The Electrical Conductivity of a Collisionless Magnetoplasma in a Weakly Turbulent Magnetic Field

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ABSTRACT

The behavior of test particles in the presence of uniform, parallel, and steady electric and magnetic fields in addition to a weakly turbulent magnetic field is studied. The quasi-linear approach is used to find a diffusion equation for the distribution function describing the test particles; the diffusion equation describes the pitch-angle scattering experienced by the test particles due to the random fluctuations of the magnetic field.

In the weak-electric-field limit the equation is analyzed in detail and an expression for the electrical conductivity in terms of the correlation tensor of the fluctuating field is found. Finally, for the case of a particular model of the turbulence and a Maxwellian distribution of test particles, an explicit formula for the conductivity is given.

I. INTRODUCTION

It has been shown in recent works[1,2] that test particles in the presence of turbulent electromagnetic fields exhibit such behavior as acceleration and diffusion. It is therefore evident that turbulent fields will influence the transport properties of a plasma, which are usually determined solely by binary collisions. In many plasmas which arise in the study of astrophysics the density of particles is so small that the effects of binary collisions may be completely overshadowed by those of the fluctuating fields. Hence it would be useful to obtain formulas for such transport properties as viscosity, electrical conductivity, and thermal conductivity in collisionless turbulent plasmas. Indeed, the viscosity for such a plasma has recently been calculated in a heuristic fashion by Tsuda.[3]

Of the other two quantities, that which lends itself more readily to calculation is the electrical conductivity since it does not require consideration of inhomogeneities as would the thermal conductivity. We therefore consider a homogeneous plasma subject to uniform, constant, and parallel electric and magnetic fields and, in addition, a weakly turbulent magnetic field. We shall find an equation governing the distribution function for each particle species in the presence of these fields, and we shall analyze this equation in the static, weak-electric-field limit which will yield a formula for the electrical conductivity in terms of the properties of the turbulent plasma in the absence of the electric field. Finally, an explicit formula for the conductivity will be given for a specific model of the turbulence in the small-gyroradius limit.

We employ the quasi-linear approach developed by Hall and Sturrock,[2] relevant results of which we now state briefly: We assume the particles of one species to be described by a distribution function $F(X_\mu, t)$ giving their density in phase space. $X_\mu (\mu = 1,2,\ldots,6)$ stands six orthogonal coordinates which specify position in phase space. A particle at the point X_μ at time t obeys the equation of motion

$$\frac{dX_\mu}{dt} = G_\mu(X_\nu, t) + g_\mu(X_\nu, t) \tag{1.1}$$

where G_μ describes the effect on the motion of the large-scale, steady fields and g_μ the effects of the turbulent magnetic field. We next consider an ensemble of distribution functions F, all of which are identical at some initial time $t = t_o$ but which are subject to different realizations of the fields g_μ thereafter, so that at any time $t > t_o$ the distribution functions for different members of the ensemble will have different values. We need an equation for $\langle F \rangle$, where $\langle \rangle$ denotes an ensemble average. Under the assumptions that $\langle g_\mu \rangle = 0$ and that there is a time scale T satisfying

$$T_c \ll T \ll \frac{\langle F \rangle}{\left| g_\mu \frac{\partial F}{\partial X_\mu} \right|} \tag{1.2}$$

where T_c is the correlation time for the turbulent magnetic field, we find that, to lowest order in g_μ, $\langle F \rangle$ obeys the following equation:

$$\frac{\partial \langle F \rangle}{\partial t} + G_\mu \frac{\partial \langle F \rangle}{\partial X_\mu} = \frac{1}{h} \frac{\partial}{\partial X_\mu} \left[h D_{\mu\nu} \frac{\partial \langle F \rangle}{\partial X_\nu} \right] \tag{1.3}$$

where h is the determinant of the metric tensor in phase space for the chosen coordinate system. $D_{\mu\nu}$ is given by

$$D_{\mu\nu}(X_\alpha, t) = \int_{t_o}^{t} \langle g_\mu(X_\alpha, t)\, g_\nu(X'_\alpha(t'), t') \rangle\, dt' \ , \tag{1.4}$$

$X'_\alpha(t')$ being the position of a particle in phase space at time t' on the unperturbed orbit (i.e., the orbit calculated with $g_\mu = 0$) which passes through the point X_α at time t. The integration must be carried out in a coordinate system which has the property that

$$\frac{\partial G_\mu}{\partial t} + G_\alpha \frac{\partial G_\mu}{\partial X_\alpha} = 0 \ . \tag{1.5}$$

We may now apply the results to our particular case.

II. A HOMOGENEOUS IN PARALLEL ELECTRIC AND MAGNETIC FIELDS AND A WEAKLY TURBULENT MAGNETIC FIELD

We assume a homogeneous plasma with the uniform fields along the z-axis. Hence the total electric and magnetic fields are given by

$$\left.\begin{aligned} \vec{E} &= E\hat{z} \\ \vec{B} &= B_o\hat{z} + \vec{\delta B} \end{aligned}\right\} \tag{2.1}$$

The turbulent field is assumed to be statistically homogeneous and steady so that the correlation function

$$R_{\alpha\beta}(\vec{x},t;\vec{x}+\vec{\rho},t+\tau) = \left\langle \delta B_\alpha(\vec{x},t)\ \delta B_\beta(\vec{x}+\vec{\rho},t+\tau)\right\rangle,\ \alpha,\beta = x,y,z, \tag{2.2}$$

may be written simply as $R_{\alpha\beta}(\vec{\rho},\tau)$.

We restrict outselves to a nonrelativistic calculation, so that the equation of motion (1.1) of a test particle of charge q and mass m in the presence of these fields are given by

$$\frac{d\vec{x}}{dt} = \frac{\vec{p}}{m}, \tag{2.3}$$

$$\frac{d\vec{p}}{dt} = q\vec{E} + \vec{p}\times\vec{\Omega} + \frac{\Omega}{B_o}\vec{p}\times\delta\vec{B}, \tag{2.4}$$

where $\vec{\Omega} = \dfrac{q\vec{B}_o}{mc}$. (2.5)

Rewriting (2.4) in component form, we find

$$\left.\begin{aligned}
\frac{dp_x}{dt} &= \Omega p_y + \frac{\Omega}{B_o}\,(p_y\delta B_z - p_z\delta B_y)\ , \\
\frac{dp_y}{dt} &= -\Omega p_x + \frac{\Omega}{B_o}\,(p_z\delta B_x - p_x\delta B_z)\ , \\
\frac{dp_z}{dt} &= qE + \frac{\Omega}{B_o}\,(p_x\delta B_y - p_y\delta B_z)\ .
\end{aligned}\right\} \qquad (2.6)$$

Solving (2.6) with $\delta\vec{B} = 0$ gives the momentum along the unperturbed orbit $\vec{p}\,'(t')$. Requiring that $\vec{p}\,'(t') = \vec{p}$ when $t = t'$, we find

$$\left.\begin{aligned}
p_x'(t') &= p_\perp \cos\bigl(\phi + \Omega(t - t')\bigr), \\
p_y'(t') &= p_\perp \sin\bigl(\phi + \Omega(t - t')\bigr), \\
p_z'(t') &= p_z + qE(t' - t)\ ,
\end{aligned}\right\} \qquad (2.7)$$

or

$$\left.\begin{aligned}
p_z'(t') &= p_z + qE(t'-t)\ , \\
p_\perp'(t') &= p_\perp\ , \\
\phi'(t') &= \phi + \Omega(t - t')\ ,
\end{aligned}\right\} \qquad (2.8)$$

where ϕ is the azimuthal angle in momentum space. From (2.8) and (2.3) we then find, requiring $\vec{x}\,'(t') = \vec{x}$ at $t' = t$, that

$$\left.\begin{aligned} x'(t') &= x_g^o - r_g \sin\big(\emptyset + \Omega(t - t')\big), \\ y'(t') &= y_g^o + r_g \sin\big(\emptyset + \Omega(t - t')\big), \\ z'(t') &= z + \frac{p_z}{m}(t' - t) + \frac{qE}{2m}(t' - t)^2 , \end{aligned}\right\} \tag{2.9}$$

where $r_g = \dfrac{p_\perp}{m\Omega}$. (2.10)

The assumption of homogeneity allows us to consider momentum space diffusion alone, so that we are left with only three phase space coordinates of interest; we denote the coordinate p_z by $\mu = 1$, $p_\perp$ by $\mu = 2$, and $\emptyset$ by $\mu = 3$. Then we find that

$$\left.\begin{aligned} \frac{dp_z}{dt} &= qE + \frac{\Omega p_\perp}{B_o}\left[\delta B_y \cos\emptyset - \delta B_x \sin\emptyset\right] , \\ \frac{dp_\perp}{dt} &= \frac{\Omega p_z}{B_o}\left[\delta B_x \sin\emptyset - \delta B_y \cos\emptyset\right] , \\ \frac{d\emptyset}{dt} &= -\Omega - \frac{\Omega}{B_o}\delta B_z + \frac{\Omega p_z}{B_o p_\perp}(\delta B_x \cos\emptyset - \delta B_y \sin\emptyset) . \end{aligned}\right\} \tag{2.11}$$

Hence

$$\left.\begin{aligned} G_1 &= qE , \\ G_2 &= 0 , \\ G_3 &= -\Omega , \end{aligned}\right\} \tag{2.12}$$

so we see that for these coordinates the condition of equation (1.5) is satisfied. Defining δB_θ by

$$\delta B_\theta = \delta B_x \sin\emptyset - \delta B_y \cos\emptyset , \tag{2.13}$$

we find

$$g_1 = -\frac{\Omega p_\perp}{B_o}\,\delta B_\theta$$

and

$$g_2 = \frac{\Omega p_z}{B_o}\,\delta B_\theta \tag{2.14}$$

Proceeding as in the paper by Hall and Sturrock[2] for the case of a uniform magnetic field, we suppress the coordinate $\emptyset$, concerning ourselves with only the phase-independent part of $\langle F \rangle$. Dropping the brackets, noting that $h = p_\perp$, and calling the phase independent part $\bar{F}$, equation (1.3) becomes

$$\frac{\partial \bar{F}}{\partial t} + qE\,\frac{\partial \bar{F}}{\partial p_z} = \frac{\partial}{\partial p_z}\left[\bar{D}_{11}\,\frac{\partial \bar{F}}{\partial p_z} + \bar{D}_{12}\,\frac{\partial \bar{F}}{\partial p_\perp}\right]$$

$$+ \frac{1}{p_\perp}\,\frac{\partial}{\partial p_\perp}\left[p_\perp\,\bar{D}_{21}\,\frac{\partial \bar{F}}{\partial p_z} + p_\perp\,\bar{D}_{22}\,\frac{\partial \bar{F}}{\partial p_\perp}\right], \tag{2.15}$$

where now

$$\bar{D}_{\mu\nu}(p_z, p_\perp, t) = \frac{1}{2\pi}\int_o^{2\pi} d\emptyset \int_{t_o}^{t} \langle g_\mu(X_\alpha, t)\; g_\nu[X'_\alpha(t'), t']\rangle\, dt'; \quad \mu, \nu = 1, 2. \tag{2.16}$$

These formulas may be written more explicitly as

$$\left.\begin{aligned}
\bar{D}_{11} &= \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} p_\perp p_\perp'(t') \langle \delta B_\theta(\vec{x},t)\, \delta B_\theta(\vec{x}'(t'),t') \rangle dt' \\
&= p_\perp^2 \nu \; ; \\
\bar{D}_{21} &= - \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} p_z p_\perp'(t') \langle \delta B_\theta(\vec{x},t)\, \delta B_\theta(\vec{x}'(t'),t') \rangle dt' \\
&= -p_z p_\perp \nu \; ; \\
\bar{D}_{12} &= - \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} p_\perp p_z'(t') \langle \delta B_\theta(\vec{x},t)\, \delta B_\theta(\vec{x}'(t') \rangle dt' \\
&= -p_z p_\perp \nu - qE\triangle \; ; \\
\text{and} & \\
\bar{D}_{22} &= \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} p_z p_z'(t') \langle \delta B_\theta(\vec{x},t)\, \delta B_\theta(\vec{x}'(t'),t') \rangle dt' \\
&= p_z^2 \nu + qE\triangle \; ;
\end{aligned}\right\} \quad (2.17)$$

where we have introduced the following definitions:

$$\left.\begin{aligned}
\nu &= \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} \langle \delta B_\theta(\vec{x},t)\, \delta B_\theta(\vec{x}'(t'),t') \rangle dt' \\
\text{and} & \\
\triangle &= \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} (t' - t) \langle \delta B_\theta(\vec{x},t)\, \delta B_\theta(\vec{x}(t'),t') \rangle dt' \; .
\end{aligned}\right\} \quad (2.18)$$

Thus equation (2.15) becomes

$$\frac{\partial \overline{F}}{\partial t} + qE\frac{\partial \overline{F}}{\partial p_z} = \left(p_\perp \frac{\partial}{\partial p_z} - p_z \frac{\partial}{\partial p_\perp} - \frac{p_z}{p_\perp}\right)\left[\nu\left(p_\perp \frac{\partial \overline{F}}{\partial p_z} - p_z \frac{\partial \overline{F}}{\partial p_\perp}\right) - qE\Delta \frac{\partial \overline{F}}{\partial p_\perp}\right], \tag{2.19}$$

which, upon changing variables from p_z and $p_\perp$ to μ and p defined by

$$\mu = \frac{p_z}{\sqrt{p_z^2 + p_\perp^2}} \ , \quad p = \sqrt{p_z^2 + p_\perp^2} \ , \tag{2.20}$$

becomes

$$\frac{\partial \overline{F}}{\partial t} + qE\left[\mu \frac{\partial \overline{F}}{\partial p} + \frac{(1-\mu^2)}{p}\frac{\partial \overline{F}}{\partial \mu}\right]$$

$$= \frac{\partial}{\partial \mu}\left[(1-\mu^2)\nu \frac{\partial \overline{F}}{\partial \mu}\right] - qE\frac{\partial}{\partial \mu}\left[(1-\mu^2)\ \Delta\left(\frac{\partial \overline{F}}{\partial p} - \frac{\mu}{p}\frac{\partial \overline{F}}{\partial \mu}\right)\right] . \tag{2.21}$$

This equation is valid for any magnitude of E. However, to analyze this equation in general appears quite formidable. We therefore investigate the equation in the weak-electric-field limit. We assume that $\overline{F}$ takes the form

$$\overline{F} \approx F_o + F_1 qE + O(q^2E^2) \ . \tag{2.22}$$

We also look for an equilibrium solution such that $\frac{\partial \overline{F}}{\partial t} = 0$. However, this condition will be satisfied only approximately since some heating of the plasma will occur. F_o satisfies the equation found by Sturrock and Hall[2] with no electric field, which has the consequence that, since $\overline{F}$ is assumed stationary, F_o is necessarily isotropic. We assume further that F_1 is proportional to $P_1(\mu)$. Now ν and Δ depend on E, since the unperturbed orbits depend on E. So we expand ν and Δ in powers of qE to obtain

$$\nu = \nu_o(p,\mu) + \nu_1(p,\mu)\ qE + O(q^2E^2)$$

and (2.23)

$$\Delta = \Delta_o(p,\mu) + \Delta_1(p,\mu)\ qE + O(q^2E^2)\ .$$

Using (2.22) and (2.23) in (2.21), assuming $\frac{\partial \overline{F}}{\partial t} = 0$ and hence that F_o is isotropic, we find, after dropping terms of second and higher order in qE, that

$$qE\mu \frac{\partial F_o}{\partial p} \approx \frac{\partial}{\partial \mu}\left[(1-\mu^2)\nu_o \frac{\partial F_1}{\partial \mu}\right] qE - \frac{\partial}{\partial \mu}\left[(1-\mu^2)\ \Delta_o \frac{\partial F_o}{\partial p}\right] qE\,. \quad (2.24)$$

Taking

$$F_1(p,\mu) = f_1(p)\mu\ , \quad (2.25)$$

we obtain the equation

$$\mu \frac{\partial F_o}{\partial p} = f_1(p) \frac{\partial}{\partial \mu}\left[(1-\mu^2)\nu_o\right] - \frac{\partial}{\partial \mu}\left[(1-\mu^2)\Delta_o\right]\frac{\partial F_o}{\partial p}\ . \quad (2.26)$$

On multiplying this equation by μ and integrating from -1 to 1, we obtain

$$\left[\frac{2}{3} - \int_{-1}^{1} (1-\mu^2)\Delta_o(p,\mu)d\mu\right] \frac{\partial F_o}{\partial p} = -f_1(p)\int_{-1}^{1} (1-\mu^2)\Delta_o(p,\mu)\ d\mu\ . \quad (2.27)$$

Hence we arrive at the following expression for $f_1(p)$:

$$f_1(p) = -\frac{1-\gamma(p)}{\pi(p)}\ , \quad (2.28)$$

where

$$\left.\begin{aligned} \gamma(p) &= \frac{3}{2}\int_{-1}^{1} (1-\mu^2)\Delta_o(p,\mu)d\mu = 2\Delta_{oo} - \frac{2}{5}\Delta_{02} \\ \text{and} \qquad & \\ \pi(p) &= \frac{3}{2}\int_{-1}^{1} (1-\mu^2)\nu_o(p,\mu)d\mu = 2\nu_{oo} - \frac{2}{5}\nu_{02} , \end{aligned}\right\} \qquad (2.29)$$

and ν_{on} and Δ_{on} are the coefficients of the nth order Legendre polynomials in the expansion of ν_o and Δ_o, respectively, in spherical harmonics:

$$\left.\begin{aligned} \nu_o(p,\mu) &= \sum_{n=o}^{\infty} \nu_{on}(p)P_n(\mu) , \\ \Delta_o(p,\mu) &= \sum_{n=o}^{\infty} \Delta_{on}(p)P_n(\mu) . \end{aligned}\right\} \qquad (2.30)$$

Now

$$\langle p_z \rangle = \frac{1}{N}\int p_z \, F(\vec{p})d^3\vec{p} , \qquad (2.31)$$

where N is the mean particle number density. In spherical coordinates, this becomes

$$\langle p_z \rangle = 2\pi N^{-1}\int_o^{\infty} p^3 \, dp \int_{-1}^{1} \overline{F}(p,\mu)\mu d\mu . \qquad (2.32)$$

On using (2.22) and (2.25), this becomes

$$\langle p_z \rangle = \frac{4\pi}{3} qEN^{-1} \int_o^\infty f_1(p) p^3 dp . \tag{2.33}$$

Equation (2.28) enables us to write this expression as

$$\langle p_z \rangle = - \frac{4\pi}{3} qEN^{-1} \int_o^\infty \frac{1-\gamma(p)}{\pi(p)} \frac{\partial F_o}{\partial p} p^3 dp . \tag{2.34}$$

Since the electrical conductivity (in e.s.u.) is given by

$$\sigma = \frac{Nq \langle p_z \rangle}{mE} , \tag{2.35}$$

we finally obtain the formula

$$\sigma = - \frac{4\pi q^2}{3m} \int_o^\infty \frac{1-\gamma(p)}{\pi(p)} \frac{\partial F_o}{\partial p} p^3 dp , \tag{2.36}$$

which may be written alternatively as

$$\sigma = \frac{4\pi q^2}{3m} \int_o^\infty \frac{1}{p^2} \frac{\partial}{\partial p} \left[p^3 \frac{1-\gamma(p)}{\pi(p)} \right] F_o(p) p^2 dp . \tag{2.37}$$

Equation (2.37) is interesting in that it shows how various "spherical shells" of particles contribute to the conductivity, but equation (2.36) is the more useful form.

The total conductivity is to be obtained, of course, by summing the contributions from all particle species.

III. THE CONDUCTIVITY IN THE CASE OF A GAUSSIAN CORRELATION FUNCTION AND A MAXWELLIAN PARTICLE DISTRIBUTION

From (2.36), (2.29), (2.18), (2.13), and (2.2) we see that the electrical conductivity is determined by the correlation functions $R_{\alpha\beta}(\vec{x}-\vec{x}'(t'),\ t-t')$, where α and β can take on the indicial values x and y. If the turbulence is to be statistically axially symmetric about the z-axis, the most general form for the correlation tensor $R_{\alpha\beta}$ can be shown[4] to be

$$R_{\alpha\beta}(\vec{\rho},t) = \begin{bmatrix} a\xi^2+b & a\xi\eta \\ a\xi\eta & a\eta^2+b \end{bmatrix}, \tag{3.1}$$

where the components of $\vec{\rho}$ along the x, y, and z axes are ξ, η, and ζ respectively, and a and b are functions of $\sqrt{\xi^2+\eta^2}$, ζ, and τ. If δB_x and δB_y are uncorrelated, we have an "isotropic" correlation tensor for which $a = 0$, so that

$$\left.\begin{aligned} R_{xx} &= R_{yy} = b\ , \\ R_{xy} &= R_{yx} = 0\ . \end{aligned}\right\} \tag{3.2}$$

A simple yet physically interesting form for b is a Gaussian distribution:

$$b(\vec{\rho},\tau) = \delta^2 B_o^{\ 2} \exp\left[-\frac{\rho^2}{2L_c^{\ 2}}\right] \exp\left[-\frac{\tau^2}{2T_c^{\ 2}}\right] \tag{3.3}$$

where L_c is the correlation length, T_c the correlation time and

$$\delta^2 = \frac{\langle \delta B_x^{\ 2}\rangle}{B_o^{\ 2}} = \frac{\langle \delta B_y^{\ 2}\rangle}{B_o^{\ 2}}\ . \tag{3.4}$$

In this case we have, from (2.18),

$$\left.\begin{aligned} \nu &= \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} b(\vec{x}-\vec{x}'(t'), t-t') \cos\left(\emptyset-\emptyset'(t')\right) dt' , \\ \triangle &= \frac{\Omega^2}{2\pi B_o^2} \int_o^{2\pi} d\emptyset \int_{t_o}^{t} (t'-t)\, b(\vec{x}-\vec{x}'(t'), t-t') \cos\left(\emptyset-\emptyset'(t') \quad dt\right) . \end{aligned}\right\} \tag{3.5}$$

From (2.9) we see that

$$\left.\begin{aligned} x-x'(t') &= 2r_g \sin \tfrac{1}{2}\, \Omega\tau \cos(\emptyset + \tfrac{1}{2}\, \Omega\tau) , \\ y-y'(t') &= 2r_g \sin \tfrac{1}{2}\, \Omega\tau \sin(\emptyset + \tfrac{1}{2}\, \Omega\tau) , \\ z-z'(t') &= -\frac{p_z}{m}\, \tau + \frac{qE}{2m}\, \tau^2 ; \end{aligned}\right\} \tag{3.6}$$

where $\tau = t-t'$. Now ν_o and $\triangle_o$ are the formulas for ν and $\triangle$ with $E = O$, so that

$$\left.\begin{aligned} \nu_o &= \frac{\delta^2 q^2 B_o^2}{2\pi m^2 c^2} \int_o^{2\pi} d\emptyset \int_o^{t-t_o} d\tau \cos \Omega\tau \exp - \left[\frac{2r_g^2}{L_c^2} \sin^2 \frac{1}{2}\, \Omega\tau + \frac{p_z^2}{2m^2 L_c^2}\, \tau^2 + \frac{\tau^2}{2T_c^2}\right], \\ a_o &= \frac{\delta^2 q^2 B_o^2}{2\pi m^2 c^2} \int_o^{2\pi} d\emptyset \int_o^{t-t_o} \tau d\tau \cos \Omega\tau \exp - \left[\frac{2r_g^2}{L_c^2} \sin^2 \frac{1}{2}\, \Omega\tau + \frac{p_z^2}{2m^2 L_o^2}\, \tau^2 + \frac{\tau^2}{2T_c^2}\right]. \end{aligned}\right\} \tag{3.7}$$

For $t-t_o \gg T_c$ we may extend the upper limits of the above integral to infinity and so obtain

$$\nu_o(p,\mu) = \sqrt{\frac{\pi}{2}}\, \frac{\delta^2 q^2 B_o^2 T_c}{m^2 c^2} \left(1 + \frac{p^2 \mu^2}{p_o^2}\right)^{-\frac{1}{2}} e^{-r_g^2/L_c^2} \sum_{n=\infty}^{\infty} I_n\left(\frac{r_g^2}{L_c^2}\right) \exp - \frac{(n+1)^2 \Omega^2 T_c^2}{2\left(1 + \frac{p^2 \mu^2}{p_o^2}\right)} , \tag{3.8}$$

and

$$a_o(p,\mu) = \frac{\delta^2 q^2 B_o^2 T_c^2}{m^2 c^2}\left(1 + \frac{p^2\mu^2}{p_o^2}\right)^{-1}, \tag{3.9}$$

where

$$p_o = \frac{mL_c}{T_c} . \tag{3.10}$$

For a very weakly turbulent field, $\delta \ll 1$, and, in consequence, $a_o \ll 1$ so that we may neglect $\gamma(p)$ in equations (2.36) and (2.37). In the case that, for all moments which contribute significantly to F_o, $\frac{r_g^2}{L_c^2} \ll 1$, i.e., in the small gyroradius limit, we may use the following expression for ν_o instead of the more complicated (3.8) in evaluating the conductivity by equation (2.36):

$$\nu_o(p,\mu) \approx \sqrt{\frac{\pi}{2}}\ \frac{\delta^2 q^2 B_o^2 T_c}{m^2 c^2}\left(1 + \frac{p^2\mu^2}{p_o^2}\right)^{-1/2} \exp -\left[\frac{\Omega^2 T_c^2}{2\left(1 + \frac{p^2\mu^2}{p_o^2}\right)}\right] \tag{3.11}$$

If we now assume a Maxwellian momentum distribution corresponding to a temperature T for F_o,

$$F_o(p) = N(2\pi mkT)^{-3/2} \exp -\left[\frac{p^2}{2mkT}\right], \tag{3.12}$$

and define the quantity

$$C(v;w) = \frac{3}{2}\int_{-1}^{1} \frac{(1-\mu^2)d\mu}{(1+w^2\mu^2)^{1/2}} \exp -\left[\frac{v^2}{2(1+w^2\mu^2)}\right], \tag{3.13}$$

we find, from (2.36) and (2.29), the following formula for the electrical conductivity:

$$\sigma = \frac{8\sqrt{2}\,Nmc^2}{3\pi\,\delta^2\,B_o^2\,T_c}\int_o^\infty \frac{x^4\,e^{-x^2}\,dx}{C\left(\Omega^2 T_c^2;\ \frac{x^2}{x_o^2}\right)}\ , \tag{3.14}$$

where

$$x_o^2 = \frac{p_o^2}{2mkT} = \frac{mL_c^2}{2kTT_c^2} \tag{3.15}$$

As noted above, we must have $r_g^2/L_c^2 \ll 1$ for all momenta contributing to F_1 in (3.12) in order for this analysis to be valid. From (2.10), this condition obtains if

$$\frac{2kT}{m\Omega^2 L_c^2} = \frac{1}{\Omega^2 T_c^2 x_o^2} \ll 1 \tag{3.16}$$

or, equivalently,

$$\Omega^2 T_c^2 x_o^2 \gg 1\ . \tag{3.17}$$

Thus we see that σ depends on the density of carriers, the carrier gyrofrequency, the mass of the carriers, the magnitude of the ambient magnetic field, the degree of turbulence, the correlation time and length, and also the temperature of the plasma.

For $x_o^2 \gg 1$, i.e. for sufficiently small temperature only the range $x \ll x_o$ contributes to (3.14), so that we may approximate C by

$$C_o = C(\Omega^2 T_c^2;\ 0) = 2e^{-\frac{1}{2}\Omega^2 T_c^2} \tag{3.18}$$

to obtain

$$\sigma = \frac{4\sqrt{2}\,Nmc^2}{3\pi\,\delta^2\,B_o^{\,2}\,T_c}\,e^{\frac{1}{2}\Omega^2 T_c^{\,2}}\int_o^\infty x^4 e^{-x^2}\,dx\ , \tag{3.19}$$

i.e.

$$\sigma = \frac{Nmc^2}{(2\pi)^{\frac{1}{2}}\,\delta^2\,B_o^{\,2}\,T_c}\,e^{\frac{1}{2}\Omega^2 T_c^{\,2}}\ . \tag{3.20}$$

It is interesting to note that, in this case, the conductivity of a plasma has a larger contribution from ions than from electrons. Even in the general case given by (3.14), for which the condition $x_o^{\,2} \gg 1$ does not necessarily hold (such as the solar wind in the neighborhood of the earth), the ions are still the principal contributors to the conductivity. The implications and applications of this analysis, including a study of the equation (3.14), which must be used for many cases of astrophysical interest, will be given at a later date.

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