplemental equations, the effect of which shows most markedly in the sectorial harmonic coefficients $\bar{C}_{33}$, $\bar{C}_{44}$, and $\bar{S}_{44}$. For these three coefficients solution J is much closer than I to the independent results based on the Doppler data of *Anderle* [1966] and *Guier and Newton* [1965]. Perhaps the differences are a reflection of the variances adopted for the direction data being too small relative to those for the close satellite data. We adopt solution J, but the preference is slight.

Seven solutions for gravitational coefficients through the eighth degree are given in Table 6, which suffices to demonstrate the more important effects of variations in weighting. The standard deviations σ_{lm} resulting from the least-squares calculation are also given for solution J; the one figure given pertains to both $\bar{C}_{lm}$ and $\bar{S}_{lm}$, since their standard deviations always agreed within 0.01 $\times 10^{-6}$. The highest correlations between different harmonics produced by the least squares occurred in the expected places: (1) between coefficients both appearing in the 24-hour satellite equations, for example -0.754 for $r(\bar{C}_{22}, \bar{C}_{33})$, -0.321 for $r(\bar{C}_{33}, \bar{S}_{33})$, -0.311 for $r(\bar{S}_{22}, \bar{C}_{33})$, and 0.240 for $r(\bar{S}_{33}, \bar{C}_{42})$; and (2) between coefficients of the same order m and degree l differing by an even number, for example -0.534 for $r(\bar{C}_{41}, \bar{C}_{61})$, 0.692 for $r(\bar{C}_{41}, \bar{C}_{81})$, 0.480 for $r(\bar{C}_{42}, \bar{C}_{82})$, and 0.446 for $r(\bar{C}_{44}, \bar{C}_{64})$. All correlation coefficients not in these two categories were less than 0.18; most of them were less than 0.08. Most correlations between gravitational coefficients and station coordinates were less than 0.05; the largest was -0.152 for $r(\bar{C}_{44}, u_{3,2})$.

The solutions for the fifteenth-degree coefficients are not shown in Table 6 because they always came out the same:

TABLE 8. Comparison of Geoid Heights (Solution J) Referred to an ellipsoid a_e = 6,378,165 m, f = 1/298.25.

Station Number	Latitude, deg	Longitude East, deg	Elevation above MSL, m	Geoid Height, m	
				Geometrical	Gravitational
1	32.4	253.4	1651	−36	−23
2	−26.0	28.2	1544	28	24
3	−31.1	136.8	162	−27	0
4	36.5	353.8	24	54	51
5	35.7	139.5	58	18	19
6	29.4	79.5	1927	−64	−49
7	−16.5	288.5	2451	23	2
8	29.6	52.5	1596	−32	−13
9	12.1	291.2	7	−47	−22
10	27.0	279.9	15	−49	−30
11	−31.9	294.9	598	26	9
12	20.7	203.7	3035	−6	−20

$$\bar{C}_{15,12} = -0.043 \pm 0.002 \times 10^{-6}$$

$$\bar{S}_{15,12} = -0.031 \pm 0.002 \times 10^{-6}$$

$$\bar{C}_{15,13} = -0.032 \pm 0.007 \times 10^{-6}$$

$$\bar{S}_{15,13} = -0.065 \pm 0.007 \times 10^{-6}$$

$$\bar{C}_{15,14} = 0.010 \pm 0.003 \times 10^{-6}$$

$$\bar{S}_{15,14} = -0.011 \pm 0.003 \times 10^{-6}$$

The geoid corresponding to solution J (plus Table 2) is shown in Figure 1. For 38 tesseral harmonic coefficients in common with the solution of *Anderle* [1966] the quadratic sum of differences in the coefficients was 1.29×10^{-12}, equivalent to ±7.3 m in geoid height, or an rms discrepancy of $\pm 0.18 \times 10^{-6}$ per coefficient. For other solutions the comparable figures are: *Guier and Newton* [1965], 38 coefficients, 1.91×10^{-12}, ±8.8 m, $\pm 0.22 \times 10^{-6}$; *Izsak* [1966], 32 coefficients, 1.94×10^{-12}, ±8.9 m, $\pm 0.25 \times 10^{-6}$; and *Gaposhkin* [1966], 40 coefficients, 1.00×10^{-12}, ±6.4 m, $\pm 0.16 \times 10^{-6}$.

The results for station coordinate shifts are given in Table 7, together with the standard deviations for the preferred solution J. The

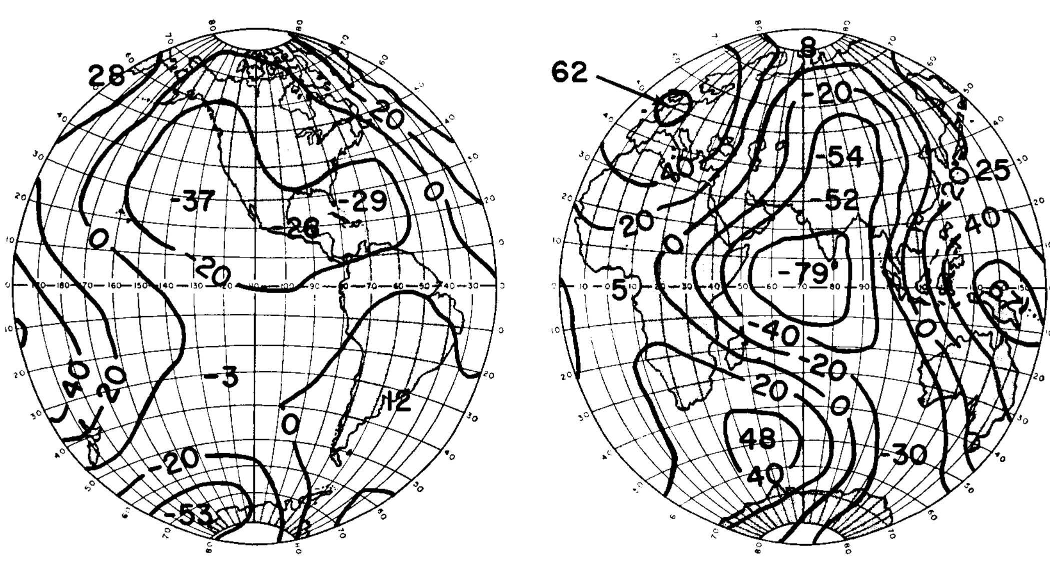

Fig. 1. Geoid heights in meters referred to an ellipsoid of flattening 1/298.25. Based on solution J, Table 6.

ill-conditioning and orientation problems occurring when the stations are allowed to move freely are evident from the results for solution N: formal standard deviations for station coordinates generated by the least-squares solutions were about ±11 m, but the rms difference between solutions A and N is ±25 m. Covariance between different stations also appears to be high; for example, the solution N $\Delta u_{1,12}$ has 16 correlation coefficients that are higher than 0.20. The fluctuation of station positions between different solutions in Table 7 is considerably more than that implied by the fluctuation of gravitational coefficients in Table 6. Multiplying the range of variation of a coefficient in Table 6 (e. g., 0.10×10^{-6} for $\bar{C}_{31}$) by the average partial derivative of satellite position with respect to the coefficient yields a range of about 6 m in orbital position. From this we would expect a range of about $\sqrt{12} \times 6$, or 20 m, in station position, since a station coordinate appears in 1/12 as many equations. This is about equal to the absolute average discrepancy between coordinates for solutions O and J, which utilize the two alternative methods of fixing orientation. It is also about equal to the rms deviation of the coordinate shifts of solution J, ±22 m, from the iterated solution of *Gaposhkin* [1966].

Geometrical geoid heights with respect to an ellipsoid of equatorial radius 6,378,165 m and flattening 1/298.25 were calculated from the final positions for solution J. These geoid heights, together with gravitational geoid heights obtained from Figure 1, are given in Table 8. If the mean value of a geometrical minus gravitational geoid height is taken as a correction to the semimajor axis, a value of 6,378,160 ± 5 m is obtained. Using this radius with the *GM* of 3.986009×10^{14} m³/sec² gives an equatorial gravity γ_e of 978.0262 cm sec^{-2}, which is somewhat lower than terrestrial solutions previously obtained [*Kaula*, 1966*b*]. The geometrical-gravitational geoid-height equations have probably had the effect of pulling the stations outward a few meters from the correct radius toward the starting values based on 6,378,165 m.

Conclusions

This investigation demonstrates that a good solution for the nonzonal harmonics of the gravitational field can be obtained from a relatively small amount of data. The agreement of the gravitational coefficients with other solutions using different data or methods of analysis is also quite satisfying; it indicates that the amplitudes of persistent oscillations in the orbits are being determined to within about ±5 m. The results for station coordinate shifts are not so satisfactory: the limitations on directions with respect to inertial space in which observations can be made for a given orbital arc of approximately 18 days apparently results in poor separation of station coordinates from orbital parameters. Some constraint in orientation is needed for the entire system, as well as considerably more data, to gain an improvement over the accuracy of ±20 m obtained in this study.

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