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FUNCTIONAL ANALYTIC ANALYSIS OF THE PURSUIT PROBLEM OF KELENDZERIDZE

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FOREWORD

The research described in this report, "Functional Analytic Analysis of the Pursuit Problem of Kelendzeridze," Number 68-6, by Rahul Chattopadhyay, was carried out under the direction of C. T. Leondes, Principal-Investigator, Department of Engineering, University of California, Los Angeles.

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CHAPTER 1

"I do not know." – Joseph L. Lagrange (1736-1813).

1.1. Introduction

The present dissertation is concerned with some new applications of the theory of moments. As such, a description of the basic problem of this theory is a relevant part of the introduction.

Problem Statement:

Let E be a normed, linear space. A set $\{x_i\}$ $i = 1, \ldots, n$ of linearly independent elements of E are given. Then given a set of scalars $\{c_i\}$ $i = 1, \ldots, n$, $\sum c_i^2 > 0$, and $L > 0$ find the necessary and sufficient condition for the existence of a linear functional f on E such that

$$f(x_i) = c_i \qquad i = 1, \ldots, n,$$

and

$$||f|| \leq L$$

The above problem has a very long history of investigation. The necessary and sufficient condition for the existence of minimum normed linear functionals was first obtained by Banach [8]. Akheizer and Krein [7] studied the above problem in great detail and derived, among other results, the conditions for the existence of a unique solution. In [7] it is also indicated that some of the basic ideas originated with A. A. Markov.† The book by Vorobyev [20] contains some applications of the theory of moments to operator equations. Results of a more general nature were presented by Ky Fan [9] at a later date.

Application of the theory of moments to optimal control of linear systems was carried out originally by Krasovskii [13]. Later on Kulikowski [18] applied the theory of moments to optimal control of

†The basic results are summarized in the appendix.

systems with amplitude, fuel, power and energy constraints. However, this method was introduced to the western world as late as 1962 by Kranc and Sarachik [19]. At about the same time, Antosiewicez published an article [10] on the theory of linear control systems based on the results of Ky Fan and others. It was followed by Neustedt's article [3] in which he made a rigorous and detailed application of the theory of moments to linear control systems.

Familiarity with the theory of moments convinces one that it has important applications in the theory of linear control systems. The questions of controllability and optimization of norm of control functions for linear systems can be most conveniently formulated in the context of the theory of moments. Also, it is not difficult to see (see appendix) that the basic maximization problem related to the theory of moments is a problem of nonlinear programming. Hence, it is possible to treat a linear variational problem by the methods of nonlinear programming (see [3]). Recently Swiger [14] and Harget [15] have applied the theory of moments to the problem of controllability of linear, distributed parameter systems. The present dissertation is an effort in the direction of finding some new applications of the theory of moments.

1.2. Scope of the Dissertation

The problem of optimal pursuit, as developed by Kelendzeridze, is presented in detail in the next chapter. The assumptions made by Kelendzeridze are analyzed and the concept of interceptibility is introduced. Necessary and sufficient conditions of interceptibility are then developed. Next, an estimate is provided for the set of initial states of the pursuer from which interception is possible. The concepts of optimal pursuit time and optimal strategies of pursuit are defined and methods of determining them are also presented.

To the author's knowledge, the entire text of this dissertation concerning the applications is original and new. The author would like to stress the fact that no new and "deep" mathematical theorems are developed here. This is contrary to the stated purpose of the dissertation. No "novel" proofs of well-known theorems are presented either. It is the author's belief that pure mathematics abounds in "deep" theorems whose potentials have not been fully explored for application. Hence, the present dissertation is strictly oriented towards application. Parts of the dissertation may not appear to be very precise to mathematically oriented readers. To them the author has only the following quotation to offer.

"The mathematicians who are merely mathematicians reason correctly, but only when everything has been explained to them in terms of definitions and principles. Otherwise they are limited and insufferable, for they only reason correctly when they are dealing with very clear principles." – Blaise Pascal (1623-1662). To an engineer, clear definitions and principles are sometimes luxuries.

CHAPTER 2

2.1 The Pursuit Problem of Kelendzeridze

The present section contains a description of the pursuit problem proposed originally by Kelendzeridze [1]. Let X and Y denote the pursuer and the pursued respectively. The equations of motion of X and Yare given as follows:

$$\frac{dx}{dt} = Ax + Bu , \quad x(0) = x^o \tag{2.1}$$

$$\frac{dy}{dt} = g(y, v) \quad , \quad y(0) = y^o , \tag{2.2}$$

where at time t, $x(t)$, $y(t) \epsilon R^n$ denote the states and $u(t) \epsilon R^r$, $v(t) \epsilon R^s$ denote the control variables of X and Y respectively. Each component of the real valued vector function $g(y, v)$ is assumed to be continuous in (y, v) and continuously differentiable with respect to each component of the n-vector y. Control functions u and v are called admissible if they are sectionally continuous vector functions of t such that $u(t) \epsilon \Omega^r \subseteq R^r$ and $v(t) \epsilon \Omega^s \subset R^s$. The subset Ω^r of R^r is assumed to be a closed, convex polyhedron. For given admissible control functions, the solutions of Equations (2.1) and (2.2) satisfying the given initial conditions will be called paths of X and Y and denoted by $x(t)$ and $y(t)$, $t > 0$ respectively.

For an arbitrary admissible control function v it is assumed that there exists an admissible u such that the paths $x(t)$ and $y(t)$, $t > 0$ satisfy the condition $x(t_1) = y(t_1)$ for some $t_1 > 0$. It is also assumed that for the chosen u and v, $x(t) \neq y(t)$ for $0 \leq t < t_1$. The pursuit time denoted by T_{uv} is defined to be $T_{uv} = t_1$.

It is assumed that for an admissible v, T_{uv} attains a minimum for some admissible u. Let $T_v = \underset{u}{\text{Min}}\, T_{uv}$. It is further assumed that an admissible v exists which maximizes T_v. This maximum, when it exists, is denoted by $T = \underset{v}{\text{Max}}\, \underset{u}{\text{Min}}\, T_{uv}$. The pair of control functions (u, v) is defined to be optimal if and only if $T_{uv} = T$. The corresponding paths of X and Y are called optimal paths.

2.2 Necessary Condition of Optimal Pursuit

The following necessary condition of optimal pursuit is due to Kelendzeridze.

Theorem 1: Let (u, v) be an optimal pair of control functions, x(t) and y(t), $t > 0$ corresponding optimal paths and T optimal pursuit time. Then there exist continuous, nonzero, covariant vector functions of time p and q $\left(p(t),\ q(t) \in R^n,\ 0 \le t \le T\right)$ such that

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} \,, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial x} \,,$$

$$\frac{dy}{dt} = \frac{\partial H}{\partial q} \,, \qquad \frac{dq}{dt} = -\frac{\partial H}{\partial y} \,,$$

and $p(T) = -q(T)$ where

$$H(t, x, p, q) = \underset{u}{\text{Max}}\ \underset{v}{\text{Min}}\ \ p'(t)\left(A\,x(t) + B\,u(t)\right) + q'(t)\,f(y, v) \,,$$

and $H(t, x, p, q) = \text{constant} \ge 0 \,,\ 0 \le t \le T$.

The proof of the above theorem can be found in [1].

2.3. Critique of the Analysis of Kelendzeridze

The analysis of the pursuit problem presented in [1], though the first of its kind, lacks completeness. The formulation of the problem also calls for considerable generalization.

It is to be noted that one of the basic assumptions in [1] is that for an arbitrary admissible v there exists admissible u such that $x(t_1) = y(t_1)$ for some $t_1 > 0$. No necessary and/or sufficient condition is presented in [1] under which such an assumption may be valid. It will be shown later that even for simple examples such an assumption may be incorrect. This assumption leads to the concept of interceptibility. The astute reader will notice the possibility of a close relation between the concept of controllability and the basic assumption of Kelendzeridze.

Another limitation of [1] is the assumption that the admissible control functions are sectionally continuous. This requirement on the admissibility of control functions will be relaxed to allow every measurable function to be admissible. Note that Theorem 1 cannot be applied without modification to cases where the control functions are restricted by isoperimetric constraints over $[0, t_1]$. This is so because the $\underset{u}{\text{Max}}\ \underset{v}{\text{Min}}$ operation in Theorem 1 is applied for each fixed t.

2.4. Some Recent Results

Since the publication of Kelendzeridze's articel, a considerable amount of work has been done on this pursuit problem. The basic assumption of Kelendzeridze (noted previously) has led Pontryagin [5] to introduce the concept of completion of linear differential games.

Let the equation of motion of a point $z(t) \in R^n$ be given as follows

$$\frac{dz}{dt} = Cz + U(u) - V(v)\ , \quad z(0) = z^o\ , \tag{2.3}$$

where u and v are control functions taking values in a (y-1) dimensional unit sphere K, C is a constant matrix, and U and V are analytic mappings of the sphere K to a space L. Let M be a (n-y) dimensional subspace of R^n.

By definition, the linear differential game can be completed on some set $A \subset R^n$ if, for any initial value $z^o \in A$ there exists a number $T(z^o) \geq 0$ such that for an arbitrary piecewise continuous admissible control v, it is possible to choose an admissible control u such that $z(t) \in M$ for some t, $0 \leq t \leq T(z^o)$.

In [5], Pontryagin provides sufficient conditions for completion of linear differential games and also for $A = R^n$. In this case, these conditions are developed for control functions which are analytic mappings of piecewise continuous functions. Moreover, no simple and explicit estimate of the set A is provided when $A \neq R^n$. Pontryagin's result is evidently pertinent to the pursuit problem. However, it cannot be applied without modification to the most general form of linear pursuit problem, let alone the problem discussed in Section 2. 1.

Incidentally, note that the basic problem of a linear differential game (and not a linear pursuit problem) is not that of "completion" but that of the existence of "playable pairs" (see [11], [12]). Hence, Pontrygin's results do not constitute a general theory of linear differential games as claimed in [5]. Moreover, since nothing is mentioned in [5] about the existence of "value" of the class of linear differential games considered, the title of the paper is somewhat misleading.

Necessary and sufficient condition for the completion of the pursuit problem discussed in Section 2. 1. is given in the next chapter. An estimate of the set A is provided. Methods for determining T and optimal (u, v) pairs are also discussed.

CHAPTER 3

3.1. Formulation of Problem

Let the pursuer be denoted by X and the pursued by Y as before. Their equations of motion are assumed to be as follows:

$$\frac{dx}{dt} = A(t)x + B(t)u, \quad x(t_o) = x^o \neq 0 \tag{3.1}$$

$$\frac{dy}{dt} = f(t, y, v), \quad y(t_o) = 0 \tag{3.2}$$

where at time t, $x(t), y(t) \in R^n$ are the state variables, and $u(t) \in R^r$ and $v(t) \in R^m$ are the control variables of X and Y respectively. X is assumed to be a proper system as defined by La Salle [2].

$A(t)$ and $B(t)$ are $(n \times n)$ and $(n \times r)$ matrices respectively. The matrices are defined almost everywhere with respect to a Lebesgue measure on $[t_o, \infty]$. For every bounded $[t_o, t]$ elements of $A(t)$ are members of $L^1[t_o, t]$ and elements of $B(t)$ are members of $L^q[t_o, t]$, $1 \leq q \leq \infty$. u is a Lebesgue measurable vector mapping of a compact interval $[t_o, t_1]$ to R^r and its components are members of $L^p[t_o, t_1]$ where $\frac{1}{p} + \frac{1}{q} = 1$. (Method of choosing t_1 will be described later on in this section.) Thus Equation (3.1) is defined almost everywhere on $[t_o, t_1]$ and has a unique solution x which is an absolutely continuous mapping of $[t_o, t_1]$ to R^n.

The function $f(t, y, v)$ is defined on $[t_o, \infty] \times R^n \times V$ where V is a compact set in R^m. $f(t, y, v)$ is continuous in (y, v) and integrable in t for fixed (y, v). Furthermore $f(t, y, v)$ satisfies the Lipschitz condition and is such that Equation (3.2) does not have any solutions with finite escape time. The control function v is measurable and for each t, $v(t) \in V$. Also the set $\{f(t, y, v) : v \in V\}$ is assumed to be convex. Thus Equation (3.2) has a unique solution y for every admissible v and the attainable set of Y in R^n is closed and bounded (see [4]).

For given admissible u and v, the solutions of Equations (3.1) and (3.2) with the given initial conditions are called the paths of X and Y respectively and are denoted by x and y. The set of admissible controls u is denoted by U and defined as follows:

$$U \triangleq \{u : ||u||_p \leq K_1, K_1 > 0\} \tag{3.3}$$

where

$$||u||_p \triangleq \left[\int_{t_o}^{t_1} \sum_1^r |u_i(t)|^p \, dt\right]^{1/p}, \quad 1 \leq p \leq \infty,$$

and

$$||u||_\infty = \max_{1 \leq i \leq r} |u_i|$$

Definition 1: The pursued Y is defined to be interceptible by the pursuer X at time t_1 if and only if for arbitrary admissible v and given x^o, there exists at least one admissible u, such that $x(t_1) = y(t_1)$, $t_1 > t_o$. Y is defined to be interceptible by X if it can be intercepted by X for some finite t_1 in the sense of Definition 1. The importance of the concept of interceptibility of Y by X should now be clear in relation to the discussion of the previous chapter. It is part of the basic assumption of Kelendzeridze. Pontryagin's concept of completion of linear differential games is related to the concept of interceptibility as shown below.

Suppose the equation of motion of Y is as follows:

$$\frac{dy}{dt} = A(t)y + D(t)v, \quad y(t_o) = 0$$

Let $z(t) \triangleq x(t) - y(t)$. Then it follows that,

$$\frac{dz}{dt} = A(t)z + B(t)u - D(t)v \ , \ z(t_o) = z^o = x^o$$

Then for given x^o, Y is interceptible by X implies $z^o \epsilon A$ where $A \subset R^n$ is the set on which the linear differential game described above can be completed (defined in §2. 4.) Thus for pursuit problems, the concept of interceptibility is more general than that of completion. Necessary and sufficient conditions of interception are given in the next two sections.

3. 2. Conditions of Interception

It is assumed in this section that x^o is given. Let t_1, $t_o < t_1 < \infty$ be an arbitrarily chosen instant of time. Then for a given v, Y is interceptible by X at t_1 if and only if there exists at least one $u \epsilon U$, such that

$$\int_{t_o}^{t_1} \Phi(t_1, \tau) B(\tau) \, du(\tau) = - \Phi(t_1, t_o) x^o + y_v(t_1)$$
$$\triangleq z(t_1, x^o, v)$$

where $\Phi(t, \tau)$ denotes the transition matrix of Equation (3. 1) (u=0), $y_v(t_1)$ denotes the solution of Equation (3. 2) at t_1 for given admissible v, and $z(t_1, x^o, v)$ is a n-vector, often abbreviated as z. (Note: It is necessary to embed the space of control functions u into a suitable Banach space of functions of strong bounded p-variation. For details see [3]).

Define $h_i(t_1, \tau) \triangleq i^{th}$ row of the matrix $\Phi(t_1, \tau)B(\tau), i=1, \ldots, n$. Since X is assumed to be a proper system, the row vectors $h_i(t_1, \tau)$ are linearly independent functions on each interval of positive length. Define

$$\lambda[t_1, z] \triangleq \operatorname*{Max}_{\xi} \frac{<\xi, z>}{||\Sigma \xi_i h_i||_q} \ ,$$

where $< , >$ denotes the scalar product in R^n and

$$\|\Sigma \xi_i h_i\|_q \triangleq \left[\int_{t_o}^{t_1} \sum_k \left|\sum_i \xi_i h_i^k\right|^q d\tau\right]^{1/q} , \quad \frac{1}{p} + \frac{1}{q} = 1$$

Then in view of the theory of moments [3], [7] the following result can be easily established.

Result 1: For given admissible v and x^o, necessary and sufficient condition for Y to be interceptible by X at t_1 is,

$$\lambda[t_1, z] \le K_1 \tag{3.4}$$

Define

$$\bar{\lambda}[t_1, x^o] = \max_{y_v \in R[t_o, t_1]} \lambda[t_1, z] \quad ,$$

where $R[t_o, t_1]$ denotes the attainable set of Y in the time interval $[t_o, t_1]$. This set is a closed and bounded subset of R^n as shown in [4]. For $p = q = 2$, the function $\bar{\lambda}[t_1, x^o]$ can be given a physical interpretation. For this special case, $\lambda[t_1, z]$ equals the minimum energy required by X to intercept Y at time t_1 for the given admissible v. Hence, $\bar{\lambda}[t_1, x^o]$ equals the maximum energy required by X to intercept Y at t_1 when X is moving along minimum-energy paths. Next result now follows easily from Result 1.

Result 2: Necessary and sufficient condition for Y to be interceptible by X at t_1 in the sense of Definition 1 is,

$$\bar{\lambda}[t_1, x^o] \le K_1$$

The above inequality can be given the following interpretation in terms of the attainable sets of X and Y respectively. Since x^o is assumed to be given, $\Phi(t_1, t_o)x^o$ is a fixed vector for given t_1. Let $S[t_o, t_1] \triangleq \{y: \lambda[t_1, z] \le K_1, \; z = -\Phi(t_1, t_o)x^o + y\}$. Then by definition

$S[t_o, t_1]$ is the attainable set of X from x^o in the time interval $[t_o, t_1]$ when $||u||_p \leq K_1$.

Result 3: $\bar{\lambda}[t_1, x^o] \leq K_1$ if and only if $R[t_o, t_1] \subseteq S[t_o, t_1]$.

Proof: Let $\bar{\lambda}[t_1, x^o] \leq K_1$. This implies $\lambda[t_1, z] \leq K_1$, $\forall y_v \in R[t_o, t_1]$ Hence by definition $R[t_o, t_1] \subseteq S[t_o, t_1]$.

Let $R[t_o, t_1] \subseteq S[t_o, t_1]$. Again by definition $\lambda[t_1, z] \leq K_1$, $\forall y \in R[t_o, t_1]$ where $z = -\Phi(t_1, t_o)x^o + y$. Hence $\bar{\lambda}[t_1, x^o] \leq K_1$.

Example: The following example shows that even in simple cases, when X and Y are identical dynamical systems and have identical constraints on control functions, interception in the sense of Definition 1 is not possible for finite t_1. Let the equations of motion of both X and Y be as follows:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

with both control functions limited in magnitude as shown below

$$\max_{t \in [t_o, t_1]} |u(t)| \leq 1$$

The initial state is $x^o = [1, 0]'$ at time $t_o = 0$. For the special case when $v(t) = -1$, $y_1(t) = -t_1^2/2$, $y_2(t) = -t_1$ and $z(t_1, x^o, v) = -[1 + t_1^2/2, t_1]'$. Evidently $\Phi(t_1, \tau)B(\tau) = [t_1 - \tau, 1]'$ and hence

$$||\Sigma \xi_i h_i||_q = \frac{|\xi_1|}{2} t_1^2 + |\xi_2| t$$

From the definition it can be easily shown that

$$\lambda[t_1, z] = 1 + \frac{2}{t_1^2}$$

and evidently $\bar{\lambda}[t_1, x^o] \geq 1 + 2/t_1^2 > 1$, $\forall t_1 < \infty$. Hence, according to Result 2, Y cannot be intercepted by X for any finite t_1 in the sense of Definition 1. This result can be easily checked by computing the attainable sets of X and Y. For this example, it becomes clear from the superposition of these attainable sets that $R[t_o, t_1] \not\subset S[t_o, t_1]$. Thus the motion of Y corresponding to v(t) = - 1 cannot be intercepted by any admissible motion of X from $[1, 0]'$ for any finite t_1.

Condition of Escape: By definition, Y can always escape from X at t_1 if for arbitrary admissible v, the path of Y cannot be intercepted by any admissible path of X in the sense that $y(t_1) = x(t_1)$.

The function $\lambda[t_1, z]$, introduced earlier, can be used to determine a necessary and sufficient condition of escape. Define

$$\underline{\lambda}[t_1, x^o] \triangleq \min_{y_v \in R[t_o, t_1]} \lambda[t_1, z]$$

Result 4: Necessary and sufficient condition for Y to escape from X at t_1 is,

$$\underline{\lambda}[t_1, x^o] > K_1$$

The following result describes the monotonic character of $\underline{\lambda}[t, x^o]$ with respect to t when X is a proper system.

Result 5: Given X to be a proper system (see [2]),

$$\underline{\lambda}[t_1, x^o] > \underline{\lambda}[t_2, x^o], \quad t_2 > t_1$$

Proof: Let $z(t) = -\Phi(t, t_o)x^o + y$ where y is a constant vector. If X is a proper system it follows that (see [14], p. 60)

$$\lambda[t_1, z(t_1)] > \lambda[t_2, z(t_2)], \quad t_2 > t_1$$

Therefore

$$\min_{y \in R[t_o, t_2]} \lambda[t_1, z(t_1)] > \min_{y \in R[t_o, t_2]} \lambda[t_2, z(t_2)] \triangleq \underline{\lambda}[t_2, x^o]$$

Also $R[t_o, t_1] \subseteq R[t_o, t_2]$, $t_2 > t_1$.

Therefore $$\min_{y \in R[t_o, t_1]} \lambda[t_1, z(t_1)] \geq \min_{y \in R[t_o, t_2]} \lambda[t_1, z(t_1)]$$

Hence, $$\underline{\lambda}[t_1, x^o] > \underline{\lambda}[t_2, x^o], \quad t_2 > t_1 .$$

The implication is that if X is a proper system, Y cannot escape from X for large values of t_1.

<u>Result 6</u>: $\underline{\lambda}[t_1, x^o] > K_1$ if and only if $R[t_o, t_1] \cap S[t_o, t_1] = \phi$

Proof: Suppose $\underline{\lambda}[t_1, x^o] > K_1$. Then $\lambda[t_1, z] > K_1$, $\forall y \in R[t_o, t_1]$. Let $y \in R[t_o, t_1]$. Then by definition of the set $S[t_o, t_1]$, $y \notin S[t_o, t_1]$. Since y is arbitrarily chosen $R[t_o, t_1] \cap S[t_o, t_1] = \phi$

Suppose $R[t_o, t_1] \cap S[t_o, t_1] = \phi$. Then for every $y \in R[t_o, t_1]$, $\lambda[t_1, z] > K_1$. Hence, $\underline{\lambda}[t_1, x^o] > K_1$.

3.3 Effect of Initial State on Interception

In the previous section x^o was assumed to be given. Thus all the results depended either explicitly or implicitly on x^o. In the present section, effect of x^o on interception is discussed. Let $Q[t_o, t_1]$ denote the set of initial states of X from which it can intercept Y at t_1. Recall that for given admissible v, interception at t_1 is possible if and only if inequality (3.4) is satisfied. Define

$$M(t_1) \triangleq \max_{\xi} \frac{|\xi|}{||\Sigma \xi_i h_i||_q}, \quad |\xi| = \left[\sum_{i=1}^{n} \xi_i^2\right]^{\frac{1}{2}}$$

Then an upper bound on $\lambda[t_1, z]$ can be provided as follows:

$$\lambda[t_1, z] \leq M(t_1)\, |z| ,$$

where $|z|$ denotes the Euclidean norm of the n-vector z. Also

$$|z| = |-\Phi(t_1, t_o)x^o + y_v(t_1)|$$

$$\leq |\Phi(t_1, t_o)x^o| + |y_v(t_1)| \ .$$

Hence, inequality (3.4) is satisfied whenever

$$|\Phi(t_1, t_o)x^o| \leq \frac{K_1}{M(t_1)} - |y_v(t_1)| \tag{3.5}$$

Let $N(t_1) \triangleq \underset{y_v \in R[t_o, t_1]}{\text{Max}} |y_v(t_1)|$

Then for any arbitrary admissible v, inequality (3.5) is satisfied whenever

$$|\Phi(t_1, t_o)x^o| \leq \frac{K_1}{M(t_1)} - N(t_1)$$

Hence, from the definition

$$Q[t_o, t_1] \supseteq \overline{Q}[t_o, t_1] \triangleq \left\{x^o : |\Phi(t_1, t_o)x^o| \leq \frac{K_1}{M(t_1)} - N(t_1)\right\}$$

and the following result is obtained.

Result 7: Sufficicient conditions for Y to be interceptible by X at t_1 in the sense of Definition 1 are,

(i) $\overline{Q}[t_o, t_1] \neq \phi$

(ii) $x^o \in \overline{Q}[t_o, t_1]$.

Condition (i) above can be put into a more interesting form if Y is assumed to be a linear system as given below.

$$\frac{dy}{dt} = C(t)y + D(t)v, \quad y(t_o) = 0 \tag{3.6}$$

$$V \triangleq \{v : ||v||_p \leq K_2\} \tag{3.7}$$

Let $g_i(t_1, \tau) \triangleq i^{th}$ row of the matrix $\Psi(t_1, \tau) D(\tau)$ where $\Psi(t_1, \tau)$ denotes the transition matrix of Equation (3.6) (v = 0). Then

$$N(t_1) = K_2 \max_{\eta} \frac{\|\Sigma \eta_i g_i\|_q}{|\eta|}$$

and using the definition of $M(t_1)$, Condition (i) can be rewritten as

$$\text{(iii)} \qquad \min_{\xi} \frac{\|\Sigma \xi_i h_i\|_q}{|\xi|} \geq \frac{K_2}{K_1} \max_{\eta} \frac{\|\Sigma \eta_i g_i\|_q}{|\eta|}$$

For the special case p = q = 2, Condition (iii) can be given yet another form. For q = 2, $\|\Sigma \xi_i h_i\|_q = \left[\xi' W_x(t_o, t_1) \xi\right]^{\frac{1}{2}}$, where $W_x(t_o, t_1)$ is the Kalman controllability matrix [16] for the system X. Thus in the special case of p = q = 2, Condition (iii) becomes

$$\text{(iv)} \qquad \min_{\xi} \left\{ \frac{[\xi' W_x(t_o, t_1) \xi]}{\xi' \xi} \right\}^{\frac{1}{2}} \geq \frac{K_2}{K_1} \max_{\eta} \left\{ \frac{[\eta' W_y(t_o, t_1) \eta]}{\eta' \eta} \right\}^{\frac{1}{2}}$$

Evidently, Condition (iv) implies

$$\text{(v)} \qquad \xi' W_x(t_o, t_1) \xi \geq \left(\frac{K_2}{K_1}\right)^2 \xi' W_y(t_o, t_1) \xi$$

It has been shown in [17], that inequality (v) (for $K_2 = K_1$) is a sufficient condition for the existence of optimal strategies for a special class of pursuit problems. However, as is clear from the derivation, inequality (v) is not a sufficient condition for interception of Y by X at t_1 in the sense of Definition 1. Condition (iv) is more stringent than (v) because the definition of interception used here is more stringent than that used in [17]. Note also that in [17], interception does not depend on x^o whereas in the present case Condition (ii) along with (i) has to be satisfied for interception. The problem of completion for the pursuit problem formulated in [17] is trivial. Conditions (ii) and

(iv) together satisfy the intuitive notions, that for interception, the initial state of the intercepter should be favorable and the intercepter should be more controllable than the target in some sense.

3.4 Optimal Pursuit Problem

This section is devoted to the problem of optimal pursuit. It is assumed that (a) Y is interceptible by X, and (b) $\lim_{t_1 \to t_o^+} \underline{\lambda}[t_1, x^o] > K_1$.

The last assumption implies that Y cannot be intercepted by X at t_o^+. Let the pursuit time be denoted by T_{uv} where $T_{uv} = t_1$ and t_1 is such that $x(t_1) = y(t_1)$, $x(t) \neq y(t)$, to $t_o \leq t < t_1$. It is assumed that for an arbitrary admissible v, T_{uv} attains a minimum for some admissible u. Let $T_v = \min_u T_{uv}$. It is further assumed that an admissible v exists which maximizes T_v. This maximum, when it exists, is denoted by $T = \max_v \min_u T_{uv}$. The pair (u, v) is defined to be optimal if and only if $T_{uv} = T$.

From assumptions (a) and (b) it follows that there exists a t_1 such that

$$\text{(i)}\quad \bar{\lambda}\left[t_1^+, x^o\right] \leq K_1 \leq \bar{\lambda}\left[t_1^-, x^o\right],$$

and $$\text{(ii)}\quad \bar{\lambda}[t, x^o] > K_1, \forall t,\ t_o \leq t < t_1^-$$

<u>Result 8:</u> $T = t_1$, where t_1 satisfies Conditions (i) and (ii).

Proof: From Condition (ii) and Result 2 it is clear that Y cannot be intercepted by X for any $t < t_1$. Thus $T \nless t_1$. Also from Condition (i) and Result 2 it is clear that Y can be intercepted by X at t_1 in the sense of Definition 1. Thus for arbitrary admissible v there exists admissible u such that $T_v = \min_u T_{uv} \leq t_1$. It follows that

$T = \max_v \min_u T_v \leq t_1$. Hence $T = t_1$.

Result 9: Suppose $\bar{\lambda}[T^-, x^o] = K_1$. Let v be an admissible control such that

$$\lambda[T^-, z(T^-, v, x^o)] = \bar{\lambda}[T^-, x^o]$$

Then v is an optimal control for Y.

Proof: Consider a variation of v denoted by $v + \delta v$ which does not satisfy the hypothesis. Then from definition of $\bar{\lambda}[T^-, x^o]$ it follows that

$$\lambda[T^-, z(T^-, v+\delta v, x^o)] < \bar{\lambda}[T^-, x^o] = K_1$$

Since $\underline{\lambda}[t, x^o] \leq \lambda[t, z(t, v+\delta v, x^o)]$ and by assumption $\lim_{t \to t_o^+} \underline{\lambda}[t, x^o] > K_1$ there exists $t < T$ such that

$$\lambda[t^+, z(t^+, v+\delta v, x^o)] \leq K_1,$$

$$\text{and} \quad \lambda[t^-, z(t^-, v+\delta v, x^o)] \geq K_1$$

Note that $t \neq T$ because otherwise the last inequality with $t^- = T^-$ implies that $v + \delta v$ satisfies the hypothesis, which is contrary to the previous assumption. Thus according to Result 1 the path of Y corresponding to $v + \delta v$ can be intercepted by an admissible path of X at $t < T$. Hence v is an optimal control for Y.

Result 10: For linear systems, optimal v satisfies the condition $||v||_p = K_2$.

Proof: Define $S[t_o, t_1] \triangleq \{z : z = -\Phi(t_1, t_o)x^o + y, \ y \in R[t_o, t_1]\}$

Then $$\bar{\lambda}[t_1, x^o] = \max_{z \in S[t_o, t_1]} \lambda[t_1, z]$$

The set $S[t_o, t_1]$ is closed and convex and $\lambda[t_1, z]$ is a convex function of z for fixed t_1 (see [7] or appendix). Thus either $\lambda[t_1, z]$ attains its maximum on the boundary $\partial S[t_o, t_1]$ or it is a constant on $S[t_o, t_1]$. Thus the optimal $y \in \partial R[t_o, t_1]$ and hence, for the optimal control $||v||_p = K_2$.

3. 5 Examples

Let the equations of motion of X and Y be as follows:

$$\frac{dx}{dt} = A(t)\,x + B(t)\,u, \quad x(t_o) = x^o \neq 0 \ ,$$

$$\frac{dy}{dt} = C(t)\,y + D(t)\,v, \quad y(t_o) = 0 \ ,$$

with the sets of admissible control functions defined by Equations (3. 3) and (3. 7) respectively. It is assumed that $p = q = 2$. This example cannot be solved by the method of [1] because of the nature of constraints on the control functions. By definition

$$\lambda[t_1, z] = \underset{\xi}{\mathrm{Max}} \ < \xi, z > \ ,$$

where $\xi' W_x(t_o, t_1)\, \xi = 1$.

This maximization problem can be easily solved by means of the Lagrange multiplier technique with the following result.

$$\lambda[t_1, z] = \left[z' W_x^{\#} (t_o, t_1)\, z \right]^{\frac{1}{2}} ,$$

where $z = - \Phi(t_1, t_o)\, x^o + y_v(t_1)$.

Since $\lambda[t_1, z] \geq 0$, the functions $\lambda[t_1, z]$ and $\lambda^2[t_1, z]$ has the same points of extremum. The attainable set of Y is given as,

$$R[t_o, t_1] = \left\{ y: \ y' W_y^{\#} (t_o, t_1)\, y \leq K_2^2 \right\}$$

Since by assumption X is a proper system, $\lambda^2[t_1, z]$ is a convex function of $y_v(t_1)$. Also $R[t_o, t_1]$ is a closed, bounded and convex set. Hence $\lambda^2[t_1, z]$ has a minimum in $R[t_o, t_1]$. The maximum of $\lambda^2[t_1, z]$ is attained on the boundary $\partial R[t_o, t_1]$ because otherwise it will be constant on $R[t_o, t_1]$. Let $\eta(t_1) = W_x^{\#}(t_o, t_1)\Phi(t_1, t_o)x^o$. Then

$$\bar{\lambda}[t_1, x^o] = \left[x^{o'}\Phi' W_x^{\#}\Phi x^o + \mu^2 \eta' W_x^{\#} \eta - 2\mu \eta'\eta\right]^{\frac{1}{2}}, \qquad (3.8)$$

where $$\mu(t_1) = -\frac{K_2}{\left[\eta' W_y^{\#} \eta\right]^{\frac{1}{2}}},$$

and some of the arguments are suppressed for convenience. The function $\bar{\lambda}[t_1, x^o]$ as given by Equation (3.8) is a continuous function of t_1 and T can be obtained from Result 8 as,

$$\bar{\lambda}[T, x^o] = K_1$$

Optimal value of $y_v(t_1)$ is then given as

$$y_v(t) = \mu(T)\,\eta(T)$$

The optimal control for Y becomes

$$v(t) = \mu(T)\,D'(t)\Psi'(T, t)\,W_y^{\#}(T, t_o)\,\eta(T),$$

and that for X becomes

$$u(t) = \mu(T)\,B'(t)\Phi'(T, t)\,W_x^{\#}(T, t_o)\,\eta(T).$$

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APPENDIX

The L-problem of Krein in an abstract, linear, normed space:

Let E denote a normed, linear space. Given n linearly independent elements $x_1, \ldots, x_n$ of E find necessary and sufficient conditions on the numbers $c_1, \ldots, c_n$, $L \left(\Sigma c_i^2 > 0,\ L > 0 \right)$, such that there exists a linear functional f defined on E which satisfies the relations

$$f(x_i) = c_i, \quad i = 1, \ldots, n,$$

and $$||f|| \leq L$$

Define $$\lambda(c) = \text{Max} \frac{\Sigma \xi_i c_i}{||\Sigma \xi_i x_i||},$$

where $$c = (c_1, \ldots, c_n)'.$$

Then necessary and sufficient condition for the existence of a solution to the above problem is

$$\lambda(c) \leq L$$

Properties of $\lambda(c)$:

1) $\lambda(c_1, \ldots, c_n) > 0$ if $\Sigma c_i^2 > 0$

2) $\lambda(t c_1, \ldots, t c_n) = |t| \lambda(c_1, \ldots, c_n)$

3) $\lambda(c_1 + \bar{c}_1, \ldots, c_n + \bar{c}_n) \leq \lambda(c_1 \ldots, c_n) + \lambda(\bar{c}_1, \ldots, \bar{c}_n)$

4) There exist $M > 0$ and $m > 0$ such that

$$m\, \Sigma c_i^2 \leq \lambda(c_1, \ldots, c_n) \leq M \Sigma c_i^2$$

5) $\lambda(c_1, \ldots, c_n) \leq \lambda(c_1, \ldots, c_m)$ for $m > n$

6) Define $S \triangleq \{c : \lambda(c) \leq 1\}$. Then S is a convex, bounded, closed set having interior points.