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SID 67-498

STUDY OF APOLLO WATER IMPACT

FINAL REPORT

VOLUME I

HYDRODYNAMIC ANALYSIS OF APOLLO WATER IMPACT

(Contract NAS9-4552, G.O. 5264)

MAY 1967

Prepared by

T. Li

T. Sigimura

(Authors)

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MAY 31 1967

MANNED SPACECRAFT CENTER

HOUSTON, TEXAS

Approved by

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Assistant Manager

Science and Technology

NORTH AMERICAN AVIATION, INC.

SPACE DIVISION

N68-19743 (ACCESSION NUMBER)

(THRU)

60 (PAGES)

1 (CODE)

NASA-CR-92019 (NASA CR OR TMX OR AD NUMBER)

32 (CATEGORY)

FACILITY FORM 602

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FOREWORD

This report was prepared by North American Aviation, Inc., Space Division, under NASA Contract NAS9-4552, for the National Aeronautics and Space Administration, Manned Space Flight Center, Houston, Texas, with Dr. F.C. Hung, Program Manager and Mr. P.P. Radkowski, Assistant Program Manager. This work was administered under the direction of Structural Mechanics Division, MSC, Houston, Texas with Dr. F. Stebbins as the technical monitor.

This report is presented in eleven volumes for convenience in handling and distribution. All volumes are unclassified.

The objective of the study was to develop methods and Fortran IV computer programs to determine by the techniques described below, the hydroelastic response of representation of the structure of the Apollo Command Module immediately following impact on the water. The development of theory, methods and computer programs is presented as Task I Hydrodynamic Pressures, Task II Structural Response and Task III Hydroelastic Response Analysis.

Under Task I - Computing program to extend flexible sphere using the Spencer and Shiffman approach has been developed. Analytical formulation by Dr. Li using nonlinear hydrodynamic theory on structural portion is formulated. In order to cover a wide range of impact conditions, future extensions are necessary in the following items:

a. Using linear hydrodynamic theory to include horizontal velocity and rotation.

b. Nonlinear hydrodynamic theory to develop computing program on spherical portion and to develop nonlinear theory on toroidal and conic sections.

Under Task II - Computing program and User's Manual were developed for nonsymmetrical loading on unsymmetrical elastic shells. To fully develop the theory and methods to cover realistic Apollo configuration the following extensions are recommended:

a. Modes of vibration and modal analysis.

b. Extension to nonsymmetric short time impulses.

c. Linear buckling and elasto-plastic analysis

These technical extensions will not only be useful for Apollo and future Apollo growth configurations, but they will also be of value to other aeronautical and spacecraft programs.

The hydroelastic response of the flexible shell is obtained by the numerical solution of the combined hydrodynamic and shell equations. The results obtained herein are compared numerically with those derived by neglecting the interaction and applying rigid body pressures to the same elastic shell. The numerical results show that for an axially symmetric impact of the particular shell studied, the interaction between the shell and the fluid produces appreciable differences in the overall acceleration of the center of gravity of the shell, and in the distribution of the pressures and responses. However the maximum responses are within 15% of those produced when the interaction between the fluid and the shell is neglected. A brief summary of results is shown in the abstracts of individual volumes.

The volume number and authors are listed on the following page.

INDEX FOR FINAL REPORT DRAFT

"Apollo Water Impact"

Volume No.	Volume Title	Authors
1	Hydrodynamic Analysis of Apollo Water Impact	T. Li and T. Sugimura
2	Dynamic Response of Shells of Revolution During Vertical Impact Into Water - No Interaction	A.P. Cappelli, and J.P.D. Wilkinson
3	Dynamic Response of Shells of Revolution During Vertical Impact Into Water - Hydroelastic Interaction	J.P.D. Wilkinson A.P. Cappelli, R.N. Salzman
4	Comparison With Experiments	J.P.D. Wilkinson
5	User's Manual - No Interaction	J.P.D. Wilkinson
6	User's Manual - Interaction	J.P.D. Wilkinson and R.N. Salzman
7	Modification of Shell of Revolution Analysis	A.P. Cappelli and S.C. Furuike
8	Unsymmetric Shell of Revolution Analysis	A.P. Cappelli, T. Nishimoto, P.P. Radkowski and K.E. Pauley
9	Mode Shapes and Natural Frequencies Analysis	A.P. Cappelli
10	User's Manual for Modification of Shell of Revolution Analysis	A.P. Cappelli and S.C. Furuike
11	User's Manual for Unsymmetric Shell of Revolution Analysis	E. Carrion, S.C. Furuike and T. Nishimoto

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ABSTRACT

The analysis presented in this volume is confined to the impact of the Apollo Command Module on water using a rigid sphere as a mathematical model. A general potential for the oblique impact is derived using a method developed by the first author of this volume. This method differs from the existing approaches in that the condition for the flow to be tangential to the varying wetted portion of the spherical surface is satisfied for all values of t, where t denotes time; while using the conventional method, the potential has to be calculated for each increment of time.

Particular series expansions are obtained in the case of vertical impact for the velocity of penetration, the free surface, and the pressure. These are power series in the depth of penetration. They are obtained from the nonlinear differential equations instead of from the linearized ones.

A computer feasibility study is included in this volume. No numerical calculation is attempted at this time. An analytical formulation has also been extended to the conic and toroidal section of the Command Module.

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CONTENTS

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ILLUSTRATIONS

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LIST OF SYMBOLS

$A_m^{(i)}, A_{mn}^{(i)}$	arbitrary functions of time
a_ν	coefficients in Equation (106)
B	nondimensional depth of penetration
$B^{(1)}, B^{(2)}$	arbitrary functions
b_ν	coefficients in Equation (106)
c	speed of sound
$C_n(\cdot\,\cdot)$	arbitrary functions
$C_\nu(\ell)$	coefficients of $C_1\ (\ell, t)$
$F\ (\omega, \beta)$	function defined by Equation (110)
$F\ (\alpha, \beta, \gamma, x)$	hypergeometric function
$f_\nu\ (\omega)$	coefficients in Equation (110)
g	gravity
g_0	gravity at earth sea level
g^*	gravity ratio
$g_\nu(\omega)$	coefficients defined by Equation (114)
I_2	modified Bessel function of the first kind
J_2	Bessel function of the first kind

K_2	modified Bessel function of the second kind
ℓ	dummy variable
$\overline{M}$	momentum
M	virtual mass
M_0	mass of impacting body
m	positive integer
P	dimensional pressure
P_o	initial dimensional pressure
p	nondimensional pressure
R	dimensional body-fixed radial coordinates
R_0	radius of the sphere
r	nondimensional body-fixed radial coordinates
V	dimensional velocity of penetration
V_0	initial dimensional velocity
$w_o(t)$	nondimensional velocity of penetration
$\overline{U}, \overline{V}, \overline{W}$	dimensional velocity components of the fluid
$u(\xi)$	function defined by Equation (97)
X, Y, Z	dimensional body-fixed rectangular coordinates
x, y, z	nondimensional body-fixed rectangular coordinates
$\overline{X}, \overline{Y}, \overline{Z}$	dimensional space-fixed rectangular coordinates
$\overline{x}, \overline{y}, \overline{z}$	nondimensional space-fixed coordinates

$x(B)$, $y(x)$	functions defined by Equation (94)
Y_2	Bessel function of the second kind
α	nondimensional constant defined by Equation (9)
α_ν	coefficients defined by Equation (120)
β	function of time defined by Equation (62)
θ	body-fixed meridan angle
ξ_1, ξ_2, ξ_3	generalized coordinates
$\bar{\rho}$	dimensional density
σ	function of ω defined by Equation (32)
ϕ	relative potential
$\bar{\phi}$	absolute potential
ϕ_0	classic potential for a sphere moving in an infinite medium
ϕ_1	function of θ defined by Equation (24)
ϕ_2, ϕ_3	functions defined by Equation (29)
$\phi_\nu(r, \omega)$	coefficients of ϕ
$\Phi_\nu(\omega)$	coefficients defined by Equation (111)
Φ_ν^*	coefficients defined by Equation (119)
χ_m	function defined by Equation (34)
ψ	function defined by Equation (24)
ψ_ν	coefficients defined by Equation (112)

ψ^*	coefficients defined by Equation (119)
ω	body-fixed cone half angle
$\psi^*_{n\mu}$	operator with respect to body-fixed coordinates
$\bar{\nabla}$	operator with respect to space-fixed coordinates

INTRODUCTION

The water impact problem has been of interest to the U. S. Navy for many, many years because of its concern to understand the physics of ship slamming, seaplane landing and torpedo launching. A large amount of money has been spent to finance both theoretical and experimental studies of this problem.

As a result of these studies, many papers have been written and published on water impact of bodies of simple geometry such as wedges, cones and spheres. These publications cover both the experimental and the theoretical aspects of the problem. A short list of papers is given in a review article by V.G. Szebehely (Applied Mechanics Reviews, ASME Journal, February 1960). In reviewing all the publications on this subject one can conclude that some progress has been made but that the main problem remains.

Recently the water impact problem has aroused the interest of the National Aeronautics and Space Administration and the spacecraft industry for the reason that a spacecraft, such as Apollo, is adapted to land mainly on water. Because of the high cost of the payload, it is extremely important that the engineers have a workable knowledge of the impact pressure during the first few microseconds, perhaps a few milliseconds after the impact, in order to achieve an economical and safe design.

To gain knowledge of pressure distribution of landing impact, experiments have been funded by the space agencies and data have been obtained. We know, however, that any instrumentation has a time lag. Furthermore, the instruments mounted near the body nose, which strikes the water first, are always knocked out of their positions. As a consequence, no records on impact pressure during the first few micro- to milliseconds have been obtained.

In order to obtain these important values, one has to resort to theoretical studies and to check the theoretical results against experimental data taken a few milliseconds after impact. In the case of concrete experimental verification, confidence in the theory is established and the theoretical pressure distribution for the first few micro- to milliseconds can be used for design.

As pointed out, however, by G. Birkhoff in his book on hydrodynamic stability, the water impact problem is one of the knottiest problems, with which neither mathematicians nor hydrodynamicists have had any luck. This indicates that the physics of the problem has not been fully understood, hence none of the approaches used so far to attack this problem is perfect.

The approach advanced in this report is based upon the assumption of impact of a sphere on an incompressible calm sea. Under the assumption of irrotational flow, a velocity potential is introduced which has to satisfy the following conditions:

1. The flow has to be tangential to the surface of the sphere.

2. The potential has to be dependent on the impact velocity.

3. The far away quiescent free surface has to be a part of the solution of the kinematic equation for the simple reason that the potential has to depend on the initial free surface.

4. After submergence, the solution has to reduce to the classical solution for a sphere moving with the velocity of penetration in an infinite medium.

Conditions 1 and 2 have been imposed by various authors, while 3 and 4 are introduced here for the first time.

As pointed out previously, the major obstacle to the solution of the water impact problem is the development of the mathematical theory. Many writers believe that any solution to this problem, even with the exclusion of Conditions 3 and 4, would reduce to a constant, since it concerns a boundary value problem with a closed boundary. However, one should note that this is only true when we limit our discussion to functions regular in all variables.

The solution presented in this report consists of two parts, both analytic in the angular coordinates but not in the radial coordinate. As the radial coordinate is concerned, one is a Laurent's series, the other is an analytic function of $\ell n r$ multiplied by the $-1/2$ power of r. Since the second part is not regular in r, our findings do not contradict the classic theory which states that the (regular) solution to a Neumann problem with a closed boundary is a constant.

This new solution contains a class of functions given as definite integrals. These functions solve problems for a sphere just as Bessel functions solve problems for a cylinder.

A second difficulty of the problem of water impact is the determination of the arbitrary functions contained in the general solution from the boundary conditions. It is clear that the four conditions listed in this INTRODUCTION are not sufficient to determine all the arbitrary functions. It is also clear this physical problem has a unique solution, but it is not clear what physical phenomena take place immediately before the impact.

It is true that a completely incompressible approach will lead to infinite initial impact pressure which is physically absurd. To avoid this it is advisable to take the compression wave into consideration. This is done in this report by demand that the potential function satisfies the wave equation near the first point of impact.

The results obtained from this procedure lead to solutions containing only one arbitrary constant and two arbitrary functions of the single variable of integration. These functions are fixed by momentum balancing.

The final potential, the free surface, the impact pressure and impact force are all in the form of power series of the depth of penetration, the coefficients of which can be evaluated from recursion formulae. A numerical procedure to compute these coefficients is outlined in this report. No calculation has been made at this time.

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I. MATHEMATICAL FORMULATION

The problem of a sphere impacting on water can best be treated by using nondimensionalized moving spherical coordinates. We will describe the coordinate system first, then give the fundamental equations of motion in these coordinates. Finally, we will discuss the boundary conditions for the potential thoroughly.

1. MOVING COORDINATES

Denote the dimensional coordinates of a point in space referred to a body-fixed coordinate system (moving system) by

$$X, \ Y, \ Z \ \text{(rectangular)}$$

$$R, \ \theta, \ \omega \ \text{(spherical)}$$

and those referred to a space fixed system by

$$\bar{X}, \ \bar{Y}, \ \bar{Z} \ \text{(rectangular)}$$

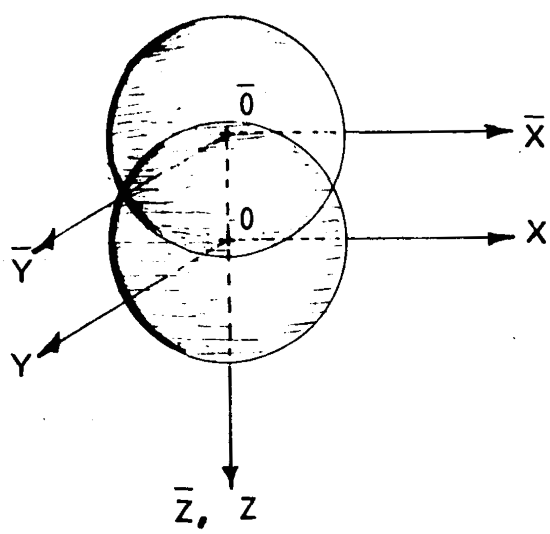

Figure 1. Coordinate System

We will assume that the moving system and the (space) fixed system coincide at time $T = 0$.

For our study, it is convenient to nondimensionalize all the quantities. Thus, we set for the coordinates

$$\bar{X} = R_0 \bar{x}, \ \bar{Y} = R_0 \bar{y}, \ \bar{Z} = R_0 \bar{z}$$

$$X = R_0 x, \ Y = R_0 y, \ Z = R_0 z \tag{1}$$

where R_0 is the radius of the sphere, and for the physical quantities

$$T = \frac{R_0}{V_0} t, \ V(t) = V_0 w_0(t), \ P = \bar{P}_0 p \tag{2}$$

where $V(t)$ is the velocity of the sphere $V_0 = V(0)$ is the initial impact velocity, $\bar{P}$ the dimensional pressure, $\bar{P}_0$ its initial value. It is clear that the nondimensional rectangular coordinates are related by

$$\bar{x} = x, \ \bar{y} = y, \ \bar{z} = z + \int_0^t w_0(t)dt \tag{3}$$

and the moving rectangular and spherical coordinates by

$$x = r\sin\omega\cos\theta, \ y = r\sin\omega\sin\theta, \ 2 = r\cos\omega \tag{4}$$

2. FUNDAMENTAL EQUATIONS

Under the assumption of irrotational flow one can assume a velocity potential ϕ which is nondimensional in the nondimensional system. For an incompressible fluid the dimensional continuity equation is

$$\frac{\partial \bar{U}}{\partial \bar{X}} + \frac{\partial \bar{V}}{\partial \bar{Y}} + \frac{\partial \bar{W}}{\partial \bar{Z}} = 0 \tag{5}$$

where $\bar{U}$, $\bar{V}$, and $\bar{W}$ are the dimensional velocity components referred to the space-fixed system. If ϕ represents the velocity potential for the relative motion in the body-fixed coordinate system, the continuity equation becomes

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r\sin\omega}\frac{1}{r}\frac{\partial}{\partial\omega}\left(\sin\omega\frac{\partial\phi}{\partial\omega}\right) + \frac{1}{r^2\sin^2\omega}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{6}$$

The dimensional momentum equations

$$\frac{\partial\overline{U}}{\partial\overline{T}} + \overline{U}\frac{\partial\overline{U}}{\partial\overline{X}} + \overline{V}\frac{\partial\overline{U}}{\partial\overline{Y}} + \overline{W}\frac{\partial\overline{U}}{\partial\overline{Z}} = -\frac{1}{\bar{\rho}}\frac{\partial\overline{P}}{\partial\overline{X}} \tag{7a}$$

$$\frac{\partial\overline{V}}{\partial\overline{T}} + \overline{U}\frac{\partial\overline{V}}{\partial\overline{X}} + \overline{V}\frac{\partial\overline{V}}{\partial\overline{Y}} + \overline{W}\frac{\partial\overline{V}}{\partial\overline{Z}} = -\frac{1}{\bar{\rho}}\frac{\partial\overline{P}}{\partial\overline{Y}} \tag{7b}$$

$$\frac{\partial\overline{W}}{\partial\overline{T}} + \overline{U}\frac{\partial\overline{W}}{\partial\overline{X}} + \overline{V}\frac{\partial\overline{W}}{\partial\overline{Y}} + \overline{W}\frac{\partial\overline{W}}{\partial\overline{Z}} = \frac{1}{\bar{\rho}}\frac{\partial\overline{P}}{\partial\overline{Z}} - g \tag{7c}$$

where $\bar{\rho}$ denotes the dimensional density, assume the following forms in the nondimensional moving rectangular coordinate system:

$$\frac{\partial}{\partial x}\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]\right\} = -\alpha\frac{\partial p}{\partial x} \tag{8a}$$

$$\frac{\partial}{\partial y}\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]\right\} = -\alpha\frac{\partial p}{\partial y} \tag{8b}$$

$$\frac{\partial}{\partial z}\left\{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right]\right\} = -\alpha\frac{\partial p}{\partial z} - \dot{w}_0(t) - g^* \tag{8c}$$

where

$$\alpha = \frac{\overline{P}_0}{\bar{\rho}V_0^2} = \left(\frac{c}{V_0}\right)^2 \tag{9}$$

is a nondimensional constant and

$$g_0 = g/g^* \tag{10}$$

is the gravity at earth-sea level, c, the velocity of sound in water.

Integrating Equations (8) one obtains the Bernoulli equation

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\} = -\alpha p - z\dot{w}_0(t) - g^*z \tag{11}$$

By virtue of the relations

$$r^2 = x^2 + y^2 + z^2 \tag{12a}$$

$$\tan\theta = y/x \tag{12b}$$

$$\tan\omega = \frac{1}{z}\left(x^2+y^2\right)^{\frac{1}{2}} \tag{12c}$$

one can write the Bernoulli equation in moving spherical coordinates as follows:

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}\left\{\left(\frac{\partial\phi}{\partial r}\right)^2+\frac{1}{r^2\sin^2\omega}\left(\frac{\partial\phi}{\partial\theta}\right)^2+\frac{1}{r^2}\left(\frac{\partial\phi}{\partial\omega}\right)^2\right\}$$
$$= -\alpha p - \left|\dot{w}_0(t) + g^*\right| r\cos\omega \tag{13}$$

3. PARTIAL DIFFERENTIAL EQUATION FOR THE FREE SURFACE

Let us assume, for simplicity, that the initial free water surface is quiescent immediately before the impact. After the impact, the disturbed or raised free surface is given in the moving spherical coordinates by

$$f(r, \theta, \omega, t) \equiv r - r(\theta, \omega, t) = 0 \tag{14}$$

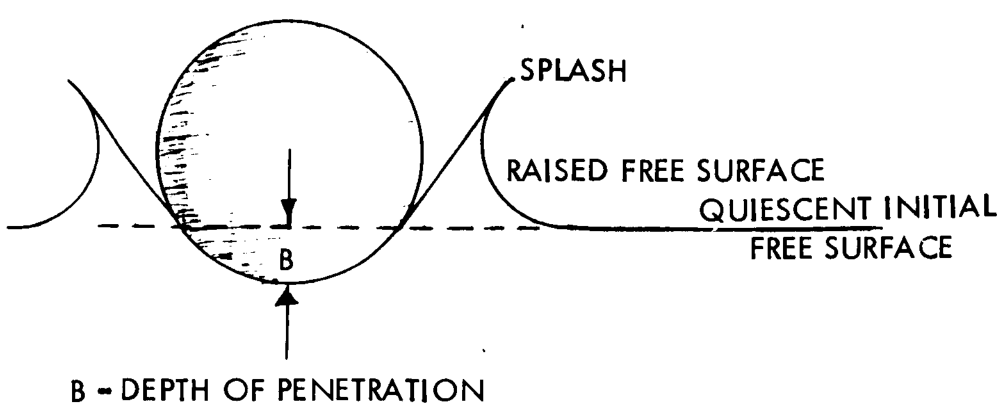

Figure 2. Free Surface

This surface has to satisfy the kinematic condition

$$\frac{Df}{Dt} = 0 \tag{15}$$

which, when expressed in terms of moving spherical coordinates, becomes

$$\frac{\partial \phi}{\partial r} - \frac{1}{r^2 \sin^2 \omega} \frac{\partial \phi}{\partial \theta} \frac{\partial r}{\partial \theta} - \left(\frac{1}{r} \frac{\partial \phi}{\partial \omega}\right) \frac{1}{r} \frac{\partial r}{\partial \omega} - \frac{\partial r}{\partial t} = 0 \tag{16}$$

Equation (16) asserts that, once a fluid particle reaches the free surface, it can only move on this surface. It is clear that the far away quiescent free surface is part of the free surface and has to satisfy Equation (16) in an asymptotic manner.

4. PRESSURE CONDITION

For an incompressible, irrotational fluid the Bernoulli equation (Equation (13)) determines the pressure for a given potential ϕ. This pressure is a function of location and time within the liquid mass. It reduces, however, to a constant at the free surface. This means that for r defined by Equation (14) and satisfying Equation (16) one will have

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} \left\{ \left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{1}{r^2 \sin^2 \omega} \left(\frac{\partial \phi}{\partial \theta}\right)^2 + \frac{1}{r^2} \left(\frac{\partial \phi}{\partial \omega}\right)^2 \right\}$$

$$+ \left(\dot{w}_0 (t) + g^*\right) r \cos\omega = \text{constant} \tag{17}$$

for $r = r(\theta, \omega, t)$ all θ, all ω and all t.

Equation (17) is called the pressure condition for the free surface. It will be used to determine the arbitrary function contained in the integral of the kinematic equation. The problem of the determination of the arbitrary function using pressure condition is nonlinear.

5. BOUNDARY CONDITIONS FOR THE POTENTIAL

The conditions for the potential listed in the INTRODUCTION need some justification. Let us take up the first condition and assume that the sphere is rigid. Then any liquid particle moving with the induced (perturbation) velocity will not penetrate the spherical surface. Instead, it will move tangentially to this surface at every point of the wetted portion, of course, at any time. Mathematically this means

$$\left(\frac{\partial \phi}{\partial r}\right)_{r = 1} = 0 \tag{18}$$

identically everywhere on the wetted portion of the spherical surface, i. e., for every value of θ, of ω and of t in that region.

Now we can discuss the second condition given in the INTRODUCTION. Under the assumption that the body is moving with velocity $w_0(t)$ towards the space-fixed point at infinity, for an observer in a body-fixed reference system the point at infinity is moving with a velocity $w_0(t)$ towards him. This means mathematically that there exists an absolute potential $\overline{\Phi}$ for which

$$|\nabla\overline{\Phi}|_{r = \infty} = \left(\frac{\partial \overline{\Phi}}{\partial z}\right)_{z = \infty} = w_0(t) \tag{19}$$

Since the velocity $w_0(t)$ of the sphere is proportional to the impact velocity, Equation (19) shows the dependence of the potential on impact velocity.

Conditions (18) and (19) are not sufficient to uniquely define the potential. To achieve this we will need additional conditions. One of these is given by the third of the conditions listed in the INTRODUCTION.

We note at first the difference in disturbances created by a sphere impacting on a calm sea from that created by one impacting on a rough sea. Undoubtedly the potentials in these cases are not the same. In other words, the potential has to depend on the initial free surface. This condition is expressed mathematically in that the initial free surface

$$r = r_0(\theta, \omega, t) \tag{20}$$

has to satisfy the kinematic condition

$$\left(\frac{\partial \phi}{\partial r}\right)_{r = r_0} - \frac{1}{r_0^2 \sin^2 \omega} \left(\frac{\partial \phi}{\partial \theta}\right)_{r = r_0} \frac{\partial r_0}{\partial \theta} - \left\{\frac{1}{r_0} \left(\frac{\partial \phi}{\partial \omega}\right)_{r = r_0}\right\} \frac{1}{r_0} \frac{\partial r_0}{\partial \omega} - \frac{\partial r_0}{\partial t} = 0 \tag{21}$$

asymptotically. Furthermore, the pressure there has to be zero. Equation (21), when considered as an equation for ϕ, gives an additional condition to fix one of the arbitrary functions contained in it, while the pressure condition fixes another.

The fourth condition listed in the INTRODUCTION is also important, for it defines the behavior of ϕ after submergence. It is clear that, when a space capsule has penetrated a certain depth, the body will behave like one in an infinite medium. The mathematical expression for that in the case of a sphere is:

$$\left(\phi\right)_{B \geq B_0} = w_0(t) \left(r + \frac{1}{2r^2}\right) \cos\omega \tag{22}$$

where B is the nondimensional depth of penetration and B_0 a critical depth and

$$\phi_0 = w_0(t) \left(r + \frac{1}{2} \frac{1}{r^2}\right) \cos\omega \tag{23}$$

is the classic potential for a sphere moving with a velocity $w_0(t)$ in an infinite medium.

The four conditions discussed above are obtained from physical considerations. Any potential, whether analytical or numerical, used to describe the physics of impact (vertical or oblique) of a sphere on water has to satisfy all these conditions.

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II. ANALYSIS

1. INTEGRATION OF THE LAPLACE EQUATION

a. Separation of Variables

To integrate the Laplace equation in spherical coordinates

$$\frac{\partial^2 \omega}{\partial r^2} + \frac{2}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r \sin\omega}\,\frac{1}{r}\,\frac{\partial}{\partial \omega}\left(\sin\omega \frac{\partial \phi}{\partial \omega}\right) + \frac{1}{r^2 \sin^2 \omega}\,\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

we use the classic method of separation of variables. We set

$$\phi = \phi_1(\theta)\, \psi\,(r, \omega) \tag{24}$$

and demand that $\phi_1(\theta)$ be periodic in θ and that $\psi(r, \omega)$ be periodic in ω. After separation of variables, one gets

$$r^2 \sin^2 \omega \left\{ \frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\,\frac{\partial \psi}{\partial r} + \frac{1}{r \sin\omega}\,\frac{1}{r}\,\frac{\partial}{\partial \omega}\left(\sin\omega \frac{\partial \psi}{\partial \omega}\right) \right\} - h\psi = 0 \tag{25}$$

and

$$\frac{d^2 \phi_1}{d\theta^2} + h\phi_1 = 0 \tag{26}$$

where h is a parameter of separation. The periodicity of the solution of Equation (26) requires that

$$h = m^2$$

where m is an integer. Hence, we have

$$\phi_1\,(\theta) = A_m^{(1)}(t)\, \cos m\theta + A_m^{(2)}(t)\, \sin m\theta \tag{27}$$

where the A's are arbitrary functions.

Equation (25) can be written as

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r\sin\omega}\frac{1}{r}\frac{\partial}{\partial \omega}\left(\sin\frac{\partial \psi}{\partial \omega}\right) - \frac{m^2}{r^2 \sin^2\omega}\psi = 0 \tag{28}$$

Assuming

$$\psi = \phi_2(\omega)\,\phi_3(r) \tag{29}$$

one has

$$\frac{1}{\sin\omega}\frac{d}{d\omega}\left(\sin\omega\,\frac{d\phi_2}{d\omega}\right) - \left(\frac{1}{4} + k + \frac{m^2}{\sin^2\omega}\right)\phi_2 = 0 \tag{30}$$

and

$$r^2\frac{d^2\phi_3}{dr^2} + 2r\frac{d\phi_3}{dr} + \left(\frac{1}{4} + k\right)\phi_3 = 0$$

where $\left(\frac{1}{4} + k\right)$ is a second parameter of separation.

The next thing to do is to find out whether there are restrictions on k imposed by the periodicity of $\phi_2(\omega)$ in ω.

Introducing a new variable σ by

$$\sigma = \sin^2\omega \tag{32}$$

one can write Equation (30) as

$$4\sigma^2(1-\sigma)\frac{d^2\phi_2}{d\sigma^2} + 2\sigma(2-3\sigma)\frac{d\phi_2}{d\sigma} - \left(\left(\frac{1}{4} + k\right)\sigma + m^2\right)\phi_2 = 0 \tag{33}$$

Any solution of Equation (33) is a function of σ which is periodic in ω, regardless of what value k assumes. Hence, <u>there is no restriction on k imposed by the periodicity requirement of</u> ϕ_2.

To solve Equation (33) one sets

$$\phi_2 = \sigma^{m/2} \chi_m \tag{34}$$

and obtains from (33) the equation

$$\sigma(1 - \sigma)\frac{d^2 \chi_m}{d\sigma^2} + \left(m + 1 - \left(m + \frac{1}{2} + 1\right)\sigma\right) \frac{d\chi_m}{d\sigma}$$

$$- \frac{1}{4} \left(m(m + 1) + \frac{1}{4} + k \right)\chi_m = 0 \tag{35}$$

To discuss Equations (31) and (35), one distinguishes two cases. These are

$$\text{(i) } k = - \left(n + \frac{1}{2}\right)^2, \text{ n positive integer} \tag{36}$$

$$\text{(ii) } k = \ell^2, \qquad \ell > o \text{ non-integer} \tag{37}$$

We will show in the following sections that case (i) delivers the classic solution, while case (ii) gives rise to a new class of solution. A combination of these two solutions gives the solution required.

b. The Classic Solution

For $\frac{1}{4} + k = - n(n + 1)$ Equation (31) becomes

$$r^2 \frac{d^2 \phi_3}{dr^2} + 2r\frac{d\phi_3}{dr} - n(n + 1)\phi_3 = 0 \tag{38}$$

Its general solution is

$$\phi_3 = C_n^{(1)}(t)r^n + C_n^{(2)}(t)r^{-(n + 1)} \tag{39}$$

where the C's are arbitrary functions. For the same value of k Equation (35) reduces to

$$\sigma(1-\sigma)\frac{d^2\chi_m}{d\sigma^2} + \left\{ (m+1) - \left(\frac{m-n}{2} + \frac{m+n+1}{2} + 1 \right) \sigma \right\} \frac{d\chi_m}{d\sigma}$$

$$-\frac{1}{2}(m-n)\frac{1}{2}(m+n+1)\chi_m = 0 \qquad (40)$$

which has the general solution

$$\chi_m = B^{(1)}(m,n,t)\, F\left(\frac{m+n+1}{2}, \frac{m-n}{2}; \frac{1}{2}; 1-\sigma\right)$$
$$+ B^{(2)}(m,n,t)(1-\sigma)^{\frac{1}{2}}\, F\left(\frac{m+n}{2}+1, \frac{m-n}{2}+\frac{1}{2}; \frac{3}{2}; 1-\sigma\right) \qquad (41)$$

where the B's are arbitrary functions. Consequently, we have

$$\phi_2(\omega) = \sin^m\omega \left\{ B^{(1)}(m,n,t)\, F\left(\frac{m+n+1}{2}, \frac{m-n}{2}; \frac{1}{2}; \cos^2\omega\right) \right.$$

$$\left. + B^{(2)}(m,n,t)\cos\omega\, F\left(\frac{m+n}{2}+1, \frac{m-n}{2}+\frac{1}{2}; \frac{3}{2}; \cos^2\omega\right) \right\} \qquad (42)$$

For $n = m + 2s$, where s is a positive integer, the first hypergeometric series terminates, while the second series degenerates to a polynomial for $n = m + 1+2s$.

Using F. Neumann's definition for associated Legendre functions (see Reference 1,p.323, or Reference 2, p.117).

$$P_n^m(\mu) = \frac{(2n)!}{2^n(n-m)!n!}(1-\mu^2)^{\frac{m}{2}} \left\{ \mu^{n-m} - \frac{(n-m)(n-m-1)}{2(2n-1)}\mu^{n-m-2} \right.$$

$$+ \frac{(n-m)(n-m-1)(n-m-2)(n-m-3)}{2.4(2n-1)(2n-3)}\mu^{n-m-4}$$

$$\left. - \dots \right\} \qquad (43)$$

one has for $\frac{1}{4} + k = -n(n+1)$ the following solution for the Laplace equation

$$\phi = \sum_{n=0}^{\infty} \left\{ C_n^{(1)}(t) r^n + C_n^{(2)}(t) r^{-(n+1)} \right\} \sum_{m=0}^{\infty} \left\{ A_{mn}^{(1)}(t) \cos m\theta \right.$$

$$\left. + A_{mn}^{(2)}(t) \sin nm\theta \right\} P_n^m (\cos\omega) \tag{44}$$

where the arbitrary function $B^{(1)}$ and $B^{(2)}$ are absorbed into $A_{mn}^{(1)}(t)$ and $A_{mn}^{(2)}(t)$. This is the classic solution to the Laplace equation for a sphere recaptured in Lamb's book (Reference 2).

To satisfy conditions (18) and (19) at the same time one has to choose $n = 1$, $m = 0$. Thus we obtain

$$\phi_0 = w_0(t) \left(r + \frac{1}{2} r^{-2} \right) \cos\omega \tag{45}$$

This is the potential for a sphere moving with velocity $w_0(t)$ in an infinite medium.

c. The New Solution

For $k = \ell^2$ Equation (31) becomes

$$r^2 \frac{d^2 \phi_3}{dr^2} + 2r \frac{d\phi_3}{dr} + \left(\frac{1}{4} + \ell^2 \right) \phi_3 = 0 \tag{46}$$

with the following general solution

$$\phi_3 = r^{-\frac{1}{2}} \left(B\cos(\ell \, \ell nr) + A\sin(\ell \, \ell nr) \right) \tag{47}$$

As a consequence of Equations (18) and (19), the conditions for ϕ_3 are

$$\left(\frac{d\phi_3}{dr} \right)_{r=1} = 0, \qquad \left| \nabla\phi \right|_{r=\infty} = 0 \tag{48}$$

Since

$$\frac{d\phi_3}{dr} = r^{-\frac{3}{2}} \left\{ \left(A\ell - \frac{1}{2} B \right) \cos(\ell \, \ell nr) - \left(B\ell + \frac{1}{2} A \right) \sin(\ell \, \ell nr) \right\} \tag{49}$$

the second part of Equation (48) is satisfied. To satisfy the first part one chooses

$$B = 2\ell A \tag{50}$$

and obtains

$$\phi_3 = A(\ell, t) r^{-\frac{1}{2}} \left(2\ell \cos(\ell \ \ell nr) + \sin(\ell \ \ell nr)\right) \tag{51}$$

where A is an arbitrary function of ℓ and t.

For $k = \ell^2$ Equation (35) becomes

$$\sigma(1-\sigma)\frac{d^2\chi_m}{d\sigma^2} + \left\{(m+1) - \left(m + \frac{1}{2} + 1\right)\right\} \frac{d\chi_m}{d\sigma}$$

$$- \frac{1}{4}\left\{\left(m + \frac{1}{2}\right)^2 + \ell^2\right\}\chi_m = 0 \tag{52}$$

which is the hypergeometric differential equation with

$$\alpha = \frac{1}{2}\left(m + \frac{1}{2} + i\ell\right) \tag{53a}$$

$$\beta = \frac{1}{2}\left(m + \frac{1}{2} - i\ell\right) \tag{53b}$$

$$\gamma = m + 1 \tag{53c}$$

The general solution of Equation (52) is given by

$$\chi_m = C_1(m, k, t)\ F\left\{\frac{1}{2}\left(m + \frac{1}{2}\right) + \frac{i\ell}{2}\ ,\ \frac{1}{2}\left(m + \frac{1}{2}\right) - \frac{i\ell}{2};\frac{1}{2};\ (1-\sigma)\right\}$$

$$+ C_2(m, k, t)(1-\sigma)^{\frac{1}{2}}\ F\left\{\frac{1}{2}\left(m + \frac{3}{2}\right) + \frac{i\ell}{2}\ ,\ \frac{1}{2}\left(m + \frac{3}{2}\right) - \frac{i\ell}{2};\ \frac{3}{2};\ (1-\sigma)\right\} \tag{54}$$

where the C's are arbitrary functions.

Consequently, one finds

$$\phi_2 = \sigma^{m/2}\left\{C_1 F\left(\frac{1}{2}\left(m+\frac{1}{2}\right)+\frac{i\ell}{2}, \frac{1}{2}\left(m+\frac{1}{2}\right)-\frac{i\ell}{2}; \frac{1}{2}; (1-\sigma)\right)\right.$$

$$\left. + C_2(1-\sigma)^{\frac{1}{2}} F\left(\frac{1}{2}\left(m+\frac{3}{2}\right)+\frac{i\ell}{2}, \frac{1}{2}\left(m+\frac{3}{2}\right)-\frac{i\ell}{2}; \frac{3}{2}\cdot(1-\sigma)\right)\right. \quad (55)$$

Because of the relation

$$0 \le \sigma \le 1 \quad (56)$$

the integer m has to be positive while ℓ may have any real value.

However, one can also use the relation

$$F(\alpha, \beta; \gamma; \zeta) \equiv F(\beta, \alpha; \gamma; \zeta)$$

to limit ℓ to positive values. Since the coefficients of the hypergeometric function $F(\alpha, \overline{\alpha}; \gamma; \zeta)$ have factors of the form $(\alpha+\beta)(\overline{\alpha}+\beta)$ where α and $\overline{\alpha}$ are conjugates complex, the functions contained in Equation (55) are all real.

From the above discussion one concludes that the new solution to the Laplace equation satisfying Condition (48) is

$$\phi = A(t) + r^{-\frac{1}{2}} \sum_{m=0}^{\infty} (\sin\omega)^m \left\{B_1(m,t)\cos m\theta\right.$$

$$\left. + B_2(m,t)\sin m\theta\right\} \int_0^a \left\{2\,\ell\cos(\ell \quad \ell nr)\right.$$

$$\left. + \sin(\ell \quad \ell nr)\right\}\left\{C_1(m,\ell,t)\, F\left(\frac{1}{2}\left(m+\frac{1}{2}\right)+\frac{i\ell}{2}, \frac{1}{2}\left(m+\frac{1}{2}\right)\right.\right.$$

$$\left. - \frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right)$$

$$\left. + C_2(m,\ell,t)\cos\omega\, F\left(\frac{1}{2}\left(m+\frac{3}{2}\right)+\frac{i\ell}{2}, \frac{1}{2}\left(m+\frac{3}{2}\right)-\frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right)\right\} d\ell \quad (57)$$

where A, B_1, B_2, C_1 and C_2 are arbitrary functions of the variables indicated in the parentheses. It can be shown by convergence considerations that the upper limit of integration is

$$a = 1 \quad (58)$$

d. The General Solution

Now we can combine the classic solution given by Equation (45) with the new solution given by Equation (57). Thus we obtain the general solution to the Laplace equation (6) with Conditions (18) and (19) as

$$\phi = A(t) + w_0(t)\left(r + \frac{1}{2}r^{-2}\right)\cos\omega + r^{-\frac{1}{2}} \sum_{m=0}^{\infty} (\sin\omega)^m \left\{B_1(m,t)\cos m\theta + B_2(m,t)\sin m\theta\right\} \int_0^1 \left(2\ell\cos(\ell \ \ell nr) + \sin(\ell \ \ell nr)\right)$$

$$\times \left\{C_1(m,\ell,t) \ F\left(\frac{1}{2}\left(m+\frac{1}{2}\right) + \frac{i\ell}{2}, \frac{1}{2}\left(m+\frac{1}{2}\right) - \frac{i\ell}{2}, \frac{1}{2}; \cos^2\omega\right)\right.$$

$$\left. + C_2(m,\ell,t)\cos\omega \ F\left(\frac{1}{2}\left(m+\frac{3}{2}\right) + \frac{i\ell}{2}, \frac{1}{2}\left(m+\frac{3}{2}\right) - \frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right)\right\} d\ell \quad (59)$$

This potential satisfies the condition that the flow is tangential to the surface of the sphere. It also shows its dependence on the impact velocity. It would reduce to the classic solution for a sphere moving in an infinite medium after submergence, if one could show that the new contribution becomes negligible after the sphere penetrates a certain depth. Two of the arbitrary functions contained in ϕ have to be determined by the initial free surface condition. This will be discussed later.

For vertical impact one has $m = 0$ and

$$\phi = A(t) + w_0(t)\left(r + \frac{1}{2}r^{-2}\right)\cos\omega$$

$$+ r^{-\frac{1}{2}} \int_0^1 \left(2\ell\cos(\ell \ \ell nr) + \sin(\ell \ \ell nr)\right) \left\{C_1(\ell,t) \ F\left(\frac{1}{4} + \frac{i\ell}{2}, \frac{1}{4} - \frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right)\right.$$

$$\left. + C_2(\ell,t)\cos\omega \ F\left(\frac{3}{4} + \frac{i\ell}{2}, \frac{3}{4} - \frac{i\ell}{2}, \frac{3}{2}; \cos^2\omega\right)\right\} d\ell \quad (60)$$

This solution contains only three arbitrary functions $A(t)$, $C_1(\ell, t)$ and $C_2(\ell, t)$. $A(t)$ and $C_2(\ell, t)$ can be completely determined by the condition of their dependence on the initial free surface.

2. DETERMINATION OF THE ARBITRARY FUNCTIONS

a. Determination of $C_2(\ell, t)$

The initial quiescent free surface is given by

$$r_0 = \frac{\beta}{\cos\omega} \tag{61}$$

where

$$\beta = 1 - B(t) \tag{62}$$

with B(t) denoting the nondimensional depth of penetration (Figure 2). For this surface one finds

$$\frac{\partial r_0}{\partial \omega} = \beta(t)\,\frac{\sin\omega}{\cos^2\omega}\,, \quad \frac{\partial r_0}{\partial t} = \frac{\dot{\beta}}{\cos\omega} \tag{63}$$

Furthermore, one computes, from Equation (60)

$$\frac{\partial \phi}{\partial r} = w_0(t)\left(1 - r^{-3}\right)\cos\omega$$

$$-\frac{1}{2} r^{-\frac{3}{2}} \int_0^1 (1 + 4\ell^2)\sin(\ell\, \ell nr)\left\{C_1 F\left(\frac{1}{4} + \frac{i\ell}{2}, \frac{1}{4} - \frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right)\right.$$

$$\left. + C_2 \cos\omega F\left(\frac{3}{4} + \frac{i\ell}{2}, \frac{3}{4} - \frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right)\right\} d\ell \tag{64}$$

To compute $\frac{\partial \phi}{\partial \omega}$ one sets

$$a = \frac{3}{4} + \frac{i\ell}{2}, \quad \beta = \frac{3}{4} - \frac{i\ell}{2}, \quad \gamma = \frac{3}{2}$$

in Equation 9.2.12 on page 243 of Reference 3, and finds

$$\frac{1+4\ell^2}{12}\,\zeta F\left(\frac{7}{4}+\frac{i\ell}{2}, \frac{7}{4}-\frac{i\ell}{2}, \frac{5}{2}; \zeta\right) + F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}; \frac{1}{2}; \zeta\right)$$

$$= F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}; \frac{1}{2}; \zeta\right) \tag{65}$$

By virtue of Equation (65) we obtain

$$\frac{\partial \phi}{\partial \omega} = -\,w_0(t)\left(r+\frac{1}{2}r^{-2}\right)\sin\omega - r^{-\frac{1}{2}}\sin\omega\int_0^1 \Big(2\ell\cos(\ell\;\ell n r) + \sin(\ell\;\ell n r)\Big)$$

$$\times\left\{\frac{1}{4}(1+4\ell^2)C_1\cos\omega\; F\left(\frac{5}{4}+\frac{i\ell}{2}, \frac{5}{4}-\frac{i\ell}{2}; \frac{3}{2}, \cos^2\omega\right)\right.$$

$$\left. + C_2 F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}, \frac{1}{2}; \cos^2\omega\right)\right\} d\ell \tag{66}$$

At the quiescent free surface, defined by Equation (61), the first derivatives of ϕ assume the following values

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=r_0} = w_0\left(1-\frac{\cos^3\omega}{\beta^3}\right)\cos\omega - \frac{1}{2}\,\frac{\cos^{2/3}\omega}{\beta^{3/2}}\int_0^1 (1+4\ell^2)\,\times$$

$$\times\sin\Big(\ell\;\ell n\;\beta - \ell\;\ell n\;\cos\omega\Big)\left\{C_1 F\left(\frac{1}{4}+\frac{i\ell}{2}, \frac{1}{4}-\frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right)\right.$$

$$\left. + C_2\cos\omega\;\; F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right)\right\} d\ell \tag{67}$$

$$\left(\frac{\partial\phi}{\partial\omega}\right)_{r=r_0} = -\,w_0\left(\frac{\beta}{\cos\omega}+\frac{1}{2}\,\frac{\cos^2\omega}{\beta^2}\right)\sin\omega$$

$$-\frac{\cos^{\frac{1}{2}}\omega}{\beta^{\frac{1}{2}}}\sin\omega\int_0^1\Big\{2\ell\cos\;\Big(\ell\;\ell n\beta - \ell\;\ell n\cos\omega\Big)$$

(Continued on next page)

$$+ \sin\left(\ell\, \ell n\beta - \ell\, \ell n\, \cos\omega\right)\Bigg]\Bigg\{\frac{1}{4}(1 + 4\,\ell^2)\, C_1 \cos\omega \times$$

$$\times F\left(\frac{5}{4}+\frac{i\ell}{2}, \frac{5}{4}-\frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right) + C_2\, F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}, \frac{1}{2}; \cos^2\omega\right)\Bigg\} d\ell \tag{68}$$

Since B(t) is the depth of penetration, one has

$$\dot{B}(t) = -\, w_0(t) \tag{69}$$

Substituting Equations (63), (67) and (69) into Equation (21), one obtains

$$w_0 \cos^{\frac{7}{2}}\omega - \frac{1}{2}\beta^{\frac{3}{2}}\cos\omega \int_0^1 (1 + 4\,\ell^2)\sin\left(\ell\ \ell n\, \frac{\cos\omega}{\beta}\right) \times$$

$$\times \Bigg[C_1\, F\left(\frac{1}{4}+\frac{i\ell}{2}, \frac{1}{4}-\frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right) + C_2 \cos\omega\ F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right)\Bigg] d\ell$$

$$- \beta^{\frac{3}{2}}\sin^2\omega \int_0^1 \Bigg[2\ell\cos\left(\ell\, \ell n\, \frac{\cos\omega}{\beta}\right)$$

$$- \sin\left(\ell\, \ell n\, \frac{\cos\omega}{\beta}\right)\Bigg]\Bigg\{\frac{1}{4}(1 + 4\,\ell^2) C_1 \cos\omega\ F\left(\frac{5}{4}+\frac{i\ell}{2}, \frac{5}{4}-\frac{i\ell}{2}; \frac{3}{2}; \cos^2\omega\right)$$

$$+ C_2\, F\left(\frac{3}{4}+\frac{i\ell}{2}, \frac{3}{4}-\frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right)\Bigg\}\ d\ell\ - \frac{w_0}{2}\sin^2\omega\cos^{3/2}\omega = 0 \tag{70}$$

The arbitrary time function C_2 can be determined from the condition that the initial quiescent free surface should satisfy the kinematic condition asymptotically, i.e., Equation (70) should hold true for $\omega \to \pi/2$. This is the mathematical expression for Condition 2 stipulated in the INTRODUCTION to this report.

For $\omega \to \pi/2$, one obtains from Equation (70)

$$\int_0^1 C_2(\ell, t)\left\{2\ell \cos\left(\ell \ln \frac{\sin \epsilon}{\beta}\right) - \sin\left(\ell \ln \frac{\sin\epsilon}{\beta}\right)\right\} d\ell = 0 \tag{71}$$

for small ϵ and all t. Equation (71) is an integral equation of the first kind. For a given t, its kernel is a function of ℓ and $\sin\epsilon$. It can be shown that the eigenfunctions of this kernel form a complete orthogonal system. The only solution to Equation (71) is then

$$C_2 = 0 \tag{72}$$

The facts that the quiescent initial free surface

$$r \cos \omega = \beta(t)$$

is a plane parallel to the plane

$$\omega = \frac{\pi}{2} \tag{73}$$

and that two parallel planes are tangent to each other at infinity lead to the conclusion that the initial quiescent free surface is an asymptotic solution to the kinematic equation for the free surface.

Hence, we have from Equations (60) and (72) the following result: The velocity potential for vertical impact of a sphere on an incompressible fluid is of the form

$$\phi = A(t) + w_0(t)\left(r + \frac{1}{2} r^{-2}\right)\cos\omega + r^{-\frac{1}{2}} \int_0^1 \left(2\ell \cos(\ell \ \ln r) + \sin(\ell \ \ln r)\right)$$

$$\times C_1(\ell, t)\ F\left(\frac{1}{4} + \frac{i\ell}{2}, \frac{1}{4} - \frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right) d\ell \tag{74}$$

b. <u>Determination of A(t)</u>

Consider the pressure acting on the initial free surface at infinity, one has from Bernoulli's equation for

$$r \cos\omega = \beta(t), \quad r \to \infty, \omega \to \frac{\pi}{2} \tag{75}$$

and p = 0 the result

$$\left(\frac{\partial\phi}{\partial t}\right)_{\substack{r\to\infty\\ \omega\to\frac{\pi}{2}}} + \frac{1}{2}w_0^2(t) = -\left(\dot{w}_0(t) + g^*\right)\beta(t) \tag{76}$$

From Equations (60) and (75), with $C_2(\ell, t) = 0$ one has

$$\left(\frac{\partial\phi}{\partial t}\right)_{\substack{r\to\infty\\ \omega\to\frac{\pi}{2}}} = \dot{A}(t) + \dot{w}_0(t)\beta(t) \tag{77}$$

and substitution of Equation (77) into Equation (76) results in

$$A(t) = A - \int_0^t \left(2\dot{w}_0(t) + g^*\right)\beta(t)dt - \frac{1}{2}\int_0^t w_0^2(t)dt \tag{78}$$

where A is a constant.

As a consequence of Equations (74) and (78) we have the general potential satisfying Conditions (1) and (2) stipulated in the INTRODUCTION as

$$\phi = A - \int_0^t \left(2\dot{w}_0(t) + g^*(t)\right)dt - \frac{1}{2}\int_0^t w_0^2(t)dt + w_0(t)\left(r + \frac{1}{2}r^{-2}\right)\cos\omega$$

$$+ r^{-\frac{1}{2}}\int_0^1 \left(2\ell\cos(\ell\,\ell nr) + \sin(\ell\,\ell nr)\right) C_1(\ell, t)\; F\left(\frac{1}{4} + \frac{i\ell}{2}, \frac{1}{4} - \frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right)d\ell \tag{79}$$

where A is an arbitrary constant and C_1 is an arbitrary function of ℓ and t.

c. Absolute Potential

To obtain a nontrivial value for $C_1(\ell, t)$ one has to introduce the absolute potential $\overline{\Phi}$ by

$$\bar{\Phi} = \phi + w_0(t)z \tag{80}$$

It is clear from the conditions

$$\left(\frac{\partial \phi}{\partial r}\right)_{\substack{r = 1 \\ \omega = 0}} = 0 \tag{81a}$$

and

$$\left(\frac{\partial \bar{\Phi}}{\partial z}\right)_{\substack{r = 1 \\ \omega = 0}} = w_0(t) \tag{81b}$$

that ϕ is a relative potential and that $\bar{\Phi}$ gives the absolute velocity.

At the initial stage one will expect the absolute potential to satisfy the wave equation in the space-fixed system

$$\nabla^2 \bar{\Phi} - \left(\frac{V_0}{c}\right)^2 \frac{\nabla^2 \bar{\Phi}}{\partial t^2} = 0 \tag{82}$$

which is valid for $r \approx 1$, small ω and small t.

Since the absolute potential satisfies the Laplace equation in the space-fixed system, Equation (82) reduces to

$$\frac{\partial^2 \bar{\Phi}}{\partial t^2} = 0 \tag{83}$$

Equation (83), when written for the body-fixed system, reads

$$\frac{\partial^2 \bar{\Phi}}{\partial t^2} - 2w_0(t) \frac{\partial^2 \bar{\Phi}}{\partial z \partial t} - \dot{w}_0(t) \frac{\partial \bar{\Phi}}{\partial t} + w_0^2(t) \frac{\partial^2 \bar{\Phi}}{\partial z^2} = 0 \tag{84}$$

which is valid for $r \simeq 1$, small ω and small t.

Using the transformation

$$\frac{\partial}{\partial z} = \cos\omega \frac{\partial}{\partial r} - \frac{1}{r}\sin\omega \frac{\partial}{\partial \omega} \tag{85}$$

one computes for $r \simeq 1$ and small ω

$$\frac{\partial \phi}{\partial z} = \cos\omega \frac{\partial \phi}{\partial r} - \frac{1}{r}\sin\omega \frac{\partial \phi}{\partial \omega} \simeq 0 \tag{86}$$

$$\frac{\partial^2 \phi}{\partial z^2} = \cos^2\omega \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r^2}\sin\omega\cos\omega \frac{\partial \phi}{\partial \omega} - \frac{2}{r}\sin\omega\cos\omega \frac{\partial^2 \phi}{\partial r \partial \omega}$$

$$+ \frac{1}{r}\sin^2\omega \frac{\partial \phi}{\partial r} + \frac{1}{r^2}\sin^2\omega \frac{\partial \phi}{\partial \omega} \simeq \frac{\partial^2 \phi}{\partial r^2} \tag{87}$$

From Equation (64) one obtains, by differentiating with respect to r

$$\frac{\partial^2 \phi}{\partial r^2} = 3w_0(t) r^{-4} \cos\omega$$

$$+ \frac{3}{4} r^{-\frac{5}{2}} \int_0^1 (1 + 4\ell^2)\sin(\ell \;\; \ell n r)\, C_1\, F\left(\frac{1}{4} + \frac{i\ell}{2},\, \frac{1}{4} - \frac{i\ell}{2};\, \frac{1}{2};\, \cos^2\omega\right) d\ell$$

$$- \frac{1}{2} r^{-\frac{5}{2}} \int_0^1 \ell(1 + 4\ell^2)\cos(\ell \;\; \ell n r)\, C_1\, F\left(\frac{1}{4} + \frac{i\ell}{2},\, \frac{1}{4} - \frac{i\ell}{2};\, \frac{1}{2};\, \cos^2\omega\right) d\ell$$

(88)

For $r \simeq 1$ and small ω one has

$$\frac{\partial^2 \phi}{\partial r^2} \simeq 3\, w_0(t) + \pi^{\frac{1}{2}}\, \ell n \sin\omega \int_0^1 \frac{\ell(1 + 4\ell^2)\, C_1(\ell, t)}{\Gamma\left(\frac{1}{4} + \frac{i\ell}{2}\right)\Gamma\left(\frac{1}{4} - \frac{i\ell}{2}\right)} d\ell \tag{89}$$

where Equation (10) on page 114 of Reference 2 is applied.

Introducing Equations (80), (86), (87) and (89) into Equation (84) one obtains

$$3w_0^3 - 2w_0\dot{w}_0 - \ddot{w}_0 + 2B\ddot{w}_0 + g_0^* w_0$$
$$+ \frac{1}{2}\pi^{\frac{1}{2}} \ell n \ \sin\omega \int_0^1 \frac{\left((1 + 4\ell^2) w_0^2(t)\, C_1(\ell, t) - 4\frac{\partial^2 C_1}{\partial t^2}\right) d\ell}{\Gamma\left(\frac{1}{4} + \frac{i\ell}{2}\right)\Gamma\left(\frac{1}{4} - \frac{i\ell}{2}\right)} = 0 \qquad (90)$$

where

$$B(t) = \int_0^t w_0(t) dt \qquad (91)$$

denotes the nondimensional depth of penetration.

Equation (90) is only possible if and only if

$$3\dot{B}^3 - 2\dot{B}\ddot{B} - \dddot{B} + 2B\dddot{B} + g^*\dot{B} = 0 \qquad (92)$$

and

$$(1 + 4\ell^2)\dot{B}^2(t)\, C_1(\ell, t) - 4\frac{\partial^2 C_1}{\partial t^2} = \qquad (93)$$

are satisfied simultaneously.

d. Determination of B

Writing

$$\dot{B}^2 = y(x) - \frac{g^*}{3}, \quad B - \frac{1}{2} = x \qquad (94)$$

one reduces Equation (92) to the linear equation

$$x \frac{d^2 y}{dx^2} - \frac{dy}{dx} + 3y = 0 \tag{95}$$

Equation (95) can be transformed to the Bessel equation

$$\frac{d^2 u}{d\xi^2} + \frac{1}{\xi}\frac{du}{d\xi} + \left(1 - \frac{4}{\xi^2}\right) u = 0 \tag{96}$$

by setting

$$2(3x)^{\frac{1}{2}} = \xi, \quad y(x) = \xi^2 u(\xi) \tag{97}$$

Thus, we find

$$u = a\, J_2(\xi) + b\, Y_2(\xi) \tag{98}$$

where $J_2(\xi)$ is a Bessel function of the first kind and Y_2 one of the second kind. a and b are arbitrary constants.

Using Equations (94) and (97), one gets by substitution

$$\dot{B}^2 = 12\left(\frac{1}{2} - B\right)\left\{a\, I_2\left(2.\, 3^{\frac{1}{2}}\left(\frac{1}{2} - B\right)^{\frac{1}{2}}\right) + b\, K_2\left(2.\, 3^{\frac{1}{2}}\left(\frac{1}{2} - B\right)^{\frac{1}{2}}\right)\right\}$$

$$- \frac{1}{3} g^* \geq 0 \qquad \text{for } B < \frac{1}{2} \tag{99}$$

and

$$\dot{B}^2 = 12\left(B - \frac{1}{2}\right)\left\{a\, J_2\left(2\left(3\left(B - \frac{1}{2}\right)\right)^{\frac{1}{2}}\right) - \frac{\pi}{2} b\, Y_2\left(2\left(3\left(B - \frac{1}{2}\right)\right)^{\frac{1}{2}}\right)\right\}$$

$$- \frac{1}{3} g^* \geq 0 \quad \text{for } B > \frac{1}{2} \tag{100}$$

I_2 and K_2 are modified Bessel functions. The constants in Equation (99) and (100) are so chosen that the two functions defined by these equations are identical near $B = \frac{1}{2}$.

To determine a and b one takes the nondimensional initial impact velocity into consideration and obtains from Equation (99)

$$6\left(aI_2\left(6^{\frac{1}{2}}\right)+b\,K_2\left(6^{\frac{1}{2}}\right)\right)-\frac{1}{3}g^* \simeq 1 \tag{101}$$

From the nature of the functions I_2 and K_2 and the restrictions imposed by Equation (99), one concludes, by letting $B \longrightarrow \frac{1}{2}$, that it is necessary to choose

$$b > 0 \tag{102}$$

A second equation for a and b has to be obtained later.

Under the assumptions that $r \simeq 1$ and ω is small, one can solve Equation (99) to obtain

$$\int_{B_0}^{B} \frac{d\xi}{\left[12\left(\frac{1}{2}+\xi\right)\right]^{\frac{1}{2}}\left\{a\,I_2\left[2\left(3\left(\frac{1}{2}+\xi\right)\right)^{\frac{1}{2}}\right]+b\,K_2\left[2\left(3\left(\frac{1}{2}+\xi\right)\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}}}=t \tag{103}$$

where B_0 is the initial depth of penetration created by the compression wave immediately before the impact.

Equation (103), when inverted, gives the depth of penetration B as a function of t.

e. <u>Determination of C_1 (ℓ, t)</u>

Introducing $\dot{B}^2$ into Equation (93) we have a second order linear partial differential equation for C_1 which can be treated as an ordinary equation. In fact, if one considers C_1 as a function of B(t) and makes use of the relation

$$\ddot{B} = \frac{d\dot{B}}{dt} = \dot{B}\,\frac{d\dot{B}}{dB} \tag{104}$$

Equation (93) can be written as

$$\dot{B}^2\,\frac{\partial^2 C_1}{\partial B^2} + \frac{1}{2}\,\frac{d\dot{B}^2}{dB}\,\frac{\partial C_1}{\partial B} - \left(\frac{1}{4} + \ell^2\right)\dot{B}^2\,(t)C_1 = 0 \tag{105}$$

It is not hard to obtain $\frac{d\dot{B}^2}{dB}$ from Equation (99). After substitution of the values of $\dot{B}^2$ and $\frac{d\dot{B}^2}{dB}$ into Equation (105), one can integrate the result as an ordinary differential equation and obtain a solution containing two additional arbitrary constants which are functions of ℓ. These constants have to be determined by momentum balance and the pressure condition.

3. DETERMINATION OF THE FREE SURFACE - A FEASIBILITY STUDY

From Equations (99) and (101) it is clear that, for small depth of penetration, the square of the velocity of penetration and the velocity itself can be written as

$$\dot{B}^2 = 1 + \sum_{\nu=1}^{\infty} a_\nu B^\nu, \quad B = \sum_{\nu=0}^{\infty} b_\nu B^\nu \tag{106}$$

where the coefficients are linear functions of a single parameter a. Consequently one finds, from Equation (105)

$$C_1(\ell, t) = \sum_{\nu=0}^{\infty} C_\nu(\ell) B^\nu \tag{107}$$

where only $C_0(\ell)$ and $C_1(\ell)$ are arbitrary.

Substituting Equation (107) into Equation (74) results in

$$\phi = A(t) + w_0(t)\left(r + \frac{1}{2}r^{-2}\right)\cos\omega$$

$$+ r^{-\frac{1}{2}} \sum_{\nu=0}^{\infty} B^{\nu} \int_0^1 \Big(2\ell\cos(\ell \;\ell nr) + \sin(\ell\;\ell nr)\Big)\, C_{\nu}(\ell)$$

$$\times F\left(\frac{1}{4} + \frac{i\ell}{2}, \frac{1}{4} - \frac{i\ell}{2}; \frac{1}{2}; \cos^2\omega\right) d\ell \tag{108}$$

which is of the form

$$\phi = \sum_{\nu=0}^{\infty} \phi_{\nu}(r,\omega)B^{\nu} \tag{109}$$

This potential contains two arbitrary functions of the dummy variable ℓ and involves otherwise only known linear functions of the parameter a. Neither the space variables nor the time variable are involved in these functions.

Assuming

$$r = F(\omega, B) = \sum_{\nu=0}^{\infty} f_{\nu}(\omega)B^{\nu} \tag{110}$$

as the equation for the free surface generated by the impact one has, by substituting r = F into $\frac{\partial\phi}{\partial r}$, $\frac{\partial\phi}{\partial\omega}$ and $\frac{\partial\phi}{\partial t}$, the results

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=F} = \sum_{\nu=0}^{\infty} \Phi_{\nu}(\omega)B^{\nu} \tag{111}$$

$$\left(\frac{\partial\phi}{\partial\omega}\right)_{r=F} = \sum_{\nu=0}^{\infty} \Psi_{\nu}(\omega)B^{\nu} \tag{112}$$

$$\left(\frac{\partial\phi}{\partial t}\right)_{r=F} = \sum_{\nu=0}^{\infty} X_{\nu}(\omega)B^{\nu} \tag{113}$$

where Φ_{ν}, ϕ_{ν} and ψ_{ν} are known functions of $f_0, f_1, \ldots f_{\nu}$.

Writing

$$r^2 = \sum_{\nu=0}^{\infty} g_\nu(\omega) B^\nu \tag{114}$$

where $g_\nu(\omega)$ are known quadratic functions of $f_0, f_1 \ldots$ and f_ν and making use of Equation (16) – the kinematic equation for the free surface, one obtains the equation

$$\sum_{\nu=0}^{\infty}\left(\sum_{\mu=0}^{\nu} \phi_{\nu-\mu} g_\mu\right) B^\nu - \sum_{\nu=0}^{\infty}\left(\sum_{\mu=0}^{\nu} \Psi_{\nu-\mu} \frac{df_\mu}{d\omega}\right) B^\nu$$

$$- \sum_{\nu=0}^{\infty}\left(\sum_{\mu=0}^{\nu} (\nu-\mu+1) f_{\nu-\mu+1} \sum_{\lambda=0}^{\mu} b_\lambda g_{\mu-\lambda}\right) B^\nu \tag{115}$$

for vertical impact.

Comparing coefficients of B^ν one gets

$$\sum_{\mu=0}^{\nu} \Psi_{\nu-\mu} g_\mu - \sum_{\mu=0}^{\nu} \Psi_{\nu-\mu} \frac{df_\mu}{d\omega} - \sum_{\mu=0}^{\nu} (\nu-\mu+1) f_{\nu-\mu+1} \sum_{\lambda=0}^{\mu} b_\lambda g_{\mu-\lambda} = 0 \tag{116}$$

For $\nu = 0$ one has

$$b_0 g_0 f_1 = \Phi_0 g_0 - \Psi_0 \frac{df_0}{d\omega} \tag{117}$$

giving f_1 in terms of f_0 and its first derivative.

For $\nu = 1$ one finds

$$2b_0 g_0 f_2 = g_0 \Phi_1 + g_1 \Phi_0 - \Psi_0 \frac{df_1}{d\omega} - \Psi_1 \frac{df_0}{d\omega} - f_1(b_0 g_1 + b_1 g_0) \tag{118}$$

which gives f_2 in terms of f_0, f_1 and their first order derivatives.

It is not hard to calculate the coefficients of

$$\left(\frac{\partial \phi}{\partial r}\right)^2_{r = F} = \sum_{\nu = 0}^{\infty} \Phi^*_{\nu} B^{\nu}, \quad \left(\frac{\partial \phi}{\partial \omega}\right)^2_{r = F} = \sum_{\nu = 0}^{\infty} \psi^*_{\nu} B^{\nu} \tag{119}$$

from Equations (111) and (112).

Substituting Equations (113) and (119) into the pressure condition (Equation (17)) for the free surface and assuming

$$\dot{w}(t) = \dot{B}\, \frac{d\dot{B}}{dB} = \sum_{\nu = 0}^{\infty} a_{\nu} B^{\nu} \tag{120}$$

where a_{ν} and b_{ν} are related by Equation (106), one gets by comparison of coefficients of B^{ν}

$$\chi_0(\omega) + \frac{1}{2}\left(\Phi^*_0 + \Psi^*_0\right) + a_0 + g^* = 0 \tag{121}$$

and

$$\chi_{\nu}(\omega) + \frac{1}{2}\left(\Phi^*_{\nu} + \Psi^*_{\nu}\right) + a_2 = 0 \tag{122}$$

for $\nu = 1, 2, 3, \ldots$.

We note in passing the following facts:

1. The calculation of b_{ν} in Equation (106) from a_{ν} involves second degree algebraic operations.

2. The calculation of $C_2(\ell)$, $C_3(\ell)$, ... from $C_0(\ell)$ and $C_1(\ell)$ involves only linear operation (Equation (107)).

3. The determination of $\phi_{\nu}(r, \omega)$ in Equation (108) requires only write-out.

4. The calculation of $\Phi_{\nu}(\omega)$, $\Psi_{\nu}(\omega)$ and $\chi_{\nu}(\omega)$ involves differentiation, substitution and expansion of known functions.

5. The calculation of $g_{\nu}(\omega)$ from f_{ν} requires simple quadratic operation.

6. The computation of f_1, f_2, ... from Equation (117) and (118) requires only linear operation.

7. The determination of ϕ_ν^* and ψ_ν^* in Equation (119) requires substitution, expansion and multiplication.

8. The determination of a_ν in Equation (120) follows immediately from Equation (106) by multiplication of two series.

Since the evaluation of all the coefficients involves only algebraic operations with, and differentiation of known functions, a computer program for the calculation of the free surface is feasible.

4. MOMENTUM BALANCE - VIRTUAL MASS

The momentum of the fluid is given by

$$\text{Momentum} \quad \overline{M} = \rho \int_V \nabla \overline{\Phi} \, d\tau \tag{123}$$

where

$\rho \equiv$ density of the fluid

$\overline{\Phi} \equiv$ velocity potential of the absolute motion.

$V \equiv$ volume depicted below.

We will denote

unit normal to free surface (S_1) by $\vec{n_1}$ and that to the frontier surface (S_2) by $\vec{n_2}$

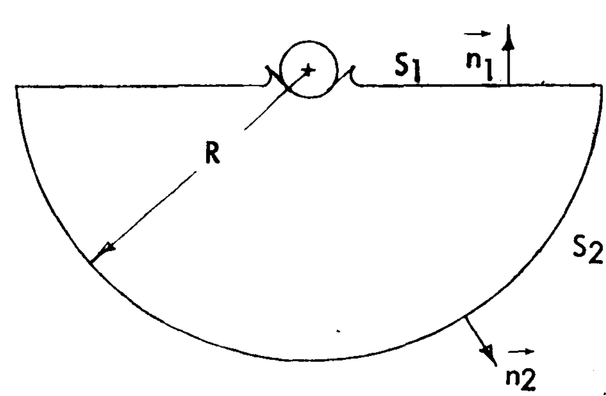

Figure 3. Virtual Mass

Writing Equation (123) in terms of a surface integral by Gauss' theorem, one has

$$\bar{M} = -\rho \int_{S_1} \vec{n}_1 \bar{\Phi}\, ds - \rho \int_{S_2} \vec{n}_2 \bar{\Phi}\, ds \tag{124}$$

The vertical component (z-component) is obtained by taking the scalar product of Equation (124) with the unit vector in the z-direction, $\vec{e}_z$.

$$(\bar{M})_z = \rho \int_V \nabla \bar{\Phi} \cdot \vec{e}_z\, d\tau = \rho \int_V \frac{\partial \bar{\Phi}}{\partial z}\, d\tau$$
$$- \rho \int_{S_1} \vec{n}_1 \cdot \vec{e}_z \bar{\Phi}\, ds - \rho \int_{S_2} \vec{n}_2 \cdot \vec{e}_z \bar{\Phi}\, ds \tag{125}$$

If the free surface is defined by

$$F_1(x, y, z, t) = 0 \tag{126}$$

the unit normal is given by

$$\vec{n}_1 = \frac{\nabla F_1}{|\nabla F_1|} \tag{127}$$

Surface S_2, the frontier boundary of the virtual mass, is given by

$$F_2(x, y, z, t) = R^2 = x^2 + y^2 + z^2 = c_2 \tag{128}$$

The scalar product of its normal vector with $\vec{e}_z$, the unit vector in the z-direction gives

$$\vec{n}_2 \cdot \vec{e}_z = \frac{\nabla F_2}{|\nabla F_2|} \cdot \vec{e}_z = \frac{z}{R} \equiv \cos(z, r) \tag{129}$$

Substituting Equations (127) and (129) into Equation (125) results in

$$(\bar{M})_z = \rho \int_V \frac{\partial \bar{\Phi}}{\partial z}\, d\tau = -\rho \int_{S_1} \frac{\nabla F_1}{|\nabla F_1|} \cdot \vec{e}_z \bar{\Phi}\, ds - \rho \int_{S_2} \cos(z, R) \bar{\Phi}\, ds \tag{130}$$

The force is given by the time rate of change of the momentum

$$F_z = \frac{d}{dt}(\bar{M})_z = -\frac{d}{dt}\int_{S_1} \rho \frac{\nabla F_1}{|\nabla F_1|} \cdot \vec{e}_z \bar{\Phi} ds - \rho \int_{S_2} \cos(z, R) \frac{\partial \bar{\Phi}}{\partial t} ds \tag{131}$$

On the other hand, the force may also be determined from the integration of the pressure.

$$F_z = F + \int_{S'} P_N ds \tag{132}$$

where

F ≡ downward force exerted by the body.

P_N ≡ pressure normal to the surface, S

S' ≡ total surface omitting the wetted surface of the body.

To evaluate the integral in Equation (132) one used Bernoulli's equation

$$p = -\rho\ (\nabla\bar{\Phi} \cdot \nabla\bar{\Phi}) - \rho \frac{\partial \bar{\Phi}}{\partial t} \tag{133}$$

Since p = 0 on the free surface, the integral in Equation (133) becomes

$$\int_{S'} P_N\ ds = \int_{S_2} \cos(z, R) p ds = -\rho \int_{S_2} \cos(z, R) \left(\nabla\bar{\Phi} . \nabla\bar{\Phi} + \frac{\partial \bar{\Phi}}{\partial t}\right) ds \tag{134}$$

Combining Equations (131), (132) and (134) we have

$$F - \rho \int_{S_2} |\nabla\bar{\Phi}|^2 \cos(z, R) ds = -\frac{d}{dt}\int_{S_1} \frac{\nabla F_1}{|\nabla F_1|} \cdot \vec{e}_z \bar{\Phi} ds \tag{135}$$

For the water impact problem we are concerned with an infinite fluid which, according to Figure (3) means that the frontier surface moves towards infinity. Recalling that the potential for the absolute motion vanishes at infinity and is proportional to the inverse of R we can assume that

$$\nabla\phi \sim R^{-\frac{3}{2}} \text{ as } R \rightarrow \infty$$

Consequently, the integral over S in Equation (135) becomes

$$\lim_{R \rightarrow \infty} \int_{S_2} \left|\nabla\overline{\phi}\right|^2 \cos(z, R) ds \cong \lim_{R \rightarrow \infty} \frac{1}{R^3} \int_{S_2} \cos(z, R) ds = 0$$

and therefore

$$F = -\frac{d}{dt} \int_{S_1} \rho\overline{\phi} \left(\frac{\nabla F_1}{\left|\nabla F_1\right|} \cdot \vec{e}_z\right) ds \tag{136}$$

The downward force exerted by the body, F, is determined by the time rate of change of the momentum of the body

$$F = \frac{d}{dt}\left(M_0 w_0(t)\right) \tag{137}$$

where

$M_0 \equiv$ mass of the body

$w_0(t) \equiv$ vertical velocity of the body

Combining Equations (136) and (137) one gets

$$\frac{d}{dt}\left\{M_0 w_0(t) + \int_{S_1} \rho\overline{\phi} \left(\frac{\nabla F_1}{\left|\nabla F_1\right|} \cdot \vec{e}_z\right) ds\right\} = 0$$

which, integrated once, gives

$$M_0 w_0(t) + \int_{S_1} \rho\overline{\phi} \left(\frac{\nabla F_1}{\left|\nabla F_1\right|} \cdot \vec{e}_z\right) ds = c_o = \text{constant} \tag{138}$$

Defining the momentum virtual mass, M, by

$$M = \frac{1}{w_0(t)} \int_{S_1} \rho\bar{\phi} \left(\frac{\nabla F_1}{|\nabla F_1|} \cdot \vec{e}_z \right) ds \tag{139}$$

one gets, by momentum balance, the result

$$\left(1 + \frac{M}{M_0}\right) w_0(t) = 1 \tag{140}$$

In Equation (139) $\bar{\phi}$ is given by Equation (80), $w_0(t)$ by Equation (99) and F_1 by Equation (110). After expansion of all the functions involved in power series of B one can obtain a series expansion for M. Substituting M into Equation (140) one can determine the parameter a contained in $w_0(t)$ and the arbitrary functions of l contained in ϕ by comparison of coefficients.

After the determination of ϕ and subsequently of $\bar{\phi}$, the virtual mass can be calculated by using Equation (139), the pressure acting on the sphere by the Bernoulli equation, and the impact force by Equation (137).

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III. SOLUTION FOR THE CONICAL AND TOROIDAL SECTIONS OF APOLLO

The Apollo vehicle is made up of three distinct geometrical parts: a spherical segment as the base, a conical section as the wall and a toroidal section as the joint of base and wall.

In the case of vertical landing it is clear that only the base experiences the critical pressure. However, one is also interested in the existence of the potential solution for the conical and the toroidal sections. That a potential does exist for each of these cases is demonstrated in the following section.

1. SEPARATION OF VARIABLES

The Laplace equation is separable when expressed in any orthogonal coordinate system (ξ_1, ξ_2, ξ_3) which can be transformed directly to the rectangular coordinate system (x, y, z) through relations of the form: $x = x(\xi_1, \xi_2, \xi_3)$ etc. In particular, the Laplace equation is separable in the conical coordinate system and the toroidal coordinate system. That is to say, in either of these coordinate systems represented by (ξ_1, ξ_2, ξ_3) the partial differential equation can be decomposed into three ordinary differential equations. This means, by setting

$$\phi(\xi_1, \xi_2, \xi_3) = \phi_1(\xi_1)\, \phi_2(\xi_2)\, \phi_3(\xi_3) \tag{141}$$

one can obtain three ordinary differential equations for $\phi_1(\xi_1)$, $\phi_2(\xi_2)$ and $\phi_3(\xi_3)$ from the Laplace equation

$$\nabla^2 \phi(\xi_1, \xi_2, \xi_3) = 0 \tag{142}$$

2. SOLUTION FOR THE CONICAL SECTION

To find the potential for the conical section of Apollo one uses conical coordinates and sets (Reference 4)

$$x = \pm \frac{\xi_1 \xi_2 \xi_3}{\alpha\beta} \tag{143a}$$

$$y^2 = \frac{\xi_1^2}{\beta^2} \frac{\left(\xi_2^2 - \beta^2\right)\left(\xi_3^2 - \beta^2\right)}{\left(\beta^2 - \alpha^2\right)} \tag{143b}$$

$$z^2 = \frac{\xi_1^2}{\alpha^2} \frac{\left(\xi_2^2 - \alpha^2\right)\left(\xi_3^2 - \alpha^2\right)}{\left(\alpha^2 - \beta^2\right)} \tag{143c}$$

The coordinate surfaces are spheres and cones given by

$$x^2 + y^2 + z^2 = \xi_1^2 \quad \text{sphere} \tag{144a}$$

$$\frac{x^2}{\xi_2^2} + \frac{y^2}{\xi_2^2 - \beta^2} + \frac{z^2}{\xi_2^2 - \alpha^2} = 0 \quad \text{cone} \tag{144b}$$

$$\frac{x^2}{\xi_3^2} + \frac{y^2}{\xi_3^2 - \beta^2} + \frac{z^2}{\xi_3^2 - \alpha^2} = 0 \quad \text{cone} \tag{144c}$$

where

$$\alpha^2 > \xi_2^2 > \beta^2 > \xi_3^2$$

It is obvious that the surface of a cone is specified by ξ_2 = constant, say $\xi_2 = 1$.

Introducing the new coordinates into the Laplace equation one obtains the following separated equations

$$\frac{d^2 \phi_1}{d\xi_1^2} + \frac{2}{\xi_1} \frac{d\phi_1}{d\xi_1} - \frac{k_1}{\xi_1^2} \phi_1 = 0 \tag{145a}$$

$$\left(\xi_2^2 - \beta^2\right)\left(\alpha^2 - \xi_2^2\right)\frac{d^2\,\phi_2}{d\xi_2^2} + \xi_2\left(\alpha^2 + \beta^2 - 2\xi_2^2\right)\frac{d\phi_2}{d\xi_2} + \left(k_1\,\xi_2^2 - k_2\right)\phi_2 = 0 \tag{145b}$$

$$\left(\beta^2 - \xi_3^2\right)\left(\alpha^2 - \xi_3^2\right)\frac{d^2\,\phi_3}{d\xi_3^2} + \xi_3\left(2\xi_3^2 - \alpha^2 - \beta^2\right)\frac{d\phi_3}{d\xi_3} + \left(k_2 - k_1\,\xi_3^2\right)\phi_3 = 0 \tag{145c}$$

where k_1 and k_2 are separation constants.

The classical potential is obtained by assuming discrete eigenvalues for k_1 and k_2. It is known that the classic potential does not satisfy the boundary condition

$$\left(\frac{\partial\phi}{\partial\xi_2}\right)_{\xi_2 = 1} = 0$$

on the surface of the conical section. This can be remedied, however, by giving discrete eignevalues to k_1 and continuous eigenvalues to k_2 following Dr. Ta Li's theory.

3. SOLUTION TO THE TOROIDAL SECTION OF APOLLO

The potential for the toroidal section can be obtained by introducing toroidal coordinates ξ_1, ξ_2, ξ_3

$$x = \frac{a\,\xi_3\left(\xi_1^2 - 1\right)^{1/2}}{\left(\xi_1 - \xi_2\right)} \tag{146a}$$

$$y = \frac{a\left[\left(\xi_1^2 - 1\right)\left(1 - \xi_3^2\right)\right]^{1/2}}{\left(\xi_1 - \xi_2\right)} \tag{146b}$$

$$z = \frac{a\left(1 - \xi_2^2\right)^{1/2}}{\left(\xi_1 - \xi_2\right)} \tag{146c}$$

Utilizing the transformations

$$\xi_1 = \cos h\ (\tau)$$

$$\xi_2 = \cos\ (\sigma)$$

$$\xi_3 = \cos\ (\phi)$$

we find that the coordinate surfaces are tores, meridian planes and spheres.

$$x^2 + y^2 + (z - \cot\sigma)^2 = \frac{a^2}{\sin^2\sigma}\ \text{(sphere)} \tag{147a}$$

$$\left(\left(x^2 + y^2\right)^{1/2} - a \cot h\ \tau\right)^2 + z^2 = \frac{a^2}{\sin h\ \tau}\ \text{(anchor ring or tore)} \tag{147b}$$

$$y = x \tan\phi\ \text{(plane through z-axis)} \tag{147c}$$

It is evident that the surface of a tore is given by τ = constant, i.e., ξ_1 = constant.

The separated equations are

$$\left(\xi_1^2 - 1\right) \frac{d^2\phi_1}{d\xi_1^2} + 2\xi_1 \frac{d\phi_1}{d\xi_1} + \left(k_1 + \frac{k_2}{\left(\xi_1^2 - 1\right)}\right)\phi_1 = 0 \tag{148a}$$

$$\left(1 - \xi_2^2\right) \frac{d^2 \phi_2}{d\xi_2^2} - \xi_2 \frac{d\phi_2}{\xi_2} - k_1 \phi_2 = 0 \tag{148b}$$

$$\left(1 - \xi_3^2\right) \frac{d^2 \phi_3}{d\xi_3^2} - \xi_3 \frac{d\phi_3}{d\xi_3} - k_2 \phi_3 = 0 \tag{148c}$$

where k_1 and k_2 are separation constants.

The classical potential was obtained by giving discrete eigenvalues to k_1 and k_2. Since this potential does not satisfy the boundary condition $(\partial\phi/\partial\tau) = 0$ for ξ_1 = constant. Continuous eigenvalues have to be assigned to k_2 to obtain the new solution.

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CONCLUSION

The study made in this report on the vertical impact of the Apollo vehicle on water is based upon the assumption that the flow is incompressible and nonviscous. The sphere is taken as the mathematical model for the Apollo capsule. This is physically realistic for vertical impact since the bottom portion of the capsule is a spherical segment. However, it is also shown that the conic portion and the toroidal portion are amenable to analysis.

The first difficulty in solving the impact problem is to find the correct nontrivial potential to satisfy the boundary conditions. This is overcome by using a new separation technique, recently developed by the author. The second difficulty lies in the determination of the arbitrary functions. This is partially accomplished by considering the initial free surface and the pressure condition at infinity, and partially achieved by combining the Laplace equation with the wave equation near the first point of impact and then by momentum balancing.

It is shown in this report that the potential, the virtual mass, the velocity of penetration, the impact pressure and the impact force are all series of the depth of penetration. It is also shown that the calculation of the coefficients of these series can be done numerically, so that a computer program is in sight.

A general potential for oblique impact of a sphere is also obtained. However, the determination of the arbitrary functions has not been attempted. How this should be done in the oblique case will depend on the outcome of the numerical program of the vertical case. The vertical entry is comparatively simple but general enough to give information which could lead to better understanding of the oblique problem.

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