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STUDY OF APOLLO WATER IMPACT

FINAL REPORT

VOLUME 8

UNSYMMETRIC SHELL OF REVOLUTION ANALYSIS

(Contract NAS9-4552, G.O. 5264)

May 1967

Prepared by

A. P. Cappelli
T. Nishimoto
P. P. Radkowski
K. E. Pauley
(Authors)

Approved by

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Program Manager
Structures and Materials

L. A. Harris
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NORTH AMERICAN AVIATION, INC.
SPACE DIVISION

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FOREWORD

This report was prepared by North American Aviation, Inc., Space Division, under NASA Contract NAS9-4552, for the National Aeronautics and Space Administration, Manned Space Flight Center, Houston, Texas, with Dr. F.C. Hung, Program Manager and Mr. P.P. Radkowski, Assistant Program Manager. This work was administered under the direction of Structural Mechanics Division, MSC, Houston, Texas with Dr. F. Stebbins as the technical monitor.

This report is presented in eleven volumes for convenience in handling and distribution. All volumes are unclassified.

The objective of the study was to develop methods and Fortran IV computer programs to determine by the techniques described below, the hydro-elastic response of representation of the structure of the Apollo Command Module immediately following impact on the water. The development of theory, methods and computer programs is presented as Task I Hydrodynamic Pressures, Task II Structural Response and Task III Hydroelastic Response Analysis.

Under Task I - Computing program to extend flexible sphere using the Spencer and Shiffman approach has been developed. Analytical formulation by Dr. Li using nonlinear hydrodynamic theory on structural portion is formulated. In order to cover a wide range of impact conditions, future extensions are necessary in the following items:

a. Using linear hydrodynamic theory to include horizontal velocity and rotation.

b. Nonlinear hydrodynamic theory to develop computing program on spherical portion and to develop nonlinear theory on toroidal and conic sections.

Under Task II - Computing program and User's Manual were developed for nonsymmetrical loading on unsymmetrical elastic shells. To fully develop the theory and methods to cover realistic Apollo configuration the following extensions are recommended:

a. Modes of vibration and modal analysis.

b. Extension to nonsymmetric short time impulses.

c. Linear buckling and elasto-plastic analysis

These technical extensions will not only be useful for Apollo and future Apollo growth configurations, but they will also be of value to other aeronautical and spacecraft programs.

The hydroelastic response of the flexible shell is obtained by the numerical solution of the combined hydrodynamic and shell equations. The results obtained herein are compared numerically with those derived by neglecting the interaction and applying rigid body pressures to the same elastic shell. The numerical results show that for an axially symmetric impact of the particular shell studied, the interaction between the shell and the fluid produces appreciable differences in the overall acceleration of the center of gravity of the shell, and in the distribution of the pressures and responses. However the maximum responses are within 15% of those produced when the interaction between the fluid and the shell is neglected. A brief summary of results is shown in the abstracts of individual volumes.

The volume number and authors are listed on the following page.

The contractor's designation for this report is SID 67-498.

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3	Dynamic Response of Shells of Revolution During Vertical Impact Into Water - Hydroelastic Interaction	J. P. D. Wilkinson, A. P. Cappelli, and R. N. Salzman
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ABSTRACT

A general numerical procedure is presented for determining the static and dynamic response of shells of revolutions with arbitrary distributions of stiffness subjected to arbitrary loads and temperatures. These formulations are based on Sander's linear first order shell theory which was modified to include the effect of transverse shear distortion. The method consists of a Fourier analysis to separate the circumferential variation in the governing equations. This results in equations with the coefficients coupled in the Fourier index. The matrix form of this equation is reduced to an algebraic form by finite difference. The unknown Fourier components of the solution are obtained by a matrix elimination procedure of this form of the governing equations.

The numerical analysis is mechanized for solution on the digital computer. Numerical examples and comparisons are presented.

The procedure is general and yields accurate solutions for complicated structural response for both static and dynamic conditions.

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CONTENTS

ILLUSTRATIONS

SYMBOLS

FOURIER COEFFICIENTS

u_n, v_n, w_n, $\phi_{\xi n}$, $\phi_{\theta n}$	= displacements and rotations
$p_{\xi n}$, $p_{\theta n}$, p_n	= loads
$t^T_{\xi n}$, $t^T_{\theta n}$, $m^T_{\xi n}$, $m^T_{\theta n}$	= temperature induced force terms
t^n_{ξ}, t^n_{θ}, $t_{\xi\theta}$, $\hat{t}_{\xi\theta}$, q^n_{ξ} q^n_{θ}, m^n_{ξ}, m^n_{θ}, $m^n_{\xi\theta}$,	= force resultants, modified and effective
b_{mj}, c_{mj}, d_{mj}, q_{mj}, q_{12j}, q_{13j}	= stiffnesses
$f_{5k+1,\ 5j+1}$ - - - - - $f_{5k+5,\ 5j+5}$ $g_{5k+1,\ 5j+1}$ - - - - - $g_{5k+5,\ 5j+5}$ $h_{5k+1,\ 5j+1}$ - - - - - $h_{5k+5,\ 5j+5}$ $\alpha_{5k+1,\ 5j+1}$ - - - - - $\alpha_{5k+5,\ 5j+5}$ $\beta_{5k+1,\ 5j+1}$ - - - - - $\beta_{5k+5,\ 5j+5}$ p_{5k+1} - - - - - - - - - p_{5k+5} $r_{5k+1,\ 5j+1}$ - - - - - $r_{5k+5,\ 5j+5}$ $s_{5k+1,\ 5j+1}$ - - - - - $s_{5k+5,\ 5j+5}$ l_{5k+1} - - - - - - - - - l_{5k+5}	= elements of F, G, H, α, β, p, R, S, and l

COORDINATES± CONSTANTS AND VARIABLES

ξ, θ, ζ	=	coordinates
r	=	normal distance from shell axis
s	=	meriodional shell coordinate
Δ, δ	=	spacial and time finite difference increments
a, h_o, σ_o, E_o	=	reference constants
$\omega_\theta, \omega_\xi$	=	nondimensional curvatures
$U_\xi, U_\theta, W, \Phi_\xi, \Phi_\theta$	=	displacements and rotations
$\sigma_\xi, \sigma_\theta, \tau_{\xi\theta}, \tau_{\xi\zeta}, \tau_{\theta\zeta}$	=	stresses
$\epsilon_\xi, \epsilon_\theta, \epsilon_{\xi\theta}, \gamma_{\xi\zeta}, \gamma_{\theta\zeta}$	=	strains
$k_{\xi\theta}, k_\xi, k_\theta$	=	bending distortion
B_i, C_i, D_i, G_i	=	stiffness functions
$E_\xi, E_\theta, \nu_{\xi\theta}, \nu_{\theta\xi}, \alpha_\xi, \alpha_\theta$	=	material properties
q_ξ, q_θ, ζ	=	loads
T	=	temperature change
μ	-	mass distribution
η	=	external damping
ψ	=	elastic foundation parameter
$N_\xi, N_\theta, \overline{N}_{\xi\theta}, M_\xi, M_\theta, M_{\xi\theta}, Q_\xi$ $Q_\theta, \hat{N}_{\xi\theta}, N_\xi^T, N_\theta^T, M_\xi^T, M_\theta^T$	=	force resultants, modified, effective, and temperature induced

MATRICES

F, G, H, Ω, Λ, R, S, α, β, L, M, N	5K x 5K order
P, 1, x	5K x 1 order

INDICES

i, j, n	Dummy
k	k^{th} Fourier component

1.0 ANALYSIS

1.1 INTRODUCTION

A general numerical procedure is presented for determining the response of shells of revolutions with arbitrary distributions of stiffness subjected to arbitrary loads and temperatures. The dynamic response problem will be presented and the specialization for the static analysis will be made.

The analysis is based on a modified form of the general first order linear shell theory of Sanders. These equations have been modified to include transverse shear distortion, see Appendix 1-1. The modified equilibrium equations are extended to include time dependence by D'Alembert's principle. Fourier analysis is used to separate variables in the circumferential direction and a system of finite difference approximations are used to reduce the partial differential equations to an algebraic set. This set is solved by using a direct matrix elimination procedure.

The material presented is an extension of the work by Sanders, Budiansky and Radkowski, and Johnson and Greif - Ref 1, 2, 3. The notation used is identical to that of Ref. 2 except where noted.

1.2 LIMITATIONS

The shell theory on which these programs are based is restricted to linear, elastic, thin shell theory.

1. The thickness of the shell at any point is small compared to the other dimensions.

2. Deformations of the shell are small compared to the dimensions of the shell.

3. All material points of the shell deform elastically, obeying Hooke's law for ortholropic materials.

4. The shell is "complete", i.e., its only boundaries are at meridian ends and inner and outer surfaces.

5. The class of shells considered was a surface of revolution reference surface which is within or in close proximity of the shell walls.

6. The parameters of stiffness, e.g., in-plane stiffnesses are permitted to vary in both the meridional and circumferential directions. Implied is that parameters such as thickness, Young's modulus, etc., are permitted to vary in both the meridional and circumferential directions.

7. Arbitrary loads and temperature distributions are permissible.

8. The effects of transverse shear is included.

9. Instability is not considered.

10. Arbitrary distribution of mass, elastic foundation and external damping is included.

1.3 SHELL COORDINATE SYSTEM

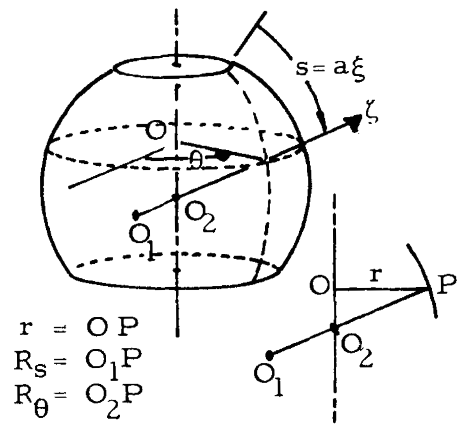
The class of shells considered must have a surface of revolution reference surface lying within or in close proximity of the shell walls. Material points of the shell are then described by an orthogonal coordinate system (s, θ, ζ) based on this reference. The meridional distance (s) is a mreasure from a boundary along the reference surface. The circumferential angle (θ) is a measure from some convenient reference and the normal measure (ζ) is the distance from the reference surface measured along the outward normal to the reference surface.

The geometry of the reference surface is given by r(s), when r is the distance from the axis, the principal radii of curvature are

$$R_\theta = r\left[1 - \left(\frac{dr}{ds}\right)^2\right]^{-1/2} \qquad 1\text{-}1$$

$$R_s = -\left[1 - \left(\frac{dr}{ds}\right)^2\right]^{-1/2} \Bigg/ \left(\frac{d^2r}{ds^2}\right) \qquad 1\text{-}2$$

Introduce the nondimensional meridional coordinate $\xi = s/a$, where a is a reference length; then, with $\rho = r/a$, the nondimensional curvatures $\omega_\xi = a/R_s$ and $\omega_\theta = a/R_\theta$ can be found from the formulas



$$\omega_\theta = \left[i - (\rho')^2 \right]^{1/2} \Big/ \rho \qquad 1\text{-}3$$

$$\omega_\xi = -(\gamma' + \gamma^2)/\omega_\theta \qquad 1\text{-}4$$

where

$$\gamma = \rho'/\rho \qquad 1\text{-}5$$

In these equations, and henceforth, $(\ )' \ \dfrac{\partial(\ )}{\partial \xi}$

Finally from the Codazzi relation we obtain

$$\omega'_\theta = \gamma\,(\omega_\xi - \omega_\theta) \qquad 1\text{-}6$$

and the relation

$$\frac{\rho''}{\rho} = -\omega_\xi\,\omega_\theta \qquad 1\text{-}7$$

1.4 EQUATIONS OF MOTION

The general equilibrium equations for an arbitrary shell based on the first-order linear shell theory of Sanders are given in Ref. 1. These equations are modified to include the effect of transverse shear distortion by the procedure suggested by Sanders. These equilibrium equations are extended to equations of motion by use of D'Alembert's principle. These equations specialized for a shell whose reference surface is a surface of revolution are given as,

$$a\left[\frac{\partial \rho}{\partial \xi} N_\xi + \rho \frac{\partial N_\xi}{\partial \xi} + \frac{\partial \overline{N}_{\xi\theta}}{\partial \theta} - \frac{\partial \rho}{\partial \xi} N_\theta \right] + a\rho\,\omega_\xi\, Q_\xi$$

$$+ 1/2\,(\omega_\xi - \omega_\theta)\frac{\partial \overline{M}_{\xi\theta}}{\partial \theta} + a^2 \rho\, q_\xi$$

$$= a^2 \rho \left[\mu \frac{\partial^2 U_\xi}{\partial t^2} + \eta_\xi \frac{\partial U_\xi}{\partial t} + \psi_\xi U_\xi \right] \qquad 1\text{-}9.1$$

$$a\left[\frac{\partial N_\theta}{\partial \theta} + 2\frac{\partial \rho}{\partial \xi}\,\overline{N}_{\xi\theta} + \rho\,\frac{\partial \overline{N}_{\xi\theta}}{\partial \xi}\right] + a\,\rho\,\omega_\theta\,Q_\theta + \frac{\rho}{2}\,\frac{\partial}{\partial \xi}\left[(\omega_\theta - \omega_\xi)\,M_{\xi\theta}\right]$$

$$+ a^2 \rho q_\theta = a^2 \rho \left[\mu\,\frac{\partial^2 U_\theta}{\partial t^2} + \eta_\theta\,\frac{\partial U_\theta}{\partial t} + \psi_\theta U_\theta\right] \qquad 1\text{-}9.2$$

$$a\left[\frac{\partial \rho}{\partial \xi}\,Q_\xi + \rho\,\frac{\partial Q_\xi}{\partial \xi} + \frac{\partial Q_\theta}{\partial \theta} - \rho\,(\omega_\xi\,N_\xi + \omega_\theta\,N_\theta)\right] + a^2 \rho\, q_\zeta$$

$$= a^2 \rho \left[\mu\,\frac{\partial^2 W}{\partial t^2} + \eta_\zeta\,\frac{\partial W}{\partial t} + \psi_\zeta\,W\right] \qquad 1\text{-}9.3$$

$$\frac{\partial \rho}{\partial \xi}\,M_\xi + \rho\,\frac{\partial M_\xi}{\partial \xi} + \frac{\partial \overline{M}_{\xi\theta}}{\partial \theta} - \frac{\partial \rho}{\partial \xi}\,M_\theta - a\rho Q_\xi$$

$$= a\rho\left[\mu^*\,\frac{\partial^2 \Phi_\xi}{\partial t^2} + \eta^*_\xi\,\frac{\partial \Phi_\xi}{\partial t} + \psi^*_\xi\,\Phi\right] \qquad 1\text{-}9.4$$

$$\frac{\partial M_\theta}{\partial \theta} + \frac{2\,\partial \rho}{\partial \xi}\,\overline{M}_{\xi\theta} + \rho\,\frac{\partial \overline{M}_{\xi\theta}}{\partial \xi} - a\,\rho\,Q_\theta$$

$$= a\rho\left[\mu^*\,\frac{\partial^2 \Phi_\theta}{\partial t} + \eta^*_\theta\,\frac{\partial \Phi_\theta}{\partial t} + \psi^*_\theta\,\Phi_\theta\right] \qquad 1\text{-}9.5$$

Where the components of membrane force, transverse force and moment (about the reference surface) per unit length, and load per unit area (assumed to be applied at the reference surface) are shown in Figures 1.2.

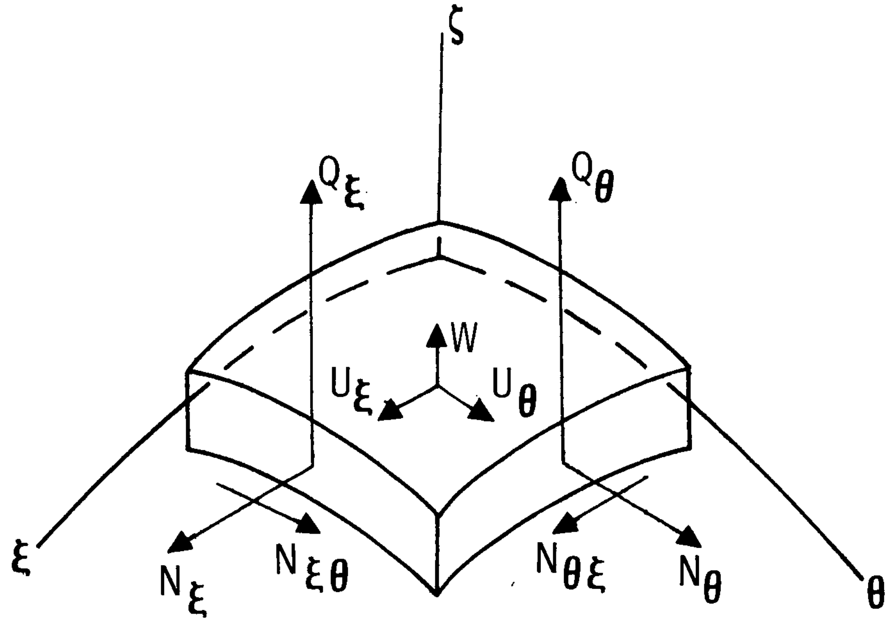


Figure 1-1. Displacements, Membrane Forces, Transverse

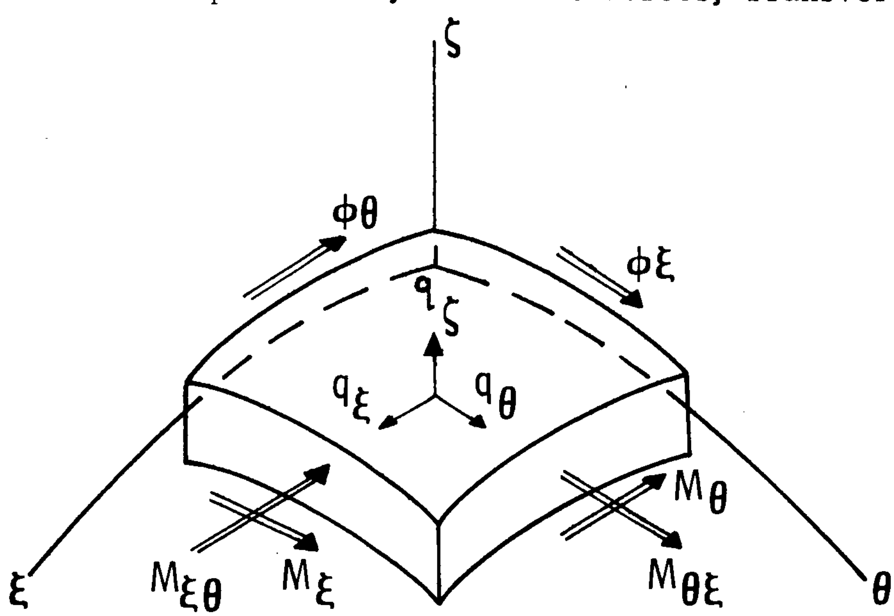


Figure 1-2. Moments, Loads, Rotations

In the Sanders' first-order theory the inplane shearing forces $N_{\xi\theta}$ and $N_{\theta\xi}$ as well as twisting moments $M_{\xi\theta}$ and $M_{\theta\xi}$ are not handled separately, but instead are combined to form modified variables

$$\overline{N}_{\xi\theta} = 1/2\ (N_{\xi\theta} + N_{\theta\xi})$$

$$\overline{M}_{\xi\theta} = 1/2\ (M_{\xi\theta} + M_{\theta\xi}) \qquad 1\text{-}10$$

and

$$\overline{k}_{\xi\theta} = 1/2\,(k_{\xi\theta} + k_{\theta\xi})$$

It is necessary when including the effects of transverse shear distortion to consider five equilibrium equations. Recall in Ref. 1 that when shear deformation is neglected the shear forces are eliminated and resulting equilibrium equations are reduced to the consideration of three equations. The neglecting of transverse shear strains implies that normals to middle surface of the shell remain normal after deformation. The degree of error introduced by this assumption naturally depends on the magnitude of transverse shearing forces and shear rigidity of the shell. For discontinuous loads and shells having low shear rigidity (sandwich shells), shear deformations may be comparable to bending and axial deformations and cannot be ignored.

It is necessary when including the effects of transverse shear distortion to consider five equilibrium equations rather than the reduced set of three equations (see Ref. 1) that can be used when shear deformation is neglected.

It should be noted that for shells which do not possess a common reference surface of revolution the more general form of Sanders equations for an arbitrary shell must be utilized.

1.5 FORMULATION INTO SOLUTION VARIABLES

The equations of motion are now expressed in terms of the solution variables, displacements and rotations.

The force and moment expressions in the equations of motion are determined by evaluating the following integrals through the thickness.

$$\begin{aligned}
N_\xi &= \int \sigma_\xi \, d\zeta & M_\xi &= \int \sigma_\xi \, \zeta d\zeta \\
N_\theta &= \int \sigma_\theta \, d\zeta & M_\theta &= \int \sigma_\theta \, \zeta d\zeta \\
\overline{N}_{\xi\theta} &= \int \tau_{\xi\theta} \, d\zeta & \overline{M}_{\xi\theta} &= \int \tau_{\xi\theta} \, \zeta d\zeta \\
Q_\xi &= \int \tau_{\xi\zeta} \, d\zeta & Q_\theta &= \int \tau_{\theta\zeta} \, d\zeta
\end{aligned} \qquad \text{1-11}$$

where in the above integrals we have neglected terms of order ζ/R , R is the minimum radius of curvature. The stresses used above are defined as:

σ_ξ , σ_θ are normal stresses, acting on the faces

$\tau_{\xi\theta}$ is an inplane shear stress acting parallel to the reference surface

$\tau_{\xi\zeta}$, $\tau_{\theta\zeta}$ are transverse shear stresses acting normal to the reference surface

By assuming that plane sections before remain plane after deformation the strains at a distance ζ from the reference surface can be expressed in terms of the reference surface strains as follows:

$$\epsilon_\xi(\zeta) = \epsilon_\xi + \zeta K_\xi$$

$$\epsilon_\theta(\zeta) = \epsilon_\theta + \zeta K_\theta \qquad \text{1-12}$$

$$\epsilon_{\xi\theta}(\zeta) = \epsilon_{\xi\theta} + \zeta K_{\xi\theta}$$

where ϵ_ξ, ϵ_θ and $\epsilon_{\xi\theta}$ are the strain of the reference surface and $\epsilon_{\xi\theta}(\zeta)$ is one-half the usual engineering strain.

The stress-strain-temperature relations for an orthotropic material are

$$\sigma_\xi = \frac{E_\xi}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}\left\{\epsilon_\xi + \nu_{\theta\xi}\epsilon_\theta + \zeta(K_\zeta + \nu_{\theta\xi}K\ ) - (\alpha_\xi + \nu_{\theta\xi}\alpha_\theta)T\right\}$$

$$\sigma_\theta = \frac{E_\theta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}\left\{\epsilon_\theta + \nu_{\xi\theta}\epsilon_\xi + \zeta(K_\theta + \nu_{\xi\theta}K_\xi) - (\alpha_\theta + \nu_{\xi\theta}\alpha_\xi)T\right\}$$

$$\tau_{\xi\theta} = G(\epsilon_{\xi\theta} + \zeta_\xi k_{\xi\theta})$$

$$\tau_{\xi\zeta} = G_\xi \gamma_{\xi\zeta}$$

$$\tau_{\xi\zeta} = G_\theta \gamma_{\theta\zeta} \qquad \text{1-13}$$

From a consideration of orthotropic materials an identity $\nu_{\xi\theta} E_{\theta} = \nu_{\theta\xi} E_{\xi}$ will be utilized. (See expressions for B_3, C_3 and D_3 Eq 1-15.)

Substituting these equations into Eqs. 1-11 employing appropriate integrations through the thickness yield the following stress/resultants-strain relationships.

$$N_{\xi} = B_1 \epsilon_{\xi} + B_3 \epsilon_{\theta} + C_1 k_{\xi} + C_3 k_{\theta} - N_{\xi}^{T}$$

$$N_{\theta} = B_3 \epsilon_{\xi} + B_2 \epsilon_{\theta} + C_3 k_{\xi} + C_2 k_{\theta} - N_{\theta}^{T}$$

$$\overline{N}_{\xi\theta} = G_1 \epsilon_{\xi\theta} + G_{12} k_{\xi\theta}$$

$$Q_{\xi} = G_2 \gamma_{\xi\zeta}$$

$$M_{\xi} = C_1 \epsilon_{\xi} + C_3 \epsilon_{\theta} + D_1 k_{\xi} + D_3 k_{\theta} - M_{\xi}^{T}$$

$$M_{\theta} = C_3 \epsilon_{\xi} + C_2 \epsilon_{\theta} + D_3 k_{\xi} + D_2 k_{\theta} - M_{\theta}^{T}$$

$$\overline{M}_{\xi\theta} = G_{12} \epsilon_{\xi\theta} + G_{13} k_{\xi\theta}$$

$$Q_{\theta} = G_3 \gamma_{\theta\zeta} \qquad \text{1-14}$$

where in the above equations the shell stiffnesses are given by

$$B_1 = \int \frac{E_{\xi}\, d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} \qquad C_1 = \int \frac{E_{\xi}\zeta\, d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$B_2 = \int \frac{E_{\theta}\, d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} \qquad C_2 = \int \frac{E_{\theta}\zeta\, d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$B_3 = \int \frac{\nu_{\xi\theta} E_\theta d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} = \int \frac{\nu_{\theta\xi} E_\xi d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} \quad D_1 = \int \frac{E_\xi \zeta^2 d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$C_3 = \int \frac{\nu_{\xi\theta} E_\theta \zeta d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} = \int \frac{\nu_{\theta\xi} E_\xi \zeta d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} \quad D_2 = \int \frac{E_\theta \zeta^2 d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$D_3 = \int \frac{\nu_{\xi\theta} E_\theta \zeta^2 d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} = \int \frac{\nu_{\theta\xi} E_\xi \zeta^2 d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} \quad G_{12} = \int G\zeta d\zeta$$

$$G_1 = \int G d\zeta \quad G_2 = \int G_\xi d\zeta \quad G_3 = \int G_\theta d\zeta \quad G_{13} = \int G\zeta^2 d\zeta \qquad 1\text{-}15$$

and thermal loads are

$$N_\xi^T = \int \frac{(\alpha_\xi + \nu_{\theta\xi}\alpha_\theta) E_\xi T d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$N_\theta^T = \int \frac{(\alpha_\theta + \nu_{\xi\theta}\alpha_\xi) E_\theta T d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$M_\xi^T = \int \frac{(\alpha_\xi + \nu_{\theta\xi}\alpha_\theta) E_\xi T \zeta d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})}$$

$$M_\theta^T = \int \frac{(\alpha_\theta + \nu_{\xi\theta}\nu_\xi) E_\theta T \zeta d\zeta}{(1 - \nu_{\xi\theta}\nu_{\theta\xi})} \qquad 1\text{-}16$$

In evaluating the stiffness quantities, the reference surface is chosen at a convenient location within the shell wall. It is not possible to simplify these expressions for the general case of shells having varying meridional and circumferential stiffness properties. (For shells of revolution, it was possible (see Ref. 2) to select a convenient location for reference such that C, G_{12} stiffness quantities described above vanished.

For the case of multilayer shells the integration is taken layer piecewise through the thickness because of the discontinuities caused by different properties of such layer. The shell stiffness and thermal loads take the form,

$$A = \int F(\zeta) d\zeta = \sum_j \int_{0_j}^{h_j} F_j(\zeta_j) d\zeta_j \qquad 1\text{-}17$$

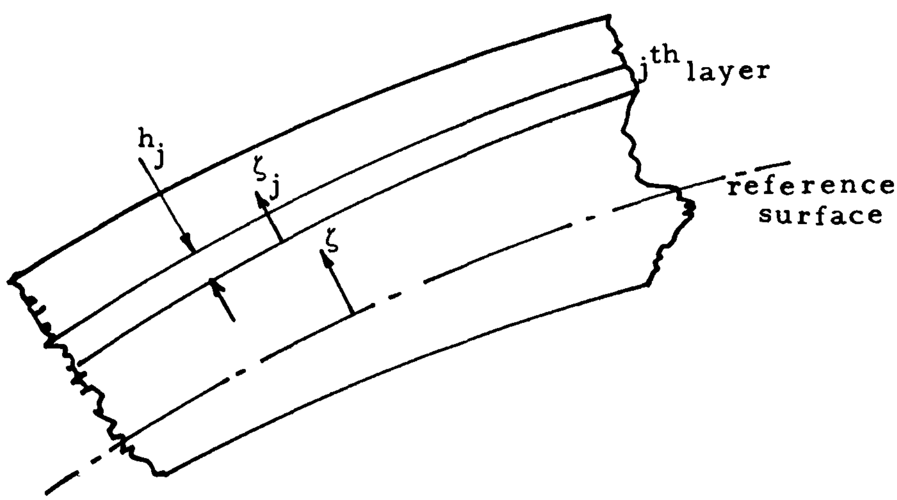


Figure 1-3. Multilayer Configuration

The reference surface strains may be defined in terms of displacements and rotations by the following expressions. The membrane strains of the reference surface are given by

$$\epsilon_\xi = \frac{1}{a}\left[\frac{\partial U_\xi}{\partial \xi} + \omega_\xi W\right]$$

$$\epsilon_\theta = \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + \gamma U_\xi + \omega_\theta W\right]$$

$$\epsilon_{\xi\theta} = \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial U_\xi}{\partial \theta} + \frac{\partial U_\theta}{\partial \xi} - \gamma U_\theta\right] \qquad 1\text{-}18$$

where U, V, W are displacements in the ξ, θ and ζ directions respectively. Transverse shear strains are given by

$$\gamma_{\xi\zeta} = \Phi_\xi - \frac{1}{a}\left[-\frac{\partial W}{\partial \xi} + \omega_\xi U_\xi\right]$$

$$\gamma_{\theta\zeta} = \Phi_\theta - \frac{1}{a}\left[-\frac{1}{\rho}\frac{\partial W}{\partial \theta} + \omega_\theta U_\theta\right] \qquad 1\text{-}19$$

where Φ_ξ, Φ_θ are rotations.

The bending distortion terms are given by

$$k_\xi = \frac{1}{a}\frac{\partial \Phi_\xi}{\partial \xi}$$

$$k_\theta = \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial \Phi_\theta}{\partial \theta} + \gamma \Phi_\xi\right]$$

$$k_{\xi\theta} = \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial \Phi_\xi}{\partial \theta} + \frac{\partial \Phi_\theta}{\partial \xi} + \gamma \Phi_\theta + \frac{1}{2a}(\omega_\xi - \omega_\theta)\left(\frac{1}{\rho}\frac{\partial U_\xi}{\partial \theta} - \frac{\partial U_\theta}{\partial \xi} - \gamma U_\theta\right)\right] \qquad 1\text{-}20$$

Substituting equations 1-18, 1-19, 1-20) into equation 1-14 the force terms in the equations of motion can be expressed in terms of the displacements

$$N_\xi = \frac{1}{a}\left\{B_1 \frac{\partial U_\xi}{\partial \xi} + B_3 \gamma U_\xi + B_3 \frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + (B_1 \omega_\xi + B_3 \phi_\xi) W\right.$$

$$\left. + C_1 \frac{\partial \Phi_\xi}{\partial \xi} + C_3 \gamma \Phi_\xi + C_3 \frac{1}{\rho} \frac{\partial \Phi_\theta}{\partial \theta}\right\} - N_\xi^T$$

$$N_\theta = \frac{1}{a}\left\{B_3 \frac{\partial U_\xi}{\partial \xi} + B_2 \gamma U_\xi + B_2 \frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + (B_3 \omega_\xi + B_2 \omega_\theta) W\right.$$

$$\left. + C_3 \frac{\partial \Phi_\xi}{\partial \xi} + C_2 \gamma \Phi_\xi + C_2 \frac{1}{\rho} \frac{\partial \Phi_\theta}{\partial \theta}\right\} - N_\theta^T$$

$$\overline{N}_{\xi\theta} = \frac{1}{2a}\left\{\frac{1}{\rho}\left[G_1 + G_{12}\frac{1}{2a}(\omega_\xi - \omega_\theta)\right]\frac{\partial U_\xi}{\partial \theta} + \left[G_1 - G_{12}\frac{1}{2a}(\omega_\xi - \omega_\theta)\right]\frac{\partial U_\theta}{\partial \xi}\right.$$

$$\left. - \gamma\left[G_1 + G_{12}\frac{1}{2a}(\omega_\xi - \omega_\theta)\right] U_\theta + G_{12}\frac{1}{\rho}\frac{\partial \Phi_\xi}{\partial \theta} + G_{12}\frac{\partial \Phi_\theta}{\partial \xi} - G_{12}\gamma \Phi_\theta\right\}$$

$$Q_\xi = G_2\left\{-\frac{1}{a}\omega_\xi U_\xi + \frac{1}{a}\frac{\partial W}{\partial \xi} + \Phi_\xi\right\}$$

$$M_\xi = \frac{1}{a}\quad C_1 \frac{\partial U_\xi}{\partial \xi} + C_3 \gamma U_\xi + C_3 \frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + (C_1 \omega_\xi + C_3 \omega_\theta) W$$

$$\left. + D_1 \frac{\partial \Phi_\xi}{\partial \xi} + D_3 \gamma \Phi_\xi + D_3 \frac{1}{\rho} \frac{\partial \Phi_\theta}{\partial \theta}\right\} - M_\xi^T$$

$$M_\theta = \frac{1}{a}\left\{C_3 \frac{\partial U_\xi}{\partial \xi} + C_2 \gamma U_\xi + C_2 \frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + (C_3 \omega_\xi + C_2 \omega_\theta) W\right.$$

$$\left. + D_3 \frac{\partial \Phi_\xi}{\partial \xi} + D_2 \gamma \Phi_\xi + D_2 \frac{1}{\rho} \frac{\partial \Phi_\theta}{\partial \theta}\right\} - M_\theta^T$$

$$\overline{M}_{\xi\theta} = \frac{1}{2a}\left\{\frac{1}{\rho}\left[G_{12} + G_{13}\left(\frac{1}{2a}\right)(\omega_\xi - \omega_\theta)\right]\frac{\partial U_\xi}{\partial \theta} + \left[G_{12} - G_{13}\frac{1}{2a}(\omega_\xi - \omega_\theta)\right]\frac{\partial U_\theta}{\partial \xi}\right.$$

$$\left. -\gamma\left[G_{12} + G_{13}\frac{1}{2a}(\omega_\xi - \omega_\theta)\right]U_\theta + G_{13}\frac{1}{\rho}\frac{\partial \Phi_\xi}{\partial \theta} + G_{13}\frac{\partial \Phi_\theta}{\partial \xi} - G_{13}\gamma\Phi_\theta\right\}$$

$$Q_\theta = G_3\left\{-\frac{1}{a}\omega_\theta U + \frac{1}{\sigma\rho}\frac{\partial W}{\partial \theta} + \Phi_\theta\right\} \qquad 1\text{-}21$$

By employing the relations of Eq. 1-21, the equation of motion can be expressed in terms of the dependent variables, displacements and rotations.

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2.0 NUMERICAL ANALYSIS

2.1 FOURIER EXPANSIONS

The analysis utilized is a Fourier approach which will permit separation of variables and yield a more tractable set of shell equations. The procedure involves expanding of the pertinent variables in Fourier series with appropriate normalization to provide nondimensional Fourier coefficients of roughly comparable magnitudes for different variables. Letting σ_o, E_o, h_o be a reference stress level, Young's modulus, and thickness respectively, solutions for the field equations are sought in the form

$$U_\xi = \frac{a\sigma_o}{E_o} \sum_{h=0}^{\infty} u_n(\xi, t) \cos n\theta$$

$$U_\theta = \frac{a\sigma_o}{E_o} \sum_{n=1}^{\infty} v_n(\xi, t) \sin n\theta$$

$$W = \frac{a\sigma_o}{E_o} \sum_{n=0}^{\infty} w_n(\xi, t) \cos n\theta$$

$$\Phi_\xi = \frac{\sigma_o}{E_o} \sum_{n=0}^{\infty} \phi_{\xi n}(\xi, t) \cos n\theta,$$

$$\Phi_\theta = \frac{\sigma_o}{E_o} \sum_{n=1}^{\infty} \phi_{\theta n}(\xi, t) \sin n\theta \qquad (2-1)$$

These Fourier expansions are consistent with loadings of the form

$$q_\xi = \frac{\sigma_o h_o}{a} \sum_{n=0}^{\infty} p_{\xi n}(\xi, t) \cos n\theta$$

$$q_\theta = \frac{\sigma_o h_o}{a} \sum_{n=1}^{\infty} p_{\theta n}(\xi, t) \sin n\theta$$

$$q_\zeta = \frac{\sigma_o h_o}{a} \sum_{n=0}^{\infty} p_n(\xi, t) \cos n\theta \qquad (2\text{-}2)$$

The above Fourier expansions are not the most general form that can exist. The expansions q_ζ, q_ξ are symmetrical expansions about $\theta = 0$. For full generality, they must be augmented by additional sine series expansions. The form q_θ in turn would be supplemented by cosine series. Similarily, a convenient set of sine expansions must hold for displacements and rotations. For ease of presentation the condensed form of expansions (Eqs. 2-1, 2-2) will be used. The contribution of augmented terms in the series expansion will be described later.

Expansions for the temperature distributions may be described in a similar manner; however, since the thermal coefficients and Young's modulus can vary in the circumferential direction, it will be more convenient to expand the thermal load in Fourier series as follows

$$N_2^T = \sigma_o h_o \sum_{n=0}^{\infty} t_{\xi n}^T \cos n\theta$$

$$N_\theta^T = \sigma_o h_o \sum_{n=0}^{\infty} t_{\theta n}^T \cos n\theta$$

$$M_\xi^T = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} M_{\xi n}^T \cos n\theta$$

$$M_\theta^T = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} m_{\theta n}^T \cos n\theta \qquad (2\text{-}3)$$

Where the Fourier components $t_{\xi n}^T$, $t_{\theta n}^T$, $m_{\xi n}^T$ and $m_{\theta n}^T$ are given by

$$t_{\xi n}^T = \frac{2}{\pi} \int_o^\pi \frac{N_\xi^T}{\sigma_o h_o} \cos n\theta \, d\theta$$

$$t_{\theta n}^T = \frac{2}{\pi} \int_o^\pi \frac{N_\theta^T}{\sigma_o h_o} \cos n\theta \, d\theta$$

$$m_{\xi n}^T = \frac{2}{\pi} \int_o^\pi \frac{aM_{\xi_1}^T}{\sigma_o h_o^3} \cos n\theta \, d\theta$$

$$m_{\theta n}^T = \frac{2}{\pi} \int_o^\pi \frac{aM_\theta^T}{\sigma_o h_o^3} \cos n\theta \, d\theta \qquad (2\text{-}4)$$

Since the stiffness parameters are variable in the circumferential directions these will also be expanded in a Fourier series. For example, the expansion for the extensional stiffness parameter is of the form

$$B = \sum_{j=0}^{\infty} b_j \cos j\theta + \sum_{j=1}^{\infty} \bar{b}_j \sin j\theta \qquad (2\text{-}5)$$

In the Apollo heat shield shell, there exists a plane of symmetry with respect to planform geometry. See Fig. (2-1).

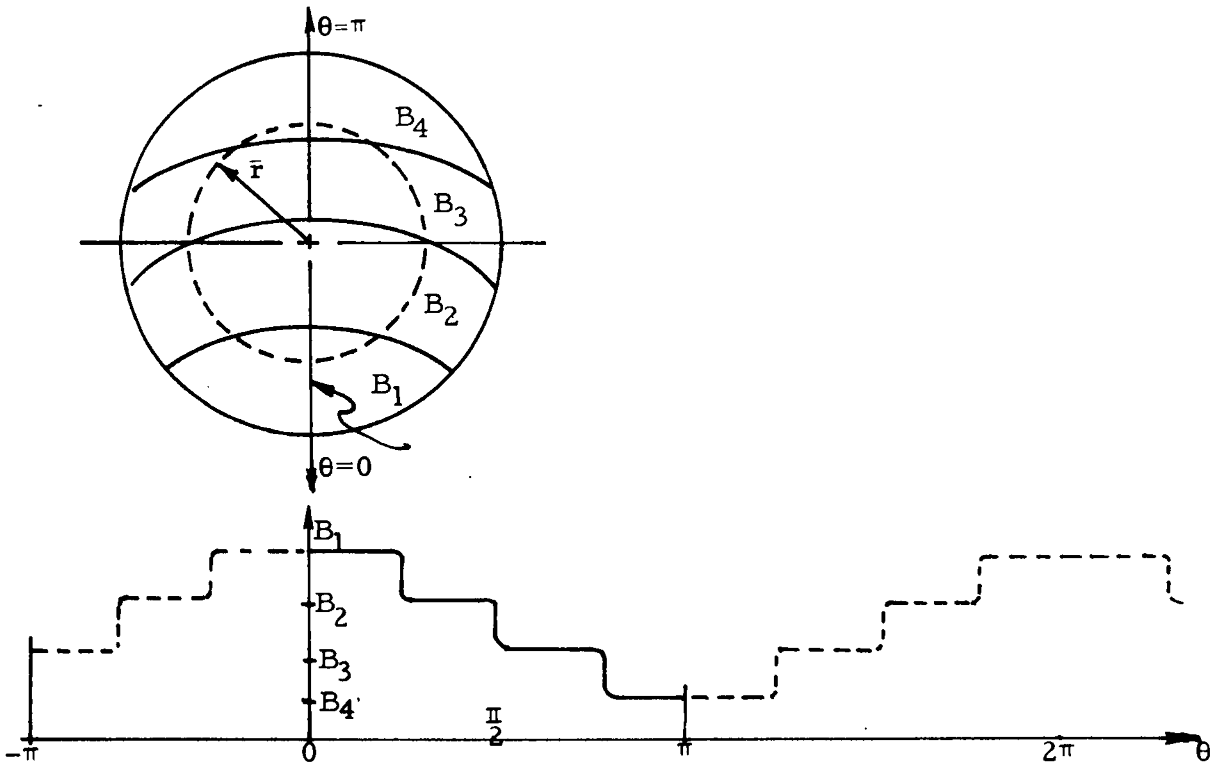

Figure 2-1. Stiffness Profile

A plane of symmetry will be also assumed in this study. In such a case the sine terms in Eq. 2-5 are dropped and the coefficients of B viz. b_j are found by integrations of the form

$$b_j = \frac{2}{\pi} \int_0^{\pi} B\ (\xi\ ,\theta)\ \cos j\theta\, d\theta \qquad (2\text{-}6)$$

In order to determine the accuracy of the Fourier series representation of the Apollo heat shield configuration several numerical examples were run. The results of the comparison can be seen in Fig. 2-2 where extensional rigidity versus circumferential coordinate is plotted for four trial cases using 10, 15, 20 and 30 Fourier components of solution.

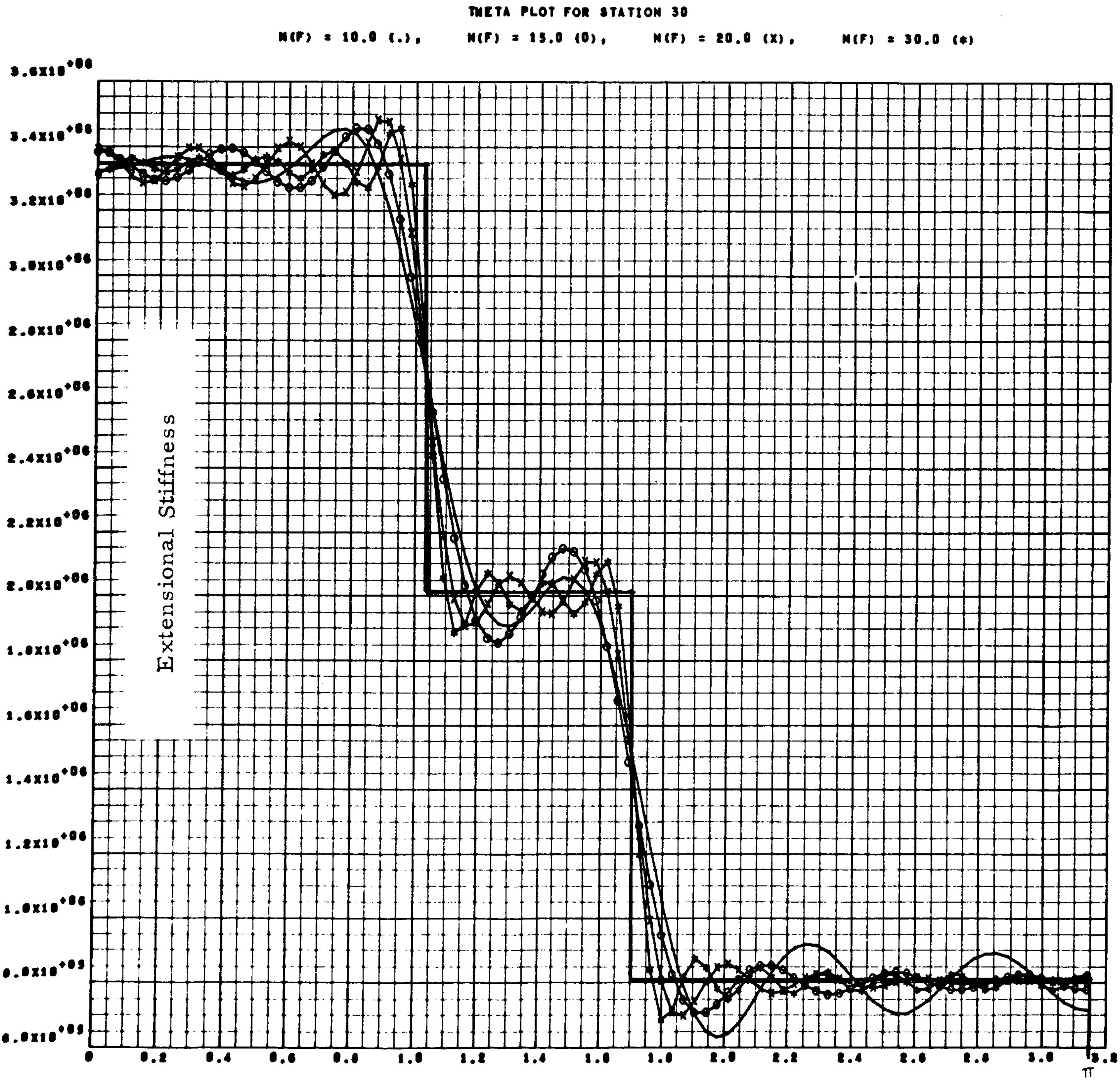

Figure 2-2. Fourier Discription of Extensional Rigidity Versus Circumferential Coordinate Comparison

The Fourier expansions of the shell stiffness parameters (Eq. 1-15), consistent with previous formulation are given by

$$B_m = E_o h_o \sum_{j=0}^{\infty} b_{m_j}(\xi) \cos j\theta, \quad G_m = E_o h_o \sum_{j=0}^{\infty} g_{m_j}(\xi) \cos j\theta$$

$$C_m = E_o h_o^2 \sum_{j=0}^{\infty} c_{m_j}(\xi) \cos j\theta, \quad G_{12} = E_o h_o^2 \sum_{j=0}^{\infty} g_{12_j}(\xi) \cos j\theta$$

$$D_m = E_o h_o^3 \sum_{j=0}^{\infty} d_{m_j}(\xi) \cos j\theta, \quad G_{13} = E_o h_o^3 \sum_{j=0}^{\infty} g_{13_j}(\xi) \cos j\theta \quad (2\text{-}7)$$

Substitution of the displacement (rotation) series expansions (Eqs. 2-1) and the above stiffness expansions into Eqs. (1-21) and employing the proper trigonometric identities yields the following series expressions relating forces (moments) in terms of the Fourier coefficients of the displacement variables and stiffness parameters:

$$N_\xi = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} \left\{ B_1^{nj} \left(\frac{\partial u_j}{\partial \xi} + \omega_\xi \omega_j \right) + B_3^{kj} \left(\gamma u_j + \frac{n}{\rho} v_j + \omega_\theta \omega_j \right) \right. \right.$$

$$\left. \left. + \lambda C_1^{nj} \frac{\partial \phi}{\partial \xi} \xi j + \lambda C_3^{kj} \left(\gamma \phi_{\xi j} + \frac{n}{\rho} \phi_{\theta j} \right) \right\} - t_{\xi n}^T \right] \cos n\theta$$

$$N_\theta = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} \left\{ B_3^{nj} \left(\frac{\partial u_j}{\partial_\xi} + \omega_\xi \omega_j \right) + B_2^{nj} \left(\gamma U_j + \frac{U}{\rho} v_j + \omega_\theta \omega_j \right) \right. \right.$$

$$\left. \left. + \lambda C_3^{nj} \frac{\partial \phi}{\partial_\xi} \xi j + \lambda C_2^{nj} \left(\gamma \phi_{\xi j} + \frac{n}{\rho} \phi_{\theta j} \right) \right\} - t_{\theta n}^T \right] \cos n\theta$$

$$\overline{N}_{\xi\theta} = \sigma_o h_o \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} \left\{ \frac{G_1^{nj}}{2} \left(\frac{\partial v_j}{\partial_\xi} - \frac{n}{\rho} uj - \gamma v_j \right) \right\} \right.$$

$$+ \lambda \sum_{j=0}^{\infty} \left\{ \frac{G_{12}^{nj}}{2} \left(-\frac{n}{\rho} \phi_{\xi j} + \frac{\partial \phi_{\theta j}}{\partial_\xi} - \gamma \phi_{\theta j} \right. \right.$$

$$\left. \left. \left. + \frac{1}{2} (\omega_\xi - \omega_\theta) \left(-\frac{n}{\rho} u_j - \frac{\partial v_j}{\partial_\xi} - \gamma v_j \right) \right) \right\} \right] \sin n\theta$$

$$Q_\xi = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} \left\{ G_2^{nj} \left(-\omega_\xi n_j + \frac{\partial \omega_j}{\partial_\xi} + \phi_{\xi j} \right) \right\} \right] \cos n\theta$$

$$M_\xi = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} \left[\frac{1}{\lambda} \sum_{j=0}^{\infty} C_1^{nj} \left(\frac{\partial n_j}{\partial_\xi} + \omega_\xi \omega_j \right) + C_3^{nj} \left(\gamma n_j + \frac{n}{\rho} v_j + \omega_\theta \omega_j \right) \right.$$

$$\left. \sum_{j=0}^{\infty} \left\{ D_1^{nj} \frac{\partial \phi_{\xi j}}{\partial_\xi} + D_3^{nj} \left(\gamma \phi_{\xi j} + \frac{n}{\rho} \phi_{\theta j} \right) \right\} - m_{\xi n}^T \right] \cos n\theta$$

$$M_\theta = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} \left[\frac{1}{\lambda} \sum_{j=0}^{\infty} \left\{ C_3^{nj} \left(\frac{\partial n_j}{\partial_\xi} + \omega_\xi \, \omega_j \right) + C_2^{nj} \left(\gamma n_j + \frac{n}{\rho} \, v_j + \omega_\theta \, \omega_j \right) \right\} \right.$$

$$\left. + \sum_{j=0}^{\infty} \left\{ D_3^{nj} \, \frac{\partial \phi_{\xi j}}{\partial_\xi} + D_3^{nj} \left(\gamma \phi_{\xi j} + \frac{n}{\rho} \, \phi_{\theta j} \right) \right\} - m_{\theta n}^T \right] \cos n\theta$$

$$\overline{M}_{\xi\theta} = \frac{\sigma_o h_o^3}{a} \sum_{n=1}^{\infty} \left[\frac{1}{\lambda} \sum_{j=0}^{\infty} \left\{ \frac{G_{12}^{nj}}{2} \left(\frac{\partial v_j}{\partial_\xi} - \frac{n}{\rho} \, u_j - \gamma v_j \right) \right\} \right.$$

$$+ \sum_{j=0}^{\infty} \left\{ \frac{G_{13}^{nj}}{2} \left(-\frac{n}{\rho} \, \phi_{\xi j} + \frac{\partial \phi_{\theta j}}{\partial_\xi} \right) - \gamma \, \phi_{\theta j} \right.$$

$$\left. \left. + \frac{1}{2} \left(\omega_\xi - \omega_\theta \right) \left(-\frac{n}{\rho} n_j - \frac{\partial v_j}{\partial_\xi} - \gamma v_j \right) \right) \right\} \right] \sin n\theta$$

$$Q_\theta = \sigma_o h_o \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} \left\{ G_3^{kj} \left(-\omega_\theta \, v_j - \frac{n}{\rho} \, \omega_j + \phi_{\theta j} \right) \right\} \right] \sin n\theta$$

$$\mu \ddot{U}_\xi = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} M_1^{nj} \ddot{U}_j \right] \cos n\theta$$

$$\mu \ddot{U}_\theta = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} M_2^{nj} \ddot{v}_j \right] \sin n\theta$$

$$\mu \ddot{W} = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} M_3^{nj} \ddot{w}_j \right] \cos n\theta$$

$$\mu^* \ddot{\Phi}_\xi = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} M_4^{nj} \ddot{\phi}_j \right] \cos n\theta$$

$$\mu^* \ddot{\Phi}_\theta = \frac{\sigma_o h_o^3}{a} \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} M_5^{nj} \ddot{\phi}_{\theta j} \right] \sin n\theta$$

$$\eta_\xi \dot{U}_\xi = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} C_1^{nj} \dot{u}_j \right] \cos n\theta$$

$$\eta_\theta \dot{U}_\theta = \sigma_o h_o \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} C_2^{uj} \dot{v}_j \right] \sin n\theta$$

$$\eta_\zeta \dot{W} = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=1}^{\infty} C_3^{nj} \dot{w}_j \right] \cos n\theta$$

$$\eta^*_{\xi}\dot{\Phi}_{\xi} = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} C_4^{nj} \dot{\phi}_{\xi j} \right] \cos n\theta$$

$$\eta^*_{\theta}\dot{\Phi}_{\theta} = \frac{\sigma_o h_o^3}{a} \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} C_5^{nj} \dot{\phi}_{\theta j} \right] \sin n\theta$$

$$\psi_{\xi} U_{\xi} = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} K_1^{nj} n_j \right] \cos n\theta$$

$$\psi_{\theta} U_{\theta} = \sigma_o h_o \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} K_2^{nj} n_j \right] \sin n\theta$$

$$\psi_{\xi} W = \sigma_o h_o \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} K_3^{nj} W_j \right] \cos n\theta$$

$$\psi^*_{\xi} \Phi_{\xi} = \frac{\sigma_o h_o^3}{a} \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} K_4^{nj} \phi_{\xi j} \right] \cos n\theta$$

$$\psi^*_{\theta} \Phi_{\theta} = \frac{\sigma_o h_o^3}{a} \sum_{n=1}^{\infty} \left[\sum_{j=0}^{\infty} K_5^{nj} \phi_{\theta_j} \right] \sin n\theta$$

where

$$\lambda = \frac{h_o}{a}$$

The stiffness recursion relationships above are described in the form

$$A^{nj} = \frac{1}{2} \left\{ a^{(n+j)} + \left[1 - \delta^2 (j-n) + \delta (n) \right] a^{|n-j|} \right\}$$

$$\bar{A}^{nj} = \frac{1}{2} \left\{ -\bar{a}^{(n+j)} + \left[1 - \delta^2 (j-n) + \delta (n) \right] \bar{a}^{|n-j|} \right\} \qquad (2\text{-}9^*)$$

where the specific coefficients of interest (dropping nj superscript) are given by

$A = B_1, B_2, B_3, C_1, C_2, C_3, D_1, D_2, D_3, G_2, \mu_1, \mu_4, \eta_1, \eta_4, \psi_1, \psi_4$

$a = b_1, b_2, b_3, c_1, c_2, c_3, d_1, d_2, d_3, g_2, M_1, M_4, C_1, C_4, K_1, K_4$

and

$\bar{A} = G_1, G_3, G_{12}, G_{13}, \mu_2, \mu_5, \eta_2, \eta_5, \psi_2, \psi_5$

$\bar{a} = g_1, g_3, g_{12}, g_{13}, M_2, M_5, C_2, C_5, K_2, K_5$

(*See Appendix 1.2 (multiplication of series expressions) for a more detailed description of A^{nj} and $\bar{A}^{nj}$.)

In the above expressions the symbolic function δ (m) is defined as

$$\delta(m) = \begin{cases} -1 & m < 0 \\ 0 & m = 0 \\ +1 & m > 0 \end{cases} \qquad (2\text{-}10)$$

2.1.1 Reduced Equations

Substitution of the stress resultant expressions (Eqs. 2.8) into the Equations of Motion (Eqs. 1-9) yields five finite series expressions in the circumferential coordinate relating the Fourier coefficients u_j, v_j, w_j, $\phi_{\xi j}$, and $\phi_{\theta j}$ of the displacement and rotation variables. For practical considerations we truncate the series solution of the dependent variables to K terms in the Fourier component. Employing the appropriate orthogonality relationships of Fourier series to these equilibrium expressions yields a system of 5K ordinary differential equations relating the 5K unknown Fourier coefficients. These equations are presented in a form amenable for computer programming and are given as follows:

$$\sum_{j=0}^{K-1} \left[f_{(5k+1,\ 5j+1)} u_j'' + f_{(5k+1,\ 5j+2)} v_j'' + f_{(5k+1,\ 5j+3)} w_j'' \right.$$

$$+ f_{(5k+1,\ 5k+4)} \phi_{\xi j}'' + f_{(5k+1,\ 5k+5)} \phi_{\theta j}'' + g_{(5k+1,\ 5j+1)} u_j'$$

$$\left. + \cdots + g_{(5k+1,\ 5j+5)} \phi_{\theta j} + h_{(5k+1,\ 5j+1)} u_j + \cdots + h_{(5k+1,\ 5j+5)} \phi_{\theta j} \right]$$

$$= p_1^k + \alpha_{(5k+1,\ 5j+1)} \ddot{u}_j + \beta_{(5k+1,\ 5j+1)} \dot{u}_j$$

$$\sum_{j=0}^{K-1}\Big[f_{(5k+2,\ 5j+1)}u''_j + f_{(5k+2,\ 5j+2)}v''_j + \cdots + f_{(5k+2,\ 5j+5)}\phi''_{\theta j}$$

$$+ g_{(5k+2,\ 5j+1)}u'_j + \ldots + g_{(5k+2,\ 5j+5)}\phi_{\theta j} + h_{(5k+2,\ 5j+1)}u_j$$

$$+ \ldots + h_{(5k+2,\ 5j+5)}\phi_{\theta j}\Big] = p_2^k + \alpha_{(5k+2,\ 5j+2)}\dot{v}_j + \beta_{(5k+2,\ 5j+2)}\dot{v}_j$$

$$\sum_{j=0}^{K-1}\Big[f_{(5k+3,\ 5j+1)}u''_j + f_{(5k+3,\ 5j+2)}v''_j + \ldots + f_{(5k+3,\ 5j+5)}\phi''_{\theta j}$$

$$+ g_{(5k+3,\ 5j+1)}u'_j + \ldots + g_{(5k+3,\ 5j+1)}\phi'_{\theta j} + h_{(5k+3,5j+1)}u_j$$

$$+ \ldots - + h_{(5k+2,\ 5j+5)}\phi_{\theta j}\Big] = p_3^k + \alpha_{(5k+3,\ 5j+3)}\ddot{w}_j + \beta_{(5k+3,5j+3)}w_j$$

$$\sum_{j=0}^{K=1}\Big[f_{(5k+4,\ 5j+1)}u''_j + \ldots + .\ f_{(5k+4,\ 5j+5)}\phi''_{\theta j} + g_{(5k+4,5j+1)}u'_j$$

$$+ \ldots + g_{(5k+4,\ 5j+5)}\phi'_{\theta j} + h_{(5k+4,\ 5j+1)}u_j + \ldots$$

$$+ h_{(5k+4,\ 5j+5)}\phi_{\theta j}\Big] = p_4^k + \alpha_{(5k+4,\ 5j+4)}\phi_\xi + \beta_{(5k+4,\ 5j+4)}\phi_{\xi j}$$

$$\sum_{j=0}^{K-1} \Big[f_{(5k+5,\ 5j+1)}\ u_j'' + \dots + f_{(5k+5,\ 5j+5)}\ \phi_{\theta j} + g_{(5k+5,\ 5j+1)}\ u_j' +$$

$$+ \dots + g_{(5k+5,\ 5j+5)}\ \phi_{\theta j} + h_{(5k+5,\ 5j+1)} + \dots +$$

$$+ \dots + h_{(5k+5,\ 5j+5)}\ \phi_{\theta j} \Big] = p_5^k = \alpha_{(5k+5,\ 5j+5)}\ \ddot{\phi}_{\theta j} + \beta_{(5k+5,\ 5j+5)}\ \dot{\phi}_{\theta j}$$

$$(k = 0, 1, 2, \dots\dots K-1) \qquad (2\text{-}11)$$

where the f, g, h and p coefficients are described in Appendix 1-3. (It should be noted that the form of the above equations is more complicated than was obtained in Ref. (2) for analysis of shells of revolution. This complexity arises from the fact that the equilibrium equations cannot be decoupled for each Fourier component of displacement variables for the case of unsymmetric shell.)

The above equation can be conveniently written in matrix form as follows:

$$F z'' + G z' + (H + K) z = \alpha \ddot{z} + \beta \dot{z} + p \qquad (2\text{-}12)$$

where F, G, H, (5K x 5K) and z, p, (5K x 1) are defined as follows

$$F = \begin{bmatrix} f_{11} & f_{12} & - & - & - & - \\ f_{21} & & & & & \\ | & & & & & \\ | & & & & & \\ | & & & & & \\ & - & - & - & - & f_{5k,\,5K} \end{bmatrix} \qquad G = \begin{bmatrix} g_{11} & g_{12} & - & - & - \\ g_{21} & & & & \\ | & & & & \\ | & & & & \\ | & & & & \\ & & & & \end{bmatrix}$$

$$H = \begin{bmatrix} h_{11} & h_{12} & - & - & - & - \\ h_{21} & & & & & \\ | & & & & & \\ | & & & & & \\ | & & & & & \\ | & & & & & \end{bmatrix} \qquad K = \begin{bmatrix} k_{11} & k_{12} & - & - & - & - \\ k_{21} & & & & & \\ | & & & & & \\ | & & & & & \\ | & & & & & \\ | & & & & & \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \alpha_{11} & \alpha_{12} & - & - & - \\ \alpha_{21} & & & & \\ | & & & & \\ | & & & & \\ | & & & & \\ | & & & & \end{bmatrix} \qquad \beta = \begin{bmatrix} \beta_{11} & \beta_{12} & - & - & - \\ \beta_{21} & & & & \\ | & & & & \\ | & & & & \\ | & & & & \\ | & & & & \end{bmatrix}$$

$$p = \begin{bmatrix} p_1^o \\ p_2^o \\ p_4^o \\ p_5^o \\ p_1^1 \\ p_5^{K-1} \end{bmatrix} \qquad z = \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \phi_{\xi 1} \\ \phi_{\theta 1} \\ u_2 \\ \vdots \\ \phi_{\theta K} \end{bmatrix}$$

The elements of the F, G and H matrices are given in Appendix 1.2 and are presented there in a format which is designed to ease computer programming. The coefficients of P are the Fourier components of the applied external load and are known quantities for a specific loading case.

The analysis developed assumed that the loads q_ξ, q_ζ were symmetrical and q_θ antisymmetric about o = o (eqs. 2-2). For complete generality, these expansions should be augmented by the following:

$$q_{\xi(\xi,\theta,t)} = \sum_{n=1}^{\infty} \bar{p}_{\xi n}(\xi, t) \text{ Sin } n\theta$$

$$q_{\xi(\xi,\theta,t)} = \sum_{n=0}^{\infty} \bar{p}_{\theta n}(\xi, t) \text{ Cos } n\theta$$

$$q_{\zeta(\xi,\theta,t)} = \sum_{n=1}^{\infty} p_\zeta(\xi, t) \text{ Sin } n\theta \qquad (2\text{–}13)$$

The more general case of shells having arbitrary distribution of stiffness, distribution of loads damping, and elastic foundation cannot be uncoupled in the same manner as the case of plane of symmetry of stiffness and arbitrary loads. In this case the total Fourier series representation of all the variables, displacements, rotations, stiffness and loads, must be carried in the analysis. The analysis will follow the same format of the special case formulated previously.

For consistency, the dependent variables must similarly be modified. For the case of shells having a plane of symmetry in stiffness, the analysis would proceed in a similar manner as that previously described. Now the solution represents Fourier coefficients for the augmented series expansions. The coefficients of Eqs. 2–11 are similar except for an appropriate change in sign in some of the coefficients. This complete set of coefficients will be presented at a later date. The solutions obtained can be superimposed for the case of a general loading condition.

The more general case of shells having arbitrary distribution of stiffness, distribution of loads, damping and elastic foundation cannot be uncoupled in the same manner as the case of plane of symmetry of stiffness and arbitrary loads. In this case the total Fourier series representation of all the variables, displacements, rotations, stiffness and loads, must be carried in the analysis. The analysis will follow the same format of the special case formulated previously.

2.1.2 Boundary Conditions

Consistent with Sander's equilibrium equations, the boundary conditions for the specification of the forces or displacements, or constraint between them are described below. On the edge where ξ = constant (i.e., $\xi = 0$ and $\xi = s$)

$$
\begin{aligned}
&N_\xi \text{ or } U_\xi \\
&\hat{N}_{\xi\theta} \text{ or } U_\theta \\
&Q_\xi \text{ or } W \\
&M_\xi \text{ or } \Phi_\xi \\
&\overline{M}_{\xi\theta} \text{ or } \Phi_\theta
\end{aligned}
\tag{2-14}
$$

where

$$
\hat{N}_{\xi\theta} = \bar{N}_{\xi\theta} + \frac{1}{2a} (\omega_\theta - \omega_\xi) \dot{M}_{\xi\theta}
$$

These conditions can be expressed in matrix form by,

$$
\overline{\Omega}\,\overline{y} + \overline{\Lambda}\,\overline{z} = \overline{\ell} \tag{2-15}
$$

where $\overline{y}$, $\overline{\ell}$ $\overline{z}$ are column matrices and $\overline{\Omega}$, $\overline{\Lambda}$ are appropriate diagonal matrices

$$
\bar{y} = \begin{bmatrix} N_\xi \\ N_{\xi\theta} \\ Q_\xi \\ M_\xi \\ \bar{M}_{\xi\theta} \end{bmatrix} \qquad \bar{z} = \begin{bmatrix} U_\xi \\ U_\theta \\ W \\ \Phi_\xi \\ \Phi_\theta \end{bmatrix}
$$

$$
\bar{\Omega} = \begin{bmatrix} \omega_1 & & & & \\ & \omega_2 & & \text{o} & \\ & & \omega_3 & & \\ & \text{o} & & \omega_4 & \\ & & & & \omega_5 \end{bmatrix} \qquad \bar{\Lambda} = \begin{bmatrix} \lambda_1 & & & & \\ & \lambda_2 & & \text{o} & \\ & & \lambda_3 & & \\ & \text{o} & & \lambda_4 & \\ & & & & \lambda_5 \end{bmatrix} \qquad \bar{\ell} = \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \end{bmatrix}
$$

The logic which connects $\bar{\Omega}$, $\bar{\Lambda}$, $\bar{\ell}$ and the conditions desired are given in the following table:

Conditions at Boundary	Matrix Elements		
	ω_i	ω_i	ω_i
Displacement Specified	0	1	C_1
Force Specified	1	0	C_2
Constraint Conditions	1	C_3	0

C_1 = value of displacement

C_2 = value of force

C_3 = constant relating force and displacement

For example, if Φ_ξ is given as a boundary condition then $\Lambda_4 = 1$ $\omega_4 = 0$ and ℓ_4 is the prescribed value of Φ_ξ. Note C_i is nondimensionalized with the appropriate reference constants.

It will be convenient to expand forces and moments in Fourier series in manner consistent with the previous developments. Letting

$$N_\xi = \sum t_\xi^n \cos n\theta$$

$$\hat{N}_{\xi\theta} = \sum t_{\xi\theta}^n \sin n\theta$$

$$Q_\xi = \sum q_\xi^n \cos n\theta$$

$$M_\xi = \sum m_\xi^n \cos n\theta$$

$$\bar{M}_{\xi\theta} = \sum m_{\xi\theta}^n \sin n\theta \qquad (2\text{-}16)$$

and

$$\ell_1 = \sum \ell_1^n \cos n\theta$$

$$\ell_2 = \sum \ell_2^n \sin n\theta$$

$$\ell_3 = \sum {}_1\ell_3^n \cos n\theta$$

$$\ell_4 = \sum \ell_4^n \cos n\theta$$

$$\ell_5 = \sum \ell_5^n \sin n\theta \qquad (2\text{-}17)$$

The above series expressions together with Eqs. (2-1) are substituted into Eq. (2-15) and can be uncoupled for each Fourier coefficient yielding the following matrix form

$$\Omega y + \Lambda z = \ell$$

where

$$y = \begin{vmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_K \end{vmatrix} \quad y_i = \begin{vmatrix} t_\xi^i \\ \hat{t}_{\xi\theta}^i \\ q_\xi^i \\ m_\xi^i \\ m_{\xi\theta}^i \end{vmatrix} \quad z = \begin{vmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ z_K \end{vmatrix} \quad z_i = \begin{vmatrix} u^i \\ v^i \\ w^i \\ \phi_\xi^i \\ \phi_\xi^i \end{vmatrix} \tag{2-18}$$

$$\Omega = \begin{vmatrix} \bar{\Omega} & & & & 0 \\ & \bar{\Omega} & & & \\ & & \ddots & & \\ 0 & & & & \bar{\Omega} \end{vmatrix} \quad \Lambda = \begin{vmatrix} \bar{\Lambda} & & & & 0 \\ & \bar{\Lambda} & & & \\ & & \ddots & & \\ 0 & & & & \bar{\Lambda} \end{vmatrix} \quad \ell = \begin{vmatrix} \bar{\ell}_1 \\ \bar{\ell}_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \bar{\ell}_K \end{vmatrix} \quad \tilde{\ell}_i = \begin{vmatrix} \ell_1^i \\ \ell_2^i \\ \ell_3^i \\ \ell_4^i \\ \ell_5^i \end{vmatrix}$$

It will be desirable to express boundary conditions in terms of the substitution of Eqs. (2-16) in Eq. (2-8) with appropriate orthogonality operations yields a set of recursion expressions relating Fourier coefficients of forces and moments to the displacement and stiffness coefficients. These relationships are given by

$$t_\xi^k = \sum_{j=0}^{K-1} \Big[r_{(5k+1,\ 5j+1)}\, u_j' + r_{(5k+1,\ 5j+4)}\, \phi_{\xi j}' + s_{(5k+1,\ 5j+1)}\, v_j$$

$$+ s_{(5k+1,\ 5j+2)}\, v_j + s_{(5k+1,\ 5j+5)}\, w_j + s_{(5k+1,\ 5j+4)}\, \phi_{\xi j}$$

$$+ s_{(5k+1,\ 5j+5)} \phi_{\theta j} \Big] + a_{5k+1}$$

$$\hat{t}^k_{\xi\theta} = \sum_{j=0}^{K-1} \Big[r_{(5k+2,5j+2)} v_j + r_{(5k+2,5j+5)} \ {}'_j + s_{95k+2,5j+1)} u_j$$

$$+ s_{(5k+2,5j+2)} v_j + s_{(5k+2,5j+4)} \phi_{\theta j} + s_{(5k+2,5j+5)} \phi_j \Big]$$

$$q^k_\xi = \sum_{j=0}^{K-1} \Big[r_{(5k+3,5j+3)} w'_j + s_{(5k+3,5j+1)} u_j + s_{(5k+3,5j+4)} \phi_{\xi j} \Big]$$

$$m^k_\xi = \sum_{j=0}^{K-1} \Big[r_{(5k+4,5j+1)} u'_j + r_{(5k+4,5j+4)} \phi'_{\xi j} + s_{(5k+4,5j+1)} u_j$$

$$+ s_{(5k+4,5j+2)} v_j + s_{(5k+4,5j+3)} w_j + s_{(5k+4,5j+4)} \phi_{\xi j} + s_{(5k+4,5j+5)} \phi_{\theta j} \Big]$$

$$+ a_{5k+4}$$

$$m^k_{\xi\theta} = \sum_{j=0}^{K-1} \Big[r_{(5k+5,5j+3)} v'_j + r_{(5k+5,5j+5)} \phi'_{\theta j} + s_{(5k+5,5j+1)} u_j$$

$$+ s_{(5k+5,5j+2)} v_j + s_{(5k+5,5j+4)} \phi_{\xi j} + s_{(5k+5,5j+5)} \phi_{\theta j} \Big]$$

$$k = o,\ 1,\ 2,\ - - - - -\ K-1 \qquad (2\text{-}19)$$

Coefficients are given in Appendix 1.3

Equation 6 can be written in matrix notation as,

$$y = Rz' + Sz' + x \tag{2-20}$$

Hence, the boundary conditions (37) become

$$\Omega R z' + (\Lambda + \Omega S)\, z = \ell - \Omega x \tag{2-21}$$

The form of Eq. (2-21) modified if the shell has a pole (i.e., $r = o$) because the coefficients of the differential equations become singular for this case. Following a similar limiting process as described by Greenbaum the conditions supplied at the pole are:

For Fourier index = 0

$$u_0 = v_0 = q_{\xi 0} = \phi_{\xi 0} = \phi_{\theta 0} = 0$$

For Fourier index = 1

$$u_1 \pm v_1 = w_1 = \phi_{\xi 1} + \phi_{\theta 1} = t_{\xi 1} - t_{\xi\theta 1} = m_{\xi 1} - m_{\xi\theta 1} = 0$$

For Fourier index $\geq$ 2

$$u_i = v_i = w_i = \phi_{\xi i} = \phi_{\theta_i} = 0$$

2.2 FINITE DIFFERENCE FORMULATION FOR THE MERIDIONAL VARIATION

In a manner similar to that described in Reference 2 the partial differential equation in the matrix form (Eq. 2-12) is reduced by a system of finite difference approximations. The variation in the meridional coordinate of the Fourier coefficients are described pointwise in Eq. (2-12). The following are finite difference forms for the partial differentials in the meridional coordinate at interior points.

$$\frac{\partial^2 f}{\partial \xi^2} = \frac{1}{\Delta 2} (f_{i+1} - 2 f_i + f_{i-1})$$

$$\frac{\partial f}{\partial \xi} = \frac{1}{\Delta} (f_{i+1} - f_{i-1}) \qquad 2\text{-}23$$

where Δ is the increment along ξ and the subscripts devote the discrete values of the function taken. The forms at boundary points (initial)

$$\frac{\partial f}{\partial \xi} = \frac{1}{2\Delta} (3 f_1 - 4 f_2 + f_3) \qquad 2\text{-}24$$

terminal

$$\frac{\partial f}{\partial \xi_1} = \frac{1}{2\Delta} (- f_{N-2} + 4 f_{N-1} - 3 f_N) \qquad 2\text{-}25$$

The result of the application of the various finite difference forms can be stated compactly at the following set of equations:

$$A_o z_2 + B_o z_1 + C_o z_o = g_o$$

$$A_i f_{i+1} + B_i z_i + C_i z_{i-1} = g_i + 2\Delta (\alpha_i \ddot{z}_i + \beta_i \dot{z}_i)$$

$$A_N z_N + B_N z_{N-1} + C_N z_{N-2} = g_N \qquad 2\text{-}26$$

Where

$$A_o = - \Omega_o R_o$$

$$B_o = 2 \Omega_o / \Delta R_o$$

$$C_o = \Lambda_o + \Omega_o S_o - \frac{3\,\Omega_o R_o}{z\Delta}$$

$$g_o = l_o - \Omega_o a_o \qquad 2\text{-}27$$

The subscript (o) refers to the conditions at the initial boundary. For $i \neq o$, N

$$A_i = \frac{2 F_i}{\Delta} + G_i$$

$$B_i = \frac{-4 F_i}{\Delta} + 2\Delta (H_i + K_i)$$

$$C_i = \frac{2 F_i}{\Delta} - G_i$$

$$g_i = 2 \Delta p_i \qquad 2\text{-}28$$

Finally for $i = N$ or conditions at the terminal boundary

$$A_N = \Lambda_N + \Omega_N S_N + \frac{3\,\Omega_N R_N}{2\Delta}$$

$$B_N = \frac{-2\Omega_N R_N}{\Delta}$$

$$C_N = \frac{\Omega_N R_N}{2\Delta}$$

$$g_N = l_N - \Omega_N a_N \qquad 2\text{-}29$$

2.3 FINITE DIFFERENCE FORMULATION IN THE TIME VARIABLE

By the use of difference equations the above differential equation in matrix form may be transformed into a set of algebraic equations involving the variable z_i at successive values of time.

The most commonly used are the central difference forms; however, from a numerical stability aspect, the difference forms of Houbolt (Ref 3 and 4) are used. These forms are

$$\ddot{z} = \frac{2 z_j - 5 z_{j-1} + 4 z_{j-2} - z_{j-3}}{\delta^2}$$

$$\dot{z} = \frac{11 z_j - 18 z_{j-1} + 9 z_{j-2} - 2 z_{j-3}}{6\delta} \qquad 2\text{-}30$$

Where the subscript j refers to the time interval j = 0, 1, 2, ... and δ is the time increment.

Introducing these expressions in Eq. 2-26 results in the following set of algebraic equations for the shell response problem.

$$A_o z_{2,j} + B_o z_{1,j} + C_o z_{o,j} = g_o$$

$$A_i^* z_{i+1, j} + B_{i,j} z_{i,j} + C_i^* z_{i-1, j} = g_{i,j}^*$$

$$A_N z_{N,j} + B_N z_{N-1,j} + C_N z_{N-2, j} = g_N \qquad 2\text{-}31$$

where

$$A_i^* = \delta A_i$$

$$B_{i,j}^* = \delta B_i + 4 \frac{\Delta}{\delta} \alpha_i + \frac{11}{3} \Delta\beta_i$$

$$C_i^* = \delta C_i$$

$$g_{i,j}^* = \delta g_i + \left(10 \frac{\Delta}{\delta} \alpha_i + 6\Delta \beta_i\right) z_{i,j-1} + \left(-8 \frac{\Delta}{\delta} \alpha_i - 3\Delta\beta_i\right) z_{i,j-2}$$

$$+ \left(2 \frac{\Delta}{\delta} \alpha_i + \frac{2}{3} \Delta\beta_i\right) z_{i, j-3}$$

In the real problem no values of z_i exists for less than zero. The assumption that z_i does exist before t = o is a means of allowing the recurrence from Eq. (2-31) to apply at the origin as well as later values of time. Furthermore, no violation is made as long as the initial conditions of t = o are satisfied.

To obtain values for the fictitious terms j = -1, -2 a procedure similar to that described by Houbolt is used. The procedure will require a modification of $B^*_{i,j}$, $g^*_{i,j}$ for j = 1, 2.

The difference equations for the first and second derivatives at the third increment of four successive increments are given by

$$\ddot{z}_{i,j} = \frac{1}{\delta^2}(z_{i,\ j+1} - 2\ z_{i,\ j} + z_{i,\ j-1})$$

$$\dot{z}_{i,j} = \frac{1}{6\delta}(2\ z_{i,j} + 3\ z_{i,j} - 6\ z_{i,j} + z_{i,\ j-2}) \qquad 2\text{-}32$$

Applying the equations at t = o, i.e. j = o

$$\ddot{z}_{i,o} = \frac{1}{\delta^2}(z_{i,1} - 2\ z_{i,o} + z_{i\ -1})$$

$$\dot{z}_{i,o} = \frac{1}{6\delta}(2\ z_{i,1} + 3\ z_{i,o} - 6\ z_{i,-1} + z_{i,\ j-2}) \qquad 2\text{-}33$$

The initial conditions are the displacements and velocities are prescribed. (at t = 0). By application of Newton's second law, s secondary initial condition can be established, i.e., acceleration immediately following application of the initial forces. These conditions are

$$z_{i,\ o} = d_{i,\ o}$$

$$z_{i,\ o} = v_{i,\ o}$$

$$z_{i,\ o} = a_{i,\ o} \qquad 2\text{-}34$$

Where $d_{i,o}$, $v_{i,o}$, $a_{i,o}$ are column n matrices formed of the respective coefficients of the Fourier expansions on θ of the initial displacements, velocities, and accelerations at the meridional location i.

Substitution of these values into Eq. (2.32) yields the following relations

$$z_{i,o} = d_{i,o}$$

$$z_{i,-1} = \delta^2\ a_{i,o} + 2d_{i,o} - z_{i,1}$$

$$z_{i,-2} = 6\delta^2\ a_{i,o} + 6\delta v_{i,o} + 9d_{i,o} - 8z_{i,1} \qquad 2\text{-}35$$

Substitution of these relations in Eq. (2-31) for j = 1 yields the following change in the definitions of the arrays in Eq. (2-31)

$$B^*_{i,1} = \delta B_i + 12 \frac{\Delta}{\delta}\alpha_i + 6\Delta\beta_i$$

$$g^*_{i,1} = \delta g_i + \left(12\frac{\Delta}{\delta}\alpha_i + 6\Delta\beta_i\right) d_{i,o} + \left(4\Delta\delta\alpha_i - \frac{7}{3}\delta^2\Delta\beta_i\right) a_{i,o}$$

$$+ (12\Delta\alpha_i + 4\Delta\delta\beta_i) v_{i,o} \qquad 2\text{-}36$$

Substitution of definitions Eq. (2-31) for j = 2 yields the following change in definitions of the arrays for Eq. (2-31)

$$g^*_{i,2} = \delta g_i + \left(8\frac{\Delta}{\delta}\alpha_i + \frac{16}{3}\beta_i\right) z_{i,1} + \left(-4\frac{\Delta}{\delta}\alpha_i \frac{5}{3}\Delta\beta_i\right) d_{i,o} + \left(2\frac{\Delta}{\delta}\alpha_i + \frac{2}{3}\Delta\beta_i\right)\delta^2 a_{i,o} \qquad 2\text{-}37$$

The set of Eq. (2-31) and the additional definitions at the first two time intervals is now the algebraic statement of the dynamic response problem.

2.4 MATRIX SOLUTION OF THE DIFFERENCE EQUATIONS

The set of matrix equations (Eq. 2-31) will be solved by the same procedure described in Ref. 2. This procedure is essentially a Gaussian elimination performed on the partitioned arrays. A slight modification of the elimination procedure described in Ref. (3) is used here. Considering the first and second equations of Eq. (2-31) at the j^{th} time interval

$$A_o z_{2,j} + B_o z_{i,j} + C_o z_{o,j} = g_o$$

$$A_1 z_{2,j} + B^*_1 z_{i,j} + C_1 z^*_{o,j} = g^*_{i,j} \qquad 2\text{-}38$$

Eliminating $Z_{o,j}$ from the equations and solving for $z_{i,j}$

$$z_{1,j} = -\left[B_o - C_o C_1^{-1} B^*_{1,j}\right]^{-1} \left[A_o - C_o C_1^{-1} A_1\right] Z_{2,j}$$

$$+ \left[B_o - C_o C_1^{-1} B^*_{1,j}\right]^{-1} \left[g_o - C_o C_1^{-1} g^*_{i,j}\right] \qquad 2\text{-}39$$

which becomes the form of Ref. (2)

$$z_{1,j} = - P_{i,j} z_{2,j} + x_{i,j} \qquad 2\text{-}40$$

where

$$P_{1,j} = \left[B_o - C_o C_1^{-1} B_{1,j}\right]^{-1} \left[A_o - C_o C_1^{-1} A_1\right]$$

$$x_{1,j} = \left[B_o - C_o C_1^{-1} B^*_{1,j}\right]^{-1} \left[g_o - C_o C_1^{-1} g^*_{1,j}\right] \qquad 2\text{-}41$$

Retaining the form of Eq. (2-40) for all meridional locations the general results become

$$z_{i,j} = - P_{i,j} z_{i+1,j} + x_{i,j}$$

$$(i = 1, 2, 3 \ldots N-1) \qquad 2\text{-}42$$

Analogous to Ref. (2) forms of P and x are

$$P_{i,j} = \left[B^*_{i,j} - C_i P_{i-1,j}\right]^{-1} A_i$$

$$x_{i,j} = \left[B^*_{i,j} - C_i P_{i-1,j}\right]^{-1} \left[g^*_{i,j} - C_i x_{i-1,j}\right]$$

$$(i = 2, 3, 4, \ldots N-1) \qquad 2\text{-}43$$

Substituting the recursive relation Eq. (2-42) into the last of Eq. (2-31) yields

$$z_{N,j} = \left[A_N + (C_N P_{N-2,j} - B^*_{N,j}) P_{N-1,j}\right]^{-1}$$

$$\left[g_{N,j} - C_N x_{N-2,j} + (C_N P_{N-2,j} - B^*_{N,j}) x^*_{N-1,j}\right] \qquad 2\text{-}44$$

Having $z_{N,j}$ the solutions at previous stations, $z_{N-1,j}$, $z_{N-2,j}$, etc., can be found by using the recursive relation Eq. (2-42). Finally $z_{o,j}$ calculated from the second equation of Eq. (2-31) and is given by

$$z_{o,j} = C_1^{-1} \left[g^*_{i,j} - A_1 z_{2,j} - B^*_{1,j} z_{1,j} \right] \qquad 2\text{-}45$$

The elimination procedure described as for the general time interval j. In the procedure for solution of the dynamic response beginning at j = 1 and using the initial conditions to define the arrays $B^*_{u,1}$ and $g^*_{i,1}$ Eq. (2-36) the solution $z_{i,1}$ is obtained by the above outlined procedure. For the time interval j = 2 the definitions of Eq. (2-37) are used to derive a closed set on $z_{i,2}$. Successive solutions for $z_{i,j}$ are obtained from the general forms which use the previous solutions $z_{i,j-1}$, $z_{i,j-2}$ to give an algebraic set for the unknown $z_{i,j}$.

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3.0 NUMERICAL COMPARISONS

3.1 INTRODUCTION

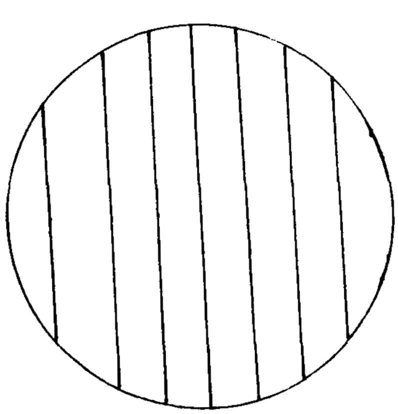
The preceding analysis was mechanized for a digital computer. The analysis was specialized in order to conform to failure limitations. The following limitations were made:

1. Single region shells
2. Isotropic stiffness relations
3. Reference surface must be the middle surface i.e. $\int E\zeta d\zeta = 0$.
4. Thermal variations circumferentially are such that limitation(s) is not distributed significantly.

The computer programs with these limitations were used to study several problems.

3.2 CASE 1

A comparison was made on the ATR209 Block 1 test. This problem was studied because test data and several other numerical solutions were available. The ATR209 configuration was an axisymetric spherical cap under an unsymmetric loading. The problem is defined below.



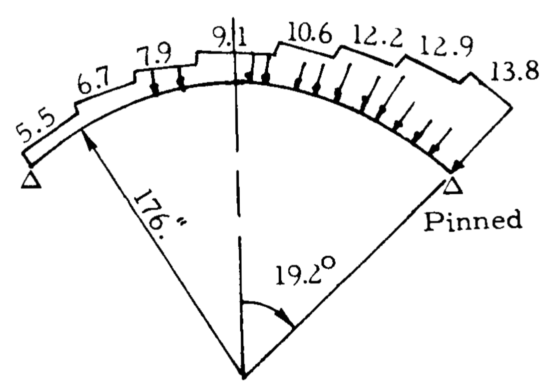


Figure 3-1. ATR 209 Geometry

The results of NAA unsymmetrical program are compared with several other investigators using other numerical techniques and the test results in Figure 3.2.

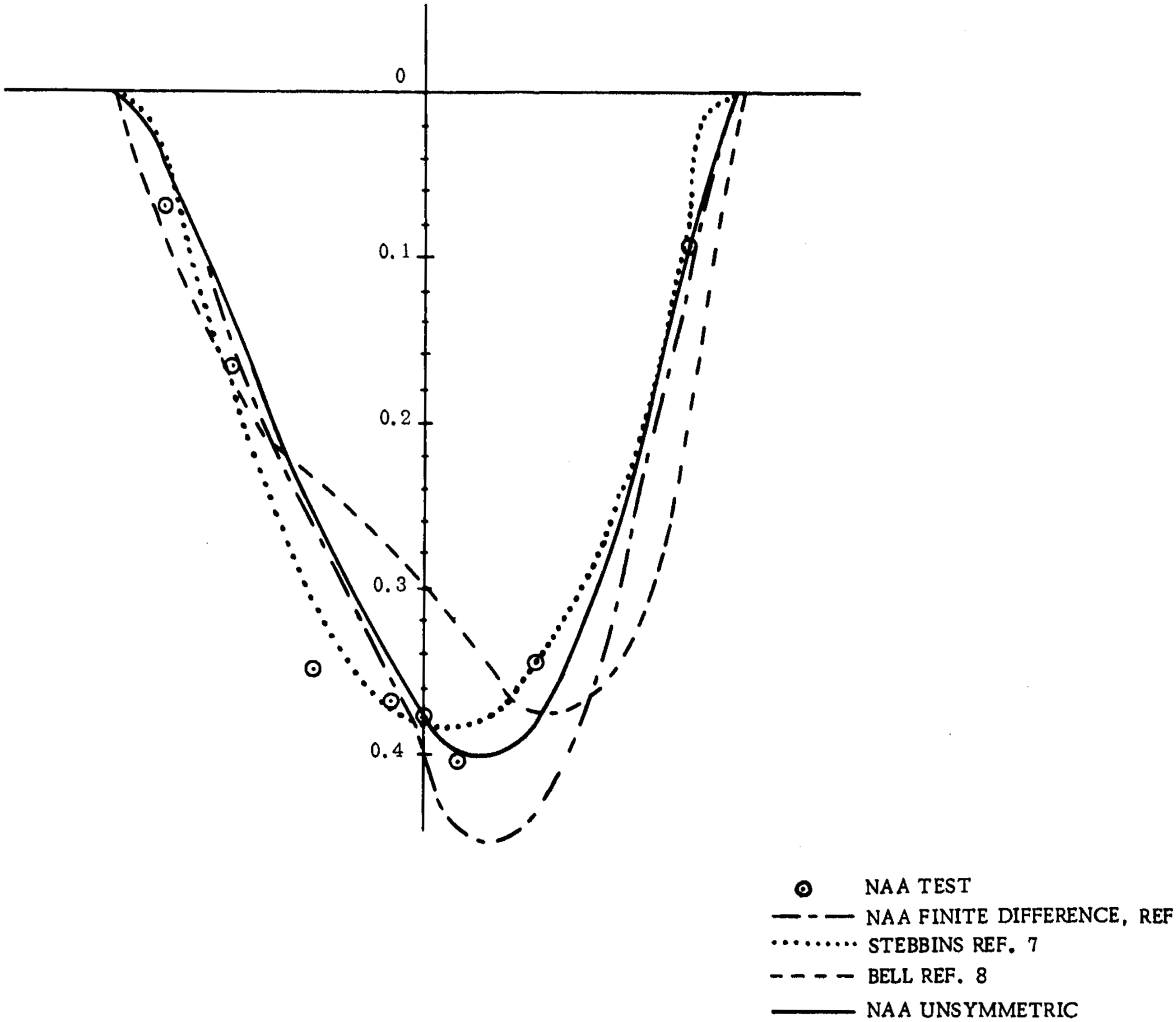


Figure 3-2. ATR 209 Test Normal Displacement Comparison

3.3 CASE 2

A second problem was considered to study the properties of the solution for an unsymmetric shell. This problem was a clamped unsymmetrical cap under uniform pressure with a stiffer region on one side. See figure below.

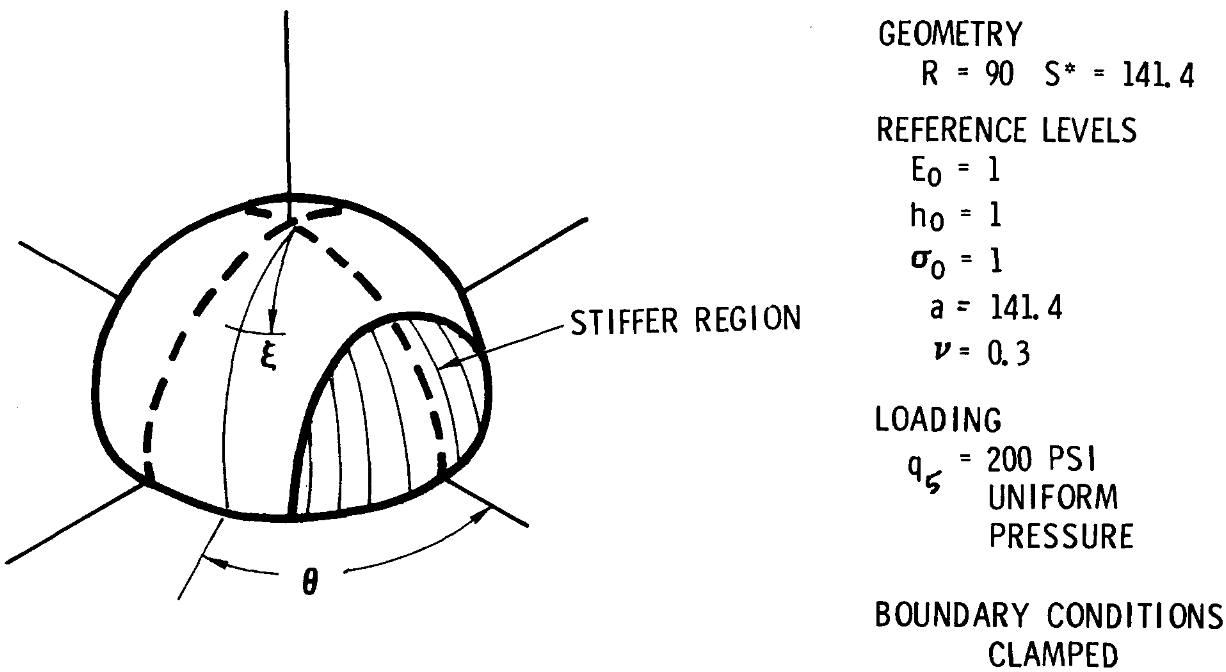


Figure 3-3. Step Hemisphere Geometry

For this problem, the Fourier series approximatly the discontinuous stiffness distribution was truncated after 10 terms. For a uniform pressure loading, individual Fourier coefficient solutions (w_n) for normal deflection are given in Figure 3-5 as a function of the meridional coordinate. It can be seen that the general character of the displacement pattern reflects the Fourier stiffness distribution profile. A convergence trend is exhibited because the higher order harmonics of the solution diminish in magnitude. The total deflection obtained by summing the Fourier contributions is plotted along the plane of symmetry (θ = 0-180 deg) in Figure 3-6. Also shown are the deflections obtained for axisymmetric shells with uniform stiffness distributions corresponding to the magnitude of the stiff and weak sides. As expected, these results bracket the solution for the variable-rigidity shell.

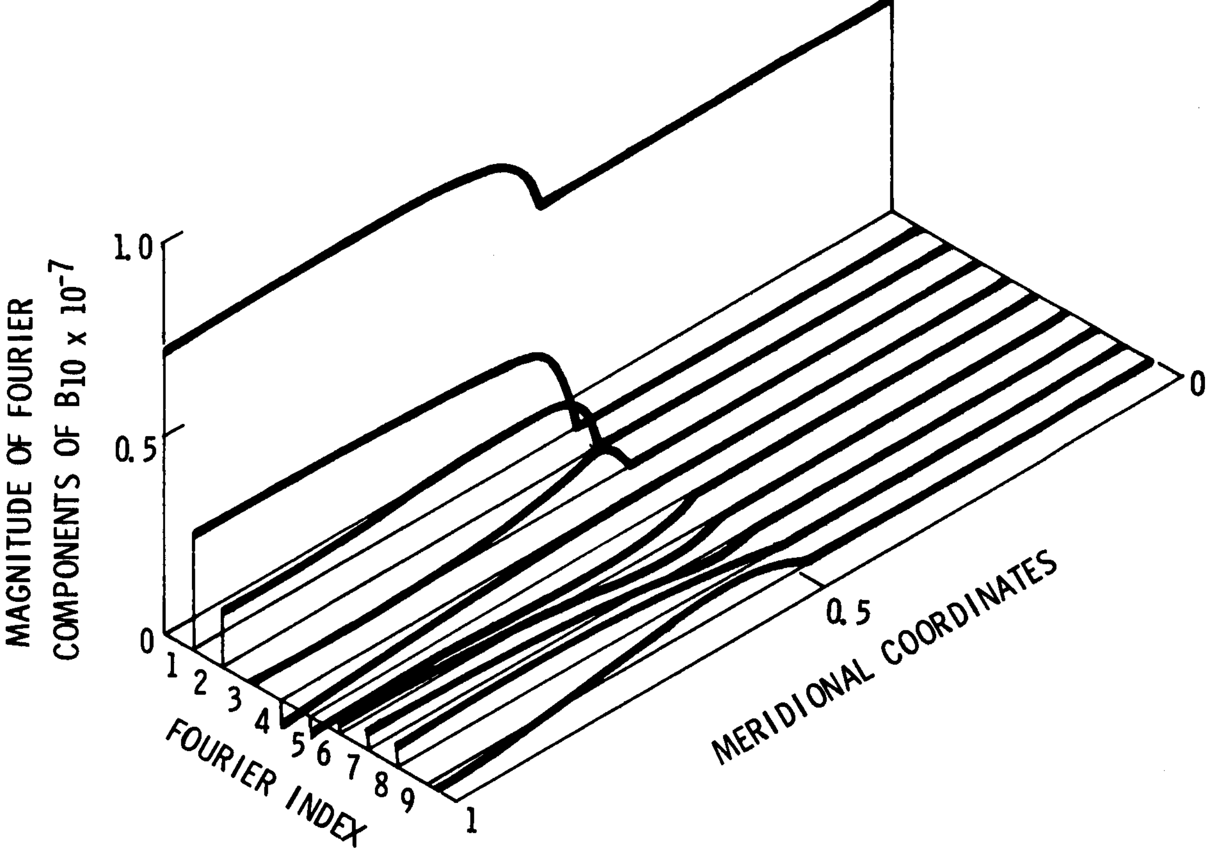


Figure 3-4. Fourier Components of the Membrane Stiffness Distribution for the Hemispherical Shell Problem

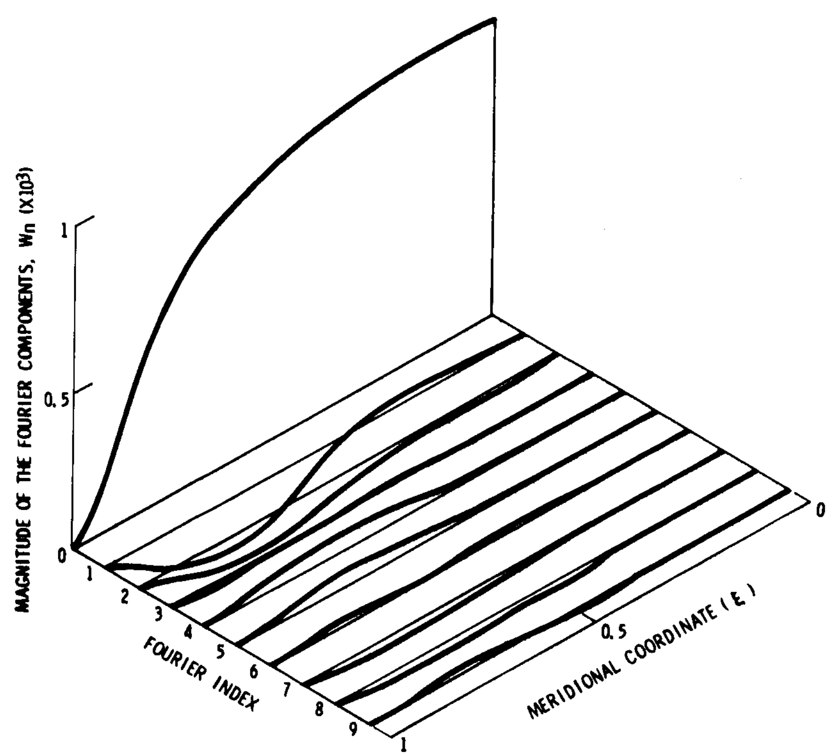


Figure 3-5. Fourier Component Distribution of Normal Displacement

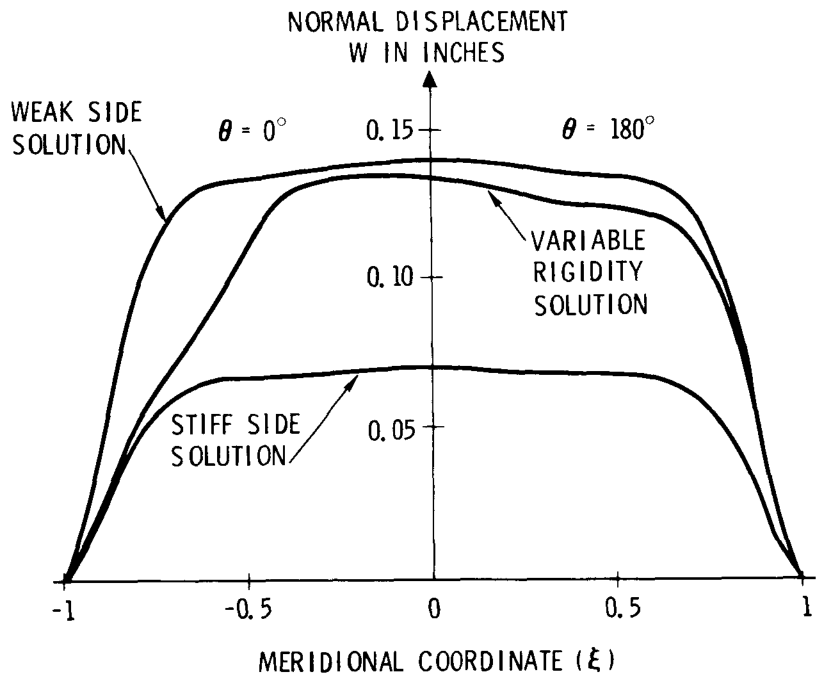


Figure 3-6. Meridional Distribution of Total Normal Displacement

The same problem was a run with only three and five terms retained in the stiffness series expansions. Variation in the third harmonic of deflection (w_2) along the meridian is plotted in Figure 3-7 for the cases where K is set equal to 3, 5, and 10. The higher harmonic coupling effects appear to diminish as more terms are retained in the solution (i.e., as K increases).

For economic purposes, the number of integration points used in obtaining the above numerical results was N = 26 for Case II. The basic character of the solutions is illustrated for these relatively coarse grids. Additional studies have shown increased accuracy when a finer finite difference mesh is taken.

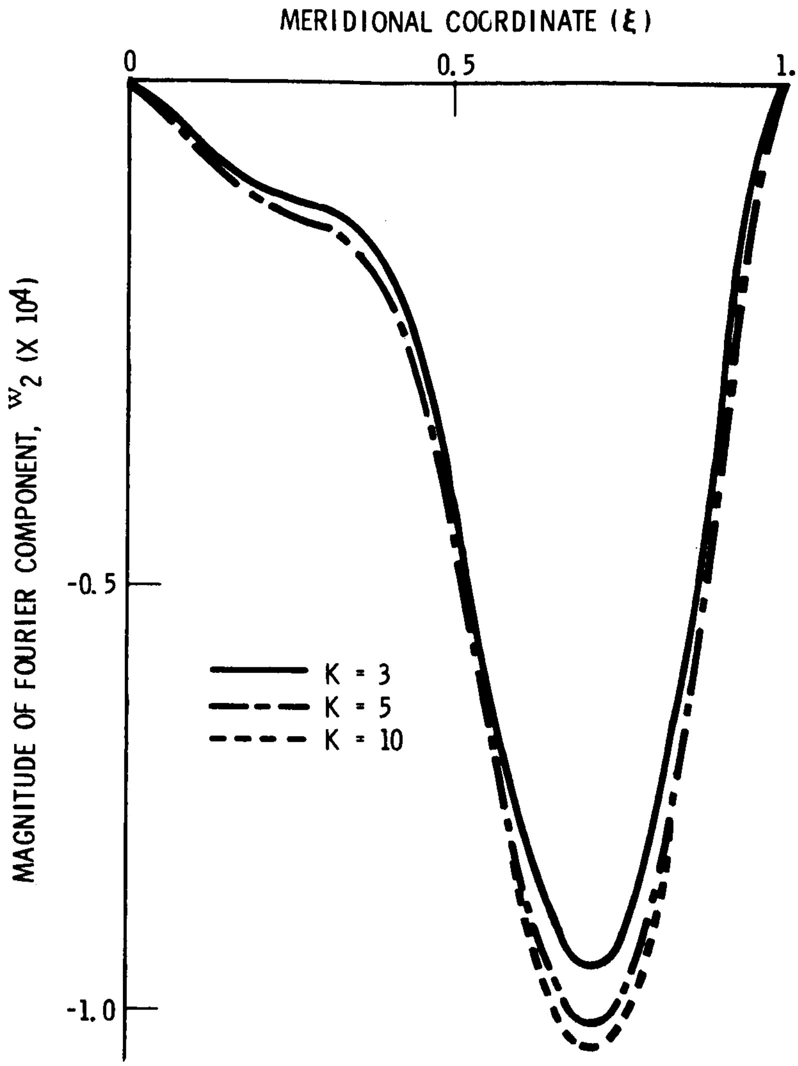


Figure 3-7. Meridional Distribution of Third Fourier Component of Solution for K = 3, 5, and 10

3.4 CASE 3.

The third problem studied was the Apollo test shield under a static pad load similar to the input load distribution. The geometry is given below.

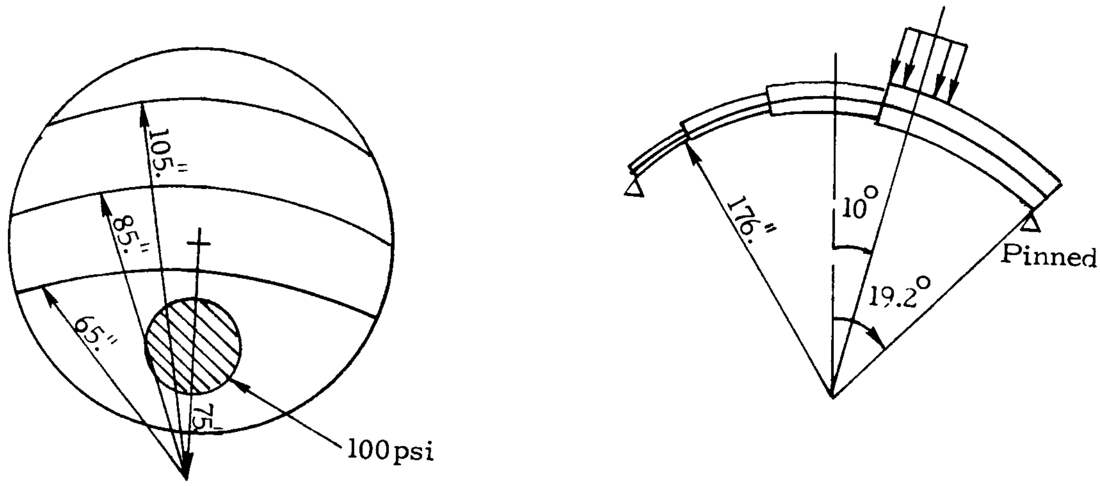


Figure 3-8. Apollo Test Shield Geometry

A comparison with several other techniques is given in the following Figure 3-9.

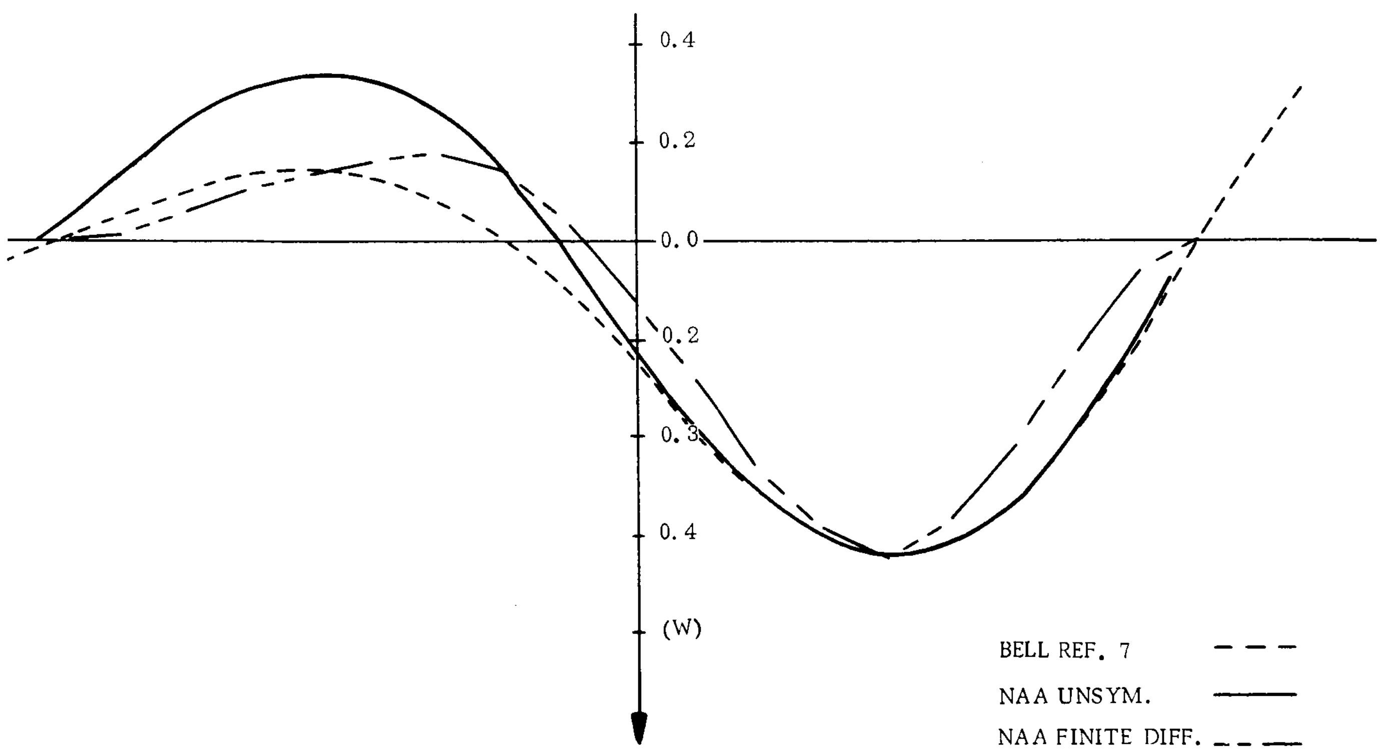


Figure 3-9. Normal Displacement Along the Plane of Symmetry

3.5 CASE 4

The fourth problem considered was a simply supported cylinder subjected to a uniform semisinuoidal pulse. The configuration and loading is given below.

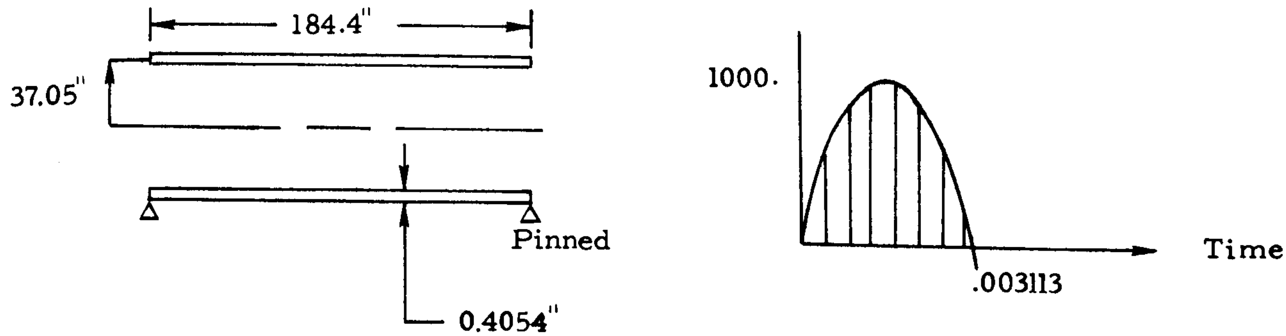


Figure 3-10. Cylinder Geometry and Loading

Several solutions were obtained using very crude spacial and time grids. A comparison is made to other analytic and numerical solutions in the following Figure 3-11.

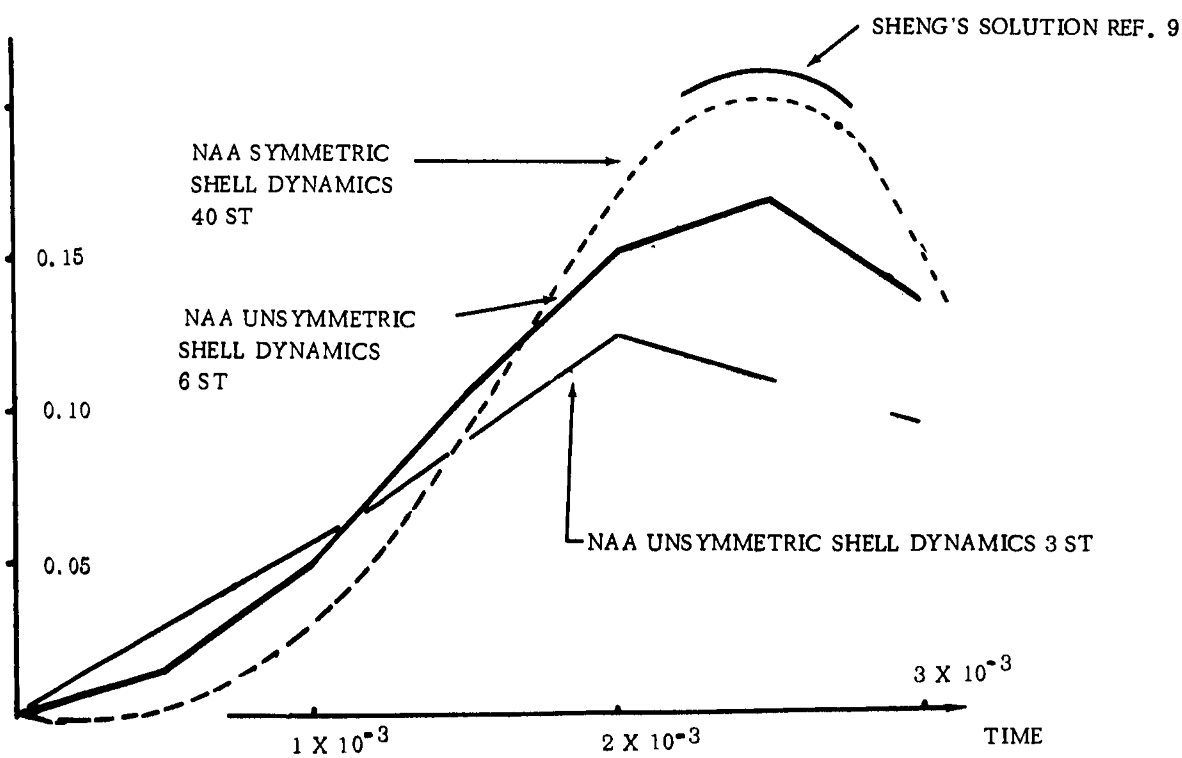


Figure 3-11. Dynamic Response of Cylinder to a Uniform Semi-Sinsoidal Pulse

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1.1 APPENDIX

Modification of Sander's Equations

Virtual change in the strain energy within C

$$\delta\bar{U} = \iint (N_{11}\delta\epsilon_{11} + N_{12}\delta\epsilon_{12} + N_{21}\delta\epsilon_{21} + N_{22}\delta\epsilon_{22} + M_{11}\delta k_{11} + M_{12}\delta k_{12} + M_{21}\delta k_{21} + M_{22}\delta k_{22} + Q_1\delta\gamma_1 + Q_2\delta\gamma_2)\, ds \quad (1)$$

$$\epsilon_{12} = \epsilon_{21} \quad (2)$$

$$k_{12} - k_{21} = \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\epsilon_{12} + \frac{1}{\alpha_1\alpha_2}\,\partial\frac{\alpha_2\gamma_2}{\partial\xi_1} - \frac{1}{\alpha_1\alpha_2}\,\partial\frac{\alpha_1\gamma_1}{\partial\xi_2} \quad (3)$$

Substituting (2) → (1)

$$= \iint \Big| N_{11}\delta\epsilon_{11} + (N_{12} + N_{21})\,\delta\epsilon_{12} + N_{22}\delta\epsilon_{22} + M_{11}\delta k_{11} + \frac{1}{2}M_{12}\delta k_{12} + \frac{1}{2}M_{12}\delta k_{12} + \frac{1}{2}M_{21}\delta k_{21} + \frac{1}{2}M_{21}\delta k_{21} + \frac{1}{2}M_{21}\delta k_{12} - \frac{1}{2}M_{21}\delta k_{12} + \frac{1}{2}M_{12}\delta k_{21} - \frac{1}{2}M_{12}\delta k_{21} + M_{22}\delta k_{22} + Q_1\delta\gamma_1 + Q_2\delta\gamma_2 \Big| \alpha_1\alpha_2 d_{\xi 1} d_{\xi 2} \quad (4)$$

Rearrange

$$= \iint \Big| N_{11}\delta\epsilon_{11} + (N_{12} + N_{21})\,\delta\epsilon_{12} + N_{22}\delta\epsilon_{22} + M_{11}\delta k_{11} + \frac{1}{2}(M_{12} - M_{21})\,\delta\,(k_{12} - k_{21}) + \frac{1}{2}(M_{12} + M_{21})\,\delta k_{21} + \frac{1}{2}(M_{12} + M_{21})\,\delta k_{12} + M_{22}\delta k_{22} + Q_1\delta\gamma_1 + Q_2\delta\gamma_2 \Big| \alpha_1\alpha_2 d_{\xi 1} d_{\xi 2} \quad (5)$$

Substitute 3 → 5.

$$\iint \Big\{ N_{11}\delta\epsilon_{11} + (N_{12} + N_{21})\,\delta\epsilon_{12} + N_{22}\delta\epsilon_{22} + M_{11}\delta k_{11} +$$

$$\frac{1}{2}(M_{12} - M_{21})\,\delta\left(\left(\frac{1}{R_2} - \frac{1}{R_1}\right)\epsilon_{12} + \frac{1}{\alpha_1\alpha_2}\frac{\partial\alpha_2\gamma_2}{\partial\xi_1} -\right.$$

$$\left.\frac{1}{\alpha_1\alpha_2}\frac{\partial\alpha_1\gamma_1}{\partial\xi_2}\right) + \frac{1}{2}(M_{21} + M_{12})\,\delta\,(k_{12} + k_{21}) + M_{22}\delta k_{22} +$$

$$Q_1\delta\gamma_1 + Q_2\delta\gamma_2 \Big|\, \alpha_1\alpha_2 d\xi_1 d\xi_2$$

Group.

$$\iint \Big\{ N_{11}\delta\epsilon_{11} + 2\left(\frac{1}{2}(N_{12} + N_{21}) + \frac{1}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)(M_{12} - M_{21})\right)\delta\epsilon_{12} +$$

$$N_{22}\delta\epsilon_{22} + M_{11}\delta k_{11} + \frac{1}{2\alpha_1\alpha_2}(M_{12} - M_{21})\,\delta\left(\frac{\partial\alpha_2\gamma_2}{\partial\xi_1} - \frac{\partial\alpha_1\gamma_1}{\partial\xi_2}\right) +$$

$$\frac{1}{2}(M_{21} + M_{12})\,\delta\,(k_{12} + k_{21}) + M_{22}\delta k_{22} + Q_1\delta\gamma_1 + Q_2\delta\gamma_2 \Big\}\, \alpha_1\alpha_2 d\xi_1 d\xi_2$$

Expanding $\frac{\partial}{\partial\xi}(\alpha\gamma)$

$$\delta\left(\frac{\partial\alpha_2\gamma_2}{\partial\xi_1} - \frac{\partial\alpha_1\gamma_1}{\partial\xi_2}\right) = \delta\left(\gamma_2\frac{\partial\alpha_2}{\partial\xi_1} + \alpha_2\frac{\partial\gamma_2}{\partial\xi_1} - \gamma_1\frac{\partial\alpha_1}{\partial\xi_2} - \alpha_1\frac{\partial\gamma_1}{\partial\xi_2}\right)$$

Substitute and Group.

$$\iint \Big\{ N_{11}\delta\epsilon_{11} + 2\left[\frac{1}{2}(N_{12} + N_{21}) + \frac{1}{4}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)(M_{12} - M_{21})\right]\delta\epsilon_{12} +$$

$$N_{22}\delta\epsilon_{22} + M_{11}\delta k_{11} + \frac{1}{2}(M_{21} + M_{12})\,\delta\,(k_{12} + k_{21}) + M_{22}\delta k_{22} +$$

$$\left[Q_1 - \frac{1}{2\alpha_1\alpha_2}\frac{\partial\alpha_1}{\partial\xi_2}(M_{12} - M_{21})\right]\delta\gamma_1 + \left[Q_2 + \frac{1}{2\alpha_1\alpha_2}\frac{\partial\alpha_2}{\partial\xi_1}(M_{12} - M_{21})\right]$$

$$\delta\gamma_2 + \frac{1}{2}(M_{12} - M_{21})\,\delta\left(\frac{1}{\alpha_1}\frac{\partial\gamma_2}{\partial\xi_1} - \frac{1}{\alpha_2}\frac{\partial\gamma_1}{\partial\xi_2}\right)\Big\}\, \alpha_1\alpha_2 d\xi_1 d\xi_2$$

Assumption

$$M_{12} - M_{21} \rightarrow o, \quad \frac{\delta \partial \gamma_i}{\partial \xi_j} << \delta (\epsilon, \ k)$$

$$\delta U = \iint \Big| N_{11} \delta \epsilon_{11} + 2\overline{N}_{12} \delta \epsilon_{12} + N_{22} \delta \epsilon_{22} + M_{11} \delta k_{11} + M_{22} \delta k_{22} +$$

$$2\overline{M}_{12} \delta \overline{k}_{12} + Q_1 \delta \gamma_1 \ Q_2 \delta \gamma_2 \Big| \alpha_1 \alpha_2 d_{\xi 1} d_{\xi 2}$$

$$\overline{N}_{12} = \frac{1}{2} (N_{12} + N_{21})$$

$$\overline{M}_{12} = \frac{1}{2} (M_{12} + M_{21})$$

$$\overline{k}_{12} = \frac{1}{2} (k_{12} + k_{21})$$

APPENDIX 1.2

Multiplication of Series Expansions

The relationships A^{nj} and A^{nj} used in the text are found by multiplying term by term the series expansions of stiffness and strain and noting a recurring sequence of the resulting expressions

$$\text{e.g.}\quad \left(\sum_{j=o}^{K-1} b_j \cos_j \theta\right)\left(\sum_{n=o}^{K-1} \epsilon_{\xi n} \cos n\theta\right) = \sum_{n=o}^{K-1}\left(\sum_{j=o}^{K-1} A^{nj} \epsilon_{\xi j}\right) \cos n\theta$$

$$= \sum_{n=o}^{K-1}\left\{\sum_{j=o}^{K-1} \frac{1}{2}\left[b^{(n+j)} + \left\{1 - \delta^2(j-k) + \delta(k)\right\} b^{|n-j|}\right]\epsilon_{\xi j}\right\} \cos n\theta$$

$$\left(\sum_{j=o}^{K-1} b_j \cos_j \theta\right)\left(\sum_{n=1}^{K-1} \bar{\epsilon}_{\xi n} \sin n\theta\right) = \sum_{n=1}^{K-1}\left(\sum_{j=o}^{K-1} A^{nj} \bar{\epsilon}_{\xi j}\right) \sin n\theta$$

$$= \sum_{n=1}^{K-1}\left\{\sum_{j=o}^{K-1} \frac{1}{2}\left[-b^{(n+j)} + \left\{1 - \delta^2(j-k) + \delta(k)\right\} b^{|n-j|}\right]\bar{\epsilon}_{\xi j}\right\} \sin n$$

where

$$\delta(m) = \begin{cases} -1 & m < o \\ o & m = o \\ +1 & m > o \end{cases}$$

APPENDIX 1.3 - COEFFICIENTS

$$f_{5k+1,\,5j+1} = B_1^{kj}$$

$$f_{5k+1,\,5j+4} = \lambda C_1^{kj}$$

$$f_{5k+1,\,5j+2} = \frac{G_1^{kj}}{2} + \frac{\lambda}{2}(\omega_\theta - \omega_\xi)\, G_{12}^{kj} + \frac{\lambda^2}{8}(\omega_\theta - \omega_\xi)^2 G_{13}^{kj}$$

$$f_{5k+2,\,5j+5} = \frac{\lambda}{2} G_{12}^{kj} + \frac{\lambda^2}{4}(\omega_\theta - \omega_\xi)\, G_{13}^{kj}$$

$$f_{5k+3,\,5j+3} = G_2^{kj}$$

$$f_{5k+4,\,5j+1} = \lambda C_1^{kj}$$

$$f_{5k+4,\,5j+4} = \lambda^2 D_1^{kj}$$

$$f_{5k+5,\,5j+2} = \frac{\lambda}{2} G_{12}^{kj} + \frac{\lambda^2}{4}(\omega_\theta - \omega_\xi)\, G_{13}^{kj}$$

$$f_{5k+5,\,5j+5} = \frac{\lambda^2}{2} G_{13}^{kj}$$

$$g_{5k+1,\,5j+1} = B_1^{kj'} + \gamma B_1^{kj}$$

$$g_{5k+1,\,5j+2} = \frac{k}{\rho} B_8^{kj} + \frac{k}{2\rho} G_1^{kj} - \frac{\lambda^2 k}{8\rho}(\omega_\theta - \omega_\xi)^2 G_{13}^{kj}$$

$$g_{5k+1,\,5j+3} = \omega_\xi B_1^{kj} + \omega_\theta B_3^{kj} + \omega_\xi G_2^{kj}$$

$$g_{5k+1,\,5j+4} = \lambda C_1^{kj'} + \gamma\lambda C_1^{kj}$$

$$g_{5k+1,\,5j+5} = \lambda \frac{k}{\rho} C_3^{kj} + \frac{\lambda k}{2\rho} G_{12}^{kj} + \frac{\lambda^2 k}{4\rho}(\omega_\xi - \omega_\theta)\, G_{13}^{kj}$$

$$g_{5k+2,\,5j+1} = \frac{k}{2\rho} G_1^{kj} + \frac{\lambda^2 k}{8\rho} (\omega_\theta - \omega_\xi)^2 G_{13}^{kj} - \frac{k}{\rho} B_3^{kj}$$

$$g_{5k+2,\,5j+2} = \frac{G_1^{kj'}}{2} + \frac{\lambda}{2} (\omega_\theta - \omega_\xi) G_{12}^{kj'} - \frac{\lambda}{2} \omega_\xi' G_{12}^{kj} + \frac{\lambda}{2} G_1^{kj} +$$

$$+ \frac{\lambda^2}{8} (\omega_\theta - \omega_\xi)^2 G_{13}^{kj'} - \frac{\lambda^2}{8} \gamma (\omega_\theta - \omega_\xi)^2 + 2\omega_\xi' (\omega_\theta - \omega_\xi) \;\; G_{13}^{kj}$$

$$g_{5k+2,\,5j+4} = -\frac{\lambda k}{2\rho} G_{12}^{kj} - \lambda \frac{k}{\rho} C_3^{kj} - \frac{\lambda 2}{4} \frac{k}{\rho} (\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$g_{5k+2,\,5j+5} = \frac{\lambda}{2} G_{12}^{kj'} + \frac{\lambda\gamma}{2} G_{12}^{kj} + \frac{\lambda^2}{4} (\omega_\theta' - \omega_\xi) G_{13}^{kj'} +$$

$$+ \frac{\lambda^2}{4} \;\; 2\gamma(\omega_\xi - \omega_\theta) - \omega_\xi' \;\; G_{13}^{kj}$$

$$g_{5k+3,\,5j+1} = -\omega_\xi G_2^{kj} - \omega_\xi B_1^{kj} - \omega_\theta B_3^{kj}$$

$$g_{5k+3,\,5j+3} = G_2^{kj'} + \gamma G_2^{kj}$$

$$g_{5k+3,\,5j+4} = G_2^{kj} - \lambda \omega_\xi C_1^{kj} - \lambda \omega_\xi C_3^{kj}$$

$$g_{5k+4,\,5j+1} = \lambda C_1^{kj'} + \lambda \gamma C_1^{kj}$$

$$g_{5k+4,\,5j+2} = \lambda \frac{k}{\rho} C_3^{kj} + \frac{\lambda k}{2\rho} G_{12}^{kj} + \frac{\lambda^2}{4} \frac{k}{\rho} (\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$g_{5k+4,\,5j+3} = \lambda \omega_\xi C_1^{kj} + \lambda \omega_\theta C_3^{kj} - G_2^{kj}$$

$$g_{5k+4,\,5j+4} = \lambda^2 D_1^{kj'} + \gamma \lambda^2 D_1^{kj}$$

$$g_{5k+4,\,5j+5} = \lambda^2 \frac{k}{\rho} D_3^{kj} + \frac{\lambda^2}{2} \frac{k}{\rho} G_{13}^{kj}$$

$$g_{5k+5,\,5j+1} = -\frac{\lambda}{2} \frac{k}{\rho} G_{12}^{kj} + \frac{\lambda^2}{4} \frac{k}{\rho} (\omega_\theta - \omega_\xi) G_{13}^{kj} - \frac{\lambda k}{\rho} C_3^{kj}$$

$$g_{5k+5,\,5j+2} = \frac{\lambda}{2} G_{12}^{kj'} + \frac{\lambda^2}{4}(\omega_\theta - \omega_\xi) G_{13}^{kj'} + \frac{\lambda^2}{4}\left(2\gamma(\omega_\theta - \omega_\xi) - \omega_\xi'\right) G_{13}^{kj}$$

$$+ \frac{\lambda\gamma}{2} G_{12}^{kj}$$

$$g_{5k+5,\,5j+4} = - \frac{\lambda^2}{2}\frac{k}{\rho} G_{13}^{kj} - \lambda^2 \frac{k}{\rho} D_3^{kj}$$

$$g_{5k+5,\,5j+5} = \frac{\lambda^2}{2} G_{13}^{kj'} + \frac{\lambda^2}{2}\gamma G_{13}^{kj}$$

$$h_{5k+1,\,5j+1} = \gamma B_3^{kj'} - \omega_\xi \omega_\theta B_3^{kj} - \lambda^2 B_2^{kj} - \frac{k^2}{2\rho^2} G_1^{kj}$$

$$+ \frac{\lambda}{2}\frac{k^2}{\rho^2}(\omega_\theta - \omega_\xi) G_{12}^{kj} - \frac{\lambda^2}{8}\frac{k^2}{\rho^2}(\omega_\theta - \omega_\xi)^2 G_{13}^{kj} - \omega_\xi^2 G_2^{kj}$$

$$h_{5k+1,\,5j+2} = \frac{k}{\rho} B_3^{kj'} - \gamma\frac{k}{\rho} B_2^{kj} - \frac{\gamma}{2}\frac{k}{\rho} G_1^{kj} + \frac{\gamma}{2}(\omega_\theta - \omega_\xi)\lambda\frac{k}{\rho} G_{12}^{kj}$$

$$- (\omega_\xi - \omega_\theta)^2 \frac{\gamma\lambda^2}{8}\frac{k}{\rho} G_{13}^{kj}$$

$$h_{5k+1,\,5j+3} = \omega_\xi B_1^{kj'} + \omega_\theta B_3^{kj'} + (\omega_\xi' + \gamma\omega_\xi) B_1^{kj} - \gamma\omega_\theta B_2^{kj}$$

$$h_{5k+1,\,5j+4} = - \lambda\omega_\xi\omega_\theta C_3^{kj} + \gamma\lambda C_3^{kj'} - \gamma^2\lambda C_2^{kj} + \omega_\xi G_2^{kj} - \frac{\lambda}{2}\frac{k^2}{\rho^2} G_{12}^{kj}$$

$$+ \frac{\lambda^2}{4}\frac{k^2}{\rho^2}(\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$h_{5k+1,\,5j+5} = \lambda\frac{k}{\rho} C_3^{kj'} - \lambda\gamma\frac{k}{\rho} C_2^{kj} - \frac{\lambda}{2}\gamma\frac{k}{\rho} G_{12}^{kj} + \frac{\lambda^2\gamma}{4}\frac{k}{\rho}(\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$h_{5k+2,\,5j+1} = - \frac{k}{2\rho} G_1^{kj'} - \frac{\gamma k}{2\rho} G_1^{kj} + \frac{\lambda\gamma}{2}\frac{k}{\rho}(\omega_\theta - \omega_\xi) G_{12}^{kj} - \frac{k}{\rho}\gamma B_2^{kj}$$

$$+ \frac{\lambda^2}{8}\frac{k}{\rho}(\omega_\theta - \omega_\xi)^2 G_{13}^{kj'} - \frac{\lambda^2}{8}\frac{k}{\rho}(\omega_\theta - \omega_\xi)\left(3\gamma(\omega_\theta - \omega_\xi) - 2\omega_\xi'\right) G_{13}^{kj}$$

$$h_{5k+2,\,5j+2} = \frac{1}{2}(\omega_\xi\omega_\theta - \gamma^2)G_1^{kj} - \frac{\gamma}{2}G_1^{kj'} + \frac{\lambda\gamma^2}{2}(\omega_\theta - \omega_\xi)G_{12}^{kj} - \frac{k^2}{\rho 2}B_2^{kj}$$

$$- \omega_\theta^2 G_3^{kj} + \frac{\lambda^2\gamma}{8}(\omega_\theta - \omega_\xi)^2 G_{13}^{kj'}$$

$$- \frac{\lambda^2}{8}\Big((\omega_\theta - \omega_\xi)^2(\omega_\xi\omega_\theta + 3\gamma^2) - (\omega_\xi - \omega_\theta)\,2\gamma\omega_\xi'\Big)G_{13}^{kj}$$

$$h_{5k+2,\,5j+3} = -\frac{k}{\rho}\omega_\xi B_3^{kj} - \frac{k}{\rho}\omega_\theta B_2^{kj} - \omega_\theta\frac{k}{\rho}G_3^{kj}$$

$$h_{5k+2,\,5j+4} = -\frac{\lambda}{2}\frac{k}{\rho}\gamma G_{12}^{kj} - \frac{\lambda}{2}\frac{k}{\rho}G_{12}^{kj'} - \frac{k}{\rho}\gamma\lambda C_2^{kj}$$

$$+ \frac{\lambda^2}{4}\frac{k}{\rho}\Big(G2(\omega_\theta - \omega_\xi) + \omega_\xi'\Big)G_{13}^{kj} - \frac{\lambda^2}{4}\frac{k}{\rho}(\omega_\theta - \omega_\xi)G_{13}^{kj'}$$

$$h_{5k+2,\,5j+5} = -\frac{\lambda\gamma}{2}G_{12}^{kj'} - \lambda\frac{k^2}{\rho 2}C_2^{kj} + \omega_\theta G_3^{kj} + \frac{\lambda}{2}(\omega_\xi\omega_\theta - \gamma^2)G_{12}^{kj}$$

$$- \frac{\lambda^2\gamma}{4}(\omega_\theta - \omega_\xi)G_{13}^{kj'} + \frac{\lambda^2}{4}(\omega_\theta - \omega_\xi)(\omega_\xi\omega_\theta + 2\gamma^2)G_{13}^{kj}$$

$$+ \frac{\lambda^2\gamma}{4}\omega_\xi' G_{13}^{kj}$$

$$h_{5k+3,\,5j+1} = -(\omega_\xi' + \gamma\omega_\xi)G_2^{kj} - \omega_\xi\gamma B_3^{kj} - \omega_\theta\gamma B_2^{kj} - \omega_\xi G_2^{kj'}$$

$$h_{5k+3,\,5j+2} = -\frac{k}{\rho}\omega_\xi B_3^{kj} - \frac{k}{\rho}\omega_\theta B_2^{kj} - \omega_\theta\frac{k}{\rho}G_3^{kj}$$

$$h_{5k+3,\,5j+3} = -\frac{k^2}{\rho^2}G_3^{kj} - \omega_\xi^2 B_1^{kj} - 2\omega_\theta\omega_\xi B_3^{kj} - \omega_\theta^2 B_2^{kj}$$

$$h_{5k+3,\,5j+4} = G_2^{kj'} + \gamma G_2^{kj} - \lambda\gamma\omega_\xi C_3^{kj} - \lambda\gamma\omega_\xi C_2^{kj}$$

$$h_{5k+3,\,5j+5} = \frac{k}{\rho}G_3^{kj} - \lambda\frac{k}{\rho}\omega_\xi C_3^{kj} - \lambda\frac{k}{\rho}\omega_\theta C_2^{kj}$$

$$h_{5k+4,\,5j+1} = -\lambda\omega_\xi\omega_\theta C_3^{kj} - \gamma^2\lambda C_2^{kj} - \frac{\lambda}{2}\frac{k^2}{\rho^2} G_{12}^{kj} + \frac{k^2}{4\rho^2}\lambda^2(\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$+\omega_\xi G_2^{kj} + \lambda\gamma C_3^{kj'}$$

$$h_{5k+4,\,5j+2} = \lambda\frac{k}{\rho} C_3^{kj'} - \frac{\lambda\gamma}{2}\frac{k}{\rho} G_{12}^{kj} + \frac{\lambda^2}{4}\frac{k}{\rho}\gamma(\omega_\theta - \omega_\xi) G_{13}^{kj} - \lambda\gamma\frac{k}{\rho} C_2^{kj}$$

$$h_{5k+4,\,5j+3} = \lambda\omega_\xi' C_1^{kj} + \lambda\gamma\omega_\xi C_1^{kj} - \lambda\gamma\omega_\theta C_2^{kj} + \lambda\omega_\xi C_1^{kj'} + \lambda\omega_\theta' C_3^{kj'}$$

$$h_{5k+4,\,5j+4} = -\lambda^2\omega_\xi\omega_\theta D_3^{kj} + \lambda^2\gamma D_3^{kj'} - \frac{\lambda^2 k^2}{2\rho^2} G_{13}^{kj} - \lambda^2\left({}^2D_2^{kj} - G_2^{kj}\right)$$

$$h_{5k+4,\,5j+5} = \lambda^2\frac{k}{\rho} D_3^{kj'} - \frac{\lambda^2\gamma}{2}\frac{k}{\rho} G_{13}^{kj} - \lambda^2\gamma\frac{k}{\rho} D_2^{kj}$$

$$h_{5k+5,\,5j+1} = -\frac{\lambda}{2}\frac{k}{\rho}\gamma G_{12}^{kj} - \frac{\lambda}{2}\frac{k}{\rho} G_{12}^{kj'} + \frac{\lambda^2}{4}\frac{k}{\rho}(\omega_\theta - \omega_\xi) G_{13}^{kj'}$$

$$-\frac{\lambda^2}{4}\frac{k}{\rho}\omega_\xi' G_{13}^{kj} - \lambda\frac{k}{\rho}\gamma C_2^{kj}$$

$$h_{5k+5,\,5j+2} = \frac{\lambda}{2}(\omega_\xi\omega_\theta - \gamma^2) G_{12}^{kj} - \frac{\lambda\gamma}{2} G_{12}^{kj'} + \frac{\lambda^2\gamma}{4}(\omega_\theta - \omega_\xi) G_{13}^{kj'}$$

$$-\frac{\lambda^2}{4}\gamma\omega_\xi' + \omega_\xi\omega_\theta(\omega_\theta - \omega_\xi)\ G_{13}^{kj} + \omega_\theta G_{13}^{kj} - \frac{k^2}{\rho^2}\lambda C_2^{kj}$$

$$h_{5k+5,\,5j+3} = -\lambda\frac{k}{\rho}\omega_\xi C_3^{kj} - \lambda\frac{k}{\rho}\omega_\theta C_2^{kj} + \frac{k}{\rho} G_3^{kj}$$

$$h_{5k+5,\,5j+4} = -\frac{\lambda^2}{2}\frac{k}{\rho} G_{13}^{kj'} - \lambda^2\frac{k}{\rho}\gamma D_2^{kj} - \frac{\lambda^2}{2}\frac{k}{\rho}\gamma G_{13}^{kj}$$

$$h_{5k+5,\,5j+5} = -\frac{\lambda^2\gamma}{2} G_{13}^{kj'} - \lambda^2\frac{k^2}{\rho^2} D_2^{kj} - G_3^{kj} + \frac{\lambda^2}{2}(\omega_\xi\omega_\theta - \gamma^2) G_{13}^{kj}$$

$$p_{5k+1} = - p_{\xi}^{k} + t_{\xi T}^{k'} + \gamma\left(t_{\xi T}^{k} - t_{\theta T}^{k}\right)$$

$$p_{5k+2} = - p_{\theta}^{k} - \frac{k}{\rho} t_{\theta T}^{k}$$

$$p_{5k+3} = - p^{k} - \omega_{\xi} t_{\xi T}^{k} - \omega \ t_{\theta T}^{k}$$

$$p_{5k+4} = \lambda^{2} m_{\xi T}^{k'} + \lambda_{\xi}^{2} m_{T}^{k} - {}^{2}\lambda m_{\theta T}^{k}$$

$$p_{5k+5} = - \frac{k}{\rho} \lambda^{2} m_{\theta T}^{k}$$

$$r_{5k+1,\,5j+1} = B_{1}^{kj}$$

$$r_{5k+1,\,5j+4} = \lambda C_{1}^{kj}$$

$$r_{5k+2,\,5j+2} = \frac{1}{2} G_{1}^{kj} + \frac{\lambda}{2} G_{12}^{kj} (\omega_{\theta} - \omega_{\xi}) + \frac{\lambda^{2}}{8} (\omega_{\theta} - \omega_{\xi})^{2} G_{13}^{kj}$$

$$r_{5k+2,\,5j+5} = \frac{\lambda}{2} G_{12}^{kj} + \frac{\lambda^{2}}{4} (\omega_{\theta} - \omega_{\xi}) G_{13}^{kj}$$

$$r_{5k+3,\,5j+3} = G_{2}^{kj}$$

$$r_{5k+4,\,5j+1} = \frac{1}{\lambda} C_{1}^{kj}$$

$$r_{5k+4,\,5j+4} = D_{1}^{kj}$$

$$r_{5k+5,\,5j+2} = \frac{1}{2\lambda} G_{12}^{kj} + \left(\frac{\omega_{\theta} - \omega_{\xi}}{4}\right) G_{13}^{kj}$$

$$r_{5k+5,\,5j+5} = \frac{1}{2} G_{13}^{kj}$$

$$s_{5k+1,\,5j+1} = \lambda B_3^{kj}$$

$$s_{5k+1,\,5j+2} = \frac{k}{\rho}\, B_3^{kj}$$

$$s_{5k+1,\,5j+3} = \omega_\xi B_1^{kj} + \omega_\theta B_3^{kj}$$

$$s_{5k+1,\,5j+4} = \gamma\lambda C_3^{kj}$$

$$s_{5k+1,\,5j+5} = \frac{k}{\rho}\,\lambda C_3^{kj}$$

$$s_{5k+2,\,5j+1} = -\,\frac{k}{2\rho}\, G_1^{kj} + \frac{\lambda^2}{8}\,\frac{k}{\rho}\,(\omega_\theta - \omega_\xi)^2 G_{13}^{kj}$$

$$s_{5k+2,\,5j+2} = -\,\frac{\gamma}{2}\, G_1^{kj} + \frac{\lambda^2}{8}\,\gamma\,(\omega_\theta - \omega_\xi)^2 G_{13}^{kj}$$

$$s_{5k+2,\,5j+4} = -\,\frac{\lambda}{2}\,\frac{k}{\rho}\, G_{12}^{kj} - \frac{\lambda^2}{4}\,\frac{k}{\rho}\,(\omega_\theta - \omega_\xi)\, G_{13}^{kj}$$

$$s_{5k+2,\,5j+5} = -\,\frac{\lambda}{2}\,\gamma G_{12}^{kj} - \gamma\,\frac{\lambda^2}{4}\,(\omega_\theta - \omega_\xi)\, G_{13}^{kj}$$

$$s_{5k+3,\,5j+4} = -\,\omega_\xi G_2^{kj}$$

$$s_{5k+3,\,5j+4} = G_2^{kj}$$

$$s_{5k+4,\,5j+1} = \frac{\gamma}{\lambda}\, C_3^{kj}$$

$$s_{5k+4,\,5j+2} = \frac{k}{\lambda\rho}\, C_3^{kj}$$

$$s_{5k+4,\,5j+3} = \frac{\omega_\xi}{\lambda}\, C_1^{kj} + \frac{\omega_\theta}{\lambda}\, C_3^{kj}$$

$$s_{5k+4,\,5j+4} = \gamma D_3^{kj}$$

$$s_{5k+4,\,5j+5} = \frac{k}{\rho} D_3^{kj}$$

$$s_{5k+5,\,5j+1} = - \frac{1}{2\lambda} \frac{k}{\rho} G_{12}^{kj} + \frac{k}{4\rho} (\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$s_{5k+5,\,5j+2} = - \frac{\gamma}{2\lambda} G_{12}^{kj} + \frac{\gamma}{4} (\omega_\theta - \omega_\xi) G_{13}^{kj}$$

$$s_{5k+5,\,5j+4} = - \frac{k}{2\rho} G_{13}^{kj}$$

$$s_{5k+5,\,5j+5} = - \frac{\gamma}{2} G_{13}^{kj}$$

$$a_{5k+1} = - t_{\xi T}^{k}$$

$$a_{5k+4} = - m_{\xi T}^{k}$$

All other r, s, a are equal to zero.

Index k, j 0, 1----K

$$\alpha_{5k+1,\,5j+1} = m_1^{kj}$$

$$\alpha_{5k+2,\,5j+2} = m_2^{kj}$$

$$\alpha_{5k+3,\,5j+3} = m_1^{kj}$$

$$\alpha_{5k+4,\,5j+4} = m_4^{kj}$$

$$\alpha_{5k+5,\,5j+5} = m_5^{kj}$$

$$\beta_{5k+1,\,5j+1} = C_1^{kj}$$

$$\beta_{5k+2,\,5j+2} = C_2^{kj}$$

$$\beta_{5k+3,\,5j+3} = C_1^{kj}$$

$$\beta_{5k+4,\ 5j+4} = C_4^{kj}$$

$$\beta_{5k+5,\ 5j+5} = C_5^{kj}$$

$$k_{5k+1,\ 5j+1} = k_1^{kj}$$

$$k_{5k+2,\ 5j+2} = k_2^{kj}$$

$$k_{5k+3,\ 5j+3} = k_1^{kj}$$

$$k_{5k+4,\ 5j+4} = k_4^{kj}$$

$$k_{5k+5,\ 5j\ \ 5} = k_5^{kj}$$

All other α's, β's, and k's are equal to zero.