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ROTATIONS AND OSCILLATIONS OF A NEUTRON STAR

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ABSTRACT

We consider in the first part of this thesis gravitational radiation from various pulsating and rotating objects using the formula obtained with the weak-field limit of general relativity. The cases of rotation and oscillation are first considered separately and then the effects of rotation on radial oscillations are investigated. Numerical estimates are made with data relevant to a neutron star and it is concluded that most of the energy a neutron star may acquire during its formation is dissipated rapidly, unless the rotation is quite slow.

We then consider the various means by which rotation can be slowed down, and in the classical case, we arrive at the conclusion that whether a neutron star can bifurcate to form a non-axisymmetric Jacobi type configuration is quite uncertain, owing to a lack of knowledge of the equation of state. We remarked, however, that the problem should be considered in the framework of Post-Newtonian approximation, and by making use of a variational approach, we obtain a set of equations determining the structure of a fast-rotating polytrope. We solve the equations for the case of unity polytropic index and with the assumption of axisymmetry.

We pointed out that despite its simplicity in the classical case, a variational approach is not the most convenient one in the Post-Newtonian approximation. A direct solution of the hydrostatic equations would be simpler in the end. We concluded by suggesting that one should look into the effects of viscosity and problems arising from internal motion.

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I. INTRODUCTION

The energy content of a neutron star has been the subject for much discussion in recent years, in connection with the energy supply and mechanism of the various X-ray sources. If one thinks of a neutron star as being formed catastrophically, probably through the free-fall collapse of a supernova, then one observes that large dynamical velocities should be developed and there should be oscillations about the stable configuration, since it is highly unlikely that the neutron star would arrive at the stable configuration gently. Furthermore, the dynamical speed of the oscillating object would not be much less than that of light since the dimensions of the equilibrium structure of a neutron star are not more than 3-10 times the Schwarzschild radius. Thus a large quantity of energy of oscillation, amounting to about 10^{53-54} ergs for a star of mass ≈ 1 M⊙ (solar mass) must be dissipated before the final stable structure can be arrived at.

Various mechanisms have been suggested for the dissipation of this amount of energy. Hoyle, Narlikar and Wheeler (1964) proposed that the emission of electromagnetic waves could lead to a very large rate of energy loss. It is pointed out that the dynamic oscillation of the main body of the neutron star implies an oscillation of the associated magnetic field. Such a field exists in the neutron star, because electrons and protons are also present inside despite the dominance of the neutrons, in a ratio determined by the condition that the sum of the top of their Fermi-distribution be equal to that of the neutrons. Furthermore, the magnitude of the magnetic intensity at the surface of a typical neutron-star (density 10^{15} gm/cc and radius 10^6 cm) could be as large as 10^{10} gauss because even a star with such a small dipole field as the sun's (1 gauss at a mean density 1 gm/cm^3) would develop a polar field of ~ 10^{10} gauss if it were catastrophically contracted to a density of about 10^{15} gm/cm^3. With such a large magnetic field, the total emission rate might be as large as ~ 10^{42} ergs/sec (just the Poynting vector times the surface area of the star). Furthermore, it is pointed out that the strong gravitational field of the star creates a near vacuum immediately outside itself, and hence the electromagnetic waves are free to travel out into space. Thus, with no propagation difficulty arising, an emission rate of ~ 10^{42} ergs/sec is sufficient to dissipate the available store of ~ 10^{54} ergs in ~ 10^{12} secs or 10^5 years.

The other mechanisms for energy dissipation which has been intensively studied are the various neutrino processes. Neutrino cooling of non-oscillating neutron stars via the Urca neutrino processes, i.e. neutron-neutron scattering followed by beta-decay and its inverse, has been investigated by various authors (e.g. Chiu and Salpeter 1964, Bahcall and Wolf, 1965a, Finzi 1965a) Finzi (1965b), however, seemed to be the first one to point out that the vibrational energy of a neutron star could also be dissipated through ν-processes. He considered the following reaction

$$n + n \rightarrow n + p + e + \bar{\nu}_e \tag{1.1}$$

and its inverse reaction

$$p + e + n \rightarrow n + n + \nu_e \ . \tag{1.2}$$

In his calculation, Finzi assumed a constant density of $\rho \sim 6 \times 10^{14}$ gm/cm^3 for the neutron star and complete degeneracy for all the particles, which were described as free fermions. At the density used, only neutrons, protons, and electrons are present, and the Fermi level En of the degenerate neutron gas is about 93 Mev. The Fermi levels of the proton and electrons, Ee and Ep, can be determined from the condition of chemical equilibrium and charge conservation. One finds in this way E_e = 89.5 MeV and E_p = 4.3 MeV, and the ratio of proton-electron pairs to the total number of particles to be about 0.009.

Thus, the equilibrium concentration of neutrons, protons and electons in the degenerate gas of the neutron star is a function of the density. In a pulsating star, the density varies periodically with time and therefore the concentration is most of the time different from the equilibrium concentration. For densities slightly higher than nuclear density, the equilibrium concentration of e's and protons increases with increasing density. Therefore, when the gas is compressed, reaction (1.1) will transform some neutrons and protons and electrons, plus anti-neutrinos which will escape. Conversely, when the gas is expanded reaction (1.2) will transform some protons and electrons into neutrons, plus neutrinos which will escape. These two reactions proceed at the expense of the vibrational energy of the star, which is very effectively damped at a rate given by $\sim 10^{52}\ \alpha^8$ ergs/sec where α is the ratio of the radial displacement of a point during oscillation to the distance of that point from the centre of the star.

The calculation has been carried out for the general case of non-uniform density and non-zero temperature, and applied to some actual neutron star models by Hansen (1966) and Hansen and Tsuruta (1967). The result of the calculation indicates vibrational energy still ranging from about 10^{47} ergs to 10^{51} ergs at 10^3 years (the supernova explosion leading to the Crab Nebula is believed to have occurred 910 years ago at a distance of about 1100 pc). This calculation thus does not rule out the possibility of a vibrational energy source for the Crab Nebula. This, however, is the case if only there is no other major loss mechanism. We may, for example, have to take into account neutrino-reactions other than the aforementioned Urca processes (and their μ-meson analogue with e and ν_e replaced by μ and ν_μ). In fact, Bahcall and Wolf (1965b) suggested the following $n\pi$ processes which gives a vastly increased rate of neutrino emission via the transformation:

$$\pi^- + n \rightarrow n + e^- + \bar{\nu}_e \tag{1.3}$$

$$n + e^- \rightarrow n + \pi^- + \nu_e \tag{1.4}$$

and their μ-meson analogue. Whether this actually occurs depends on whether nuclear matter at very high densities contains "quasi free pions", as suggested by the idealized model of superdense matter as consisting of independent, non-interacting gases (Ambartsumyan and Saakyan (1960), Cameron (1957)). On the basis of this model, high pressure raises the fermi energy of neutrons, protons and electrons to such a point that Σ^-, Λ^o, μ^- and π^- and other particles are formed. No pions, however, would appear until the density exceed about 300 times the nuclear density, although there may be interactions strongly lowering the critical density required for pion-production and for the initiation of the $n\pi$-processes.

If such a process is really effective, then the temperature and energy content of the neutron star in the Crab Nebula should be now far too low to be of significance for the X-ray radiation currently observed. However, Tsuruta and Cameron (1966) suggested that the pion-neutron interaction is such as to make it unlikely that the $n\pi$-process can operate at all. We shall therefore make no further remark on the ν-processes and shall pass on to the next important mechanism of energy loss--gravitational radiation--which forms the main body of our work in this thesis.

The existence of graviational radiation was predicted by Einstein (the usually quoted references are: Sb. Preuss. Akad. Wiss. 688, 1916 and Sb. Preuss. Akad. Wiss. 154, 1918) shortly after he formulated his general theory of relativity. Systems of moving masses should emit gravitational waves in analogy with the emission of electromagnetic waves by a system of moving charges. There has, however, been some skeptism about the reality of gravitational radiation (Pirani, 1962) and no gravitational waves have as yet been observed. However, some experimental work now seems feasible (Weber, 1961, 1963, 1964, 1967), and with the resolution of some theoretical issues in recent years, it has been possible for a number of physicists to conclude that general relativity does predict the existence of gravitational waves. Such fundamental problems are, however, beyond the scope of our present thesis. Instead, we shall consider, in the spirit of Landau and Liftschitz (1962) the problem of energy radiated by moving bodies in the form of gravitational waves.

For a system of bodies moving with velocities small compared with that of light, the energy loss by gravitational radiation is given by the weak field limit to general relativity as: (Landau and Liftshitz, 1962)[†]

$$-\frac{d\mathcal{E}}{dt} = \frac{G}{45c^5}\left(\frac{\partial^3 D_{\alpha\beta}}{\partial t^3}\right)^2 \tag{1.5}$$

where $\mathcal{E}$ is the energy of the system and $D_{\alpha\beta}$ is the quadrupole moment of the mass distribution, defined as:

$$D_{\alpha\beta} = \int \rho(\underset{\sim}{x})\left(3\,x_\alpha x_\beta - \delta_{\alpha\beta} x_\gamma^2\right) d\underset{\sim}{x} \tag{1.6}$$

Angular momentum can also be lost through graviational radiation, at a rate given by (Peters, 1964)

$$-\frac{dL_i}{dt} = \frac{2G}{45c^5}\,\epsilon_{ijk}\,\frac{\partial^2 D_{jm}}{\partial t^2}\,\frac{\partial^3 D_{km}}{\partial t^3} \tag{1.7}$$

where L_i is the i-th component of the angular momentum vector $\underset{\sim}{L}$ and ϵ_{ijk} is a completely antisymmetric unit pseudotensor.

Making use of formulae (1.6) and (1.9), we have calculated the loss of energy through emission of gravitational radiation of various oscillating and rotating systems (Chau, 1967). A rotating, non-axisymmetric body is found to be slowed down very fast, provided that the asymmetry is large. For a purely oscillating system, the quadrupole mode is damped exponentially, with the higher modes surviving.

† Greek subscripts range from 1→3, and summation over repeated indices is understood, unless stated otherwise.

Besides these directly radiating modes, we have to consider also the radial mode of oscillation. Although a purely radial mode does not radiate, the rotation which is almost surely present would destroy the spherical symmetry of the system, again leading to radiation. We have carried out a detail analysis of this mechanism of drawing energy from a radial mode to other non-radial ones, and found that after a time of about 10^3 years, the surviving radial mode would only be significant if $\Omega \lesssim 10\ \sec^{-1}$.

Such a small angular velocity for effective energy storage poses a serious problem with regard to the angular momentum of the neutron star. A very significant amount of angular momentum must be lost if we wish to have any significant radial mode of oscillation in the star at all. This can, of course, be achieved by the emission of gravitational radiation but the rate would be extremely insignificant unless the neutron star can form a non-axisymmetric Jacobi-ellipsoid of large asymmetry. Even then, the difficulty is not completely resolved, because with the loss of angular momentum, the asymmetry would become smaller and smaller; and the rotating configuration would become axisymmetric at a still very large angular velocity.

Nevertheless, we shall consider whether a rotating neutron star can form a non-axisymmetric Jacobi type configuration or not. A problem of this nature has been dealt with very extensively by Chandrasekhar and Lebovitz (1962a) and Chandrasekhar (1962b), in the classical case for an incompressible fluid. It is shown that for a slowly rotating object, the axisymmetric sequence of Maclaurin ellipsoids represents the equilibrium shape of configurations, but at a certain angular velocity Ω_b and beyond, a non-axisymmetric Jacobi sequence branches off. Since Ω_b is smaller than that when equilibrium is no longer possible, an incompressible object does really exhibit a point of bifurcation at the angular velocity Ω_b.

Similar investigations have been made for the case of polytropes (James 1964, Roberts 1963). James method is much more exact, and he obtained the result that only polytropes with an index $n \leq n_c = 0.808$ can exhibit a point of bifurcation. Roberts' approach, on the other hand, is much more approximate. However, his is the one we shall follow because of its simplicity.

Cameron and Tsuruta (1966) have applied these ideas to the rotating neutron star by the assignment of effective polytropic indices. We also make some more general estimates of effective polytropic indices, but point out that the problem should be attacked at least within the framework of Post-Newtonian Approximation (to be abbreviated as PNA for the rest of the thesis).

Again assuming that a neutron star can be approximated by a polytrope with a certain effective index, we are thus led to the determination of the structure of a fast-rotating polytrope in the PNA. We use a variational approach developed by Roberts (1063) and Krefetz (1966), and obtain the solution for the case $n = 1$. We point out, however, that a variational formulation like Roberts' is extremely cumbersome and perhaps unwarranted in the PNA. A direct generalization of Jame's method may probably be more profitable.

II. Graviational Radiation from Rotating and Non-Radially Oscillating Object

We begin by considering gravitational radiation from a rotating ellipsoid of mass m, uniform density ρ and semi-axes (a_1, a_2, a_3). This has actually been considered by Chin (1965), but our method here seems to be simpler, and in any case illustrative of the one we shall use in our later discussions.

Let the angular velocity $\underset{\sim}{\Omega}$ be in the z-direction. In the body set of axes (x' y' z'), all the off-diagonal elements of the quadrupole moment tensor about the origin vanish because of reflection symmetry. The diagonal elements can be simply evaluated, and are given by:

$$D'_{ii} = \frac{m}{5}\left(3a_i^2 - a_\alpha a_\alpha\right) \tag{2.1}$$

where i = 1, 2, 3 and is not summed over.

In terms of the space-set of axes (x, y, z), we have:

$$\begin{aligned} x' &= x\cos\Omega t + y\sin\Omega t \\ y' &= -x\sin\Omega t + y\cos\Omega t \\ z' &= z \end{aligned} \tag{2.2}$$

and from the theory of tensor transformation, we have for D_{ij} in the space set given by

$$D_{ij} = D'_{\alpha\beta}\frac{\partial X'_\alpha}{\partial X_i}\frac{\partial X'_\beta}{\partial X_j} \tag{2.3}$$

Substituting the D_{ij} calculated from Eq. (2.3) into the equation for energy loss (1.5), we immediately have

$$-\frac{d\mathcal{E}}{dt} = \frac{32G}{125c^5} m^2 \Omega^6 (a_2^2 - a_1^2)^2 \tag{2.4}$$

For slow rotation, the configuration does not differ much from that of a sphere of radius R. Thus, we can put

$$a_2 - a_1 = \eta R \tag{2.5}$$

where η is a small quantity. Eq. (2.4) can then be simplified to

$$-\frac{d\mathcal{E}}{dt} \simeq \frac{128}{125} \frac{G}{c^5} m^2 \Omega^6 R^4 \eta^2 \tag{2.6}$$

To find the damping, we need an expression for the rotational energy. This may be calculated classically, and the result is

$$\begin{aligned} \mathcal{E} &= \frac{1}{10} m (a_1^2 + a_2^2) \Omega^2 \\ &\simeq \frac{1}{5} m R^2 \Omega^2 \end{aligned} \tag{2.7}$$

Substituting this into Eq. (2.6), we readily have

$$\Omega^4 = \frac{\Omega_0^4}{1 + \kappa t} \tag{2.8}$$

where Ω_o is the angular velocity at $t = 0$, and the "damping-parameter" k is defined as

$$k = \Omega_0^4 \frac{128}{25} \frac{Gm}{c^5} \frac{(a_2^2 - a_1^2)^2}{(a_1^2 + a_2^2)}$$

$$\simeq \frac{256}{25} \frac{Gm}{c^5} \Omega_0^4 \eta^2 R^2 \tag{2.9}$$

We must remark that in the case of a Jacobi ellipsoid, η^2 also depends on Ω^2. Thus Eq. (2.8) is actually not completely correct, and should be interpreted as only giving a rough idea of the damping. For a typical neutron star of mass = 1 M☉ and radius R = 10^6 cm, even a small η = 0.1 would reduce an Ω_o = 10^2 cps to 40% of its value in about 10^9 secs or ~ 30 years. However, we must caution that it is not clear at all whether a rotating neutron star can form a Jacobi-type configuration. We shall elaborate on this point in §V.

We next consider an axisymmetric oscillation of a spherical mass of incompressible fluid of constant density ρ and radius R. Following the usual procedure for describing liquid drop oscillation, we describe the deformed surface by

$$r(\theta) = R\left[1 + \alpha_2 P_2(\theta) + \cdots + \alpha_n P_n(\theta)\right] \tag{2.10}$$

where the α_n's are functions of time and <<1, and the P_n's are Legendre functions. We note that there is no α_o term because of the assumption of incompressibility, and the α_1 term is left out because it shifts the centre of mass. Because the fluid is of constant density, Eq. (2.10) is sufficient for the calculation of the D_{ij}'s. We observe that we need calculate only one diagonal element, since the off-diagonal terms vanish because of axial-symmetry and since $D_{11} = D_{22} = -\frac{1}{2} D_{33}$ because of axisymmetry and the vanishing of trace.

Making use of the orthogonality of the P_n's, we find by direct calculation

$$D_{33} = \frac{8\pi\rho R^5}{5}\left[\alpha_2 + \sum_{n=2,\dots}\frac{10\alpha_n^2 n(n+1)}{(2n-1)(2n+1)(2n+3)}\right. \tag{2.11}$$

$$\left. + \sum_{n=2,\dots}\frac{30\alpha_n\alpha_{n+2}(n+2)(n+1)}{(2n+1)(2n+3)(2n+5)} + O(\alpha_n^3)\right]$$

We notice that the P_2-mode contributes the dominant term to the quadrupole moment tensor, and hence to the energy loss.

We now assume

$$\alpha_n = \alpha_{n0}\sin\sigma_n t \tag{2.12}$$

where α_{no} is the amplitude, assumed to vary only slowly over a period so that in calculating $\frac{\partial^3 D_{ij}}{\partial t^3}$, we can treat it as a constant.

Up to terms linear in α_n, we need only consider the P_2 mode. Substituting D_{33} from Eq. (2.11) and using $D_{11} = D_{22} = -\frac{1}{2} D_{33}$, we obtain from Eq. (1.5) the energy loss averaged over a full cycle in this linearized approximation

(2.13)

where the subscript 2 means that we are considering the P_2-mode.

Now the energy of the P_n-oscillation ($n \neq 0$) is given by Rayleigh (1945)

$$\mathcal{E}_n = \pi \rho R^5 \frac{1}{n(2n+1)} \alpha_{n0}^2 \sigma_n^2 \tag{2.14}$$

From Eqs. (2.13) and (2.14), we therefore have

$$\alpha_{20}(t) = \alpha_{20}(0)\, e^{-K_2 t} \tag{2.15}$$

where

$$K_2 = \frac{4}{25} \frac{GmR^2}{c^5} \sigma_2^4 \tag{2.16}$$

We thus see that the P_2-mode is damped exponentially. To have an idea of the damping rate, we again take our typical neutron star of mass = 1 M⊙ and R = 10^6 cm, and the classical expression for an incompressible fluid for the oscillation frequency of the non-radial modes (Lamb, 1932)

$$\sigma_n^2 = \frac{2n(n-1)}{(2n+1)} \frac{4}{3} \pi G \rho \tag{2.17}$$

With this, Eq. (2.16) gives

$$K_2 = 1.2 \times 10^1 \text{ sec}^{-1} \tag{2.18}$$

Thus, after an extremely short time (~ 1 day), the P_2-mode becomes completely insignificant and only the higher modes remain. From Eq. (2.11) we see that the coefficient of the coupled term $\alpha_n \alpha_{n+2}$ is of the same order of magnitude as the squared term α_n^2. Thus, for the mode with the largest amplitude, we can neglect the cross-term and take the squared term to be the only contribution to D_{33}. In this case, the energy loss expression for this n-th mode becomes

$$-\frac{d\mathcal{E}_n}{dt} = \frac{1024}{15}\frac{G}{c^5}\left[\frac{\pi\rho n(n+1)}{(2n-1)(2n+1)(2n+3)}\right]^2 \sigma_n^6 \alpha_{n0}^4 R^{10} \qquad (2.20)$$

Eq. (2.14) and (2.20) would then give

$$\alpha_{n0}^2(t) = \frac{\alpha_{n0}^2(0)}{1 + K_n \alpha_{n0}^2 t} \qquad (2.21)$$

where

$$K_n = 256\,\frac{Gm^2\sigma_n^4}{c^5}\,\frac{n^3(n+1)^2}{(2n-1)^2(2n+1)(2n+3)^2} \qquad (2.22)$$

and is ~ $4.3 \times 10^3\ \mathrm{sec}^{-1}$ using Eq. (2.17) and the data relevant to our "typical" neutron star.

Thus, the higher modes would be damped at a much slower rate than the P_2-mode. However, in the non-linear domain, the dynamical coupling between the various modes should be considered. This coupling could give a stronger damping because of energy transfer from the higher modes into the P_2-mode. Furthermore, the oscillations would not be purely harmonic as given by Eq. (2.12). Thus, our analysis should be interpreted as giving only an essentially qualitative description of the actual picture.

Suppose now the above pulsating object is also rotating slowly. We linearize our calculations in the small parameter (Ω/σ), to which order the energy loss expression remains unchanged because it should be even in Ω. If the rotation is about the axis of symmetry, as is probably the case for the slowly rotating case, then no angular momentum is lost, as can be seen by examining Eq. (1.7) and noticing that it vanishes for a system possessing axial symmetry and hence vanishing off-diagonal D_{ij}'s. Just to have a rough idea of how rotation can be damped, we consider an object rotating about the x-axis and at the same time pulsating in an axisymmetric manner about another axis, the z-axis, say. The loss rate can then be readily evaluated using the same technique leading to Eq. (2.6). The result when we consider only the P_2-mode is

$$-\frac{dL_1}{dt} = \frac{9}{25}\frac{Gm^2}{c^5} R^4 \alpha_{20}^2 \sigma_2^4 \Omega \qquad (2.23)$$

For higher modes, the numerical coefficient would be different and α_{no}^4 would replace α_{20}^2. In any case, it is clear from Eq. (2.23) that for small (Ω/σ), the rotation is damped out much more slowly than the pulsation. By the time the non-radial oscillations become insignificant, much of the rotation would still persist.

III. Effect of Rotation on Radial Oscillations and the Consequent Emission of Gravitational Radiation

In §II, we have been considering the directly radiating modes. Now a purely radial mode does not radiate. This can be immediately seen by noticing that a radially oscillating object maintains its spherical symmetry. Thus all the D_{ii}'s are equal and $D_{ij} = 0$ to $i \neq j$. But the trace of $\dot{D}_{ii}$ also vanishes, and hence each diagonal element also vanishes. Therefore, Eq. (1.5) gives zero for the energy loss. This, however, is only true for a purely radial vibration of an ideal spherically symmetric distribution of mass. If there should be any coupling of the radial mode to a non-zero mass quadrupole, then gravitational radiation would again be emitted. Thorne and Mettzer (1966) pointed out that the chances are overwhelming for a neution star to passess a finite amount of angular momentum, and any natural amount of angular momentum, divided by the relatively very small moment of inertial of a neution star, implies typically a very high angular velocity. We have also seen in the last section that rotation is damped away relatively slowly, and thus, we can almost certainly think of some rotation present together with the radial mode. And we shall now show in detail that this rotation, by destroying the spherical symmetry of the system, would serve as the mechanism to bring about the coupling of the radial mode to a directly radiating one.

Since the change in density distribution should be independent of the sign of $\underset{\sim}{\Omega}$, we thus have to take all effects of rotation up to terms of order $(\Omega/\sigma)^2$. We consider then a sphere of radius R consisting of a compressible fluid of uniform density pulsating in the lowest radial mode when the Lagrangian displacement is given by:

$$\xi_r = \alpha r e^{i\sigma_0 t} \tag{3.1}$$

where α is a constant. If this sphere is given a small, uniform rotation, the oscillations are altered both because the equilibrium shape is changed the pulsation equation includes additional centrifugal and Coriolis force term.

In one presence of rotation, Euler's equation can be written as:

$$\rho \frac{D\underset{\sim}{u}}{Dt} + 2\rho(\underset{\sim}{\Omega} \times \underset{\sim}{u}) + \rho \underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{r}) = -\underset{\sim}{\nabla} p - \rho \underset{\sim}{\nabla} \Phi \tag{3.2}$$

where $\underset{\sim}{u}$ is the velocity, ρ is the density, p is the pressure, Φ is the gravitational potential given by Poisson's equation:

$$\nabla^2 \Phi = -4\pi G \rho \tag{3.3}$$

and

$$\frac{D\underset{\sim}{u}}{Dt} = \frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla})\underset{\sim}{u} \tag{3.4}$$

When axisymmetric perturbations are considered, it is more advantageous to write the above equation of motion for $\underset{\sim}{u}$ in the rotating frame in spherical polar co-ordinates (Ledoux and Walraven, 1958). With the definition of velocity components in spherical co-ordinates given by $u_r = \dot{r}$, $u_\theta = r\dot{\theta}$, $u_\phi = r \sin\theta\dot{\phi}$, we have:

$$\frac{\partial u_r}{\partial t} - \frac{u_\theta^2 + u_\phi^2}{r} - 2\Omega \sin\theta\, u_\phi = -\frac{\partial \Phi}{\partial r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + \Omega^2 r \sin^2\theta \tag{3.5}$$

$$\frac{\partial u_\theta}{\partial t} + \frac{u_\theta u_r}{r} - \frac{u_\phi^2}{r}\cot\theta - 2\Omega\cos\theta\, u_\phi$$
$$= \frac{1}{r}\frac{\partial \Phi}{\partial \theta} - \frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \Omega^2 r \sin\theta\cos\phi \tag{3.6}$$

$$\frac{\partial u_\phi}{\partial t} + \frac{u_\theta u_\phi}{r}\cot\theta + \frac{u_r u_\phi}{r} + 2\Omega\sin\theta\, u_r + 2\Omega u_\theta\cos\theta$$
$$= -\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi} - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi}$$
$$= 0 \quad \text{because of axisymmetry} \tag{3.7}$$

Assuming time-dependence $e^{i\sigma t}$ for the Lagrangian displacement $\underset{\sim}{\xi}$, and linearizing in $\underset{\sim}{\xi}$, we have, by making use of the relation obtained at equilibrium:

$$\underset{\sim}{\nabla}\left(\Phi - \frac{1}{2}\Omega^2 r^2 \sin^2\theta\right) + \frac{1}{\rho}\underset{\sim}{\nabla} p = 0 \tag{3.8}$$

the following alternate forms for Eqs. (3.5) - (3.7)

$$-\sigma^2\xi_r - 2i\sigma\Omega\sin\theta\,\xi_\phi = -\frac{\partial \Phi'}{\partial r} + \frac{\rho'}{\rho^2}\frac{\partial p}{\partial r} - \frac{1}{\rho}\frac{\partial p'}{\partial r} \tag{3.9}$$

$$-\sigma^2\xi_\theta - 2i\sigma\Omega\cos\theta\,\xi_\phi = -\frac{1}{r}\frac{\partial \Phi'}{\partial \theta} - \frac{1}{\rho r}\frac{\partial p'}{\partial \theta} + \frac{\rho'}{\rho^2 r}\frac{\partial p}{\partial \theta} \tag{3.10}$$

$$-\sigma^2\xi_\phi + 2i\sigma\Omega(\xi_r \sin\theta + \xi_\theta \cos\theta) = 0 \tag{3.11}$$

Here the unprimed quantities indicate the equilibrium values of a uniformly rotating spherical of constant density, and the primed ones are the Eulerian perturbed values. The perturbed quantities ρ', p' $d\Phi'$ can be related to the equilibrium values by making use of the continuity equation, Poisson's equation and the adiabatic relation between pressure and density variations, respectively as follows:

$$\begin{aligned}\rho' &= -\operatorname{div} \rho \underset{\sim}{\xi} \\ &= -\rho \operatorname{div} \underset{\sim}{\xi}\end{aligned} \tag{3.12}$$

$$\nabla^2 \Phi' = -4\pi G \rho \operatorname{div} \underset{\sim}{\xi} \tag{3.13}$$

$$p' = -\gamma p \operatorname{div} \underset{\sim}{\xi} - \underset{\sim}{\xi} \cdot \underset{\sim}{\nabla} p \tag{3.14}$$

To solve for ξ_r, ξ_θ and ξ_ϕ we first expand them and σ^2 in powers of Ω, retaining terms up to those of order Ω^2. Thus, we write:

$$\sigma^2 = \sigma_0^2 + \Omega\sigma_1^2 + \Omega^2\sigma_2^2 \tag{3.15}$$

$$\xi_r = \xi_{r0} + \Omega\xi_{r1} + \Omega^2\xi_{r2} \tag{3.16}$$

$$\xi_\theta = \Omega\xi_{\theta 1} + \Omega^2\xi_{\theta 2} \tag{3.17}$$

$$\xi_\phi = \Omega\xi_{\phi 1} + \Omega^2\xi_{\phi 2} \tag{3.18}$$

where ξ_{ro} is the same as ξ_r defined in Eq. (3.1) and there are no zeroth order terms for ξ_θ and ξ_ϕ, since the motion is purely radial in the absence of rotation.

For oscillations that are originally axisymmetric, it can be shown that $\sigma_1 = 0$ (Clement, 1965). This can be quite readily seen if one formulates the problem in terms of a variational principle and examines the equation providing stationary values in σ for arbitrary variations in $\underset{\sim}{\xi}$. Then it becomes apparent that the first order change in σ brought about by rotation actually arises from the Coriolis term in the equation of motion, and if one writes the normal modes of the Lagrangian displacements in terms of the spherical harmonics $Y_\ell^m(\theta.\phi)$, the Coriolis term will be found to be proportional to m. Hence, for an axisymmetric oscillation where m = 0, we have $\sigma_1 = 0$ and so only a second order change to the characteristic frequency. Also, we remember that the right-hand side of Eq. (3.10) is of order Ω^2, since pressure and the density perturbations should be independent of the sign of Ω. With the above in mind, we substitute the Eqs. (3.15) - (3.18) for σ^2, ξ_r, ξ_θ, and ξ_ϕ into Eq. (3.10) and (3.11), and by comparing terms of different orders in Ω, we readily obtain:

$$\xi_{\theta 1} = \xi_{r1} = \xi_{\phi 2} = 0 \tag{3.19}$$

$$\Omega \xi_{\phi 1} = 2i\left(\frac{\Omega}{\sigma}\right)\sin\theta\ \xi_{r0} \tag{3.20}$$

$$\sin\theta\ \xi_{r2} + \cos\theta\ \xi_{\theta 2} = 0 \tag{3.21}$$

Eq. (3.20) for $\xi_{\phi 1}$ agrees with a corresponding expression given by Clement (1965). To solve for ξ_{r2} (or equivalently, $\xi_{\theta 2}$), we turn to Eq. (3.10). Although the r-equation (3.9) has not been used at all in this analysis, it will be shown later that it can be used to solve for $\sigma_2{}^2$ with known $\underset{\sim}{\xi}$, hence giving a check on the rest of the calculations.

Using Eq. (3.19) - (3.21), we can write Eq. (3.10) as:

$$-\Omega^2\sigma_0{}^2\xi_{\theta 2} + 2\Omega^2\sin 2\theta\, d\, r = -\frac{1}{r}\frac{\partial\Phi'}{\partial\theta} - \frac{1}{\rho r}\frac{\partial p'}{\partial\theta} + \frac{\rho'}{\rho^2 r}\frac{\partial p}{\partial\theta} \tag{3.22}$$

We have to evaluate first the right-hand side of Eq. (3.22), using Eqs. (3.12) - (3.14). We use for the equilibrium quantities those of a uniformly rotating mass of fluid of constant density (Chandrasekhar, 1962a). Thus, we have for a rotating ellipsoid with a surface given by:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1 \tag{3.23}$$

($a_1 = a_2$ in our present axisymmetric case)

The gravitational potential Φ is given by:

$$\Phi = \pi G \rho a_1^2 a_3 \left(I - \sum A_i x_i^2 \right) \qquad (3.24)$$

where

$$I = \int_0^\infty \frac{du}{\Delta(u)}$$

$$A_i = \int_0^\infty \frac{du}{\Delta(u)(a_i^2 + u)}$$

and $\quad \Delta^2(u) = (a_1^2+u)^2 (a_3^2+u)$

and the relation between pressure and density is given by:

$$\frac{P}{\rho} = \frac{1}{2}\Omega^2 (x_1^2 + x_2^2) - \Phi + \text{constant} \qquad (3.25)$$

where the constant is so chosen that the pressure vanishes on the surface.

We define the eccentricity e:

$$e^2 = 1 - \frac{a_3^2}{a_1^2} \qquad (3.26)$$

and the relation between e^2 and Ω^2 is given by, the virial theorem, say, as:

$$\frac{\Omega^2}{2\pi G \rho} = \frac{\sqrt{1-e^2}}{e^2}\left(3 - 2e^2\right)\sin^{-1} e - \frac{3(1-e^2)}{e^2} \qquad (3.27)$$

For small rotation in which we are interested, we can treat e as a small parameter. Expanding the various expressions listed above, and keeping terms only up to those of order e^2, we readily obtain from Eq. (3.25) and (3.24):

$$\frac{p}{\rho} = \frac{1}{2}\Omega^2 r^2 \sin^2\theta - \pi G \rho \left[\frac{2}{3} r^2 - \frac{2}{3} R^2 \left(1 - \frac{2}{3} e^2\right) - \frac{4}{15} e^2 R^2 + \frac{2}{15} e^2 r^2 \left(2 - 3\sin^2\theta\right) \right] \qquad (3.28)$$

where we have made p vanish on the surface (3.23), which we can now write as:

$$r(\theta) = R\left[1 + e^2\left(\frac{1}{2}\sin^2\theta - \frac{1}{3}\right)\right] \qquad (3.29)$$

and we have related a_1 and a_2 of the ellipsoid to R by:

$$a_1 = R\left(1 + \frac{1}{6} e^2\right) \qquad (3.30)$$

$$a_3 = R\left(1 - \frac{1}{3} e^2\right) \qquad (3.31)$$

and the relation between e^2 and Ω^2 becomes:

$$e^2 = \frac{15\,\Omega^2}{8\pi G\rho} \tag{3.32}$$

Using now Eqs. (3.12) and (3.13) for ρ' and p', and Eqs. (3.19) - (3.21) for ξ, and Eq. (3.28) for p, we can now write Eq. (3.22) as:

$$\begin{aligned} -\sigma_0^2 \xi_{\theta 2} + 2\sin 2\theta\, \alpha r = & -\frac{1}{r}\frac{\partial \Phi'}{\partial \theta} - \frac{15}{4}\alpha r \sin 2\theta \\ & + \frac{1}{r}\Big[\frac{15}{4}\alpha r^2 \sin 2\theta\left(\gamma + \frac{2}{3}\right) - \frac{4}{3}\pi G \rho r \frac{\partial \xi_{r2}}{\partial \theta} \\ & - \gamma \pi G \rho\left(\frac{2}{3}r^2 - \frac{2}{3}R^2\right)\frac{\partial}{\partial \theta}\operatorname{div} \xi\Big] \end{aligned} \tag{3.33}$$

An examination of Eq. (3.33) indicates that in the desired solution, the θ-dependence of $\xi_{\theta 2}$ can be taken as sin 2θ. Thus, by also making use of Eq. (3.21), we can write:

$$\xi_{\theta 2} = f(r)\sin 2\theta \tag{3.34}$$

$$\xi_{r2} = -2f(r)\cos^2\theta \tag{3.35}$$

and for the perturbed internal gravitational potential we write:

$$\Phi'(r,\theta) = g_o(r) + g_2(r) P_2(\theta) \tag{3.36}$$

with $g_o(r)$, $g_2(r)$, and $f(r)$ still to be determined.

By substituting Eqs. (3.34) - (3.36) into Eq. (3.33), we would then have:

$$-\sigma_o^2 f(r) + 2\alpha r = \frac{3}{2}\frac{g_2(r)}{r} - \frac{15}{4}\alpha r + \frac{15}{4}\alpha r\left(\gamma + \frac{2}{3}\right)$$
$$- \frac{8}{3}\pi G\rho f(r) - \frac{1}{r\sin 2\theta}\gamma\pi G\rho\left(\frac{2}{3}r^2 - \frac{2}{3}R^2\right)\left(\frac{\partial}{\partial\theta} \operatorname{div} \xi\right) \tag{3.37}$$

By inspection, we see that $f(r) = \alpha\lambda r$ can be a solution provided that λ satisfies:

$$\lambda\left(\frac{8}{3}\pi G\rho - \sigma_o^2\right) = -6\pi G\rho k_2 - \frac{13}{4} + \frac{15}{4}\gamma \tag{3.38}$$

where use has been made of one fact that in this case $\frac{\partial}{\partial\theta}$ div $\xi = 0$, while from Poisson's equation, we have $g_2(r) = -4\pi G\rho\alpha\, k_2 r$.

Our derivation is now nearly complete except for the constant k_2. This can be determined in quite a straightforward manner by demanding that the gravitational potential and its derivatives be continuous on the perturbed boundary. The details of the calculations we leave in Appendix A. The result is:

$$K_2 = \frac{299}{120}\frac{1}{\pi G\rho} - \frac{16}{9}\lambda \tag{3.39}$$

which therefore gives a λ from Eq. (3.38):

$$\lambda = \frac{1}{\sigma_0^2}\left(\frac{1092}{60} - \frac{15}{4}\gamma\right) \tag{3.40}$$

Our solution is now complete. Summing up, we have for the Lagrangian displacement:

$$\xi_r = \alpha r - 2\Omega^2\lambda r\alpha \cos^2\theta \tag{3.41}$$

$$\xi_\theta = \alpha\lambda r\,\Omega^2 \sin 2\theta \tag{3.42}$$

$$\xi_\phi = 2i\left(\frac{\Omega}{\sigma}\right)\alpha r \sin\theta \tag{3.43}$$

and the new boundary becomes:

$$r'(\theta) = r(\theta)\left[1 + \alpha\cos\sigma t - 2\Omega^2\lambda\cos^2\theta\cos\sigma t\right] \tag{3.44}$$

with r(θ) given by Eq. (3.29).

We remark that, although our solutions are obtained by inspection from the θ-equation (3.10), they satisfy the boundary conditions that $\underset{\sim}{\xi} = 0$ at the center, and give a surface on which the pressure vanishes.

Furthermore, as remarked earlier on P.21, we can substitute the solution for ξ from Eq. (3.41) - (3.43) into the r-equation (3.9), which would then yield an expression for σ_2^2:

$$-\alpha\sigma_2^2 + 2\sigma_0^2\lambda = -2k_2 + (1-3\gamma)\alpha + \frac{8}{3}\gamma\pi G\rho\lambda \tag{3.45}$$

On substituting for k_2 and λ the values as given by Eqs. (3.39) and (3.40), we have:

$$\sigma_2^2 = \frac{2}{3}(5-3\gamma) \tag{3.46}$$

which agrees with the corresponding expression given by Ledoux (1945).

With Eqs. (3.41) - (3.44), the quadrupole moment tensor can be readily evaluated. Limiting ourselves only to terms up to Ω^2, we have for the time-dependent part:

$$D_{33} = -\frac{8}{15}\pi\rho\alpha R^5 \cos\sigma t\left(\frac{45}{4}\gamma - \frac{9}{5}\right)\left(\frac{\Omega^2}{\sigma_0^2}\right) \tag{3.47}$$

We note here that D_{33} is of order Ω^2, which is expected since the departure from sphericity is brought about by rotation. Furthermore, $D_{33} \propto \alpha$ which is also expected, since a purely rotating axisymmetric object with no pulsation should not give any non-vanishing, time-dependent quadrupole moment.

With Eq. (3.47), we have from Eq. (1.5) for the time averaged energy loss:

$$-\frac{d\mathcal{E}}{dt} = \frac{1}{375}\frac{G}{c^5} m^2 R^4 \alpha^2 \sigma_0^6 \left(\frac{45}{4}\gamma - \frac{9}{5}\right)^2 \left(\frac{\Omega^2}{\sigma_0^2}\right)^2 \tag{3.48}$$

Since no angular momentum is lost in this case because of axi-symmetry, the angular velocity Ω will remain constant. Thus, all the energy loss is at the expense of oscillation. With ξ given by Eq. (3.41) - (3.43), we have for the total pulsation energy:

$$\mathcal{E} = \frac{3}{20} m R^2 \alpha^2 \sigma_0^2 \tag{3.49}$$

A comparison of Eqs. (3.48) and (3.49) shows that the energy decreases exponentially, given by:

$$\alpha(t) = \alpha(0)\, e^{-\kappa_0 t} \tag{3.50}$$

where $$K_0 = \frac{2}{225} \frac{GmR^2}{c^5} \Omega^4 \left(\frac{45}{4}\gamma - \frac{9}{5}\right)^2 \tag{3.51}$$

Again, using our typical neutron star, and a $\gamma \sim 1.5$ as a crude compromise between $\gamma = \frac{4}{3}$ for an extremely relativistic, completely degenerate fermi-gas, and $\gamma = \frac{5}{3}$ for a perfect, monoatomic gas, we obtain for K_o:

$$K_0 = 1.1 \times 10^{-14} \Omega^4 \tag{3.52}$$

Using $\sigma_o = 3 \times 10^3$ cps (Tsuruta, Wright, and Cameron, 1965), we find that for $\Omega = 10^2 \text{ sec}^{-1}$, the oscillation energy is reduced to insignificance after a time of about one year. Thus, radial oscillation is damped away extremely rapidly if there is a coupling with rotation.

For this and other points, we shall discuss in more detail in the next section.

IV. Results of Graviational Radiation Calculation and Consequences

We now assemble here for discussion the various expressions derived in the last two sections for damping of oscillations. We remember that we take for a typical neutron star an object of mass m = 1 M☉ and radius R = 10^6 cm. For the oscillation frequencies, we use $\sigma = 3 \times 10^3$ cps for the radial mode (Tsuruta, Wright and Cameron, 1965) and the expression (2.17) for an incompressible fluid for the non-radial ones.

For the P_2-oscillation, Eqs. (2.15) and (2.18) gives:

$$\alpha_{20}(t) = \alpha_{20}(0)\, e^{-K_2 t}$$

where $K_2 = 1.2 \times 10^1$ sec^{-1}.

For the P_4-oscillation, Eqs. (2.21) and (2.22) give:

$$\alpha_{40}^2(t) = \frac{\alpha_{40}^2(0)}{1 + K_4 \alpha_{40}^2 t}$$

where $K_4 = 4.3 \times 10^3$ sec^{-1}.

For rotation and nearly radial oscillations, we have Eq. (4.50) and (4.51)

$$\alpha(t) = \alpha(0)\, e^{-K_0 t}$$

where $K_o = 1.1 \times 10^{-14}\, \Omega^4$.

From these expressions, we can easily obtain an upper limit to the total oscillation energy after any length of time by taking the original amplitude to be unity. We remark that we have assumed uniform density for the equilibrium configuration in our work, and this is of course not true for the neutron star. However, the departure from uniformity is not too big, except for the outermost regions, and our calculations would still give a very good idea under what conditions the oscillation energy can still be large enough to be of interest in phenomena in supernova remnants. The results of damping in the various cases are tabulated as shown in Table 1. From the table, we see that the only significant surviving mode after a time of 10^3 years would be radial modes if $\Omega \lesssim 10\ \mathrm{sec}^{-1}$. This is indeed a very small angular velocity, and by comparison, the angular velocity of bifurcation for an incompressible object is about $3.5 \times 10^3\ \mathrm{sec}^{-1}$, while the centrifugal acceleration at the equator would become comparable to the gravitational acceleration for Ω about equal to $1.4 \times 10^4\ \mathrm{sec}^{-1}$.

TABLE I

Energy Remaining in the Various Modes at Different Times as a Result of Graviational Radiation Damping

Mode	Oscillation Energy (in ergs) at Time:			
	Initial ($\alpha = 1$)	1 Day	1 Year	10^3 Years
$P_o(\Omega = 10^2\ sec^{-1})$	2.7×10^{51}	2.3×10^{51}	6.0×10^{22}	0
$P_o(\Omega = 10\ sec^{-1})$	2.7×10^{51}	2.7×10^{51}	2.7×10^{51}	4.1×10^{48}
$P_o(\Omega = 1\ sec^{-1})$	2.7×10^{51}	2.7×10^{51}	2.7×10^{51}	2.7×10^{51}
* $P_o(\Omega = 0$ ν- processes)	2.7×10^{51}	2.1×10^{49}	3.0×10^{48}	3.0×10^{47}
P_2	1.8×10^{52}	0	0	0
P_4	1.4×10	3.8×10^{43}	1.1×10^{41}	1.1×10^{38}

* Taken from Finzi, A., P. R. L. 15, 509 (1965).

Thus we see that a small angular velocity is required for effective energy storage in the radial modes of oscillation, and this poses a serious problem with respect to the angular momentum of the neutron star. If we assume that the neutron star of $\Omega \sim 10$ cps is formed by contraction from an original star of mass = 1 Mo and radius ~ 10^{11} cm, then, if angular momentum is conserved in the process, the angular velocity of the original star would have to be 10^{-9} sec^{-1}, very much slower than that for normal stars. It thus appears that to save the situation, i.e. to have a significant amount of energy stored in the radial modes of a neutron star about 10^3 years after its formation, a very significant amount of angular momentum has to be lost.

One mechanism of slowing down the spin of the neutron star is suggested by Cameron and Tsuruta (1966) in which there is a loss of angular momentum when an external magnetic field exerted a torque on mass flowing away in the form of solar wind. Using the estimate of Woltjer (1964) that a neutron star may be formed with a magnetic field of the order of $10^{14} \rightarrow 10^{16}$ gauss, they estimate that the spin can be eliminated in a time scale of seconds.

On the other hand, if a fast rotating neutron star can form a Jacobi-type configuration of large asymmetry, then the resultant mass quadrupole formed can lead to a very rapid damping by emission of gravitational radiation, as discussed in §II.

Of course, this by no means takes away all the spin, for with the loss of angular momentum, the rotating configuration gradually loses its asymmetry. If finally arrives at an axisymmetric shape, when no further loss is possible and still with an angular velocity far too large for effective energy storage.

Cameron and Tsuruta (1966) have considered such a possibility and they tackled the problem in an approximate manner by the assignment of effective polytropic index. It is known that for a classical polytrope of index less than about 1 (James, 1964, Roberts 1963), a point of bifurcation does exist on the sequence of axisymmetric equilibrium configuration, and for sufficiently fast rotation, the equilibrium shape can be one of non-axisymmetry. With this in mind Tsuruta and Cameron assigned rough polytropic indices to their models of neutron stars by comparing the ratio of central to mean energy densities for their models with those for polytropes. Using nuclear interaction potentials in their equation of state, they have found that the effective polytropic index of their models with a mass in the vicinity of one solar mass is indeed near or below unity. Hence they conclude that there is a good chance that neutron stars can deform into Jacobi type configurations if they are spinning fast enough.

We wish to remark that the whole problem of bifurcation is actually a very much more involved one, and the above result is still inconclusive. We shall give in the next section a brief survey of the bifurcation problem in the classical case, and employ another method of assigning effective polytropic index. Then we shall point out that a full discussion should at least be attempted in the PNA, and we shall developed the first few steps that can lead to the final solution.

V. The Problem of Bifurcation in the Classical Case

The question of whether a point of bifurcation can exist along a sequence of axisymmetric configurations has been answered for an incompressible fluid in the classical case by Chandrasekhar and Lebovitz (1962a). They pointed out that the equations characterizing the state of equilibrium "provide no substance at all to the common expectation that symmetry about the rotational axis should be associated with any form produced by a rotational field." However, when $\Omega^2 \rightarrow 0$, an obvious way to satisfy the equation of equilibrium identically is by requiring axisymmetry. Hence the equilibrium shape is that of an axisymmetric, Maclaurin spheroid (Lamb, 1932). However, as Ω^2 increases and the configuration departs from sphericity, a point could be reached when it is possible to satisfy the equilibrium equation without the assumptions of axisymmetry. For an object rotating uniformly about the z-axis, the general condition for the occurrence of a point of bifurcation along a sequence of axisymmetric configurations is:

$$\Omega^2 I_{11} = W_{12;12} \tag{5.1}$$

where

$$I_{11} = \int \rho x_1^2 \, d\underline{x} \tag{5.2}$$

is the moment of inertia tensor, $W_{12;\,12}$ is a supermatrix defined as:

$$W_{pq;ij} = \int \rho x_p \frac{\partial V_{ij}}{\partial x_q} d\underset{\sim}{x} \tag{5.3}$$

with the potential tensor V_{ij} defined as:

$$V_{ij} = \int \frac{\rho(\underset{\sim}{x}')(x_i - x_i')(x_j - x_j')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' \tag{5.4}$$

For an incompressible fluid, simple analytic solutions can be obtained for I_{ii} and $W_{pq;ij}$ and (5.1) can be easily solved. This gives an eccentricity $e_b = 0.8126700$, and since this is smaller than $e_c = 0.9528867$ when ordinary instability of the Maclaurin spheroid sets in, one can therefore conclude that a bifurcation point does exist for an incompressible fluid.

But a neutron star is not incompressible. At the least, it should be approximated to by a polytrope with an index to be assigned. Now this problem of whether a polytrope of a given index n can possess a bifurcation point is much harder to analyse than the corresponding case for an incompressible fluid. One first has to determine the structure of a fast rotating polytrope, and this is no simple matter in itself. Then one has to determine whether the required angular velocity Ω for bifurcation for a given index n is larger or smaller than the critical velocity Ω_c when equilibrium is no longer possible before one can say bifurcation actually takes place or not. We shall postpone a more detailed discussion of this to the next section, and we shall just quote here that it was found that polytropes with an index $n \leq n_c = 0.808$

possess a point of bifurcation.

Assuming that we can treat a neutron star within this classical framework, we must first of all decide on how to assign an effective polytropic index to the star. As described earlier, Cameron and Tsuruta made the assignment by a comparison of ratios of central densities to mean density. They obtain a result of $n \leq 1$ for those models with a mass in the range 1 - 2 $M_{\odot}$ (depending on model). Now a system of degenerate fermions can be regarded as having an effective polytropic index of 1.5, and thus it seems from their calculations that the effects of general relativity and nuclear interactions can lower the effective index n_{eff} to ≤ 1. We shall examine this problem in more details here.

Effects due to general relativity alone increase the central condensation, but except for large relativity parameter q (defined to be the ratio of central pressure to central energy density) and high index n, the change is quite small. This can be seen from the following table, taken from Tooper (1964).

	$\rho_c/\bar{\rho}$		
n	q = 0.0	q = 0.1	q = 0.4
1	3.2899	3.342	3.736
1.5	5.9907	6.310	8.326
2.0	11.403	12.99	24.41
2.5	23.406	31.180	129.60
3.0	54.180	98.35	4177.0

Furthermore, since $\rho_c/\bar{\rho}$ is increased by general relativistic effects, one would be led to a larger n_{eff} than is actually the case if one makes the assignment by comparing $\rho_c/\bar{\rho}$ for a model with the corresponding value for the non-relativistic polytrope, as Cameron and Tsuruta have apparently done.

We therefore are of the opinion that relativity-effects are not the ones lowering n_{eff}. Instead, interactions brought about by a nuclear-potential would probably account for the lowering. In fact, when one takes a Harrison-Wheeler-Wakaons type of equation of state (representing matter by a non-interacting mixture of election, proton and neutron gases; more about this later), then $\rho_c/\bar{\rho}$ ranges from 2.7×10^3 to 12 from one end of the stable mass spectrum to the other (Wheeler, 1966). For a Skyrme (1959), Cameron (1959), Saakyam (1963) type, representing matter by a gas of neutrons interacting through an emprically derived nuclear potential, the corresponding range is 8.1 → 2.8. One would certainly arrive at quite a different n_{eff} when one compares the above figures with the corresponding ones for a polytrope.

There is, however, another method for the assignment of effective polytropic index. By definition, a polytrope is described by the following pressure-density relation:

$$P = k\rho^{1+\frac{1}{n}} \tag{5.5}$$

where k is a constant and n is the polytropic index. Thus:

$$1 + \frac{1}{n} = \frac{d \ln P}{d \ln \rho} \tag{5.6}$$

Graphs of $\ln P$ against $\ln \rho$ can be found in the works of Cameron and Tsuruta (1966a, b), and we have made some rough estimate of the slopes from the curves. For the two types of nuclear potentials, designated by V_p and V_γ in their work, there is a certain threshold density below which the slope corresponds to that of a polytrope with $n \sim 1.5$, and above which the corresponding value for n is $\lesssim 1$. This may be understood on the ground that nuclear potentials only come in significant at high density. At low density they become unimportant, and we simply have a collection of fermions for our system. For the V_γ potential, the threshold density is about 10^{14} gm/c.c., whereas that for the V_β is $\sim 10^{15}$ gm/c.c. Now for a typical neutron star described by a V_β type of equation of state (with $m = 0.97\ M_\odot$ and $R \sim 6$ km), the density is about $10^{15.5} - 10^{16}$ gm/c.c. throughout the star (except, of course, very near the surface); whereas one described by a V_γ equation of state (with $m = 2M_\odot$, $R = 10$ km) has a corresponding range from $10^{14} - 10^{15}$ gm/c.c. In both cases, the density is above the threshold one, and as we have stated, the corresponding neff would be $\lesssim 1$. Thus, the two different methods of effective index assignment are consistent with each other.

To see further the effect of the nuclear potential terms, we turn to an equation of state in which they are completely neglected. This is the Harrison-Wheeler equation of state (Harrison, Thorne, Wakano and Wheeler, 1966), which applies to a mixture of three ideal fermi gases--electrons, protons and neutrons--in statistical equilibrium at absolute zero. By making use of the following assumptions:

a) electrical neutrality, i.e., number of electrons = number of protons, or equivalently, fermi momentum of proton = fermi momentum of electron,

b) neutrino neutrality, i.e. the particles are in β-equilibrium with fermi energy of electron and fermi energy of proton = fermi energy of neutron,

c) sum of pressure of the three kinds of particles is equal to the external pressure,

they succeed in obtaining for ρ = 4.63 X 10^{12} gm/c.c. and up:

$$P(g/\mathrm{cm.sec}) = k_1 \rho^{5/3} \left(k_2 + \rho^{5/9}\right)^{-6/5} \tag{5.7}$$

where
$$k_1 = 2.9959 \times 10^{20}$$
$$k_2 = 9.1208 \times 10^{18}$$

From Eq. (5.7), we immediately obtain:

$$\frac{d\ln P}{d\ln\rho} = \frac{5}{3} - \frac{2}{3}\,\frac{1}{1 + \frac{k_2}{\rho^{5/9}}} \tag{5.8}$$

We evaluated the above for $\rho = 10^{14}$, 10^{15}, and 10^{16} gm/c.c. We then find, from Eq. (5.6), the values for n to be 1.59, 1.82 and 2.63 respectively. Thus, they come far short of the $n \leq 1$ necessary for the existence of a bifurcation point. This, however, is not surprising, since we are just considering a collection of non-interacting fermi particles.

It is not known what should be the "exact" equation of state. Despite the apparent unreality in the above equation of state, it is not easy "to make any improvement (in this equation of state) which will be at the same time substantial and reliable" (Harrison, Thorne, Wakans, Wheeler, 1965, p. 121). Furthermore, the masses and radii of stellar models based on various equations of state differ only by factors less than three, e.g.

Eq. of State	$\log \rho_c$	M/M_θ	R(km)
HWW	15	0.6	10.5 *
V_β	14.9626	0.4	11.66†
V_γ	15.0359	1.5	11.9 †

*Meltzer & Thorner, Ap. J. 145, 514 (1966) Fig. 2

†Camerone & Tsuruta, Can. J. Physics, 44, 1863 (1966) Table 4

However, for density in the range $10^{14} - 10^{16}$ gm/c.c, it seems appropriate to take into account in some manner the effects of nuclear interaction. In this aspect, the HW equation of state seems less realistic then the V_β or V_γ ones, although the latter ones are by no means completely reliable.

Thus nuclear potentials seem to be capable of depressing the effective polytropic index to a value when bifurcation is possible, provided that the density is high enough. This comes from the fact that the two potentials become highly repulsive at high densities, and hence disfavor the crowding together of matter and hence leading to a low central condensation. Since V_γ becomes much more repulsive than V_β at high densities, it should follow that a V_γ type model stands more chance for bifurcation than a V_β type. This is confirmed from the results of Camerone and Tsuruta (1966b).

We close this section with the observation that classically, it seems possible for a neutron star to exhibit a point of bifurcation. We must caution, however, that this depends very much on the equation of state used. Also, we remark that a typical neutron star of mass $M = 1\ M_\odot$ and radius $R = 10$ km carries a relativity parameter $GM/RC^2 \sim 0.14$ and hence should not be treated within the framework of classical hydrodynamics. At the least, we should examine the problem in the PNA recently developed by Chandrasekhar (1965).

To do this, we turn our attention to the structure of a rotating polytrope, which we shall determine using a method first developed by Roberts (1963) in the classical case. This we shall review in some details in the next section.

VI. Structure of a Rotating Polytrope—classical case

The structure of a slowly rotating polytrope has been considered by Chandrasekhar and Lebovitz (1962c), when it is possible to arrive at an analytic solution by expansion in terms of the rotation parameter. Bifurcation, however, certainly does not take place at small rotation. Thus, our interest is primarily that of a highly rotating polytrope which has been studied with different methods and degrees of accuracy by James (1964) and Roberts (1963). The latter used a very approximate method which is nevertheless extremely simple and easy, at least for the classical case. James approach, on the other hand, is direct and exact, and though we shall not use it here, we shall list it out, since it is certainly the best and most correct one to use.

James considered the axisymmetric case first, and determined its structure by expanding the density and potential near the centre of mass in a power series in the radial variable (the coefficients themselves are expanded in terms of the Legendre polynomials), and then using analytic continuation and step-by-step integration for the rest of the object. The end of an axisymmetric sequence is defined by the vanishing of the effective gravity g_e at the equator. He then considered a small non-axisymmetric perturbation and arrived at a condition for the existence of such a non-axisymmetric form adjacent to the axisymmetric one. This is used in conjunction with $g_e = 0$ to see whether bifurcation occurs before or after the end of the axisymmetric sequence. The result is that bifurcation is possible for polytropes with an index $n \lesssim n_c = 0.808$.

Robert's approach is a variational one, and the procedure is to minimize the total energy while keeping the mass and the angular momentum constant. This is used to pick the "best" trial function from a particular class of trial-function which, in Robert's case, is the class of all spheroidal distributions in which the equidensity surfaces are similar and similarly situated concentric spheroids.

Now for a rotating polytrope with a pressure (p), density (ρ) relation described by

$$p = K\rho^{1+\frac{1}{n}} \tag{6.1}$$

(where n, K are constants), the total energy E can be written as:

E = rotational energy + internal energy + graviational energy

$$= \frac{1}{2}\Omega^2\int \rho\varpi^2 d\underset{\sim}{x} + nK\int \rho^{1+\frac{1}{n}} d\underset{\sim}{x} + \left(-\frac{1}{2}\int \rho U d\underset{\sim}{x}\right)$$

$$= \frac{1}{2}\Omega^2 \mathrm{I} + \mathrm{II} - \frac{1}{2}\mathrm{III} \quad \text{, say} \tag{6.2}$$

where Ω is the angular velocity, ϖ the distance from the axis of rotation and U the gravitaional potential. I, II and III represent the various integrals, a notation very convenient later in the PNA.

The angular momentum $\underset{\sim}{L}$ and mass M are

$$L = \Omega\int \rho\varpi^2 d\underset{\sim}{x} = \Omega \mathrm{I} \tag{6.3}$$

$$M = \int \rho\, d\underset{\sim}{x} \tag{6.4}$$

Suppose now there is a change $\delta\rho(x)$ in $\rho(x)$ such that $\underset{\sim}{L}$ and M are unaltered, ie.

$$\delta L = \delta\Omega \int \rho \varpi^2 d\underset{\sim}{x} + \Omega \int \delta\rho\, \varpi^2 d\underset{\sim}{x} \\ = 0 \tag{6.5}$$

and
$$\delta M = \int \delta\rho\, d\underset{\sim}{x} = 0 \tag{6.6}$$

We suppose also that the corresponding change δU in U is given by Poisson's equation

$$\nabla^2 (\delta U) = -4\pi G (\delta\rho) \tag{6.7}$$

Then:

$$\begin{aligned} -\tfrac{1}{2}\delta \mathrm{III} &= -\tfrac{1}{2} \int (U\delta\rho + \rho\delta U)\, d\underset{\sim}{x} \\ &= -\int U \delta\rho\, d\underset{\sim}{x} + \delta Q \\ &= -\int U\delta\rho\, d\underset{\sim}{x} \end{aligned} \tag{6.8}$$

Since
$$\delta Q = \tfrac{1}{2} \int_{\text{all space}} (U\delta\rho - \rho\delta U)\, d\underset{\sim}{x} \\ = 0 \tag{6.9}$$

where use has been made of Poisson's equation and Gauss divergence theorem.

Thus, the change in E due to a change in ρ is given by:

$$\begin{aligned} \delta E &= \tfrac{1}{2}(2\Omega\,\delta\Omega\, \mathrm{I} + \Omega^2 \delta \mathrm{I}) + \delta \mathrm{II} - \tfrac{1}{2}\delta \mathrm{III} \\ &= \int \left[(n+1) K\rho^{\frac{1}{n}} - U - \tfrac{1}{2}\Omega^2 \varpi^2 \right] \delta\rho\, d\underset{\sim}{x} \end{aligned} \tag{6.10}$$

where $\delta\Omega$ has been eliminated $\delta L = 0$.

To take the remaining constraint $\delta M = 0$ into account, we introduce a Lagrangian multiplier Λ, and demand that

$$\delta(E - \Lambda M) = 0$$

which becomes, from Eqs. (6.4) and (6.10)

$$\int \left[(n+1) K \rho^{\frac{1}{n}} - U - \tfrac{1}{2}\Omega^2 \varpi^2 - \Lambda \right] \delta\rho \, d\underset{\sim}{x} = 0 \quad (6.11)$$

Foran arbitrary $\delta\rho$, we therefore have:

$$(n+1) K \rho^{\frac{1}{n}} = U + \tfrac{1}{2}\Omega^2 \varpi^2 + \Lambda \quad (6.12)$$

which is just the equation of hydrostatic equilibrium for a polytrope (Chandrasekhar 1962c). Incidentally, we notice here that $(-\Lambda)$ is just the graviational potential at the pole of the configuration.

The variational principle is then applied to a trial function:

$$\rho = \rho(\eta) \quad (6.13)$$

where

$$\eta^2 = x_1^2 + x_2^2 + \frac{x_3^2}{(1-e^2)} \quad (6.14)$$

with the eccentricity e = constant.

The theory of an inhomogenous ellipsoid in which the equidensity surfaces are similar and similarly situated ellipsoids have been discussed in great detail by Roberts (1962). Using those results and defining

$$F(\eta) = \int_{\eta^2}^{\eta_0^2} \rho(\eta) \, d\eta^2 \quad (6.15)$$

where η_o is the surface of the spheroid defined by $\rho(\eta_o) = 0$, we find

$$\frac{1}{2}\Omega^2 I = \frac{4\pi}{3}\Omega^2\sqrt{1-e^2}\int_0^{\eta_o}\rho\,\eta^4\,d\eta \tag{6.16}$$

$$II = 4\pi n K\sqrt{1-e^2}\int_0^{\eta_o}\rho^{1+\frac{1}{n}}\eta^2\,d\eta \tag{6.17}$$

$$-\frac{1}{2}III = -\frac{2\pi^2 G}{e}(1-e^2)\sin^{-1}e\int_0^{\eta_o}[F(\eta)]^2\,d\eta \tag{6.18}$$

$$M = 4\pi\sqrt{1-e^2}\int_0^{\eta_o}\rho\,\eta^2\,d\eta \tag{6.19}$$

$$L = \frac{8\pi}{3}\Omega\sqrt{1-e^2}\int_0^{\eta_o}\rho\,\eta^4\,d\eta \tag{6.20}$$

We next consider variations w.r.t. ρ, Ω and e, and we eliminate $\delta\Omega$ by $\delta L = 0$. Following the approach just described and equating the coefficients of $\delta\rho$ and δe separately to zero, we finally obtain the basic equation governing ρ:

$$(n+1)K\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\rho^{\frac{1}{n}}}{d\eta}\right) = -\left[\frac{4\pi G}{e}\sqrt{1-e^2}\sin^{-1}e\right]\rho + 2\Omega^2 \tag{6.21}$$

and the equation relating Ω^2 and e^2:

$$\Omega^2 = \frac{\pi G\sqrt{1-e^2}}{2e^3}\left[(3-2e^2)\sin^{-1}e - 3e\sqrt{1-e^2}\right]\frac{\int_0^{\eta_o}[F(\eta)]^2\,d\eta}{\int_0^{\eta_o}\eta^2 F(\eta)\,d\eta} \tag{6.22}$$

Eq. (6.22) can also be obtained in a more direct manner using the virial theorem (Chandrasekhar and Lebovitz, 1962).

In keeping with the theory of polytropes as usually presented, we now introduce the dimensionless variables λ, θ, ξ defined by:

$$\rho = \lambda \theta^{n} \tag{6.23}$$

$$\eta = \alpha \xi \tag{6.24}$$

where
$$\alpha^2 = \frac{(n+1)K\lambda^{-1+\frac{1}{n}}}{4\pi G} f(e) \tag{6.25}$$

with
$$f(e) = \frac{e}{\sqrt{1-e^2}\,\sin^{-1} e} \tag{6.26}$$

Furthermore, the rotational parameters w and v are defined by:

$$v = \frac{\Omega^2}{2\pi G \lambda} \tag{6.27}$$

and $$w = v\, f(e). \tag{6.28}$$

With all this, Eq. (6.21) becomes:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^{n} + w \tag{6.29}$$

and (6.22) can be rewritten as:

$$\frac{3}{e^2} - \lambda - \frac{3\sqrt{1-e^2}}{e \sin^{-1} e} = G(w) \tag{6.30}$$

where

$$G(w) = \frac{\lambda w^2 \xi_0^5 - 10\, w\, \xi_0^4 \theta'(\xi_0) - 60\, w\, J_1}{\lambda w^2 \xi_0^5 - 10\, w\, \xi_0^4 \theta'(\xi_0) + 15\, \xi_0^3 [\theta'(\xi_0)]^2 + 15\, w J_1 + 15 J_2} \tag{6.31}$$

with
$$J_1 = \int_0^{\xi_0} \theta\, \xi^2 d\xi \tag{6.32}$$
$$J_2 = \int_0^{\xi_0} \xi^2 \theta^{n+1} d\xi$$
$$= \frac{(n+1)}{(5-n)} \left[\xi_0^3 \, [\theta'(\xi_0)]^2 + 5 w J_1 \right] \tag{6.33}$$

and ξ_o is the first zero of the solution of Eq. (6.29) for θ.

The most convenient procedure of solution is the following:

(a) Take a $w < 1$

(b) Solve Eq. (6.29) for θ, subject to $\theta(0) = 1$ and $\theta'(0) = 0$. Let ξ_o be the first zero of this solution.

(c) Evaluate $G(w)$ from Eq. (6.32) using the now known $\theta(\xi)$ and ξ_o.

(d) By inverse interpolation , e can be determined from Eq. (6.29)

(e) v can then be obtained from Eq. (6.28).

Thus, for any given v, one can find the corresponding eccentricity. Furthermore, it is found that solutions exist only in a limited range

$$0 < v < v_c \tag{6.34}$$

We now apply to our homoeoidally striated spheroids the criterion for the existence of a point of bifurcation along a sequence of uniformly rotating axisymmetric bodies. From Chandrasekhar and Lebovitz, (1962a) we have:

$$\Omega^2 I_{11} = W_{33} - W_{11} = W_{12;12} \qquad *$$

where W_{ii}'s are the potential energy tensors and the other symbols are as defined in §V. For our particular density distribution, all the quantities can be readily evaluated and we have after simplification

$$(3+8e^2-8e^4)\sin^{-1}e = e\sqrt{1-e^2}\,(10e^2+3) \qquad \#$$

for the eccentricity of the spheroid at bifurcation. Two distinct characters of this condition # stand out. First of all, it is the same as that for the homogenous spheroid. Secondly, it is independent of $\rho(\eta^2)$. Thus, * alone cannot be used to determine whether bifurcation actually takes place. We have to determine, in addition, the eccentricity (or angular velocity) when equilibrium is no longer possible and compare it with that at bifurcation before we can say whether an object can actually bifurcate or not.

In the particular case n = 1 (which shall be the one under study in the PNA) Eq. (6.29) can be solved exactly to give:

$$\theta(\xi) = w + (1-w)\frac{\sin\xi}{\xi} \tag{6.35}$$

This function gives $\theta(\xi_o) = \theta'(\xi_o) = 0$ for $w = w_c = 0.178465$ at $\xi_o = 4.493409$. From the procedure of calculations just described, we readily find that $v_c = 0.106757$, $e_c = 0.86988$. Since $e_c > e_b$, it appears therefore that according to this variational approach, bifurcation would indeed occur for the n= 1 polytrope before the termination of the axisymmetric equilibrium sequence.

This result seems to be at variance with that arrived at by James ($n_c = 0.808$). This should, however, be expected from the very nature of our approach. Roberts himself pointed out that the variational calculation must certainly overestimate v_c, because a system loses equilibrium essentially through a local violation of the hydrostatic equation at the equator. The variational calculation, on the other hand, satisfies the hydrostatic equation in an integrated way, and the assumption of homoeoidally straited equidensity surfaces will artificially prevent matter from being ejected from the equator for as long as the rest of the surface can constrain it to the body.

With this note of caution, we shall now turn to the generalization of this approach to PNA.

VII. Structure of a Rotating Polytrope in the Post-Newtonian Approximation—a variational formulation

We shall show that a principle similar to Roberts can be formulated in the framework of the PN equations derived by Chandrasekhar (1965). In that approximation, the expressions for energy E, mass M and angular momentum M are given by

$$\begin{aligned} E = {} & \frac{1}{2}\Omega^2\int\rho\varpi^2 d\underset{\sim}{x} + \int\rho\Pi\, d\underset{\sim}{x} - \frac{1}{2}\int\rho\mathfrak{V} d\underset{\sim}{x} \\ & + \frac{1}{c^2}\Big[\frac{5}{8}\Omega^4\int\rho\varpi^4 d\underset{\sim}{x} + \frac{5}{2}\Omega^2\int\rho\varpi^2\mathfrak{V} d\underset{\sim}{x} \\ & + \Omega^2\int\rho\varpi^2\left(\Pi+\frac{P}{\rho}\right) d\underset{\sim}{x} + 2\int\rho\mathfrak{V}\Pi\, d\underset{\sim}{x} \\ & - \frac{5}{2}\int\rho\mathfrak{V}^2 d\underset{\sim}{x} - 2\Omega^2\int\rho\varpi^2\mathfrak{D}\, d\underset{\sim}{x}\Big] \end{aligned} \tag{7.1}$$

$$M = \int\rho\, d\underset{\sim}{x} + \frac{1}{c^2}\Big[\frac{1}{2}\Omega^2\int\rho\varpi^2 d\underset{\sim}{x} + 3\int\rho\mathfrak{V} d\underset{\sim}{x}\Big] \tag{7.2}$$

$$\begin{aligned} L = {} & \Omega\int\rho\varpi^2 d\underset{\sim}{x} + \frac{1}{c^2}\Big[\Omega^3\int\rho\varpi^4 d\underset{\sim}{x} + 6\Omega\int\rho\varpi^2\mathfrak{V} d\underset{\sim}{x} \\ & + \Omega\int\rho\varpi^2\left(\Pi+\frac{P}{\rho}\right) d\underset{\sim}{x} - 4\Omega\int\rho\varpi^2\mathfrak{D}\, d\underset{\sim}{x}\Big] \end{aligned} \tag{7.3}$$

where $\rho\Pi = nP$ is the internal energy for unit volume, $\mathfrak{D}$ is given for the axisymmetric case by

$$\mathfrak{D} = \frac{\mathfrak{D}_1}{x_1} = \frac{\mathfrak{D}_2}{x_2} \tag{7.4}$$

where $$\nabla^2 \mathcal{V}_i = -4\pi G \rho X_i \qquad (7.5)$$

and the other notations are the same as those defined in §VI.

As in §VI, we not write, following the notation of Krefetz (1967):

$$E = \frac{1}{2}\Omega^2 \mathrm{I} + \mathrm{II} - \frac{1}{2}\mathrm{III} + \frac{1}{c^2}\Big[\frac{5}{8}\Omega^4 \overline{\mathrm{IV}} + \frac{5}{2}\Omega^2 \overline{\mathrm{V}} + \Omega^2 \overline{\mathrm{VI}} + 2\overline{\mathrm{VII}} - \frac{5}{2}\overline{\mathrm{VIII}} - 2\Omega^2 \overline{\mathrm{IX}}\Big] \qquad (7.6)$$

$$M = M_0 + \frac{1}{c^2}\left(\frac{1}{2}\Omega^2 \mathrm{I} + 3\,\mathrm{III}\right) \qquad (7.7)$$

$$L = \Omega \mathrm{I} + \frac{\Omega}{c^2}\left(\Omega^2 \overline{\mathrm{IV}} + 6\,\overline{\mathrm{V}} + \overline{\mathrm{VI}} - 4\,\overline{\mathrm{IX}}\right) \qquad (7.8)$$

where I, II, . . ., IX represent the different integrals in Equations (7.1) → (7.3).

Our variational approach consists in taking the variation of E, subject to the conditions that M and L remain constant. We proceed by eliminating $\delta\Omega$ with the aid of $\delta L = 0$, and then solving the remainder of problem by the Lagrangian method of undetermined multiplier. ($\delta E - \Lambda\,\delta M = 0$ with Λ being the Lagrangian multiplier).

The details of the calculations are given in Krefetz (1967), and hence we shall not repeat them here. The final result is:

$$-\frac{1}{2}\Omega^2\delta\mathrm{I} + \delta\mathrm{II} - \frac{1}{2}\delta\mathrm{III} + \frac{1}{c^2}\Big[-\frac{3}{8}\Omega^2\delta\overline{\mathrm{IV}} - \frac{7}{2}\Omega^2\delta\overline{\mathrm{V}} + 2\delta\overline{\mathrm{VII}} - \frac{5}{2}\delta\overline{\mathrm{VIII}} + 2\Omega^2\delta\overline{\mathrm{IX}}\Big]$$
$$= \Lambda\Big[\delta M_0 + \frac{1}{c^2}\left(\frac{1}{2}\Omega^2\delta\mathrm{I} + 3\,\delta\mathrm{III}\right)\Big] \qquad (7.10)$$

Fora completely arbitrary $\delta\rho$, Eq. (7.10) should led to the hydrostatic equation in the PNA (Krefetz, 1966):

$$\Psi = \Pi + \frac{p}{\rho} - U - \frac{1}{2}\Omega^2\varpi^2 + \frac{1}{c^2}\Big[\frac{1}{2}U^2 - (U + \tfrac{1}{2}\Omega^2\varpi^2)(\Pi + \frac{p}{\rho}) - \frac{1}{2}(\tfrac{1}{2}\Omega^2\varpi^2)^2 - 3U(\tfrac{1}{2}\Omega^2\varpi^2) - 2\Phi + 4\Omega^2\varpi^2\mathcal{D}\Big]$$

$$= \text{constant} \tag{7.11}$$

where Φ is defined by

$$\nabla^2\Phi = -4\pi G\rho\,\phi \tag{7.12}$$

and

$$\phi = \Omega^2\varpi^2 + U + \frac{1}{2}\Pi + \frac{3}{2}\frac{p}{\rho} \tag{7.13}$$

That this is indeed the case we now proceed to demonstrate.

Before considering the variation of the various integrals in Eq. (7.10), we first of all remark that we need only take the variation of the integrands because our integrand vanishes on the boundary.

We now consider the various terms in Eq. (7.10)

$$-\frac{1}{2}\Omega^2\delta\mathrm{I} = -\frac{1}{2}\Omega^2\int\varpi^2\delta\rho\, d\underset{\sim}{x} \tag{7.14}$$

$$\delta\mathrm{II} = \int\left(\Pi + \frac{p}{\rho}\right)\delta\rho\, d\underset{\sim}{x} \tag{7.15}$$

$$-\frac{1}{2}\delta\mathrm{III} = -\int U\delta\rho\, d\underset{\sim}{x} \tag{7.16}$$

$$2\delta\,\overline{\text{VII}} - \frac{5}{2}\delta\,\overline{\text{VIII}} = \int\left[5U\left(\Pi + \frac{P}{\rho}\right) - \frac{11}{2}U^2 - 2\Phi\right]\delta\rho\, d\underset{\sim}{x}$$
$$+ \frac{7}{2}\Omega^2\int\rho\varpi^2\delta U\, d\underset{\sim}{x} - \frac{3}{2}\Omega^2\int\varpi^2 U\delta\rho\, d\underset{\sim}{x} \tag{7.17}$$

where use has been made of:

$$\int\rho\phi\,\delta U\, d\underset{\sim}{x} = \int\Phi\,\delta\rho\, d\underset{\sim}{x}$$

(ref. Chandrasekhar, 1965, P. 1500, Lemma 3)

$$-\frac{3}{8}\Omega^4\delta\,\overline{\text{IV}} - \frac{7}{2}\Omega^2\delta\,\overline{\text{V}} = -\frac{3}{8}\Omega^4\int\varpi^4\delta\rho\, d\underset{\sim}{x}$$
$$-\frac{7}{2}\Omega^2\int\varpi^2 U\delta\rho\, d\underset{\sim}{x} - \frac{7}{2}\Omega^2\int\rho\varpi^2\delta U\, d\underset{\sim}{x} \tag{7.18}$$

$$2\Omega^2\delta\,\overline{\text{IX}} = 4\Omega^2\int\varpi^2\mathfrak{D}\,\delta\rho\, d\underset{\sim}{x} \tag{7.19}$$

(having used the same technique as that in getting Eq. (6.8)).

$$\Lambda\,\delta M_0 = \Lambda\int\delta\rho\, d\underset{\sim}{x} \tag{7.20}$$

$$\frac{1}{2}\Omega^2\Lambda\,\delta I = \frac{1}{2}\Omega^2\Lambda\int\varpi^2\delta\rho\, d\underset{\sim}{x} \tag{7.21}$$

$$3\Lambda\,\delta\,\overline{\text{III}} = 6\Lambda\int U\delta\rho\, d\underset{\sim}{x} \tag{7.22}$$

Putting the various terms (7.14) → (7.22) into Eq. (7.10), and using $\Pi + \frac{P}{\rho} - U - \frac{1}{2}\Omega^2\varpi^2 = \Lambda$ in the explicitly post-Newtonian terms, we get (we should have used the Newtonian Λ_0 instead of Λ, but this doesn't matter for the explicitly PN terms)

$$\delta E - \Lambda\,\delta M = \int \delta\rho\, d\underset{\sim}{x} \Big\{ \Pi + \frac{p}{\rho} - U - \frac{1}{2}\Omega^2 \varpi^2 + \frac{1}{c^2}\Big[-\frac{11}{2} U^2 + (5U - \Omega^2\varpi^2)\Big(\Pi + \frac{p}{\rho}\Big) + \frac{1}{8}\Omega^4\varpi^4 - 4U\Omega^2\varpi^2 - 2\Phi + 4\Omega^2\varpi^2 \mathfrak{D} \Big] - \Lambda - \frac{\Lambda}{c^2}\Big(6U - \frac{1}{2}\Omega^2\varpi^2\Big) \Big\}$$

$$= 0 \tag{7.23}$$

By a simple factorization, Eq. (7.23) can be written as:

$$\int \delta\rho\,(\Psi - \lambda)\Big[1 + \frac{1}{c^2}\Big(6U - \frac{1}{2}\Omega^2\varpi^2\Big)\Big] d\underset{\sim}{x} = 0 \tag{7.24}$$

where Ψ is the same expression as that defined in Eq. (7.11). Since Eq. (7.24) is required to be true for an arbitrary $\delta\rho$, we have $\Psi = \lambda$, and this is equivalent to the equation of hydrostatic equilibrium (7.11).

Having demonstrated that our variational procedure is really a correct one, we now proceed to apply Eq. (7.10) to a rotating polytrope. Since the whole calculation is long and messy, we first outline the steps as follows.

We shall assume with Roberts that the equidensity surfaces are similar and similarly situated spheroids. We therefore write:

$$\rho(\eta) = \rho_0(\eta) + \frac{1}{c^2}\sum_{i=1,\dots n} a_i \rho_i'(\eta) \tag{7.25}$$

where

$$\eta^2 = x_1^2 + x_2^2 + \frac{x_3^2}{(1-e^2)} \tag{7.26}$$

The constant e is the eccentricity for a particular solution, ρ is the trial function, and the a_i's are the variational parameters. A subscript zero indicates that the quantity so labelled is derived from the Newtonian solution. Thus, e.g., we shall write for the eccentricity e of the PN configuration:

$$e = e_o + \frac{1}{c^2} e' \tag{7.27}$$

and for the angular velocity Ω:

$$\Omega = \Omega_o + \frac{1}{c^2} \Omega' \tag{7.28}$$

Because of the complications caused by the various PN terms, we have not been able to solve the Euler-Lagrangian equations, as Roberts has done. Instead, we use the variational principle directly, and determine the parameters in our trial functions by minimizing the energy subject to the constraints as mentioned.

Remembering that $\delta\Omega$ has already been eliminated with $\delta L = 0$, we therefore have the following Eqs. from $\delta(E - \Lambda M) = 0$:

$$\frac{\partial}{\partial a_i}(E - \Lambda M) = 0 \qquad i = 1, . ., n \tag{7.29}$$

$$\frac{\partial}{\partial e}(E - \Lambda M) = 0 \tag{7.30}$$

For a given Newtonian configuration (hence known e_o, Ω_o, ρ_o), the above becomes (n + 1) equations in the (n + 3) unknowns a_i, e', Ω' and Λ.

We thus still have two more conditions at our liberty. We choose to take the mass as defined in (7.2) and eccentricity of the PN configuration to be the same as those of the Newtonian one. Thus, our problem is completely specified. We are comparing a PN polytrope with a neighbouring Newtonian one of the same eccentricity and mass. The variational equations then enable us to calculate the change in density distribution and angular velocity.

VIII. The Variational Equations in the Post-Newtonian Approximation

We shall now set up the various equations arising from the variational formulation. Let us first take Eq. (7.29), which is actually equivalent to writing out Eq. (7.23) where we now take

$$\delta\rho = \frac{\partial\rho}{\partial a_i}\delta a_i = \delta a_i \rho_i'$$

Thus, we have from Eq. (7.23):

$$\int \rho_i' d\underset{\sim}{x} \left\{ \Pi + \frac{p}{\rho} - U - \frac{1}{2}\Omega^2\varpi^2 + \frac{1}{c^2}F - \Lambda - \frac{\Lambda}{c^2}H \right\} = 0 \tag{8.1}$$

where F and H represent the explicitly Post-Newtonian terms in Eq. (7.23).

We now expand the various terms in order of $\frac{1}{c^2}$. Thus:

$$\begin{aligned} \Pi + \frac{p}{\rho} &= (n+1)K\rho^{\frac{1}{n}} \\ &= (n+1)K\rho_0^{\frac{1}{n}} + \frac{1}{c^2}\left(\frac{1}{\rho}\frac{\partial p}{\partial \rho}\right)_0 \sum a_i \rho_i' \end{aligned} \tag{8.2}$$

$$U(\underset{\sim}{x}) = G \int \frac{\rho(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}'$$

$$= G \int \frac{\rho_0(\underset{\sim}{x}') + \frac{1}{c^2} \sum a_i \rho_i'(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' \tag{8.3}$$

$$\Omega = \Omega_0 + \frac{1}{c^2} \Omega' \tag{8.4}$$

$$\Lambda = \Lambda_0 + \frac{1}{c^2} \Lambda' \tag{8.5}$$

Substituting into (8.1), we therefore have:

$$\int \rho_i' d\underset{\sim}{x} \left[(n+1) K \rho_0^{\frac{1}{n}} - U_0 - \frac{1}{2} \Omega_0^2 \varpi^2 - \Lambda_0 \right]$$

$$+ \frac{1}{c^2} \int \rho_i' d\underset{\sim}{x} \left[\left(\frac{1}{\rho} \frac{\partial p}{\partial \rho} \right)_0 \sum a_j \rho_j' - G \int \frac{\sum a_j \rho_j'(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d\underset{\sim}{x}' \right.$$

$$\left. - \Omega_0 \Omega' \varpi^2 + F_0 - \Lambda' - \Lambda_0 H_0 \right]$$

$$= 0 \tag{8.6}$$

where a subscript zero means that the quantity denoted can be evaluated usifng the Newtonian configuration. Since the integrand of the first integral vanishes identically, because of the classical hydrostatic equation, we therefore have:

$$\sum_j a_j \int \left(\frac{1}{\rho}\frac{\partial p}{\partial \rho}\right)_0 \rho_i' \rho_j' \, d\underset{\sim}{x} - G \sum_j a_j \iint \frac{\rho_i'(\underset{\sim}{x}) \rho_j'(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \, d\underset{\sim}{x} \, d\underset{\sim}{x}'$$
$$- \Omega_0 \Omega' \int \rho_i' \varpi^2 \, d\underset{\sim}{x} - \Lambda' \int \rho_i' \, d\underset{\sim}{x}$$
$$+ F_0 - \Lambda_0 H_0 = 0 \tag{8.7}$$

where

$$F_0 = \int \rho_i' \Big[-\tfrac{11}{2} \mathfrak{V}^2 + (5\mathfrak{V} - \Omega^2 \varpi^2)\left(\Pi + \frac{p}{\rho}\right) + \tfrac{1}{8} \Omega^4 \varpi^4 - 4 \mathfrak{V} \Omega^2 \varpi^2 - 2\Phi + 4 \Omega^2 \varpi^2 \mathfrak{D} \Big]_0 \, d\underset{\sim}{x} \tag{8.9}$$

$$H_0 = \int \rho_i' \left[6\mathfrak{V} - \tfrac{1}{2} \Omega^2 \varpi^2 \right]_0 d\underset{\sim}{x} \tag{8.10}$$

For the e-Eq. (7.30), we go back to Eq. (7.10) where now $\delta = \frac{\partial}{\partial e}$. Before any variation is possible, we have to write out the integrals in such a way that the e-dependence is explicit.

We first define

$$\begin{aligned} x_1 &= \eta_1 \\ x_2 &= \eta_2 \\ x_3 &= \sqrt{1-e^2}\, \eta_3 \end{aligned} \tag{8.11}$$

Then the volume element $d\underset{\sim}{x}$ would become $\sqrt{1 - e^2}\, d\underset{\sim}{\eta}$ and the spheroidal equidensity surfaces in $\underset{\sim}{x}$ co-ordinates would become spheres in the η co-ordinates, with a radius η^* say.

With this, the various integrals become:

$$\mathrm{I} = \int \rho \varpi^2 d\underset{\sim}{x} = \frac{8\pi}{3}\sqrt{1-e^2}\int_0^{\eta^*} \rho \eta^4 d\eta \tag{8.12}$$

$$\mathrm{II} = \int \rho \Pi\, d\underset{\sim}{x} = 4\pi n K \sqrt{1-e^2}\int_0^{\eta^*} \rho^{1+\frac{1}{n}} \eta^2 d\eta \tag{8.13}$$

$$\mathrm{III} = \int \rho \mathcal{V} d\underset{\sim}{x} = 4\pi^2 G \frac{(1-e^2)\sin^{-1} e}{e}\int_0^{\eta^*} F^2 d\eta \tag{8.14}$$

$$\text{where } F(\eta) = \int_\eta^{\eta^*} 2\rho(\eta')\eta' d\eta'$$

$$\mathrm{IV} = \int \rho \varpi^4 d\underset{\sim}{x} = \frac{32\pi}{15}\sqrt{1-e^2}\int_0^{\eta^*} \rho \eta^6 d\eta \tag{8.15}$$

$$\mathrm{V} = \int \rho \varpi^2 \mathcal{V} d\underset{\sim}{x} = 2\pi\sqrt{1-e^2}\int_0^{\eta^*}\int_0^{2\pi} \rho(\eta)\mathcal{V}(\eta,\theta)\,\eta^4 d\eta \sin^3\theta\, d\theta \tag{8.16}$$

$$\overline{\mathrm{VII}} = \int \rho \mathcal{V} \Pi \, d\underset{\sim}{x} = 2\pi n K \sqrt{1-e^2} \int_0^{2\pi}\!\!\int_0^{\eta^*} \rho^{1+\frac{1}{n}} \mathcal{V} \eta^2 d\eta \sin\theta \, d\theta$$

(8.17)

$$\overline{\mathrm{VIII}} = \int \rho \mathcal{V}^2 d\underset{\sim}{x} = 2\pi \sqrt{1-e^2} \int_0^{2\pi}\!\!\int_0^{\eta^*} \rho \mathcal{V}^2 \eta^2 d\eta \sin\theta \, d\theta \quad (8.18)$$

$$\overline{\mathrm{IX}} = \int \rho \varpi^2 \mathfrak{D} \, d\underset{\sim}{x} = 4\pi^2 G (1-e^2) \left[\frac{\sin^{-1} e - e\sqrt{1-e^2}}{e^3} \right] \int_0^{\eta^*} \eta^2 F^2 d\eta \quad (8.19)$$

$$M_0 = \int \rho \, d\underset{\sim}{x} = 4\pi \sqrt{1-e^2} \int_0^{\eta^*} \rho \, \eta^2 d\eta \quad (8.20)$$

The derivation of the above is all very straightforward, except for III and IX, which we calculated using a method developed by Roberts (1962). This is shown in Appendix B. The variation in e can now be simply written down as:

$$\frac{\partial}{\partial e} (E - \Lambda M) = 0 \quad \Longrightarrow$$

$$\frac{4\pi\Omega^2 e}{3\sqrt{1-e^2}}\int_0^{\eta^*}\rho\eta^4 d\eta - \frac{4\pi n K e}{\sqrt{1-e^2}}\int_0^{\eta^*}\rho^{1+\frac{1}{n}}\eta^2 d\eta$$

$$-2\pi^2 G\left[\frac{e\sqrt{1-e^2}-(1+e^2)\sin^{-1}e}{e}\right]\int_0^{\eta^{*0}} F^2 d\eta + \frac{1}{c^2}(PN1)$$

$$-\Lambda\left[\frac{-4\pi e}{\sqrt{1-e^2}}\int_0^{\eta^*}\rho\eta^2 d\eta + \frac{1}{c^2}(PN2)\right] = 0 \qquad (8.21)$$

where

$$(PN1) = \frac{4\pi e\Omega^4}{5\sqrt{1-e^2}}\int_0^{\eta_0}\rho\eta^6 d\eta - \frac{7}{2}\Omega^2\left[\frac{-2\pi e}{\sqrt{1-e^2}}\int_0^{\eta_0}\int_0^{2\pi}\rho U\eta^4 d\eta\sin^3\theta\, d\theta\right.$$

$$\left. + 2\pi\sqrt{1-e^2}\int_0^{\eta_0}\int_0^{2\pi}\rho\frac{\partial U}{\partial e}\eta^4 d\eta\sin^3\theta\, d\theta\right]$$

$$+ 2\left[\frac{-2\pi e}{\sqrt{1-e^2}}\int_0^{\eta_0}\int_0^{2\pi} nK\rho^{1+\frac{1}{n}}U\eta^2 d\eta\sin\theta\, d\theta\right.$$

$$\left. + 2\pi\sqrt{1-e^2}\, nK\int_0^{\eta_0}\int_0^{2\pi}\rho^{1+\frac{1}{n}}\frac{\partial U}{\partial e}\eta^2 d\eta\sin\theta\, d\theta\right]$$

$$-\frac{5}{2}\left[\frac{-2\pi e}{\sqrt{1-e^2}}\int_0^{\eta_0}\int_0^{2\pi}\rho U^2\eta^2 d\eta\sin\theta\, d\theta\right.$$

$$\left. + 4\pi\sqrt{1-e^2}\int_0^{\eta_0}\int_0^{2\pi}\rho U\frac{\partial U}{\partial e}\eta^2 d\eta\sin\theta\, d\theta\right]$$

$$+ 4\pi^2 G\, g(e)\int_0^{\eta_0}\eta^2 F^2 d\eta \qquad (8.22)$$

with $$g(e) = \frac{\sin^{-1}e\,(e^2-3) + e\sqrt{1-e^2}\,(e^2+3)}{e^4}$$

$$(PN2) = \frac{4\pi e\Omega^2}{3\sqrt{1-e^2}}\int_0^{\eta_0}\rho\eta^4 d\eta + 12\pi^2 G\left[\frac{e\sqrt{1-e^2}+(1+e^2)\sin^{-1}e}{e^2}\right]\int_0^{\eta_0}F^2 d\eta \qquad (8.23)$$

In the above expressions, except of course for the explicitly PN terms $\frac{1}{c^2}$ (PN1) and $\frac{1}{c^2}$ (PN2), the integrals are taken over the PN configuration with a radius η^* where

$$\rho(\eta^*) = 0$$

$$\text{and} \quad \eta^* = \eta_0 + \frac{1}{c^2}\eta'$$

$$\text{where} \quad \rho_0(\eta_0) = 0$$

(8.24)

The expression for the 'mass' M can also be written out in terms of the variable η as follows:

$$M = \int \rho \, d\underset{\sim}{x} + \frac{1}{c^2}\left[\frac{1}{2}\Omega^2 \int \rho \varpi^2 d\underset{\sim}{x} + 3\int \rho \, U \, d\underset{\sim}{x}\right]$$

$$= 4\pi\sqrt{1-e^2}\int_0^{\eta^*} \rho \eta^2 d\eta + \frac{1}{c^2}\left[\frac{4\pi}{3}\Omega_0^2\sqrt{1-e_0^2}\int_0^{\eta_0} \rho_0 \eta^4 d\eta + \frac{12\pi^2 G(1-e_0^2)\sin^{-1}e_0}{e_0}\int_0^{\eta_0} F_0^2 d\eta\right]$$

$$= 4\pi\sqrt{1-e^2}\int_0^{\eta_0} \rho_0 \eta^2 d\eta + 4\pi\sqrt{1-e_0^2}\,\frac{1}{c^2}\sum a_j \int_0^{\eta_0} \rho_j' \eta^2 d\eta + \frac{1}{c^2}\left[\frac{4\pi}{3}\Omega_0^2\sqrt{1-e_0^2}\int_0^{\eta_0} \rho_0 \eta^4 d\eta + \frac{12\pi^2 G(1-e_0^2)\sin^{-1}e_0}{e_0}\int_0^{\eta_0} F_0^2 d\eta\right]$$

(8.25)

We note that since $\rho_o(\eta_o) = 0$, all the integrals can now be taken from $0 \to \eta_o$.

If we demand that the 'mass' M of the PN configuration as defined by Eq. (8.25) be equal to the mass of the Newtonian one as given by Eq. (6.4), we would then have, since we have chosen $e = e_o$:

$$\sum a_j \int_0^{\eta_o} \rho_j' \eta^2 d\eta + \frac{1}{3}\Omega_o^2 \int_0^{\eta_o} \rho_o \eta^4 d\eta$$

$$+ \frac{3\pi G \sqrt{1-e_o^2}\,\sin^{-1} e_o}{e_o} \int_0^{\eta_o} F_o^2 d\eta = 0 \tag{8.26}$$

We shall now render the ρ-equation (8.7) the e-equation (8.21) and the M-equation (8.26) dimensionless. We use the same transformations as listed in Eqs. (6.23) and (6.26), and we are therefore performing the scalings in terms of the classical K and λ. Thus, λ is not necessarily the central density of the PN configuration. We also choose with Tooper (1964) the ratio of the central pressure to the central energy density for the PNA parameter q. Thus,

$$q = \frac{K\lambda^{1+\frac{1}{n}}}{\lambda c^2} = \frac{K\lambda^{\frac{1}{n}}}{c^2} \tag{8.27}$$

Furthermore, we put

$$\theta(\xi) = \theta_o(\xi) + q \sum_{j=1,..,n} a_j \theta_j'(\xi) \tag{8.28}$$

and on comparing Eq. (8.28) with Eq. (7.25), we have:

$$\rho_j' = \left(n K \lambda^{1+\frac{1}{n}}\right) \theta_0^{n-1} \theta_j' \tag{8.29}$$

All the "potentials" will also be written in dimensionless form (see Appendix C). Thus:

$$\begin{aligned} \mathcal{V}_j(\underset{\sim}{x}) &= G \int \frac{\lambda\, \theta_0^{n-1}(\underset{\sim}{x}')\, \theta_j'(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}' \\ &= 2\pi G \sqrt{1-e_0^2}\; a^2 \lambda\, \mathcal{V}_j^*(\xi, \theta) \end{aligned} \tag{8.30}$$

$$\text{where} \quad \mathcal{V}_j^*(\xi,\theta) = \int_0^\infty \frac{du}{\Delta(u)} \int_{\xi^*}^{\xi_0} \theta_0^{n-1} \theta_j' \, \xi' d\xi' \tag{8.31}$$

$$\text{with} \quad (\xi^*)^2 = \frac{\xi^2 \cos^2\theta}{1+u} + \frac{\xi^2 \sin^2\theta\,(1-e_0^2)}{1-e_0^2+u}$$

$$\Delta^2(u) = (1+u)^2(1-e_0^2+u) \tag{8.32}$$

$$\text{and} \quad \theta_0(\xi_0) = 0 \tag{8.33}$$

$$\mathcal{U}(\underset{\sim}{x}) = G\int \frac{\lambda\,\theta_0^{n}(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}'$$

$$= 2\pi G \alpha^2 \lambda \sqrt{1-e_0^2}\; \mathcal{U}^*$$

(8.34)

where $$\mathcal{U}^*(\xi,\theta) = \int_0^\infty \frac{du}{\Delta(u)} \int_{\xi^*}^{\xi_0} \theta_0^{n}\, \xi' d\xi'$$

(8.35)

$$\mathcal{D}_i(\underset{\sim}{x}) = G\int \frac{\rho(\underset{\sim}{x}')\,x_i'}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d\underset{\sim}{x}'$$

$$= 2\pi G \alpha^2 \lambda\, x_i \sqrt{1-e_0^2}\; \mathcal{D}^*$$

(8.36)

where $$\mathcal{D}^* = \int_0^\infty \frac{du}{(1+u)\Delta(u)} \int_{\xi_0^*}^{\xi_0} \theta_0^{n}\, \xi' d\xi'$$

(8.37)

Since the Lagrangian multiplier actually carries the dimension of a potential, we therefore write

$$\Lambda = \Lambda_0 + \frac{1}{c^2}\Lambda'$$
$$= \Lambda_0 + q\,\Lambda_1$$

where $$\Lambda_0 = -2\pi G \alpha^2 \lambda \sqrt{1-e_0^2}\; \mathcal{U}_0^*$$ (8.38)

$$\Lambda_1 = \pi G \alpha^2 \lambda \sqrt{1-e_0^2}\, \Lambda^*$$

(8.39)

with U_o^* just being the potential U* calculate at the pole where $\xi = \xi_o$ and $\theta = 0$.

With this preamble, we can readily change Eq. (8.7) into the following dimensionless form:

$$\sum_{j=0,..,n} a_j \left[4\sqrt{1-e_0^2} \int \theta_0^{n-1} \theta_i' \theta_j' \xi^2 d\xi - n(1-e_0^2) f(e_0) D(i,j,1) \right]$$

$$- \frac{2}{3} v' \sqrt{1-e_0^2}\, f(e_0) \int \theta_0^{n-1} \theta_i' \xi^4 d\xi - f(e_0)(1-e_0^2) \Lambda^* \int \theta_0^{n-1} \theta_i' \xi^2 d\xi$$

$$= -3 (1-e_0^2)^{3/2} [f(e_0)]^2 (n+1) U_0^* D(i,2)$$

$$+ \frac{1}{3} (n+1) [f(e_0)]^2 (1-e_0^2) v_0 U_0^* \int \theta_0^{n-1} \theta_i' \xi^4 d\xi$$

$$+ \frac{11}{4} (n+1) [f(e_0)]^2 (1-e_0^2)^{3/2} D(i,15)$$

$$- 5(n+1) f(e_0) (1-e_0^2) D(i,3)$$

$$+ \frac{2}{3} v_0 (n+1) f(e_0) \sqrt{1-e_0^2} \int \theta_0^{n} \theta_i' \xi^4 d\xi$$

$$
\begin{aligned}
&-\frac{1}{15}(n+1)v_0^2[f(e_0)]^2\sqrt{1-e_0^2}\int \theta_0^{n-1}\theta_i'\xi^6 d\xi \\
&+2(n+1)v_0[f(e_0)]^2(1-e_0^2)D(i,4) \\
&+(n+1)(1-e_0^2)^{3/2}[f(e_0)]^2 D(i,5) \\
&+(n+3)(1-e_0^2)f(e_0)D(i,6) \\
&+(n+1)v_0(1-e_0^2)[f(e_0)]^2 D(i,7) \\
&-2(n+1)(1-e_0^2)[f(e_0)]^2 v_0 D(i,8)
\end{aligned}
\tag{8.40}
$$

$i = 0, \ldots, n$.

where the D's represent the various double integrals, defined as follows:

$$D(i,j,1) = \iint \theta_0^{n-1}\theta_i' U_j^* \sin\theta \, d\theta \; \xi^2 d\xi$$

$$D(i,2) = \iint \theta_0^{n-1}\theta_i' U^* \sin\theta \, d\theta \; \xi^2 d\xi$$

$$D(i,15) = \iint \theta_0^{n-1}\theta_i' (U^*)^2 \sin\theta \, d\theta \; \xi^2 d\xi$$

$$D(i,3) = \iint \theta_0^{n}\theta_i' U^* \sin\theta \, d\theta \; \xi^2 d\xi$$

$$D(i,4) = \iint \theta_0^{n-1}\theta_i' U^* \sin^3\theta \, d\theta \; \xi^4 d\xi$$

$$D(i,5) = \iint \theta_0^{n} U^* U_i^* \xi^2 d\xi \sin\theta \, d\theta$$

$$D(i,6) = \iint \theta_0^{n+1} \mathcal{V}_i^{*} \sin\theta \, d\theta \, \xi^2 d\xi$$

$$D(i,7) = \iint \theta_0^{n} \mathcal{V}_i^{*} \sin^3\theta \, d\theta \, \xi^4 d\xi$$

$$D(i,8) = \iint \theta_0^{n-1} \theta_i' \mathcal{D}^{*} \sin^3\theta \, d\theta \, \xi^4 d\xi$$

Also $\quad v_0 = \dfrac{\Omega_0^2}{2\pi G\lambda} \quad , \quad v' = \dfrac{\Omega_0 \Omega'}{\pi G K \lambda^{1+\frac{1}{n}}}$ (8.41)

The derivation of the above terms is quite straightforward, except that in calculating $\int \rho_i' \Phi_0 \, d\underset{\sim}{x}$, we use

$$\begin{aligned} \int \rho_i' \Phi_0 \, d\underset{\sim}{x} &= G \iint \rho_i'(\underset{\sim}{x}) \frac{\rho(\underset{\sim}{x}') \phi(\underset{\sim}{x}')}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \, d\underset{\sim}{x} \, d\underset{\sim}{x}' \\ &= G \int \rho(\underset{\sim}{x}') \phi(\underset{\sim}{x}') \, d\underset{\sim}{x}' \int \frac{\rho_i'(\underset{\sim}{x})}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \, d\underset{\sim}{x} \\ &= G \int \rho(\underset{\sim}{x}') \phi(\underset{\sim}{x}') \mathcal{V}_i(\underset{\sim}{x}') \, d\underset{\sim}{x}' \\ &= G \int \rho(\underset{\sim}{x}') \mathcal{V}_i(\underset{\sim}{x}') \left[\mathcal{V} + \tfrac{1}{2}\Pi + \tfrac{3}{2}\frac{p}{\rho} + \Omega^2 \varpi^2 \right] d\underset{\sim}{x} \end{aligned}$$

where we have used the definition of Φ and ϕ as given in Eqs. (7.12) and (7.13).

We shall now consider Eq. (8.21) - (8.23). Remembering that we have chosen for our problem $e = e_o$, and after separating out the zeroth order part, we have:

$$\frac{2e_0 f(e_0)}{3\sqrt{1-e_0^2}}\left[v'\int\theta_0^n\xi^4 d\xi + n v_0 \sum_{j=0,1,\ldots} a_j \int \theta_j'\,\theta_0^{n-1}\xi^4 d\xi\right]$$

$$-\frac{4ne_0}{\sqrt{1-e_0^2}}\sum_{j=0,\ldots} a_j\int\theta_0^n\theta_j'\xi^2 d\xi$$

$$-f(e_0)\left[\frac{e_0\sqrt{1-e_0^2}-(1+e_0^2)\sin^{-1}e_0}{e_0^2}\right]\int F_0^* F^{*\prime} d\xi$$

$$+(PN1)^* = 2e_0 f(e_0)\,U_0^*\, n\sum_{j=0,\ldots} a_j\int\theta_0^{n-1}\theta_j'\xi^2 d\xi$$

$$-e_0 f(e_0)\Lambda^*\int\theta_0^n\xi^2 d\xi + (PN2)^* \tag{8.42}$$

where

$$F_0^* = 2\int_\xi^{\xi_0}\theta_0^n\xi\, d\xi$$

$$F^{*\prime} = 2n\sum_{j=1,\ldots} a_j\int_\xi^{\xi_0}\theta_0^{n-1}\theta_j'\,\xi\, d\xi$$

$$(PN2)^* = \frac{1}{3}(n+1)\left[f(e_0)\right]^2 e_0 v_0 U_0^*\int\theta_0^n\xi^4 d\xi$$

$$-\frac{3}{2}(n+1)\sqrt{1-e_0^2}\,\left[f(e_0)\right]^2\left[\frac{e_0\sqrt{1-e_0^2}-(1+e_0^2)\sin^{-1}e_0}{e_0^2}\right]U_0^*\int(F_0^*)^2 d\xi$$

(8.43)

$$(PN1)^{*} = \frac{n+1}{20} v_0^2 \frac{e_0}{\sqrt{1-e_0^2}} \left[f(e_0)\right]^2 \int \theta_0^n \xi^6 d\xi$$
$$+ \frac{7}{4}(n+1)\, e_0 \left[f(e_0)\right]^2 v_0\, D(0,9)$$
$$- \frac{7}{4}(n+1)\, v_0 \left[f(e_0)\right]^2 \sqrt{1-e_0^2}\; D(0,10)$$
$$- 2n\, e_0 f(e_0)\, D(0,11)$$
$$+ 2n \sqrt{1-e_0^2}\, f(e_0)\, D(0,12)$$
$$+ \frac{5}{4}(n+1)\, e_0 \sqrt{1-e_0^2} \left[f(e_0)\right]^2 D(0,13)$$
$$- \frac{5}{2}(n+1)(1-e_0^2)\left[f(e_0)\right]^2 D(0,14)$$
$$+ (n+1)\, v_0\, g(e_0) \left[f(e_0)\right]^2 \int (F_0^{*})^2 \xi^2 d\xi \qquad (8.44)$$

where the double integrals are:

$$D(0,9) = \int_0^{\xi_0}\int_0^{\pi} \theta_0^n\, U^{*} \xi^4 d\xi \sin^3\theta\, d\theta$$

$$D(0,10) = \int_0^{\xi_0}\int_0^{\pi} \theta_0^n \frac{\partial}{\partial e}\left(\sqrt{1-e^2}\, U^{*}\right) \xi^4 d\xi \sin^3\theta\, d\theta$$

$$D(0,11) = \int_0^{\xi_0}\int_0^{\pi} \theta_0^{n+1}\, U^{*} \xi^2 d\xi \sin\theta\, d\theta$$

$$D(0,12) = \int_0^{\xi_0}\int_0^{\pi} \theta_0^{n+1} \frac{\partial}{\partial e}\left(\sqrt{1-e^2}\, U^{*}\right) \xi^2 d\xi \sin\theta\, d\theta$$

$$D(0,13) = \int_0^{\xi_0}\int_0^{\pi} \theta_0^n (\mathcal{V}^*)^2 \sin\theta \, d\theta \; \xi^2 d\xi$$

$$D(0,14) = \int_0^{\xi_0}\int_0^{\pi} \theta_0^n \, \mathcal{V}^* \frac{\partial}{\partial e}\left(\sqrt{1-e^2}\,\mathcal{V}^*\right) \xi^2 d\xi \sin\theta \, d\theta$$

and $\frac{\partial}{\partial e}\left(\sqrt{1-e^2}\,\mathcal{V}^*\right)$

is given in Appendix C.

The M-Eq. (8.26) is relatively simple. We can easily change it to:

$$n \sum_{j=0,..} a_j \int_0^{\xi_0} \theta_0^{n-1} \theta_j' \xi^2 d\xi + \frac{1}{6}(n+1) f(e_0) v_0 \int_0^{\xi_0} \theta_0^n \xi^4 d\xi$$

$$+ \frac{3}{4} \frac{\sqrt{1-e_0^2}\,\sin^{-1} e_0}{e_0} (n+1) f(e_0) \int_0^{\xi_0} (F_0^*)^2 d\xi = 0 \tag{8.44}$$

Eqs. (8.40), (8.41) - (8.43) and (8.44) then form the basic set of equations we shall use to solve for the a_i's, v' and Λ^*.

IX. Method of Solution and Results

We have stated that Robert's result can be used as the zero-th order solution. We shall consider here only the case n = 1, because of the existence of an analytic solution (refer (6.35))

$$\theta_0(\xi) = w + (1-w)\frac{\sin\xi}{\xi} \tag{9.1}$$

For a given w, Roberts' (1963) solution gives the corresponding eccentricity e_o and radius ξ_o. Thus, for example, for w = 0.178465 (corresponding to the largest possible value for equilibrium), we have v_o = 0.106757, e_o = 0.869889 and ξ_o = 4.493409.

For the trial functions, we must choose those that satisfy the boundary condition that at the origin, the first derivative must vanish. In the absence of further information, and because of its simplicity, we shall take

$$\sum a_j \theta_j' = a_0 + a_2\xi^2 + a_3\xi^3 \tag{9.2}$$

The term linear in ξ has been left out because it causes a cusp at the origin. To gain an idea of whether this will give a reasonably good approximate solution, we shall solve our equations for the special case of no rotation and compare the results with those given by Krefetz (1967).

The only difficulty in the solution of the problem lies in the evaluation of the various integrals, which are of 3 types (1) single integrals like $\int \theta_o^{\,n}\xi^2 d\xi$ (2) "potential" like U* and (3) the various double integrals D's.

Because of the analytic expression for n = 1, all the single integrals can be easily evaluated, and hence we shall not bother describing them here.

A typical "potential" integral is U*

$$U^* = \int_0^\infty \frac{dv}{\Delta(v)} \int_{\xi^*}^{\xi_0} \theta_0^n \xi' d\xi' \tag{9.3}$$

The integral over ξ can be exactly carried out because of Eq. (9.1). For the integral over v we split it up from 0 → 100 and then 100 → ∞. Thus symbolically, we have

$$\int_0^\infty = \int_0^{100} + \int_{100}^\infty$$

$$= I_1 + I_2 \text{ , say.}$$

I_1 is calculated using the 32-point Gauss-Legendre quadrature For I_2, we expand the integrand in series of $\frac{1}{v}$ and keep the first two terms. The integration can then be exactly carried out.

As a check, we consider:

$$I = \int_0^\infty \frac{dv}{(1+v)\sqrt{1-e^2+v}} = \frac{2\sin^{-1}e}{e}$$

For e = 0.869889, the exact result is 2.42554. Our method gives 2.42470, thus carrying an error of 0.03 per cent.

Our double integrals are typically of the form

$$\int_0^{\xi_0} \int_0^\pi f(\xi)\, G(\xi,\theta) \sin\theta \, d\theta \, d\xi \tag{9.4}$$

where $G(\xi,\theta)$ is one of the "potentials" discussed in Appendix C; and $f(\xi)$ is some simple function of ξ. The integral is evaluated using a 12-point 7th degree accuracy formula (Tyler, 1953). In this method:

$$\int_{-a}^{+a}\int_{-b}^{+b} F(x,y)\,dx\,dy = (ab)\Big[R_1 \sum F(\pm x_1, \pm y_1) + R_2 \sum F(\pm x_2, \pm y_2) + 2R_3 \sum F(\pm x_3, 0) + 2R_4 \sum F(0, \pm y_4)\Big]. \quad (9.5)$$

where $R_1 = 0.520593$

$R_2 = 0.237432$

$R_3 = R_4 = 0.120988$

and $\dfrac{x_1}{a} = \dfrac{y_1}{b} = 0.380555$

$\dfrac{x_2}{a} = \dfrac{y_2}{b} = 0.805980$

$\dfrac{x_3}{a} = \dfrac{y_4}{b} = 0.925820$

This method yields (0.6639) for the integral $\int_0^1\int_0^1 \frac{dx\,dy}{\sqrt{3-x^2-y^2}}$ which is correct up to the last figure when compared with the exact result.

In applying this method to our integral we need only define:

$$y = \cos\theta$$
$$x = \xi - \frac{1}{2}\xi_o$$

when eq. (9.4) would then become

$$\int_{-a}^{+a}\int_{-b}^{+b} f(x)\,G(x,y)\,dx\,dy$$

where now $a = \frac{1}{2}\xi_0$ and $b = 1$.

To see how accurate such a formula is for integrals of our type, we apply it to the special case of no rotation. For $D(0,0,1) = \iint \theta_0^{n-1}\theta_0' \,\mathcal{V}_0^* \sin\theta\, d\theta\, \xi^2 d\xi$ (see P.70), exact calculation gives 163.2126 whereas the above formula gives 163.2329.

Our procedure of solution is the following. For a given w, we calculate ξ_0, e_0 and v_0 for the classical configuration by the procedure listed on P.49. With this, we can then calculate the various integrals that appear in the variational equations. The results are then fed into another computer program that computes the coefficients for the various a_i's in equations (8.40), (8.42) and (8.44). The net result is then just a set of linear, simultaneous algebraic equations which can be readily solved for a_0 , a_2 , a_3 , v' and Λ^* .

The results are tabulated as follows, in which we remember

$$\mathcal{V}_0 = \frac{\Omega_0^2}{2\pi G\lambda} \qquad \text{from eqn. (6.27)}$$

$$\mathcal{V}' = \frac{\Omega_0\Omega'}{\pi G k \lambda^{1+\frac{1}{n}}} \qquad \text{from eqn. (8.40)}$$

$$\Omega = \Omega_0 + \frac{1}{c^2}\Omega' \qquad \text{from eqn. (8.4)}$$

$$\text{and } \frac{\rho(\xi)}{\text{Central Density}} = \theta_0 + q\left[a_0(1-\theta_0) + a_2\xi^2 + a_3\xi^3\right]$$
$$= \theta_0 + q\,\theta' \qquad \text{, say.} \qquad (9.6)$$

w	e	v_0	a_0	a_2	a_3	v'
0.00	0.00	0.00	+5.81817	−3.24599	+0.68813	0.00
0.05	0.53130	0.04466	+10.20545	−5.37697	+1.22992	+0.31657
0.075	0.63040	0.06299	+12.65229	−6.40236	+1.46923	+0.54491
0.10	0.70630	0.07860	+16.57410	−8.02304	+1.84033	+0.86105
0.11	0.73220	0.08406	+18.77620	−8.9080	+2.03850	+1.02860
0.13	0.77806	0.09358	+25.49540	−11.53340	+2.61490	+1.48610
0.15	0.81800	0.10104	+37.30940	−15.82430	+3.51400	+2.22080
0.16	0.83650	0.10385	+45.81070	−18.58440	+4.05300	+2.74100
0.17	0.85430	0.10594	+56.6500	−21.56440	+4.57130	+3.40460
0.17847	0.86989	0.10676	+70.92260	−23.48860	+4.65910	+4.45330

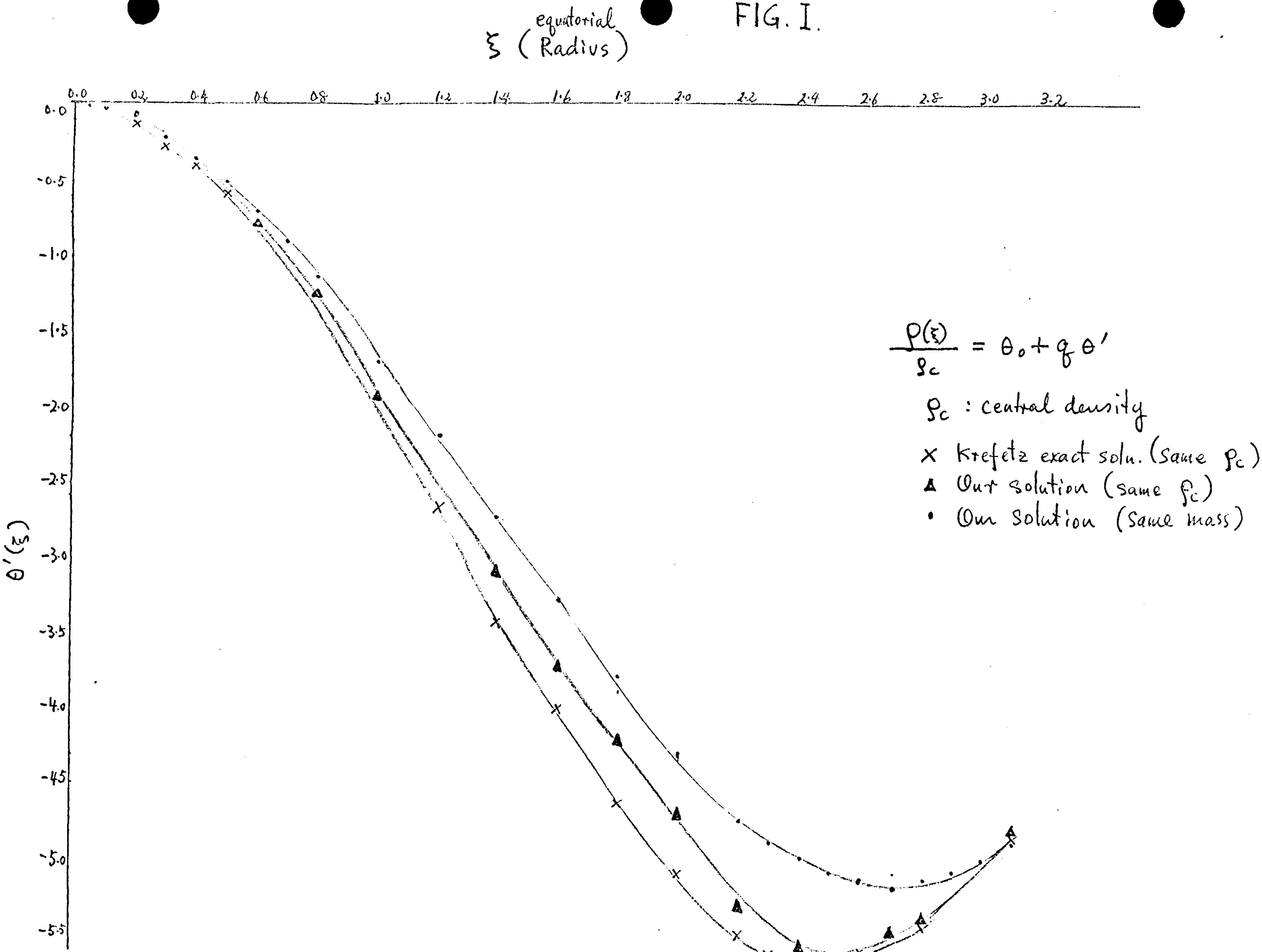
As a check on the accuracy of our results, and the validity of our approach, we compared our results with those of Krefetz (1967) for the non-rotating case. This is shown in Fig.1, where we plot $\theta'(\xi)$ from eqn.(9.6) against ξ. It can be seen that the agreement is a very close one.

The effectas of rotation and PNA on the density distribution are shown in Fig. II , where we plot $\theta'(\xi)$ against ξ for different values of w. The curves are terminated at the radius of the classical configuration. The change in angular velocity due to PN effects are, on the other hand, displayed in fig. III , where we plot v' against w .

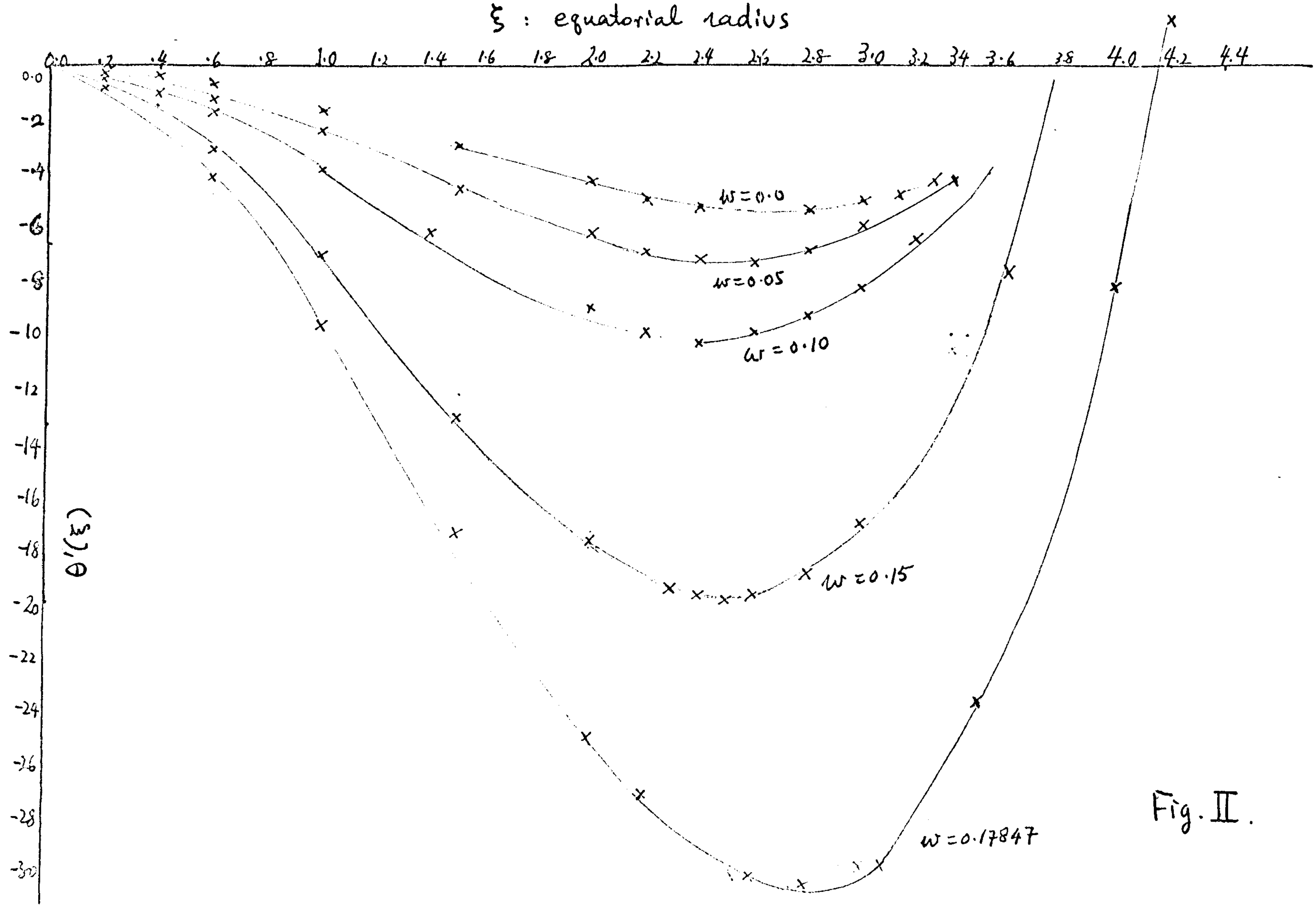
Our calculations indicate that for the same eccentricity and mass, the PN configuration has a larger angular velocity than the Newtonian one. This is in agreement with the results of Chandrasekhar (1965b) for a rotating Maclaurin spheroid in the PNA, when the figure of the rotating body in the PN theory is also approximated to by a spheroid. His results, however, are obtained without really solving the equations of equilibrium, although an exact analysis, when the PN congiguration is no longer assumed to be ellipsoidal can be found in a later paper (1966) .

We defer to Section x for further discussions on the general nature of our approach.

FIG. I.



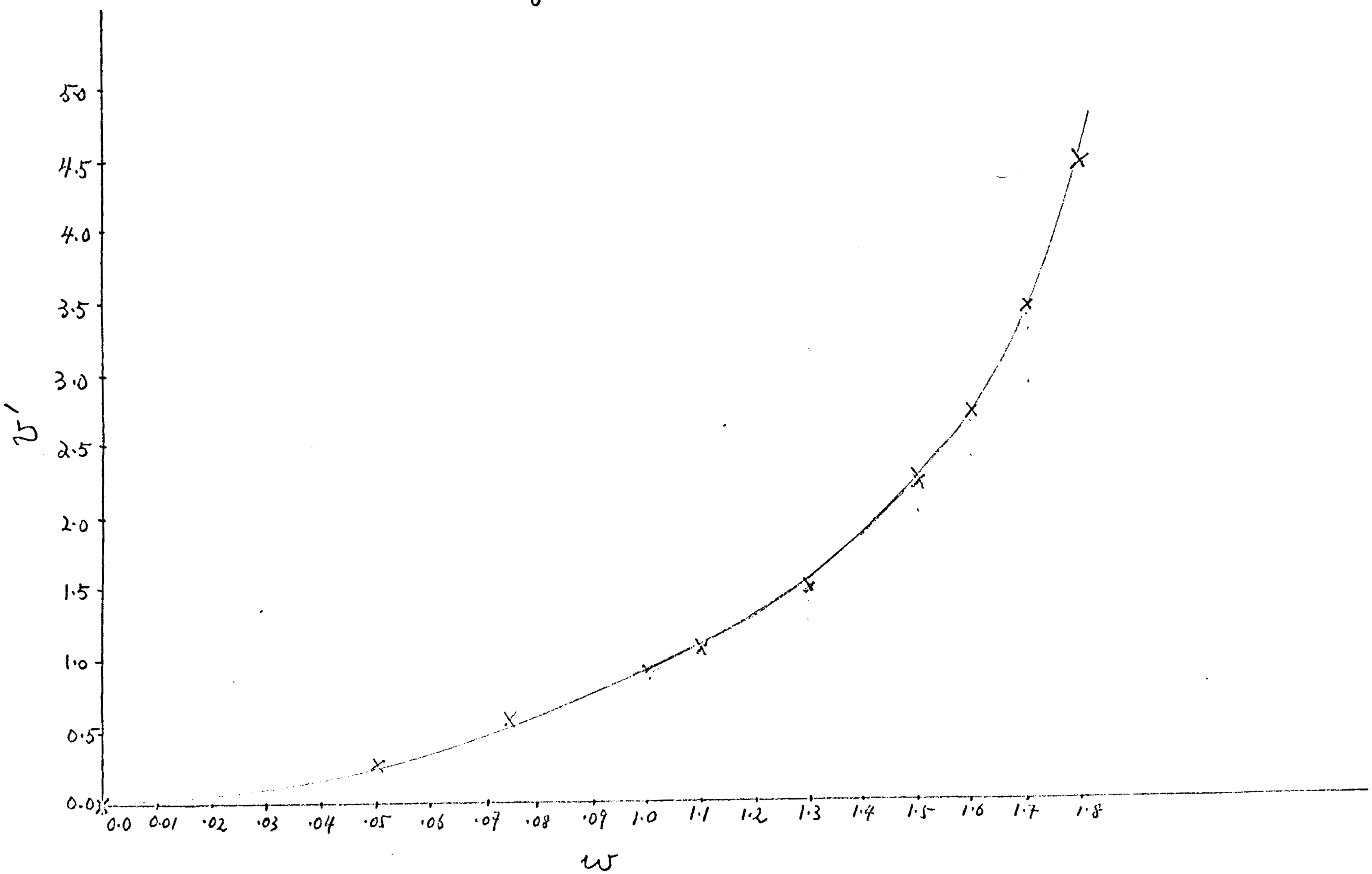
ξ (equatorial Radius)
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
3.2
0.0
-0.5
-1.0
-1.5
-2.0
-2.5
-3.0
-3.5
-4.0
-4.5
-5.0
-5.5
θ'(ξ)
ρ(ξ)/ρc = θ0 + q θ'
ρc : central density
× Krefetz exact soln. (Same ρc)
▲ Our solution (same ρc)
• Our solution (Same mass)



ξ : equatorial radius
0.0
·2
·4
·6
·8
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
4.2
4.4
0.0
-2
-4
-6
-8
-10
-12
-14
-16
-18
-20
-22
-24
-26
-28
-30
θ'(ξ)
w = 0·0
w = 0·05
w = 0·10
w = 0·15
w = 0·17847

Fig. II.

Fig. III



5.0
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
v'
0.0 0.01 ·02 ·03 ·04 ·05 ·06 ·07 ·08 ·09 1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8
w

X. Discussion of Results and Outlook

Our results should only be interpreted from a qualitative point of view, although they do indicate, in agreement with Chandrasekhar (1965b) that for the same eccentricity and mass, the PN configuration has a higher angular velocity than the Newtonian one. In the first place, our assumption of an ellipsoidal density distribution for a rotating polytrope is an extremely crude one. Its chief merit in the classical case lies in its simplicity, and its giving a simple, ordinary differential equation for the density distribution. However, all this seems to be lost when one goes to the PNA, and the calculations turned out to be tedious and time-consuming. Furthermore, Chandrasekhar (1966) has shown that even in the case of uniform density, the deformed figure in the PNA has a quartic surface instead of a quadratic one. It thus appears that it may actually be far more profitable, and even simpler in the end, to employ the direct and yet more accurate approach of James.

There are in fact many more aspects of rotation that we have not touched on at all here. We have assumed uniform rotation throughout, but it is far more probable that a state of non-uniform rotation would prevail in the stellar interior, at least in the beginning. Thus, one should take viscosity into account and investigate how rapidly non-uniformily in rotation can be smoothed out.

In addition, we have left out altogether the possibility of internal motion, which could lead to a whole variety of interesting phenomena.

Thus, for example, one can consider an ellipsoid rotating along a z-axis, with an irrotational internal motion described by a Lagrangian displacement linear in the co-ordinates (see, e.g. Lamb, 1932, p. 719-723). This is the problem first considered by Dirichlet and one obtains the interesting result that the surface can oscillate regularly between an extremely oblate and a extremely prolate form.

Further investigations were carried out by Dedekind and Riemann (for a brief review, see Chandrasekhar 1965a, 1965b, and Basset 1888). The former proved a remarkable theorem to the effect that different states of internal fluid motion can preserve the same external, ellipsoidal shape and the latter showed that the most general type of motion compatible with an ellipsoidal figure of equilibrium consists of a superposition of uniform rotation and internal motion of uniform vorticity about axes that lie in a principal plane of the ellipsoid.

Thus, we see that there is still much that can be done in this whole field of rotation. The problem surely does not end with the determination of the structure of a rotating polytrope, or the location of the bifurcation point . One can surely consider, in conjunction with the problem of neutron star and gravitational radiation, ellipsoids of the kind suggested by Dirichlet, Dedekind and Riemann. To this and related problems, we hope to return sometime in the future.

Appendix A

Matching of Boundary Conditions for the Gravitational Potential

On P. 25 , we claim that the expression for gravitational radiation is all determined one λ is found, and that this can be done by a matching of boundary conditions for the gravitational potential this we now do.

In the equilibrium configuration, the internal graviational potential can be written as:

$$\Phi_0^{int} = \pi G\rho\left[\frac{2}{3}r^2 - \frac{2}{3}(2a_1^2 + a_3^2) + \frac{2}{15}e^2r^2(2 - 3\sin^2\theta)\right] \tag{A.1}$$

where the notations are the same as those used in §III. We notice that Eq. (A.1) gives

$$\Phi_0^{int.} = \pi G\rho\left\{\frac{2}{3}r^2 - \frac{2}{3}(3R^2)\right\}$$

in the absence or rotation, agreeing with the usual formula one has for that of a sphere. The external potential Φ_0^{out} has to satisfy:

$$\nabla^2\Phi_0^{out} = 0 \tag{A.2}$$

The solution to this Laplace equation can be written as:

$$\Phi_0^{out} = \frac{K_0}{r} + \Omega^2\sum_{\ell=0}^{\infty}\frac{K_{1;\ell}}{r^{\ell+1}}P_\ell(\cos\theta) \tag{A.3}$$

To determine the unknown constants K_0 and K_1; we demand that

$$\left(\bar{\Phi}_0^{int.}\right)_S = \left(\bar{\Phi}_0^{out}\right)_S$$

$$\left(\vec{\nabla}\bar{\Phi}_0^{int}\right)_S = \left(\vec{\nabla}\bar{\Phi}_0^{out}\right)_S \tag{A.4}$$

on the boundary S defined by

$$\lambda(\theta) = a_3\left(1+\frac{1}{2}e^2\sin^2\theta\right) \tag{A.5}$$

Eqs. (A.4) can be easily solved, and we have:

$$K_0 = -\frac{4}{3}\pi G\rho\, a_1^2 a_3 \tag{A.6}$$

$$K_{1;0} = \frac{5}{6}a_3\left(a_3^2-a_1^2\right) \tag{A.7}$$

$$K_{3;2} = \frac{5}{6}a_3^3\left(a_1^2-\frac{2}{5}a_3^2\right) \tag{A.8}$$

and all other K_1; $\ell = 0$.

Substituting into (A.3), we therefore have:

$$\bar{\Phi}_0^{out} = \pi G\rho\left\{\frac{-\frac{4}{3}a_1^2a_3}{r} + \frac{4}{9}e^2\left[\frac{a_3}{r}\left(a_3^2-a_1^2\right) + \frac{a_3^2}{r^3}\,P_2\left(a_1^2-\frac{2}{5}a_3^2\right)\right]\right\} \tag{A.9}$$

We now consider the boundary conditions under perturbation. We first of all write the gravitational potential as:

$$\bar{\Phi} = \bar{\Phi}_0 + \bar{\Phi}' \tag{A.10}$$

and for ϕ^{int}, we demand that

$$\nabla^2 \Phi^{int} = -4\pi G(\rho + \rho') \tag{A.11}$$

$$\therefore \nabla^2 \Phi'^{int} = -4\pi G \rho'$$
$$= -4\pi G \rho \operatorname{div} \xi \tag{A.12}$$

On putting

$$\Phi'^{int.} = -4\pi G \rho \epsilon \left[g_o(r) + \Omega^2 \sum_{n=0,\ldots} \varphi_n(r) P_n(\theta) \right] \tag{A.13}$$

we then immediately have:

$$\nabla^2 g_o(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g_o}{\partial r}\right) = 3 \tag{A.14}$$

$$\nabla^2 \varphi_o(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi_o}{\partial r}\right) = -2\lambda \tag{A.15}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \varphi_n}{\partial r}\right) - \frac{n(n+1)}{r^2}\varphi_n = 0 \quad \forall\, n \neq 0 \tag{A.16}$$

The solutions to the above equations can be written as:

$$g_o(r) = \frac{1}{2} r^2 + C_o \tag{A.17}$$

$$\varphi_o(r) = -\frac{1}{3}\lambda r^2 + a_o \tag{A.18}$$

$$\varphi_2(r) = K_2 r^2 \tag{A.19}$$

$$\varphi_n(r) = 0 \qquad \forall\, n \neq 2, 0 \tag{A.20}$$

where the constants k_2, c_o and a_o are unknowns to be determined by a matching of boundary conditions. Thus, in the presence of perturbation, the total internal gravitational potential can be written as:

$$\Phi^{int.} = \Phi_0^{int} + \Phi^{int'}$$
$$= \pi G \rho \left\{ \frac{2}{3} r^2 - \frac{2}{3}(2a_1^2 + a_3^2) + \frac{2}{15} e^2 r^2 (2 - 3\sin^2\theta) \right\}$$
$$- 4\pi G \rho \epsilon \left\{ \left(\frac{1}{2} r^2 + c_o\right) + \Omega^2 \left(-\frac{1}{3}\lambda r^2 + a_0 + k_2 r^2 P_2(\theta)\right) \right\}$$

(A.21)

The external potential has to satisfy

$$\nabla^2 \Phi^{out} = \nabla^2 \left\{ \Phi_0^{out} + \Phi^{out'} \right\} = 0$$

and remembering that $\Phi^{out'}$ is caused by the perturbation, we can therefore write the solution as:

$$\Phi^{out} = \pi G \rho \left\{ \frac{-\frac{4}{3} a_1^2 a_3}{r} + \frac{4}{9} e^2 \left[\frac{a_3}{r}(a_3^2 - a_1^2) + \frac{a_3^2}{r^3}\left(a_1^2 - \frac{2}{5} a_3^2\right) P_2(\theta) \right] \right.$$
$$\left. + \pi G \rho \epsilon \left[\frac{A_0}{r} + \Omega^2 \left(\frac{A_{1;0}}{r} + \frac{A_{1;2}}{r^3} P_2 \right) \right] \right.$$

(A.22)

In using the boundary conditions (A.4), we remember that the unperturbed gravitational potential and its derivative has already been made continuous on the unperturbed surface ζ_o; and also the perturbed surface S can be obtained from the unperturbed S_o by the Lagrangian displacement ξ. Thus, the boundary conditions can be written in terms of quantities evaluated at S_o, e.g.,

$$(\Phi)_S = (\Phi_o)_{S_o} + (\Phi')_{S_o} + (\vec{\xi}\cdot\vec{\nabla}\Phi_o)_{S_o} + O(\xi^2)$$

for both Φ^{int} and Φ^{out}. Hence

$$(\Phi^{int})_S = (\Phi^{out})_S$$

$$\Longrightarrow (\Phi^{int\prime})_{S_o} = (\Phi^{out\prime})_{S_o} \qquad \text{(A.23)}$$

and similarly for the matching of the gradient.

Using then Eq. (A.21) and (A.22), we have by straightforward calculations:

$$-4\left(c_o + \tfrac{1}{2}a_3^2\right) = \frac{A_o}{a_3}$$

$$-4\left(a_o - \tfrac{1}{3}\lambda a_3^2 + \tfrac{5}{8}\frac{a_3^2}{\pi G\rho}\right) = \frac{A_{1;0}}{a_3} - \frac{5}{8}\frac{A_o}{\pi G\rho a_3} \qquad \text{(A.24)}$$

$$-4\left(k_2 a_3^2 - \frac{5}{8}\frac{a_3^2}{\pi G\rho}\right) = \frac{A_{1;2}}{a_3^3} + \frac{5}{8}\frac{A_o}{\pi G\rho a_3}$$

The above comes from a matching of the potentials.

$$-\frac{8}{3}a_3 = \frac{A_o}{a_3^2} + \frac{8}{3}a_3$$

$$\frac{16}{9}\lambda a_3 - \frac{5a_3}{3\pi G\rho} = \frac{A_{1;0}}{a_3^2} - \left(A_o - \frac{8}{3}a_3^2\right)\frac{5}{4\pi G\rho a_3^2} \quad \frac{16}{9}\lambda a_3 - \frac{5a_3}{\pi G\rho}$$

$$a_3\left(-8k_2 - \frac{16\lambda}{9} + \frac{1}{5}\frac{1}{\pi G\rho}\right) = \frac{3A_{1;2}}{a_3^4} + \frac{6}{\pi G\rho}a_3 - \frac{32}{9}a_3\lambda + \frac{5}{4}\frac{1}{\pi G\rho a_3^2}\left(A_0 - \frac{8}{3}a_3^3\right)$$

(A.25)

The above comes from the matching of the derivatives.

On solving the set of equations (A.24) and (A.25) we would have:

$$k_2 = \frac{299}{120}\frac{1}{\pi G\rho} - \frac{4}{9}\lambda \tag{A.26}$$

which is the expression used in Eq. (3.29).

Appendix B

Calculation of Potential Energy Tensor for a Homoeidally Striated Ellipsoid

By definition, the potential energy tensor W_{ij} is:

$$W_{ij} = -\frac{1}{2}\int \rho(\underset{\sim}{X})\, V_{ij}(\underset{\sim}{X})\, d\underset{\sim}{X} \qquad \text{(B.1)}$$

where the potential tensor V_{ij} is defined as:

$$V_{ij}(\underset{\sim}{X}) = G\int \frac{\rho(\underset{\sim}{X}')(x_i - x_i')(x_j - x_j')}{|\underset{\sim}{X} - \underset{\sim}{X}'|^3}\, d\underset{\sim}{X}' \qquad \text{(B.2)}$$

In the particular case of an ellipsoid density distribution,

$$\rho = \rho(m^2)$$

where $m^2 = \dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_1^2} + \dfrac{x_3^2}{a_3^2} \qquad m \leq 1 \qquad$ (B.3)

By noticing that the integrand in (B.1) is symmetric in $\underset{\sim}{X}$ and $\underset{\sim}{X}'$, we can evaluate the integral in a very simple fashion. (Roberts 1962). First of all, we could have divided the integral into two parts, an integration of $\underset{\sim}{X}$ and $\underset{\sim}{X}'$ over the homoeoids $(m, m + dm)$ and $(m', m' + dm')$; and then an integration over m and m'. The variables m and m' can take any values between 0 and 1. However, by making use of the symmetry of the integrand, we can adopt the equivalent alternate, yet far more convenient procedure, of integrating over $m \geq m'$ and doubling the result. Thus, we write (B.1) symbolically as:

$$W_{ij} = -\iint\limits_{m > m'} \rho(m')\, V_{ij}(m)\, dm\, dm' \tag{B.4}$$

where $V_{ij}(m)$ denotes the tensor potential produced by the homeoid (m, m + dm). Since $V_{ij}(m)$ is constant for all points inside this homoeoid, the integration of ρU_{ij} over (m', m' + dm' is trivial. It is simply $4\pi a_1 a_2 a_3 m^{12} \rho(m^{12})\, dm'\, V_{ij}(m)$. Now:

$$V_{ij}(m) = 2\pi G\rho(m)\, m\, a_i^3 a_K a_\ell\, dm\, A_i \delta_{ij} \tag{B.5}$$

(see Roberts, 1962).

$$\therefore\ W_{ij} = -8\pi^2 G a_i^4 a_K^2 a_\ell^2 A_i \delta_{ij} \int_0^1 dm \int_0^m dm'\, m\, \rho(m^2)\, m'^2 \rho(m'^2) \tag{B.6}$$

Defining $F(m^2) = \int_{m^2}^1 \rho(m^2)\, dm^2$ and making use of

$$\int_0^1 f(m)\, dm \int_0^m g(m')\, dm' = \int_0^1 g(m')\, dm' \int_{m'}^1 f(m)\, dm$$

we can easily reduce (B.6) to

$$W_{ij} = -\pi^2 G a_i^4 a_K^2 a_\ell^2 A_i \delta_{ij} \int_0^1 [F(m^2)]^2\, dm \tag{B.7}$$

We shall now calculate $\int \rho\, \bar{\omega}^2\, \mathcal{D}\, d\underset{\sim}{x}$ along the same line. We first of all note that the integral can be written as:

$$\int \rho \varpi^2 \mathcal{D}\, d\underset{\sim}{x} = \int \rho x_1 \mathcal{D}_1\, d\underset{\sim}{x} + \int \rho x_2 \mathcal{D}_2\, d\underset{\sim}{x} \tag{B.8}$$

where $\mathcal{D}_1 = x_1\mathcal{D}$; $\mathcal{D}_2 = x_2\mathcal{D}$

and
$$\nabla^2 \mathcal{D}_i = -4\pi G \rho X_i \tag{B.9}$$

in the case of axisymmetry relevant to our problem.

Now the potential d_i at a point $(X_1X_2X_3)$ produced by a homoeoid $(m, m + dm)$ with a density distribution ρX_i is given by (Routh, 1922):

$$d_i(X_1X_2X_3) = 2\pi G \rho m\,dm\, a_1^3 a_2 a_3 X_i \int_\lambda^\infty \frac{du}{(a_i^2+u)\Delta(u)} \tag{B.10}$$

where $\lambda = 0$ for interior points, and given by

$$\frac{X_1^2}{a_1^2+\lambda} + \frac{X_2^2}{a_2^2+\lambda} + \frac{X_3^2}{a_3^2+\lambda} = m^2$$

for exterior ones. $\Delta(u)$ is, as usual, defined by:

$$\Delta^2(u) = (a_1^2+u)(a_2^2+u)(a_3^2+u)$$

If we now take an ellipsoid consisting of a series of homoeoids with $m = 0 \rightarrow 1$, and consider an interior point $(X_1X_2X_3)$ lying on the surface of the ellipsoid:

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = n^2$$

then we can obtain the total potential $D_i(X_1, X_2, X_3)$ by summing $d_i(X_1, X_2, X_3)$ from $m = 0$ to $m = n$, and then $m = n$ to $m = 1$. The final result is:

$$\mathcal{D}_1 = 2\pi G x_1 a_1^3 a_2 a_3 \int_0^\infty \frac{du}{(a_1^2+u)\Delta(u)} \int_m^1 \rho(m')m'dm'$$

(B.11)

where $$m^2 = \sum_{i=1,..3} \frac{x_i^2}{(a_i^2+u)}$$

For constant ρ, we immediately have:

$$\mathcal{D}_1 = (\pi G \rho a_1 a_2 a_3) x_1 a_1^2 \left[A_1 - \sum A_{ij} x_j^2 \right] \quad \text{(B.12)}$$

with $$A_{ij\ldots\ell} = \int_0^\infty \frac{du}{(a_i^2+u)(a_j^2+u)\ldots(a_\ell^2+u)\Delta(u)}$$

Eq. (B.12) agrees with the corresponding expression for D_1 given by Chandrasekhar (see, e.g. Ap. J. 136, 1042, 1962).

We now turn to Eq. (B.8). By interpreting the integral as the potential energy of a system where the density distribution is ρX_1, and by making use of Eq. (B.10) and the method listed earlier in this Appendix, we readily obtain:

$$\int \rho \varpi^2 \mathcal{D}\, d\underset{\sim}{x} = \int \rho (x_1 \mathcal{D}_1 + x_2 \mathcal{D}_2)\, d\underset{\sim}{x}$$
$$= 2\pi^2 G a_1^8 a_3^2 A_1 \int_0^1 m^2 F^2 dm$$

(B.13)

Appendix C

Calculation of the Various Potentials

For an axisymmetric density distribution of the form

$$\rho = \rho(m^2)$$

where $m^2 = \dfrac{x_1^2 + x_2^2}{a_1^2} + \dfrac{x_3^2}{a_3^2}$ $\qquad 0 \leq m \leq 1$ (C.1)

The potential at any point (X_1, X_2, X_3) is given by:

$$U(x_1 x_2 x_3) = \pi G a_1^2 a_3 \int_0^\infty \frac{du}{\Delta(u)} \int_{m'}^1 2\rho m\, dm \tag{C.2}$$

where
$$(m')^2 = \frac{x_1^2 + x_2^2}{a_1^2 + u} + \frac{x_3^2}{a_3^2 + u} \tag{C.3}$$

$$\Delta^2(u) = (a_1^2 + u)^2 (a_3^2 + u) \tag{C.4}$$

For ρ = constant, (C.2) immediately gives:

$$U(x_1, x_2, x_3) = \pi G \rho a_1^2 a_3 \left(I - \sum A_i X_i^2\right)$$

where
$$I = \int_0^\infty \frac{du}{\Delta(u)}$$

This agrees with, say, Eq. (47) of Chandrasekhar (Ap. J. 136, 1042, 1962).

We now define a new set of dummy integration variables:

$$\eta = ma_1 \qquad v = \frac{u}{a_1^2} \tag{C.5}$$

when (C.2) can be written as:

$$U(x_1, x_2, x_3) = \pi G\sqrt{1-e^2}\int_0^\infty \frac{dv}{\Delta(v)} \int_{\eta'}^{\eta = a_1 = \eta_0,\ \text{say}} 2\rho\eta\, d\eta \tag{C.6}$$

where now $\dfrac{a_3^2}{a_1^2} = 1-e^2$

$$\Delta^2(v) = (a_1^2+v)^2(a_3+v)$$

$$(\eta')^2 = \frac{x_1^2+x_2^2}{1+v} + \frac{x_3^2}{1-e^2+v}$$

We now effect another transformation of variables,

$$x_1 = \eta_1 \qquad x_2 = \eta_2 \qquad x_3 = \sqrt{1-e^2}\ \eta_3 \tag{C.7}$$

when the ellipsoidal equidensity surfaces in the $\underset{\sim}{x}$ - co-ordinates would now become spheres in the $\underset{\sim}{\eta}$ - co-ordinates. With this, (C.6) can be written as:

$$U(\eta,\theta) = \pi G\sqrt{1-e^2}\int_0^\infty \frac{dv}{\Delta(v)} \int_{\eta^\dagger}^{\eta_0} 2\rho\eta'\, d\eta' \tag{C.8}$$

where $$(\eta^{\dagger})^2 = \frac{\eta_1^2+\eta_2^2}{1+v} + \frac{(1-e^2)\,\eta_3^2}{1-e^2+v}$$

$$= \frac{\eta^2 \sin^2\theta}{1+v} + \frac{(1-e^2)\,\eta^2\cos^2\theta}{1-e^2+v}$$

We now change to the dimensionless quantities θ and ξ defined by

$$\rho = \lambda\theta^{n}$$
$$\eta = \alpha\xi \qquad \text{(C.9)}$$

when we finally have the desired expressions

$$U(\xi,\theta) = 2\pi G\sqrt{1-e^2}\,\alpha^2\lambda\, U^*(\xi,\theta)$$

with $$U^*(\xi,\theta) = \int_0^{\infty}\frac{dv}{\Delta(v)}\int_{\xi^{\dagger}}^{\xi_0}\theta^{n}\xi'\,d\xi' \qquad \text{(C.10)}$$

& $$(\xi^{\dagger})^2 = \frac{\xi^2\sin^2\theta}{1+v} + \frac{\xi^2\cos^2\theta\,(1-e^2)}{1-e^2+v}$$

With similar manipulations, the expression (B.11) for the D_1 produced by the "density" ρX, can be written as

$$\mathcal{D}_1 = 2\pi G\alpha^2\lambda\sqrt{1-e^2}\;x_1\,\mathcal{D}^* \quad \text{where}$$

$$\mathcal{D}^* = \int_0^{\infty}\frac{dv}{(1+v)\Delta(v)}\int_{\xi^{\dagger}}^{\xi_0}\theta^{n}\xi\,d\xi \qquad \text{(C.11)}$$

Similarly, a potential U_j produced by a density distribution $\rho_j' = \lambda\theta_o^{n-1}\,\theta_j$ can be written as:

$$U_j = 2\pi G\sqrt{1-e^2}\, a^2 \lambda\, U_j^*$$

where
$$U_j^* = \int_0^\infty \frac{dv}{\Delta(v)} \int_{\zeta^\dagger}^{\zeta_0} \Theta_0^{n-1}\,\theta_j\,\zeta' d\zeta' \qquad \text{(C.12)}$$

In Eq. (8.42), we have the occasion to use $\frac{\partial U^*}{\partial e}$. From Eq. (C.10), this is simply a matter of differentiating an integral with respect to a parameter. In general, if

$$F(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

Then
$$\frac{\partial F}{\partial x} = \int_{\varphi_1}^{\varphi_2} \frac{\partial f}{\partial x} dy - \left(\frac{\partial \varphi_1}{\partial x}\right) f(x,\varphi_1) + \frac{\partial \varphi_2}{\partial x} f(x,\varphi_2)$$

In our present case, if we put

$$g(e) = \int_{\zeta^\dagger}^{\zeta_0} \theta^n \zeta' d\zeta'$$

Then
$$\frac{\partial U^*}{\partial e} = \int_0^\infty dv \frac{\partial}{\partial e}\left[\frac{1}{\Delta(v)}\, g(e)\right]$$
$$= e\int_0^\infty \frac{dv}{(1-e^2+v)\Delta(v)} \int_{\zeta^\dagger}^{\zeta_0} \theta^n \zeta' d\zeta'$$
$$+ (e\,\zeta^2 \cos^2\theta) \int_0^\infty \frac{v\,dv\;\theta^n(\zeta^\dagger)}{(1-e^2+v)^2 \Delta(v)}$$

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