SID 67-470

STUDY OF APOLLO WATER IMPACT

FINAL REPORT

VOLUME 7

MODIFICATION OF SHELL OF REVOLUTION ANALYSIS

(Contract NAS9-4552, G.O. 5264)

May 1967

Prepared by

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S.C. Furuike
(Authors)

Approved by

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Program Manager
Structures and Materials

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NORTH AMERICAN AVIATION, INC.
SPACE DIVISION

NASA CR 92023

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FOREWORD

This report was prepared by North American Aviation, Inc., Space Division, under NASA Contract NAS9-4552, for the National Aeronautics and Space Administration, Manned Space Flight Center, Houston, Texas, with Dr. F.C. Hung, Program Manager and Mr. P.P. Radkowski, Assistant Program Manager. This work was administered under the direction of Structural Mechanics Division, MSC, Houston, Texas with Dr. F. Stebbins as the technical monitor.

This report is presented in eleven volumes for convenience in handling and distribution. All volumes are unclassified.

The objective of the study was to develop methods and Fortran IV computer programs to determine by the techniques described below, the hydro-elastic response of representation of the structure of the Apollo Command Module immediately following impact on the water. The development of theory, methods and computer programs is presented as Task I Hydrodynamic Pressures, Task II Structural Response and Task III Hydroelastic Response Analysis.

Under Task I - Computing program to extend flexible sphere using the Spencer and Shiffman approach has been developed. Analytical formulation by Dr. Li using nonlinear hydrodynamic theory on structural portion is formulated. In order to cover a wide range of impact conditions, future extensions are necessary in the following items:

a. Using linear hydrodynamic theory to include horizontal velocity and rotation.

b. Nonlinear hydrodynamic theory to develop computing program on spherical portion and to develop nonlinear theory on toroidal and conic sections.

Under Task II - Computing program and User's Manual were developed for nonsymmetrical loading on unsymmetrical elastic shells. To fully develop the theory and methods to cover realistic Apollo configuration the following extensions are recommended:

a. Modes of vibration and modal analysis.

b. Extension to nonsymmetric short time impulses.

c. Linear buckling and elasto-plastic analysis

These technical extensions will not only be useful for Apollo and future Apollo growth configurations, but they will also be of value to other aeronautical and spacecraft programs.

The hydroelastic response of the flexible shell is obtained by the numerical solution of the combined hydrodynamic and shell equations. The results obtained herein are compared numerically with those derived by neglecting the interaction and applying rigid body pressures to the same elastic shell. The numerical results show that for an axially symmetric impact of the particular shell studied, the interaction between the shell and the fluid produces appreciable differences in the overall acceleration of the center of gravity of the shell, and in the distribution of the pressures and responses. However the maximum responses are within 15% of those produced when the interaction between the fluid and the shell is neglected. A brief summary of results is shown in the abstracts of individual volumes.

The volume number and authors are listed on the following page.

The contractor's designation for this report is SID 67-498.

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ABSTRACT

A numerical procedure is presented for the solution of reduced field equations for shells. The technique developed is an improved version of the numerical scheme presented by Budiansky and Radkowski for the solution of shell problems. Eccentric discontinuities and branched shells are included. The procedure is general and can be applied to any consistent set of shell equations based on any shell theory.

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SYMBOLS

MATRICES

E, F, G, H, J	$\lfloor m \times m \rfloor$ matrices representing shell material and geometric properties
e, f	$\{m \times 1\}$ column vectors for external and thermal loads
Z	$\{m \times 1\}$ column vector of the primary problem variables
Ω, Λ,	$\lfloor m \times m \rfloor$ diagonal matrices
ℓ	$\{m \times 1\}$ column vector
y	$\{m \times 1\}$ column vector of the secondary problem variables
ψ, Ψ, $\mathcal{H}$	$\lfloor m \times m \rfloor$ discontinuity matrices
A, B, C, P, $\lfloor X \rfloor$, $\lfloor Y \rfloor$	$\lfloor m \times m \rfloor$ matrices used in the numerical solution
g, X, $\lfloor L \rfloor$	$\{m \times 1\}$ column matrix used in the numerical solution

VARIABLES

u_ξ, u_θ, w	Fourier coefficients for displacements in the meridional, circumferential, and normal directions, respectively
m_ξ	Fourier coefficient for the meridional bending moment
a	reference length

Δ	internal size
j	discontinuity station
ψ	abrupt angle change at discontinuity
E_{cc}	dimensionless eccentricity at discontinuity

SUPERSCRIPTS

I	refers to values just behind a discontinuity
II	refers to values just ahead of a discontinuity

For additional information, see the text of this report or the nomenclature list of Reference 2.

1.0 INTRODUCTION

Due to the mathematical complexity of the shell field equations, it becomes necessary to employ numerical techniques in order to obtain solutions to the governing differential equations. Various numerical procedures have been presented in the literature for handling the shell problem. Of particular interest is the direct or Gaussian elimination procedure first presented by Potters (Reference 1) and successfully applied by Budiansky and Radkowski (Reference 3) to shell problems.

In this report an improved version of this numerical procedure to be used in treating general shell problems is presented. The procedure developed is similar to that described in Reference 2 expecting for the refinements introduced for treatment of boundary, discontinuity, and branched conditions. The analysis considers the case of discontinuties having continuous and discontinuous reference surfaces.

The numerical procedure is perfectly general and is applicable for solution of most appropriate sets of shell equations. The technique can be used for solution of the shear deformation shell equations described in Reference 5; the three-layer shell theory, presented in Reference 4; the unsymmetric shell analysis described in References 6, 7, and 8 and the equations of Sanders' theory treated in Reference 2.

The numerical procedure can be easily mechanized for use on the digital computer.

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2.0 SHELL EQUATIONS

The field equations describing the behavior of shells are, in general, a set of partial differential equations. By using various equation reduction techniques (e.g., Fourier series expansions (Reference 2)), it is possible to reduce the general shell problem to the consideration of a set of ordinary differential equations which can be expressed in matrix form as follows:

$$EZ'' + FZ' + GZ = e \tag{2.1}$$

where E, F, and G are $[m \times m]$ matrices representing shell material and geometric properties and e is an $\{m \times 1\}$ column matrix for external and thermal loads. The Z matrix is an $\{m \times 1\}$ column matrix of the problem variables and the primes represent differentiation with respect to the meridional coordinate.

The matrix equations described above are similar in form to Equations (39) developed in Reference 2 for shells of revolution. The form is also consistent with the three layer shell equations [(Equation 6) of Reference (4)], the shear distortion shell equations [Equation (9) of Reference (5)], and the unsymmetric shell equations [Equations (33), (41), and (20) of References (6), (7), and (8), respectively].

The order m of the matrices in Equation 1 is determined by the number of variables used to describe the shell problem; e.g., for axisymmetric three layer shell theory (Reference 4) m = 5, for shear deformation theory (Reference 5)m = 5, for unsymmetric shells (Reference 6)m = 5K and (Reference 7)m = 4K. (K represents the number of Fourier coefficients retained in the series expressions.)

For ease of presentation, it will be convenient to utilize the shell equations and nomenclature presented in Reference 2. In this case m = 4 and the matrix of variables Z is given by

$$Z = \begin{bmatrix} u_\xi \\ u_\theta \\ w \\ m_\xi \end{bmatrix} \tag{2.2}$$

The procedure developed would be similar for the shell equations described in References (4), (5), (6), and (7). (The change of nomenclature should be noted when dealing with these references.)

The boundary conditions consistent with the various shell theories can be written in matrix form as

$$\Omega H Z' + \left[\Lambda + \Omega J\right] Z = \ell - \Omega f \tag{2.3}$$

where Ω and Λ are appropriate diagonal matrices. (See Reference 2 for example.)

The shell equations (Equation 2.1) are not valid at points in the shell where discontinuities in geometry occur. In this case special transition equations are required. These equations as presented in Reference 2 are given by

$$\begin{aligned} y^{II} &= \Psi y^{I} \\ Z^{II} &= \Psi Z^{I} \end{aligned} \tag{2.4}$$

where the I and II superscripts represent values just behind and ahead of a discontinuity and Ψ is matrix describing the geometry change.

Equations 2.4 can be combined with Equation 50 of Reference 2 to yield a single equation which represents compatibility across the discontinuity. This equation can be written as

$$H^{II}(Z')^{II} + \left[J^{II}\Psi - \Psi J^{I}\right] Z^{I} - \Psi H^{I}(Z')^{I} = \Psi f^{I} - f^{II} \tag{2.5}$$

where the I and II superscripts indicate that the matrices H, J, f are determined on the basis of shell properties just behind and ahead of the discontinuity respectively.

The differential equation (2.1), the boundary conditions (2.3) and the discontinuity conditions (2.5) completely describe the boundary value problem for shells. These equations will be cast in a unified set of appropriate finite-difference equations to facilitate numerical solution.

3.0 FINITE DIFFERENCE FORMULATION

In solving a general shell problem it is convenient to divide the shell into regions. Suppose that p discontinuity locations s_1, s_2, ... s_p occur in the range (o, $\bar{s}$) of the shell; the shell could then be divided into p + 1 regions. Each of the shell regions (o_1 s_1), (s_1, s_2), ... (s_p, $\bar{s}$) can in turn be subdivided into V_1, V_2, ... V_{p+1} equal increments, respectively. The increments are identified by the index i, running from zero at s = o to $N(=\Sigma V)$ at $s = \bar{s}$. The length of the increments in the nondimensional variable ξ (see Reference 2) are then

$$\begin{aligned}
\Delta_1 &= \frac{s_1}{aV_1} \\
\Delta_2 &= \frac{(s_2 - s_1)}{aV_2} \\
&\vdots \\
\Delta_{p+1} &= \frac{(\bar{s} - s_p)}{aV_{p+1}}
\end{aligned} \tag{3.1}$$

in the successive regions bounded by the discontinuities. It should be noted that fictitious discontinuities may be inserted wherever a change in the increment size is considered desirable. It will be convenient to denote the discontinuity stations by $i = j_m$ (m = 1, 2, . . . , p).

The difference equations (2.1) can be written in finite difference form at all stations excepting the boundaries i = o, N and at discontinuties j_m (m = 1, 2, . . . , p) on the basis of the usual central difference formulas:

$$\begin{aligned}
Z_i' &= (Z_{i+1} - Z_{i-1})/2\Delta \\
Z_i'' &= (Z_{i+1} - 2Z_i + Z_{i-1})/\Delta^2
\end{aligned} \tag{3.2}$$

where Δ must be the one corresponding to the region associated with station i.

An improved procedure over that described in Reference 2 will be used to incorporate the discontinuity relations 2.5 into the analysis. To illustrate the finite differencing at a discontinuity it will be convenient to employ a model as shown in Figure 1; where, for convenience, the subscript m has been omitted on each j.

At the m^{th} discontinuity ($1 \le m \le p$) expressions for first and second derivatives, in finite difference form, are given by

$$\begin{aligned}
(Z')^{I} &= (Z_{j+1}^{I} - Z_{j-1}^{I})/2\Delta_m \\
(Z'')^{I} &= (Z_{j+1}^{I} - 2Z_j^{I} + Z_{j-1}^{I})/(\Delta_m)^2 \\
(Z')^{II} &= (Z_{j+1}^{II} - Z_{j-1}^{II})/2\Delta_{m+1} \\
(Z'')^{II} &= (Z_{j+1}^{II} - 2Z_j^{II} + Z_{j-1}^{II})/(\Delta_{m+1})^2
\end{aligned} \tag{3.3}$$

Using Equations (3.3), Equation (2.5) becomes

$$\begin{aligned}
&\frac{H^{II}}{2\Delta_{m+1}} Z_{j+1}^{II} - \frac{H^{II}}{2\Delta_{m+1}} Z_{j-1}^{II} \\
&- \frac{\Psi H^{I}}{2\Delta_m} Z_{j+1}^{I} + (J^{II}\Psi - \Psi J^{I}) Z_j^{I} \\
&+ \frac{\Psi H^{I}}{2\Delta_m} Z_{j-1}^{I} = \Psi f^{I} - f^{II}
\end{aligned} \tag{3.4}$$

Notice that Equation 3.4 contains Z matrices evaluated at the fictitious points $^{I}_{j+1}$ and $^{II}_{j-1}$ (see Figure 1). These two points can be eliminated from the analysis by writing equilibrium equations at the point $i = j^{I}$ and $i = j^{II}$ as follows:

$$E^{I}(Z'')^{I} + F^{I}(Z')^{I} + G^{I}Z^{I} = e^{I}$$

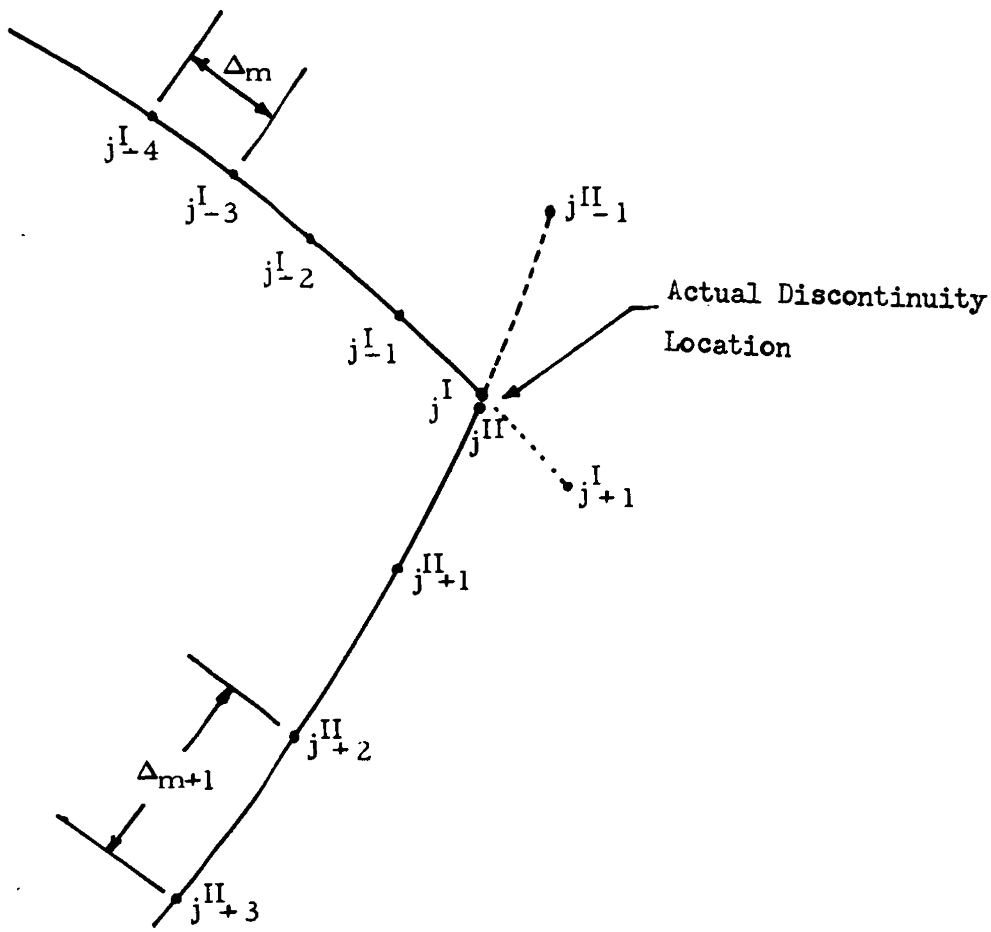

Figure 1. Finite Difference Stations in Discontinuity Region

$$E^{II}(Z'')^{II} + F^{II}(Z')^{II} + G^{II}Z^{II} = e^{II}$$

where the e^{I} and e^{II} have the same form as at other stations. In finite difference form, these become:

$$\frac{E^{I}}{(\Delta_m)}2\left[Z_{j+1}^{I} - 2Z_j^{I} + Z_{j-1}^{I}\right] + \frac{F^{I}}{2\Delta_m}\left[Z_{j+1}^{I} - Z_{j-1}^{I}\right] + G^{I}Z_j^{I} = e^{I} \tag{3.5}$$

and

$$\frac{E^{II}}{(\Delta_{m+1})^2}\left[Z_{j+1}^{II} - 2Z_j^{II} + Z_{j-1}^{II}\right] + \frac{F^{II}}{2\Delta_{m+1}}\left[Z_{j+1}^{II} - Z_{j-1}^{II}\right] + G^{II}Z_j^{II} = e^{II}$$

Solving Equations 3.5 for Z_{j+1}^{I} and Z_{j-1}^{II}, we obtain

$$Z_{j+1}^{I} = \left[\frac{2E^{I}}{\Delta_m} + F^{I}\right]^{-1} 2\Delta_m e^{I} + \left[\frac{2E^{I}}{\Delta_m} + F^{I}\right]^{-1}\left[F^{I} - \frac{2E^{I}}{\Delta_m}\right] Z_{j-1}^{I} + \left[\frac{2E^{I}}{\Delta_m} + F^{I}\right]^{-1}\left[\frac{4E^{I}}{\Delta_m} - 2\Delta_m G^{I}\right] Z_j^{I} \tag{3.6}$$

and

$$Z_{j-1}^{II} = \left[\frac{2E^{II}}{\Delta_{m+1}} - F^{II}\right]^{-1} 2\Delta_{m+1} e^{II}$$

$$- \left[\frac{2E^{II}}{\Delta_{m+1}} - F^{II}\right]^{-1} \left[\frac{2E^{II}}{\Delta_{m+1}} + F^{II}\right] Z_{j+1}^{II} \tag{3.7}$$

$$+ \left[\frac{2E^{II}}{\Delta_{m+1}} - F^{II}\right]^{-1} \left[\frac{4E^{II}}{\Delta_{m+1}} - 2\Delta_{m+1} G^{II}\right] Z_j^{II}$$

Equations 3.6 and 3.7 can be substituted into Equation 3.4 to yield a finite difference expression at a discontinuity free of fictitious points. Since the Z matrices at j^I and j^{II} are related by Equation 2.4, it will be convenient to eliminate j^{II} as an explicit point. The following expressions for derivatives at the top and bottom boundaries respectively, are of the same order of error as the central difference expressions used at all interior points:

$$\begin{aligned} Z_o' &= (-3Z_o + 4Z_1 - Z_2)/2\Delta_1 \\ Z_N' &= (3Z_N - 4Z_{N-1} + Z_{N-2})/2\Delta_p \end{aligned} \tag{3.8}$$

Different expressions were used in Reference (2).

The results of writing the various difference expressions just described can be stated compactly as the following set of algebraic equations for Z_i ($i = 0, 1, 2, \ldots, N$)

$$\begin{aligned} A_o Z_2 + B_o Z_1 + C_o Z_o &= g_o \\ A_i Z_{i+1} + B_i Z_i + C_i Z_{i-1} &= g_i \\ (i &= 1, 2, \ldots, N-1) \\ A_N Z_N + B_N Z_{N-1} + C_N Z_{N-2} &= g_N \end{aligned} \tag{3.9}$$

where the conditions at the boundary are given by

$$
\begin{aligned}
A_o &= \frac{-\Omega_o H_o}{2\Delta_1} \\
B_o &= \frac{2\Omega_o H_o}{\Delta_1} \\
C_o &= \Lambda_o + \Omega_o J_o - \frac{3\Omega_o H_o}{2\Delta_1} \\
g_o &= \ell_o - \Omega_o f_o
\end{aligned}
\tag{3.10}
$$

For $i \neq o$, j_m, j_{m+1}, $N(m = 1, 2, \ldots, p)$

$$
\begin{aligned}
A_i &= \frac{2E_i}{\Delta} + F_i \\
B_i &= -\frac{4E_i}{\Delta} + 2\Delta G_i \\
C_i &= \frac{2E_i}{\Delta} - F_i \\
g_i &= 2\Delta e_i
\end{aligned}
\tag{3.11}
$$

where the appropriate value for Δ is used. For $i = j_{m+1}$, Equation 3.11 applies except that

$$C_{j+1} = \left[\frac{2E_{j+1}}{\Delta_{m+1}} - F_{j+1}\right]\Psi \tag{3.12}$$

For $i = j_m$,

$$A_j = \left[\frac{H^{II}}{2\Delta_{m+1}} + \frac{H^{II}}{2\Delta_{m+1}}\left(\frac{2E^{II}}{\Delta_{m+1}} - F^{II}\right)^{-1}\left(\frac{2E^{II}}{\Delta_{m+1}} + F^{II}\right)\right] \tag{3.13}$$

$$B_j = \left[-\frac{\Psi H^I}{2\Delta_m}\left(\frac{2E^I}{\Delta_m} + F^I\right)^{-1}\left(\frac{4E^I}{\Delta_m} - 2\Delta_m G^I\right) + J^{II}\Psi - \right.$$

$$\left. \Psi J^I - \frac{H^{II}}{2\Delta_{m+1}}\left(\frac{2E^{II}}{\Delta_{m+1}} - F^{II}\right)^{-1}\left(\frac{4E^{II}}{\Delta_{m+1}} - 2\Delta_{m+1} G^{II}\right)\Psi \right]$$

$$C_j = \left[\frac{\Psi H^I}{2\Delta_m} - \frac{\Psi H^I}{2\Delta_m}\left(\frac{2E^I}{\Delta_m} + F^I\right)^{-1}\left(F^I - \frac{2E^I}{\Delta_m}\right) \right]$$

$$g_j = \Psi f^I - f^{II} + \frac{H^{II}}{2\Delta_{m+1}}\left(\frac{2E^{II}}{\Delta_{m+1}} - F^{II}\right)^{-1} 2\Delta_{m+1} e^{II}$$

$$+ \frac{\Psi H^I}{2\Delta_m}\left(\frac{2E^I}{\Delta_m} + F^I\right)^{-1} 2\Delta_m e^I$$

At boundary $i = N$ $(s = \bar{s})$ we find

$$A_N = \Lambda_N + \Omega_N J_N + \frac{3\Omega_N H_N}{2\Delta_p}$$

$$B_N = -\frac{2\Omega_N H_N}{\Delta p} \tag{3.14}$$

$$C_N = \frac{\Omega_N H_N}{2\Delta_p}$$

$$g_N = \ell_N - \Omega_N f_N$$

It should be noted that for fictitious discontinuities Equations 3.9 with coefficients given by Equations 3.13 may be reduced to Equations 3.9 with coefficients given by Equations 3.11 which would be expected for a continuous system. Such a relationship was not possible for the procedure used in Reference 2. In addition, the use of the central difference formulas offers greater accuracy than the backward (forward) differencing utilized at discontinuities in Reference 2.

An alternative procedure for describing Equations 3.9 at $i = 0$, N is possible. This procedure would involve writing equilibrium equations and boundary conditions (central difference) at boundary points in terms of fictitious points. The fictitious points can be eliminated by employing a procedure similar to that used in treating discontinuities. For this procedure the boundary matrices are described as follows: $(i = o)$

$$A_o = \left[\frac{\Omega H}{2\Delta} + \frac{\Omega H}{2\Delta}\left(\frac{2E}{\Delta} - F\right)^{-1}\left(\frac{2E}{\Delta} + F\right)\right]$$

$$B_o = \left[\Lambda + \Omega J + \frac{\Omega H}{2\Delta}\left(\frac{2E}{\Delta} - F\right)^{-1}\left(-\frac{4E}{\Delta} + 2\Delta G\right)\right] \tag{3.15}$$

$$g_o = \ell - \Omega f + \frac{\Omega H}{2\Delta}\left(\frac{2E}{\Delta} - F\right)^{-1} 2\Delta e$$

and at $i = N$

$$B_N = \left[\Lambda + \Omega J - \frac{\Omega H}{2\Delta}\left(\frac{2E}{\Delta} + F\right)^{-1}\left(-\frac{4E}{\Delta} + 2\Delta G\right)\right]$$

$$C_N = \left[-\frac{\Omega H}{2\Delta} - \frac{\Omega H}{2\Delta}\left(\frac{2E}{\Delta} + F\right)^{-1}\left(\frac{2E}{\Delta} - F\right)\right] \tag{3.16}$$

$$g_N = \left[\ell - \Omega f - \frac{\Omega H}{2\Delta}\left(\frac{2E}{\Delta} + F\right)^{-1} 2\Delta e\right]$$

The elimination technique for this alternative approach would follow identically to that described in Reference 2. This technique offers no significant advantages over the procedure herein presented. The elimination technique used to solve the developed equations are presented in Section 6.0.

4.0 THE ANALYSIS OF ECCENTRIC DISCONTINUITIES

The problem of eccentric discontinuities is considered using the mathematical model illustrated in Figure 2. In the case of eccentric discontinuities, the discontinuity equations (Equation 2.4) are replaced by the following:

$$
\begin{aligned}
y^{II} &= \Psi \, y^{I} \\
Z^{II} &= \mathcal{F} \, Z^{I} + \mathcal{H} \, y^{I}
\end{aligned}
\tag{4.1}
$$

The detailed development of Equations 4.1 for eccentric discontinuities is presented in Reference 9. The matrices Ψ, $\mathcal{F}$, and $\mathcal{H}$ are given by

$$
\Psi = \begin{bmatrix} \cos\Psi & o & -\sin\Psi & o \\ o & 1 & o & o \\ \sin\Psi & o & \cos\Psi & o \\ o & o & o & 1 \end{bmatrix}
\tag{4.2}
$$

$$
\mathcal{F} = \begin{bmatrix} o & o & o & o \\ o & E_{cc}(\omega_\theta)_{jI} & \dfrac{E_{cc}n}{\rho_{jI}} & o \\ o & o & o & o \\ o & o & o & o \end{bmatrix} + \Psi
\tag{4.3}
$$

$$
\mathcal{H} = E_{cc} \begin{bmatrix} o & o & o & \cos\Psi \\ o & o & o & o \\ o & o & o & \sin\Psi \\ -1 & o & o & o \end{bmatrix}
\tag{4.4}
$$

In Equations 4.3 and 4.4, E_{cc} is the dimensionless eccentricity of the participating reference surfaces measured along the radius of curvature behind the discontinuity point. (It can be noted in Figure 2 that a positive value of E_{cc} corresponds to an abrupt increase in radius of parallel circle as one proceeds in the direction of increasing i).

Using Equations 4.1 in place of Equations 2.4 and following a similar reduction procedure as indicated in the text of this report results in a set of equations analogous to Equations 3.12, 3.13 which hold for the case of eccentric discontinuities. These equations are

at $i = j_m$:

$$A_j = \frac{H^{II}}{2\Delta^{II}} + \frac{H^{II}}{2\Delta^{II}} \left(C^{II}\right)^{-1} A^{II}$$

$$B_j = \frac{H^{II}}{2\Delta^{II}} \left(C^{II}\right)^{-1} B^{II} \left[\maltese - \frac{\mathcal{H} H^{I}}{2\Delta^{I}} \left(A^{I}\right)^{-1} B^{I} + \mathcal{H} J^{I} \right] + \left[Y\right] - \left[X\right] \left(A^{I}\right)^{-1} B^{I}$$

$$C_j = - \frac{H^{II}}{2\Delta^{II}} \left(C^{II}\right)^{-1} B^{II} \mathcal{H} \frac{H^{I}}{2\Delta^{I}} \left[I + \left(A^{I}\right)^{-1} C^{I} \right] - \left[X\right] \left(A^{I}\right)^{-1} C^{I} - \left[X\right] \tag{4.5}$$

$$g_j = \frac{H^{II}}{2\Delta^{II}} \left(C^{II}\right)^{-1} \left[g^{II} - B^{II} \mathcal{H} \frac{H^{I}}{2\Delta^{I}} \left(A^{I}\right)^{-1} g^{I} - B^{II} \mathcal{H} f^{I} \right] - \left[X\right] \left(A^{I}\right)^{-1} g^{I} - \left[L\right]$$

where

$$\left[X\right] = \left(J^{II} \mathcal{H} - \Psi \right) \frac{H^{I}}{2\Delta^{I}}$$

$$\left[Y\right] = J^{II} \left[\maltese + \mathcal{H} J^{I} \right] - \Psi J^{I} \tag{4.6}$$

$$\left[L\right] = \left(J^{II} \mathcal{H} - \Psi \right) f^{I} + f^{II}$$

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

and at $i = j + 1$

$$C_{j+1} = C^{II}_{j+1} \left\{ \left[\maltese + \mathcal{H} J^{I} - \mathcal{H} \frac{H^{I}}{2\Delta^{I}} \left(A^{I}\right)^{-1} B^{I} \right] + \left[\mathcal{H} \frac{H^{I}}{2\Delta^{I}} + \mathcal{H} \frac{H^{I}}{2\Delta^{I}} \left(A^{I}\right)^{-1} C^{I} \right] \right\} P_j I_{-1} \tag{4.7}$$

$$g_{j+1} = g^{II}_{j+1} - C^{II}_{j+1} \mathcal{H} f^{I} - C^{II}_{j+1} \mathcal{H} \frac{H^{I}}{2\Delta^{I}} \left(A^{I}\right)^{-1} g^{I} + C^{II}_{j+1} \mathcal{H} \frac{H^{I}}{2\Delta^{I}} \left[I + (A^{I})^{-1} C^{I} \right] X^{I}_{j-1}$$

Note that when $E_{cc} = o$, Equations 4.5 and 4.7 reduce to Equations 3.13 and 3.12 respectively, ($\mathcal{H} = o$, $\mathcal{F} = \psi$).

In the evaluation of Equation 4.5, A^{I}, B^{I}, C^{I}, g^{I}, and A^{II}, B^{II}, C^{II}, g^{II} are the values obtained using Equations 3.11 at the points $i = j^{I}$ and $i = j^{II}$ respectively. In addition, when evaluating Equation 4.7, the C^{II}_{j+1} and g^{II}_{j+1} are matrices obtained using Equation 3.11 evaluated at $i = j + 1$.

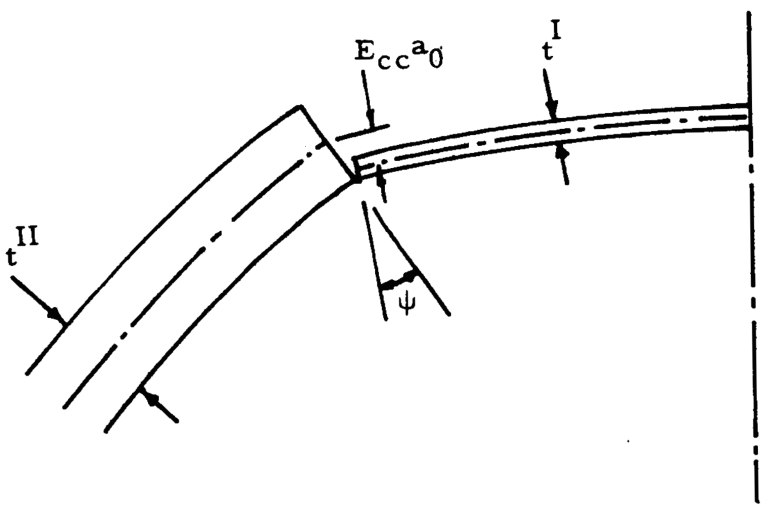

Figure 2 Eccentric Discontinuity Model

5.0 MATRIX SOLUTION OF DIFFERENCE EQUATIONS

The more sophisticated difference expressions for the first derivative at the boundary points requires a slight alteration of the Gaussian elimination procedure described in Reference 1. The elimination procedure considers the first and second of Equations 3.9 which are described by

$$\begin{aligned} A_o Z_2 + B_o Z_1 + C_o Z_o &= g_o \\ A_1 Z_2 + B_1 Z_1 + C_1 Z_o &= g_1 \end{aligned} \tag{5.1}$$

Eliminating Z_o from these equations, we obtain,

$$\left[A_o - C_o C_1^{-1} A_1\right] Z_2 + \left[B_o - C_o C_1^{-1} B_1\right] Z_1 = g_o - C_o C_1^{-1} g_1$$

From which we may solve for Z_1, as:

$$\begin{aligned} Z_1 = &- \left[B_o - C_o C_1^{-1} B_1\right]^{-1} \left[A_o - C_o C_1^{-1} A_1\right] Z_2 \\ &+ \left[B_o - C_o C_1^{-1} B_1\right] \left[g_o - C_o C_1^{-1} g_1\right] \end{aligned} \tag{5.2}$$

and therefore Z_1 can be written as

$$Z_1 = - P_1 Z_2 + X_1 \tag{5.3}$$

where

$$\begin{aligned} P_1 &= \left[B_o - C_o C_1^{-1} B_1\right]^{-1} \left[A_o - C_o C_1^{-1} A_1\right] \\ X_1 &= \left[B_o - C_o C_1^{-1} B_1\right]^{-1} \left[g_o - C_o C_1^{-1} g_1\right] \end{aligned} \tag{5.4}$$

Retaining the format of Equation 5.3 for all meridional locations, we obtain the general result

$$Z_i = -P_i Z_{i+1} + X_i \quad (i = 1, 2, 3, \ldots, N-1) \tag{5.5}$$

It is shown in Reference 2, Equation (74) that the p and x matrices are given by

$$\begin{aligned} P_i &= \left[B_i - C_i P_{i-1}\right]^{-1} A_i \\ X_i &= \left[B_i - C_i P_{i-1}\right]^{-1} \left[g_i - C_i X_{i-1}\right] \end{aligned} \tag{5.6}$$

$$(i = 2, 3, 4, \ldots, N-1)$$

At all meridional stations the A, B, C and g matrices are computed by Equation 3.11, with the exception that for the boundaries we use Equations 3.10 and 3.14 for $i = j_1, j_2, \ldots, j_p$ (i.e., at the discontinuity stations), the pertinent A, B, C and g matrices are given by Equation 3.13 and at $i = j_m + 1$ the C matrix is obtained using Equation 3.12. Thus P and X matrices are obtained for all i from 1 to N -1. Substituting the recursion relationship (Equation 5.5) into the last of Equation 3.9 yields

$$\begin{aligned} Z_N = &\left[A_N + \left(C_N P_{N-2} - B_N\right) P_{N-1}\right]^{-1} \left[g_N - C_N X_{N-2}\right. \\ &\left. + \left(C_N P_{N-2} - B_N\right) X_{N-1}\right] \end{aligned} \tag{5.7}$$

where A_N, B_N, C_N, and g_N are given by Equations 3.14. Having Z_N, we can easily obtain Z_{N-1}, Z_{N-2}, etc., using Equation 5.5. Finally, Z_o is calculated from the second equation of 3.9, and is given by

$$Z_o = C_1^{-1} \left[g_1 - A_1 Z_2 - B_1 Z_1\right] \tag{5.8}$$

Once all the Z matrices have been obtained, the calculation of stresses, moments, etc., proceeds in a straight-forward fashion for the several shell theories.

6.0 ANALYSIS OF BRANCHED SHELLS

It has been tacitly assumed that the shell under consideration has no more than two boundaries; a multiple-branch shell such as shown in Figure 3 may be analyzed, however, by applying appropriate transition conditions at the branch point.

Define separate families of auxiliary matrices P^I, P^{II}, P^{III}, X^I, X^{II} and X^{III} with the properties

$$
\begin{aligned}
Z_i^I &= - P_i^I Z_{i+1}^I + X_i^I \\
Z_i^{II} &= - P_i^{II} Z_{i+1}^{II} + X_i^{II} \\
Z_i^{III} &= - P_i^{III} Z_{i+1}^{III} + X_i^{III}
\end{aligned}
\qquad (6.1)
$$

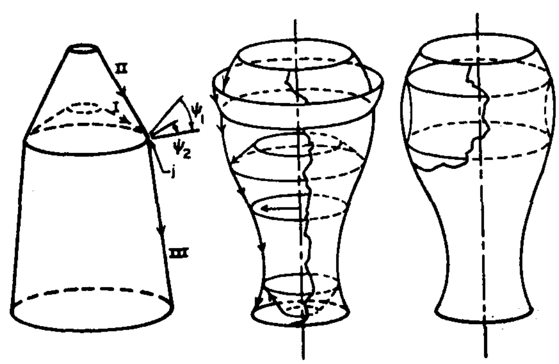

Figure 3. Branched Shells

where the superscripts refer to the separate branches shown in Figure 3a. It is possible to start the calculations of P^I, X^I and P^{II}, X^{II} at the boundaries of branches I and II and then leap across the juncture j to the calculation of P^{III}, X^{III}. The reverse sweep for the calculation of the Z's then would start at the boundary of branch III and, at the juncture j, continue independently along the branches I and II back to their respective boundaries. The details of this procedure are herein given. This method can be extended readily to handle a multiplicity of branches as in Figure 3b; it will not, however, be applicable to closed loops (Figure 3c), which must be treated separately by traditional cut-and-fit methods of indeterminate structural analysis.

The mathematical model considered for the numerical solution of branched shell problems is shown in Figure 4 with the possibility of a concentrated force P_D and M_D applied at the juncture included. The program has been set up to handle 4 shell branches meeting at a common point.

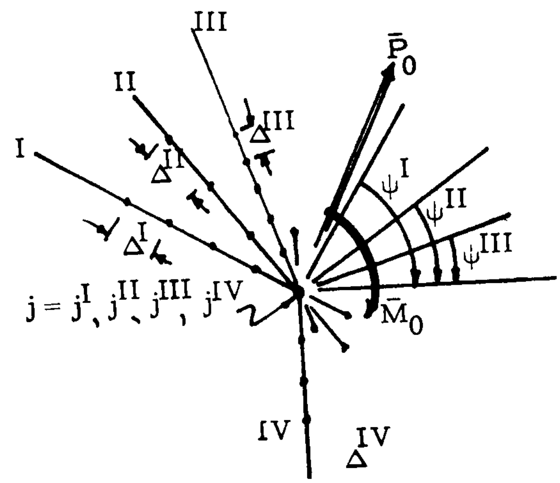

Figure 4. Mathematical Model for Branched Shell

By analogy with the previous discussion on discontinuity conditions, we may repeat here for branched shells the compatibility and equilibrium equations in the following manner:

$$
\text{Compatibility:} \quad \left.\begin{aligned}
u_\xi^{IV} &= u_\xi^M \cos\psi^M - w^M \sin\psi^M \\
u_\theta^{IV} &= u_\theta^M \\
w^{IV} &= u_\xi^M \sin\psi^M + w^M \cos\psi^M \\
\phi_\xi^{IV} &= \phi_\xi^M
\end{aligned}\right\} \quad (M = I,\ II \text{ or } III) \qquad (6.2)
$$

$$\text{Equilibrium:} \quad t_{\xi}^{IV} - \sum_{M=I}^{III} t_{\xi}^{M} \cos\psi^{M} + \sum_{M=I}^{III} \hat{f}_{\xi}^{M} \sin\psi^{M} - \bar{P}\sin\phi_0 = 0$$

$$t_{\xi\theta}^{IV} - \sum_{M=I}^{III} t_{\xi\theta}^{M} = 0$$

$$\hat{f}_{\xi}^{IV} - \sum_{M=I}^{III} t_{\xi}^{M} \sin\psi^{M} - \sum_{M=I}^{III} \hat{f}_{\xi}^{M} \cos\psi^{M} + \bar{P}\cos\phi_0 = 0$$

$$m_{\xi}^{IV} - \sum_{M=I}^{III} m_{\xi}^{M} + \bar{M} = 0 \qquad (6.3)$$

By recalling the definition of the y and z (Equation 2.2) matrices and introducing the diagonal matrices

$$\beta = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \quad \eta = \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix} \qquad (6.4)$$

Equations 3.9 and 6.3 may be recast in the formulas for compatibility

$$\beta Z^{IV} + \eta y^{IV} = \beta\Psi^{I} Z^{I} + \eta y^{I} = \beta\Psi^{II} Z^{II} + \eta y^{II} = \beta\Psi^{III} Z^{III} + \eta y^{III} \qquad (6.5)$$

and for equilibrium

$$\beta y^{IV} + \eta Z^{IV} = \sum_{M=I}^{III} \beta\Psi^{M} y^{M} + \eta Z^{M} + \bar{\Phi}\bar{P} + \bar{\eta}\bar{M} \qquad (6.6)$$

where

$$\bar{\Phi} = \begin{vmatrix} \sin\phi_0 \\ 0 \\ -\cos\phi_0 \\ 0 \end{vmatrix} \qquad \bar{\eta} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} \qquad (6.7)$$

Introducing Equations 6.3 into 6.5 and 6.6 and noting $\eta f = 0$; $\beta f = f$ and $\beta\Psi f = \Psi f$, we obtain:

for compatibility:

$$\begin{aligned}\eta H_{IV}\left(Z^{IV}\right)' + \left(\eta J_{IV} + \beta\right) Z^{IV} &= \eta H_I Z' + \left(\eta J_I + \beta\Psi^I\right) Z^I \\ &= \eta H_{II} Z_{II}' + \left(\eta J_{II} + \beta\Psi^{II}\right) Z^{II} \\ &= \eta H_{III} Z_{III}' + \left(\eta J_{III} + \beta\Psi^{III}\right) Z^{III}\end{aligned} \tag{6.8}$$

and for equilibrium

$$\beta H_{IV}\left(Z^{IV}\right)' + \left(\beta J_{IV} + \eta\right) Z^{IV} = \sum_{M=I}^{III}\left[\beta\Psi^M H_M Z^M + \left(\beta\Psi^M J_M + \eta\right) Z^M + \Psi^M f_M\right] - f_{IV} + \bar{\Phi}\bar{P} + \eta\bar{M} \tag{6.9}$$

A central finite difference scheme is used to obtain the numerical solution of Equations 6.8 and 6.9 within the framework of the Gaussian elimination procedure.

To eliminate the fictitious points (they will be used in calculating for internal forces and stresses at junction) Z^I_{j+1}, Z^{II}_{j+1}, Z^{III}_{j+1} and Z_{j+1} that appear, we utilize the equilibrium equations at the ends of the adjoining regions of the juncture in a fashion similar to that used in the discontinuity section. After substituting the expressions for fictitious points in Equations 6.8 and 6.9 and recalling the definitions of the A, B, and C matrices (Equations 2.3), we may write the recursive equation equivalent of Equation 2.4 for the branched shell. As (for j^{IV}):

$$Z_j^{IV} = - P_j^{IV} Z_{j+1}^{IV} + X_j^{IV} \tag{6.10}$$

where

$$P_j^{IV} = L_M^{-1}\left\{\frac{\beta H_{IV}}{2\Delta_{IV}}\left[I + C_{IV}^{-1} A_{IV}\right] - \left[\sum_{M=I}^{III}\left(K_M\right)\right]\left[\frac{\eta H_{IV}}{2\Delta_{IV}} I + C_{IV}^{-1} A_{IV}\right]\right\}$$

$$
\begin{aligned}
X_j^{IV} = L_M^{-1} \Bigg[\Bigg[& \frac{\beta H_{IV}}{2\Delta_{IV}} C_{IV}^{-1} g_{IV} - f_{IV} + \sum_{M=I}^{III} \Bigg\{ \beta\Psi^M \frac{H_M}{2\Delta_M} A_M^{-1} g_M + \Psi^M f_M \\
& - \frac{\beta\Psi^M H_M}{2\Delta_M}\left[A_M^{-1} C_M + I\right] X_j^{M-1} \Bigg\} - \sum_{M=I}^{III} \Bigg\{ (K_M) \Bigg[\frac{\eta H_{IV}}{2\Delta_{IV}} C_{IV}^{-1} g_{IV} \\
& + \eta \frac{H_{IV}}{2\Delta_{IV}} A_{IV}^{-1} g_{IV} - \frac{\eta H_{IV}}{2\Delta_{IV}} \left[A_{IV}^{-1} C_{IV} + I \right] X_j^{M-1} \Bigg]\Bigg\}\Bigg]
\end{aligned}
\tag{6.11}
$$

and

$$
\begin{aligned}
K_M = \Bigg\{ & \frac{\beta\Psi^M H_M}{2\Delta_M}\left[-A_M^{-1} B_M + A_M^{-1} C_M P_j^{M-1} + P_j^{M-1}\right] \\
& + \left(\beta\Psi^M J_M + \eta \right) \Bigg\} M_M^{-1}
\end{aligned}
$$

$$
\begin{aligned}
L_M = \Bigg\{\Bigg[& (\beta J_{IV} + \eta) + \frac{\beta H_{IV}}{2\Delta_{IV}} C_{IV}^{-1} B_{IV} \Bigg] - \sum_{M=I}^{III} K_M \left[\frac{\eta H_{IV}}{2\Delta_{IV}} C_{IV}^{-1} B_{IV} \right. \\
& \left. + (\eta J_{IV} + \beta) \right] \Bigg\}
\end{aligned}
\tag{6.12}
$$

where

$$
M_M = \frac{\eta H_M}{2\Delta_M}\left[-A_M^{-1} B_M + A_M^{-1} C_M P_j^{M-1} + P^{M-1}\right] + (\eta J_M + \beta\Psi^M)
$$

For the remaining branch segments (i. e., M = I, II, III), the following recursion formula is used:

$$
Z_j^M = Q_j^M Z_j^{IV} + P_j^M Z_{j+1}^{IV} + X_j^M \tag{6.13}
$$

where

$$Q_j^M = M_M^{-1} \left\{ \frac{\eta H_{IV}}{2\Delta_{IV}} C_{IV}^{-1} B_{IV} + (\eta J_{IV} + \beta) \right\}$$

$$P_j^M = M_M^{-1} \left\{ \frac{\eta H_{IV}}{2\Delta_{IV}} \left[I + C_{IV}^{-1} A_{IV} \right] \right\}$$

$$X_j^M = M_M^{-1} \left\{ - \eta \frac{H_{IV}}{2\Delta_{IV}} C_{IV}^{-1} g_{IV} - \eta \frac{H_M}{2\Delta_M} A_M^{-1} \cdot g_M \right.$$
$$\left. + \frac{\eta H_M}{2\Delta_M} \left[A_M^{-1} C_M + I \right] X_j^{M-1} \right\}$$

and (M_M) is given by Equation 6. 12.

Thus, from a knowledge of P_{j-1}^{I}, P_{j-1}^{II}, P_{j-1}^{III} - - -, P_{j-1}^{N-1} and X_{j-1}^{I}, X_{j-1}^{II}, - - -, X_{j-1}^{N-1}, the calculation can proceed directly to the determination of the Nth shell region, P_j^N, X_j^N, and then to the boundary of branch N in the standard fashion.

SUMMARY

An improved numerical procedure has been presented for the solution of the shell equations. The procedure is general and can be applied to a variety of shell problems. The report presents improvements to Reference 2 in handling boundary and discontinuity conditions.

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