SID 67-498

STUDY OF APOLLO WATER IMPACT

FINAL REPORT

VOLUME 9

MODE SHAPES AND NATURAL FREQUENCIES ANALYSIS

(Contract NAS9-4552, G.O. 5264)

May 1967

Prepared by

A. P. Cappelli
(Author)

Approved by

F. C. Hung
Program Manager
Structures and Materials

L. A. Harris
Assistant Manager
Science and Technology

NORTH AMERICAN AVIATION, INC.
SPACE DIVISION

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FOREWORD

This report was prepared by North American Aviation, Inc., Space Division, under NASA Contract NAS9-4552, for the National Aeronautics and Space Administration, Manned Space Flight Center, Houston, Texas, with Dr. F.C. Hung, Program Manager and Mr. P.P. Radkowski, Assistant Program Manager. This work was administered under the direction of Structural Mechanics Division, MSC, Houston, Texas with Dr. F. Stebbins as the technical monitor.

This report is presented in eleven volumes for convenience in handling and distribution. All volumes are unclassified.

The objective of the study was to develop methods and Fortran IV computer programs to determine by the techniques described below, the hydro-elastic response of representation of the structure of the Apollo Command Module immediately following impact on the water. The development of theory, methods and computer programs is presented as Task I Hydrodynamic Pressures, Task II Structural Response and Task III Hydroelastic Response Analysis.

Under Task I - Computing program to extend flexible sphere using the Spencer and Shiffman approach has been developed. Analytical formulation by Dr. Li using nonlinear hydrodynamic theory on structural portion is formulated. In order to cover a wide range of impact conditions, future extensions are necessary in the following items:

a. Using linear hydrodynamic theory to include horizontal velocity and rotation.

b. Nonlinear hydrodynamic theory to develop computing program on spherical portion and to develop nonlinear theory on toroidal and conic sections.

Under Task II - Computing program and User's Manual were developed for nonsymmetrical loading on unsymmetrical elastic shells. To fully develop the theory and methods to cover realistic Apollo configuration the following extensions are recommended:

a. Modes of vibration and modal analysis.

b. Extension to nonsymmetric short time impulses.

c. Linear buckling and elasto-plastic analysis

These technical extensions will not only be useful for Apollo and future Apollo growth configurations, but they will also be of value to other aeronautical and spacecraft programs.

The hydroelastic response of the flexible shell is obtained by the numerical solution of the combined hydrodynamic and shell equations. The results obtained herein are compared numerically with those derived by neglecting the interaction and applying rigid body pressures to the same elastic shell. The numerical results show that for an axially symmetric impact of the particular shell studied, the interaction between the shell and the fluid produces appreciable differences in the overall acceleration of the center of gravity of the shell, and in the distribution of the pressures and responses. However the maximum responses are within 15% of those produced when the interaction between the fluid and the shell is neglected. A brief summary of results is shown in the abstracts of individual volumes.

The volume number and authors are listed on the following page.

The contractor's designation for this report is SID 67-498.

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3	Dynamic Response of Shells of Revolution During Vertical Impact Into Water - Hydroelastic Interaction	J.P.D. Wilkinson, A.P. Cappelli, and R.N. Salzman
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ABSTRACT

A general numerical procedure is presented for the analysis of the free vibrations of shells of revolution. The analysis permits the determination of natural frequencies and mode shapes for rotationally symmetric shells having meridional variations in geometrical and mechanical properties. The analysis is based on the linear shell theory of Sanders extended to the dynamical form. A Fourier analysis is used to separate variables and uncouple the governing equations in the circumferential variable. These equations together with the boundary conditions constitute the eigenvalue problem. The natural frequencies of the shell are the roots of a determinant where elements are related to certain solutions of the homogeneous field equations. An extension of the Gaussian elimination technique is employed to obtain the characteristic frequency equation in matrix form. The determinant of this matrix is required to vanish.

The extension of the problem of shells of revolution with arbitrary distribution of mass and stiffness properties is included. The Fourier analysis of the governing equations for the separation of the circumferential variable yields equations coupled in the coefficients rather than the uncoupled form for shells of revolution. The numerical procedure for these equations in matrix form are handled in the manner outlined for the shells of revolution.

The numerical procedure is general and may be used in the solution of many types of eigenvalue problems.

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CONTENTS

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ILLUSTRATIONS

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SYMBOLS

a	= Reference length
$a_1, a_2, a_3 \ldots a_{36}$	= Coefficients defined in Appendix A. Ref. 1
b	= Nondimensional membrane stiffness
c	= Speed of sound
d	= Nondimensional bending stiffness
$e_\xi, e_\theta, e_{\xi\theta}$	= Fourier coefficients for membrane strains [Eqs. (1.17)]
$\hat{f}_\xi$	= Fourier coefficient for effective transverse shear
h	= Shell thickness
h_o	= Reference thickness
$k_\xi, k_\theta, k_{\xi\theta}$	= Fourier coefficients for bending distortion [Eqs. (1.17)]
m_o	= Mass density of the shell
$m_\xi, m_\theta, m_{\xi\theta}$	= Fourier coefficients for bending moments [Eqs. (1.19)]
r	= Normal distance from axis to shell (Figure 1.1)
s	= Meridional shell coordinate
t	= Time coordinate
$t_\xi, t_\theta, t_{\xi\theta}$	= Fourier coefficients for membrane forces [Eqs. (1.18)]

$\hat{t}_{\xi\theta}$	=	Fourier coefficient for effective membrane shear
u_ξ, u_θ	=	Fourier coefficients for meridional and circumferential displacements [Eqs. (1.14)]
w	=	Fourier coefficient for normal displacement [Eqs. (1.14)]
E	=	Young's modulus
E_o	=	Reference Young's modulus
M_ξ, M_θ, $M_{\xi\theta}$	=	Bending moments per unit length (Figure 1.2)
$\overline{M}_{\xi\theta}$	=	Modified twisting moment
N_ξ, N_θ, $N_{\xi\theta}$	=	Membrane forces per unit length (Figure 1.2)
$\overline{N}_{\xi\theta}$	=	Modified membrane shear
$\hat{N}_{\xi\theta}$	=	Effective (boundary) membrane shear [Eq. (1.23)]
Q_ξ, Q_θ	=	Transverse forces per unit length (Figure 1.2)
$\hat{Q}_\xi$	=	Effective (boundary) transverse shear
R_s, R_θ	=	Radii of curvature (Figure 1.1)
U_ξ, U_θ	=	Meridional and circumferential displacements (Figure 1.3)
W	=	Normal displacement (Figure 1.3)
γ	=	ρ'/ρ
ϵ_ξ, ϵ_θ, $\epsilon_{\xi\theta}$	=	Membrane strains [Eq. (1.7)]
ζ	=	Normal outward distance from reference surface

θ	=	Circumferential angle (Figure 1.1)
k_ξ, k_θ, $k_{\xi\theta}$	=	Bending distortions [Eq. (1.8)]
λ	=	h_o/a
ν	=	Poisson's ratio
ξ	=	Nondimensional meridional coordinate (s/a)
ρ	=	r/a
σ_o	=	Reference stress level
σ_ξ, σ_θ, $\sigma_{\xi\theta}$	=	Meridional, circumferential, and shear stresses
ϕ_ξ, ϕ_θ	=	Fourier coefficient for rotation [Eqs. (1.16)]
ω	=	Circular frequency
ω_θ ω_ξ	=	Nondimensional curvatures [Eqs. (1.3)]
Δ	=	Interval size (in units of ξ) between stations
Φ_ξ, Φ_θ	=	Rotations (Figure 1.3)

MATRICES

A, B, C, E, F, G, H, I_o, J,	=	4 x 4 matrices
Λ, Ω	=	4 x 4 diagonal matrices
z, y	=	1 x 4 column matrices

INDICES

i	=	Station
n	=	Fourier component

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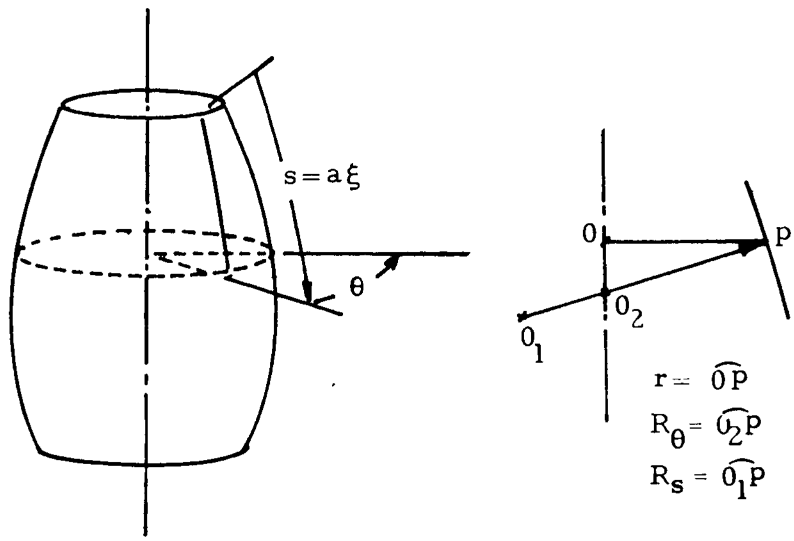

Figure 1.1 Surface Geometry and Coordinates

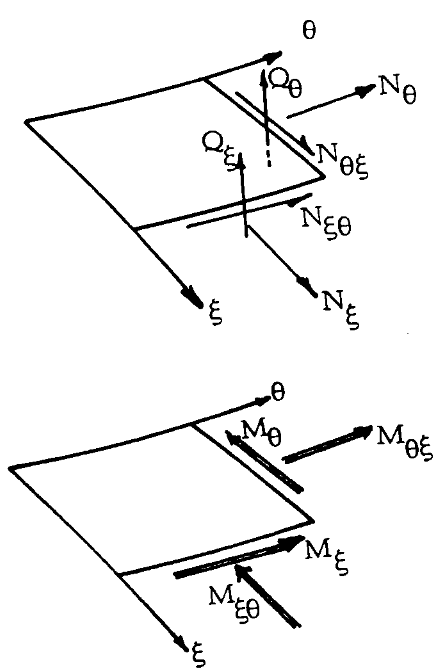

Figure 1.2 Forces, Moments and Loads

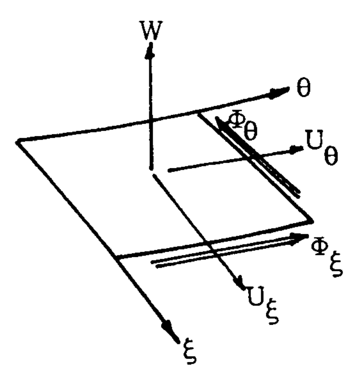

Figure 1.3 Displacements and Rotations

where

$$\gamma = \rho'/\rho$$

In these equations, and henceforth, $(\ )' \equiv (d/d\xi)(\ )$. Finally, note the Codazzi identity

$$\omega_\theta' = \gamma\,(\omega_\xi - \omega_\theta) \qquad 1.4$$

and the relation

$$\rho''/\rho = -\,\omega_\xi \omega_\theta \qquad 1.5$$

1.2.2 Equations of Motion

The components of membrane force per unit length, transverse force per unit length, moment (about the reference surface) per unit length, and load per unit area (assumed to be applied at the reference surface) are as shown in Figure 1.2. In the Sanders theory, the shearing forces $N_{\xi\theta}$ and $N_{\theta\xi}$ as well as the twisting moments $M_{\xi\theta}$ and $M_{\theta\xi}$ are not handled separately but are combined to provide the modified variables $\overline{N}_{\xi\theta}$ and $\overline{M}_{\xi\theta}$ as described in Reference 2.

With the elimination of the transverse forces Q_ξ and Q_θ, the static equilibrium equations of Sanders theory is given by Eqs. 10-a, b, c of Reference 1. Extending these equations to the dynamical form using D'Alembert's principle results in the equations of motion for the shell in the form

$$\begin{aligned} & a\left[\frac{\partial}{\partial\xi}(\rho N_\xi) + \frac{\partial}{\partial\theta}(\overline{N}_{\xi\theta}) - \rho' N_\theta\right] + \\ & \omega_\xi\left[\frac{\partial}{\partial\xi}(\rho M_\xi) + \frac{\partial}{\partial\theta}(\overline{M}_{\xi\theta}) - \rho' M_\theta\right] + \\ & \frac{1}{2}(\omega_\xi - \omega_\theta)\frac{\partial}{\partial\theta}(\overline{M}_{\xi\theta}) - a^2\rho m_o h\,\frac{\partial^2 U_\xi}{\partial t^2} = 0 \end{aligned} \qquad 1.5.1$$

$$\begin{aligned} & a\left[\frac{\partial}{\partial\theta}(N_\theta) + \frac{\partial}{\partial\xi}(\rho\overline{N}_{\xi\theta}) + \rho'\overline{N}_{\xi\theta}\right] + \\ & \omega_\theta\left[\frac{\partial}{\partial\theta}(M_\theta) + \frac{\partial}{\partial\xi}(\rho\overline{M}_{\xi\theta}) + \rho'\overline{M}_{\xi\theta}\right] + \\ & \rho/2\,\frac{\partial}{\partial\xi}\left[(\omega_\theta - \omega_\xi)\,\overline{M}_{\xi\theta}\right] - a^2\rho m_o h\,\frac{\partial^2 U_\theta}{\partial t^2} = 0 \end{aligned} \qquad 1.5.2$$

$$\frac{\partial}{\partial \xi}\left[\frac{\partial}{\partial \xi}(\rho M_\xi) + \frac{\partial}{\partial \theta}(\overline{M}_{\xi\theta}) - \rho' M_\theta\right] +$$

$$\frac{1}{\rho}\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial \theta}(M_\theta) + \frac{\partial}{\partial \xi}(\rho \overline{M}_{\xi\theta}) + \rho' \overline{M}_{\xi\theta}\right] - \quad 1.5.3$$

$$a\,\rho(\omega_\xi N_\xi + \omega_\theta N_\theta) - a^2 \rho m_o h \frac{\partial^2 W}{\partial t^2} = 0$$

where m_o is the mass density of the shell, h is the shell thickness, and t is the time coordinate.

1.2.3 Displacements, Rotations, and Strains

The displacements and rotations of the reference surface (Figure 1.3) are related by the equations

$$\Phi_\xi = \frac{1}{a}\left[-\frac{\partial W}{\partial \xi} + \omega_\xi U_\xi\right]$$

$$\Phi_\theta = \frac{1}{a}\left[-\frac{1}{\rho}\frac{\partial W}{\partial \theta} + \omega_\theta U_\theta\right] \quad 1.6$$

The membrane strains of the reference surface are given by

$$\epsilon_\xi = \frac{1}{a}\left[\frac{\partial U_\xi}{\partial \xi} + \omega_\xi W\right]$$

$$\epsilon_\theta = \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + \gamma U_\xi + \omega_\theta W\right] \quad 1.7$$

$$\epsilon_{\xi\theta} = \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial U_\xi}{\partial \theta} + \frac{\partial U_\theta}{\partial \xi} - \gamma U_\theta\right]$$

where $\epsilon_{\xi\theta}$ is half the usual engineering shear strain.

Finally, the measures of bending distortion used in the Sanders theory are

$$K_\xi = \frac{1}{a}\frac{\partial \Phi_\xi}{\partial \xi}$$

$$K_\theta = \frac{1}{a}\left[\frac{1}{\rho}\frac{\partial \Phi_\theta}{\partial \theta} + \gamma \Phi_\xi\right] \quad 1.8$$

$$K_{\xi\theta} = \frac{1}{2a}\left[\frac{1}{\rho}\frac{\partial \Phi_\xi}{\partial \theta} + \frac{\partial \Phi_\theta}{\partial \xi} - \gamma \Phi_\theta + \frac{1}{2a}(\omega_\xi - \omega_\theta)\left(\frac{1}{\rho}\frac{\partial U_\xi}{\partial \theta} - \frac{\partial U_\theta}{\partial \xi} - \gamma U_\theta\right)\right]$$

Then, by the usual Kirchhoff hypothesis ("normals remain normal") and the neglect of terms of order ζ/R_s and ζ/R_θ relative to unity, the longitudinal, circumferential, and shear strains at a distance ζ from the reference surface are

$$\begin{aligned} &\epsilon_\xi + \zeta K_\xi \\ &\epsilon_\theta + \zeta K_\theta \\ &\epsilon_{\xi\theta} + \zeta K_{\xi\theta} \end{aligned} \qquad 1.9$$

respectively.

1.2.4 Constitutive Relations

Neglecting, as usual, the effects of stresses normal to the shell permits the stress-strain relations to be written as

$$\begin{aligned} \epsilon_\xi + \zeta K_\xi &= \left[(\sigma_\xi - \nu\sigma_\theta)/E\right] \\ \epsilon_\theta + \zeta K_\theta &= \left[(\sigma_\theta - \nu\sigma_\xi)/E\right] \\ \epsilon_{\xi\theta} + \zeta K_{\xi\theta} &= \left[(1+\nu)/E\right]\ \sigma_{\xi\theta} \end{aligned} \qquad 1.10$$

The Young's modulus E will be permitted to vary only with ξ and ζ. The (modified) forces and moments are approximated closely in the shell by the following integrals through the thickness:

$$\begin{aligned} N_\xi &= \int \sigma_\xi d\zeta & M_\xi &= \int \zeta\sigma_\xi d\zeta \\ N_\theta &= \int \sigma_\theta d\zeta & M_\theta &= \int \zeta\sigma_\theta d\zeta \\ \overline{N}_{\xi\theta} &= \int \sigma_{\xi\theta} d\zeta & \overline{M}_{\xi\theta} &= \int \zeta\sigma_{\xi\theta} d\zeta \end{aligned} \qquad 1.11$$

Then, with the use of the defining relation (1.1) for the reference surface, together with the assumption of constant Poisson's ratio, it is found from (1.9-1.11) that

$$\begin{aligned} \epsilon_\xi &= \frac{N_\xi - \nu N_\theta}{\int E d\zeta} \\ \epsilon_\theta &= \frac{N_\theta - \nu N_\xi}{\int E d\zeta} \\ \epsilon_{\xi\theta} &= \frac{(1+\nu)\,N_{\xi\theta}}{\int E d\zeta} \end{aligned} \qquad 1.12$$

and

$$K_{\xi} = \frac{M_{\xi} - \nu M_{\theta}}{\int \zeta^2 E d\zeta}$$

$$K_{\theta} = \frac{M_{\theta} - \nu M_{\xi}}{\int \zeta^2 E d\zeta} \qquad 1.13$$

$$K_{\xi\theta} = \frac{(1 + \nu)\, \overline{M}_{\xi\theta}}{\int \zeta^2 E d\zeta}$$

The complete set of equations for the 17 independent variables N_{ξ}, N_{θ}, $\overline{N}_{\xi\theta}$, M_{ξ}, M_{θ}, $\overline{M}_{\xi\theta}$, U_{ξ}, U_{θ}, W, Φ_{ξ}, Φ_{θ}, ϵ_{ξ}, ϵ_{θ}, $\epsilon_{\xi\theta}$, K_{ξ}, K_{θ}, $K_{\xi\theta}$ now is given by the 17 equations in terms of shell coordinate ξ, θ (1.5 - 1.8, 1.12, 1.13) and time variable t.

1.2.5 Assumed Solution

The variables of the problem, with appropriate normalization to provide nondimensional forms, are assumed separable in the form

$$\begin{bmatrix} N_{\xi} \\ N_{\theta} \\ M_{\xi} \\ M_{\theta} \\ U_{\xi} \\ W \\ \Phi_{\xi} \\ \epsilon_{\xi} \\ \epsilon_{\theta} \\ K_{\xi} \\ K_{\theta} \end{bmatrix} = \begin{bmatrix} h_o t_{\xi}^{(n)} \\ h_o t_{\theta}^{(n)} \\ h_o^3 m_{\xi}^{(n)}/a \\ h_o^3 m_{\theta}^{(n)}/a \\ a u_{\xi}^{(n)}/E_o \\ a w^{(n)}/E_o \\ \phi_{\xi}^{(n)}/E_o \\ e_{\xi}^{(n)}/E_o \\ e_{\theta}^{(n)}/E_o \\ k_{\xi}^{(n)}/aE_o \\ k_{\theta}^{(n)}/aE_o \end{bmatrix} \sigma_o \cos n\theta \cos \omega t \qquad 1.14$$

$$
\begin{bmatrix} \bar{N}_{\xi\theta} \\ \bar{M}_{\xi\theta} \\ U_\theta \\ \Phi_\theta \\ \epsilon_{\xi\theta} \\ K_{\xi\theta} \end{bmatrix} = \begin{bmatrix} h_o t_{\xi\theta}^{(n)} \\ h_o^3 m_{\xi\theta}^{(n)}/a \\ a u_\theta^{(n)}/E_o \\ \phi_\theta^{(n)}/E_o \\ e_{\xi\theta}^{(n)}/E_o \\ k_{\xi\theta}^{(n)}/aE_o \end{bmatrix} \sigma_o \sin n\theta \cos \omega t
$$

where σ_o is a reference stress level, E_o a reference Young's modulus, and h_o a reference thickness. The circular frequency (rad/sec) is described by ω and n is the integer denoting Fourier component.

Substitution of the above in the various field equations, permits decoupling into separate sets for each Fourier index n. (For convenience, the superscript (n) in Fourier components has been omitted in the equations that follow.) The equations of motion (1.5) lead to

$$
t'_\xi + \gamma (t_\xi - t_\theta) + (n/\rho)\, t_{\xi\theta} + \lambda^2 \Big\{ \omega_\xi m'_\xi + \gamma \omega_\xi (m_\xi - m_\theta) + (n/2\rho)(3\omega_\xi - \omega_\theta) \cdot m_{\xi\theta} \Big\} + \Omega^2 u_\xi = 0 \qquad 1.15
$$

$$
t'_{\xi\theta} + 2\gamma t_{\xi\theta} - (n/\rho)\, t_\theta + \lambda^2 \Big\{ -(n/\rho)\, \omega_\theta m_\theta + \tfrac{1}{2}(3\omega_\theta - \omega_\xi)\, m'_{\xi\theta} + \tfrac{1}{2}\big[\gamma (3\omega_\theta + \omega_\xi) - \omega'_\xi\big] m_{\xi\theta} \Big\} + \Omega^2 u_\theta = 0
$$

$$
-\omega_\xi t_\xi - \omega_\theta t_\theta + \lambda^2 \Big\{ m''_\xi + 2\gamma m'_\xi - \omega_\xi \omega_\theta m_\xi + \big[\omega_\xi \omega_\theta - (n^2/\rho^2)\big] m_\theta - \gamma m'_\theta + (2n/\rho)\, m'_{\xi\theta} + (2\gamma n/\rho)\, m_{\xi\theta} \Big\} + \Omega^2 w = 0
$$

where $\lambda = \frac{h_o}{a}$ and use has been made of the geometrical identities (1.4) and (1.5). Ω is a nondimensional frequency parameter defined by $\Omega^2 = \omega^2 a^2 \bar{h}/c^2$ where $c^2 = E_o/m_o$ and $\bar{h} = h/h_o$. (Note: when E_o = elastic modulus of the shell, c represents the speed of sound.) The relations (1.6 - 1.8) give

$$
\phi_\xi = -w' + \omega_\xi u_\xi \qquad 1.16
$$

$$
\phi_\theta = (n/\rho)\, w + \omega_\theta u_\theta
$$

$$e_\xi = u'_\xi + \omega_\xi w$$

$$e_\theta = (n/\rho)\, u_\theta + \gamma u_\xi + \omega_\theta w$$

$$e_{\xi\theta} = \frac{1}{2}\left[u'_\theta - \gamma u_\theta - (n/\rho)\, u_\xi\right] \qquad 1.17$$

$$k_\xi = \phi'_\xi \qquad\qquad k_\theta = (n/\rho)\,\phi_\theta + \gamma\phi_\xi$$

$$k_{\xi\theta} = \frac{1}{2}\left\{-(n/\rho)\,\phi_\xi + \phi'_\theta - \gamma\phi_\theta + \frac{1}{2}(\omega_\theta - \omega_\xi)\left[(nu_\xi/\rho) + u'_\theta + \gamma u_\theta\right]\right\}$$

and, finally, the constitutive relations (1.12) and (1.13), inverted to give forces and moments in terms of strains and bending distortions, lead to

$$t_\xi = b\,(e_\xi + \nu e_\theta);\; t_\theta = b\,(e_\theta + \nu e_\xi)$$

$$t_{\xi\theta} = b\,(1-\nu)\, e_{\xi\theta} \qquad 1.18$$

and

$$m_\xi = d\,(k_\xi + \nu k_\theta)$$

$$m_\theta = d\,(k_\theta + \nu k_\xi) \qquad 1.19$$

$$m_{\xi\theta} = d\,(1-\nu)\, k_{\xi\theta}$$

where

$$b = \frac{\int E\, d\zeta}{E_o h_o\,(1-\nu^2)}$$

$$d = \frac{\int \zeta^2 E\, d\zeta}{E_o h_o^3\,(1-\nu^2)}$$

For each n, the set of equations for the 17 Fourier components t_ξ, $t_{\xi\theta}$, t_θ, m_ξ, m_θ, $m_{\xi\theta}$, u_ξ, u_θ, w, ϕ_ξ, ϕ_θ, e_ξ, e_θ, $e_{\xi\theta}$, k_ξ, k_θ, $k_{\xi\theta}$ of the separated form of the assumed solutions are now given by the 17 equations (1.15 - 1.19).

In the general case of nonuniform boundary conditions, the frequency of modal vibrations will depend on all of the Fourier harmonics, n, and the assumed solutions (1.14) must be expressed by Fourier series expansions. In this case the equations cannot be decoupled into separate sets for each Fourier index, n. When the boundary conditions depend on one harmonic or the boundary conditions are homogeneous, the modal frequencies will be a function of a single value of the Fourier index, n, and summation with respect to n can be discarded. This special type of boundary condition is treated in this paper. The index n thus represents a circumferential wave number.

1.2.6 Eigenvalue Problem

The set of homogeneous equations obtained constitutes an eighth-order system that can be reduced, in a conventional fashion, to three equations in u_ξ, u_θ, and w. But a more attractive procedure is to derive four differential equations, each of second order, in the variables u_ξ, u_θ, w and m_ξ. In so doing, it is necessary to eliminate by means of the relation

$$m_\theta = \nu m_\xi + d\,(1 - \nu^2)\,k_\theta \qquad 1.20$$

in order to prevent the ultimate appearance of derivatives of w of order higher than two. Then, substituting (1.20, 1.19, and 1.18) into (1.15) and using (1.16 and 1.17) to eliminate the membrane strain and bending distortion gives three of the desired equations; the fourth equation is given by (1.19), again with k_ξ and k_θ expressed in terms of the displacements. The resultant set then can be written as

$$a_1 u_\xi'' + a_2 u_\xi' + (a_3 + \Omega^2)\,u_\xi + a_4 u_\theta' + a_5 u_\theta + a_6 w' + a_7 w + a_8 m_\xi' + a_9 m_\xi = 0$$

$$a_{10} u_\xi' + a_{11} u_\xi + a_{12} u_\theta'' + a_{13} u_\theta' + (a_{14} + \Omega^2)\,u_\theta + a_{15} w'' + a_{16} w' + a_{17} w + a_{18}\, m_\xi = 0$$

$$a_{19} u_\xi' + a_{20} u_\xi + a_{21} u_\theta'' + a_{22} u_\theta' + a_{23} u_\theta + a_{24} w'' + a_{25} w' + (a_{26} + \Omega^2)\,w + a_{27} m_\xi'' + a_{28} m_\xi' + a_{29} m_\xi = 0$$

$$a_{30} u_\xi' + a_{31} u_\xi + a_{32} u_\theta + a_{33} w'' + a_{34} w' + a_{35} w + a_{36} m_\xi = 0$$

1.21

where the a's are given in Appendix A, Reference 1. These equations can be written in the matrix form

$$E z'' + F z' + (G - I_o \Omega^2)\,z = 0 \qquad 1.22$$

$$E = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_{12} & a_{15} & 0 \\ 0 & a_{21} & a_{24} & a_{27} \\ 0 & 0 & a_{33} & 0 \end{bmatrix} \quad F = \begin{bmatrix} a_2 & a_4 & a_6 & a_8 \\ a_{10} & a_{13} & a_{16} & 0 \\ a_{19} & a_{22} & a_{25} & a_{28} \\ a_{30} & 0 & a_{34} & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} a_3 & a_5 & a_7 & a_9 \\ a_{11} & a_{14} & a_{17} & a_{18} \\ a_{20} & a_{23} & a_{26} & a_{29} \\ a_{31} & a_{32} & a_{35} & a_{36} \end{bmatrix} \quad I_o = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad z = \begin{bmatrix} u_\xi \\ u_\theta \\ w \\ m_\xi \end{bmatrix}$$

1.2.7 Boundary Conditions

In the Sanders theory, the expressions for virtual work per unit length at the boundaries $s = 0, \bar{s}$ are

$$\mp (N_\xi U_\xi + \hat{N}_{\xi\theta} U_\theta + \hat{Q}_\xi W + M_\xi \Phi_\xi) \qquad 1.23$$

where

$$\hat{N}_{\xi\theta} = \overline{N}_{\xi\theta} + \left[(3/2R_\theta) - (1/2R_\xi) \right] \overline{M}_{\xi\theta}$$

and

$$\hat{Q}_\xi = (1/a\rho) \left[(\partial/\partial\xi)(\rho M_\xi) + 2(\partial\overline{M}_{\xi\theta}/\partial\theta) - \rho' M_\theta \right]$$

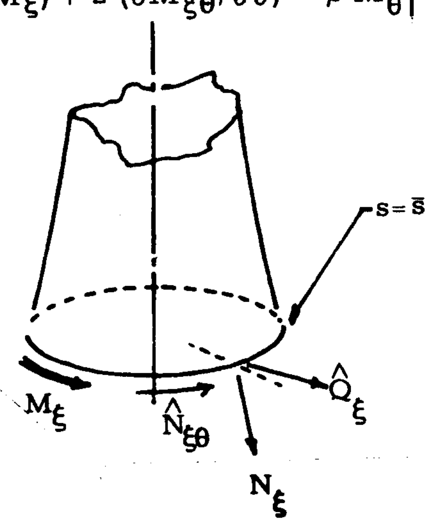

Figure 1.4

Effective Boundary Forces and Moments

are "effective" membrane and transverse shears, respectively, per unit length (see Figure 1.4). This form of the virtual work indicates the kinds of boundary conditions that can be imposed; thus, either N_ξ or U_ξ may be prescribed, either $\hat{N}_{\xi\theta}$ or U_θ may be prescribed, and so on.

Letting

$$\hat{N}_{\xi\theta} = \sigma_o h_o \hat{t}^{(n)}_{\xi\theta} \sin n\theta \cos \omega t$$

$$\hat{Q}_\xi = \sigma_o h_o \hat{f}^{(n)}_\xi \cos n\theta \cos \omega t$$

gives (dropping superscripts)

$$\hat{t}_{\xi\theta} = t_{\xi\theta} + (\lambda^2/2)(3\omega_\theta - \omega_\xi)\, m_{\xi\theta}$$
$$\hat{f}_\xi = \lambda^2 \left[m'_\xi + \gamma (m_\xi - m_\theta) + (2n/\rho)\, m_{\xi\theta} \right] \qquad 1.24$$

Then the boundary conditions just discussed always can be written (for the nth Fourier components) as

$$\Omega y + \Lambda z = 0 \qquad 1.25$$

where

$$y = \begin{bmatrix} t_\xi \\ \hat{t}_{\xi\theta} \\ \hat{f}_\xi \\ \phi_\xi \end{bmatrix}$$

and where Ω and Λ are appropriate diagonal matrices. (For example, if u_ξ is given, the first diagonal element of Ω is zero, that of Λ is unity.) But now it is desirable to express the boundary conditions entirely in terms of z; from Eqs. (1.16-1.20), it follows that

$$t_\xi = b_1 u'_\xi + b_2 u_\xi + b_3 u_\theta + b_4 w$$
$$\hat{t}_{\xi\theta} = b_5 u_\xi + b_6 u'_\theta + b_7 u_\theta + b_8 w' + b_9 w \qquad 1.26$$
$$\hat{f} = b_{10} u_\xi + b_{11} u'_\theta + b_{12} u_\theta + b_{13} w' + b_{14} w + b_{15} m'_\xi + b_{16} m_\xi$$

where the b's are given in Appendix A, Reference 1. These equations, together with (1.16), then give

$$y = Hz' + Jz \qquad 1.27$$

where

$$H = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_6 & b_8 & 0 \\ 0 & b_{11} & b_{13} & b_{15} \\ 0 & 0 & -1 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} b_2 & b_3 & b_4 & 0 \\ b_5 & b_7 & b_9 & 0 \\ b_{10} & b_{12} & b_{14} & b_{16} \\ \omega_\xi & 0 & 0 & 0 \end{bmatrix}$$

Hence, the boundary conditions (1.25) can be written as

$$\Omega H z' + (\Lambda + \Omega J)\, z = 0 \qquad 1.28$$

1.2.8 Discontinuity Conditions

The differential Eqs. (1.22) are not valid at points in the shell where discontinuities in geometry (and hence in the coefficients) occur; furthermore, z itself is ambiguous at a discontinuity in the inclination of the reference surface, where the directions of u_ξ and w change abruptly. Accordingly, special transition equations must be derived which relate z and its derivative on either side of a discontinuity.

These conditions can be described in similar manner as discussed in Reference 1. The inclusion of the discontinuity conditions in this analysis can be performed in a straightforward manner following the procedure described in Reference 1. For ease of presentation, however, the discontinuity condition will not be included in the subsequent development.

1.3 NUMERICAL ANALYSIS

1.3.1 Finite Difference Formulation

The differential equations (1.22) will be written in finite difference form at all stations except i = 0, and N on the basis of the usual central difference formulas:

$$z_i'' = (z_{i+1} - 2z_i + z_{i-1})/\Delta^2 \qquad 1.29$$

$$z_i' = (z_{i+1} - z_{i-1})/2\Delta \qquad 1.30$$

where the Δ must, of course, be the one corresponding to the region associated with the station i.

The boundary conditions (1.28) will be written at i = 0 and i = N with the help of

$$z_o' = (z_1 - z_o)/\Delta_1 \qquad 1.31$$

$$z_N' = (z_N - z_{N-1})/\Delta_N \qquad 1.32$$

Then, the results of writing the various difference equations just described can be stated compactly as the following set of algebraic equations for z_i (i=0, 1, 2, . . . N):

$$\begin{aligned} A_o z_1 + B_o z_o &= 0 \\ A_i z_{i+1} + B_i^* z_i + C_i z_{i-1} &= 0 \; (i=1,\ 2,\ \ldots\ N-1) \\ B_N z_N + C_N z_{N-1} &= 0 \end{aligned} \qquad 1.33$$

where

$$A_o = \Omega_o H_o/\Delta_1$$

$$B_o = \Lambda + \Omega_o \left[J_o - (H_o/\Delta_1) \right]$$

where the subscript zero refers, of course, to the conditions at s = 0. For i ± 0,

$$\begin{aligned} B_i^* &= B_i - 2\Delta I_o \Omega^2 \\ A_i &= (2E_i/\Delta) + F_i \\ B_i &= -(4E_i/\Delta) + 2\Delta G_i \\ C_i &= (2E_i/\Delta) - F_i \end{aligned} \qquad 1.34$$

where the appropriate value for Δ is used. The * notation indicates matrices affected by dynamic terms in the shell equation. Finally,

$$B_N = \Lambda_N + \Omega_N \left[J_N + (H_N/\Delta_p) \right]$$
$$C_N = - (\Omega_N H_N/\Delta_p) \qquad 1.35$$

where N refers to the conditions at $s = \overline{s}$.

1.3.2 Matrix Solution of Characteristic Frequency Equation

A special extension of Gaussian elimination technique presented in References 1 and 2 will be used to obtain the characteristic frequency equation in matrix form. The elimination technique proceeds by simultaneous solution for z_0, z_1 in terms of z_2 using the first two of Eq. (1.33).

Substitution of this result into the next equation (i = 2) solves for z_2 in terms of z_3. Repeating this procedure results in the general result for z_i in terms of z_{i+1} as

$$z_i = - P_i z_{i+1} \qquad 1.36$$

where

$$P_i = \left[B_i^* - C_i P_{i-1} \right]^{-1} A_i \qquad (i=2, 3, \ldots N-1)$$
$$P_1 = \left[B_o C_1^{-1} B_1^* - A_o \right]^{-1} B_o C_1^{-1} A_1 \qquad i=1$$

At station N - 1, z_{N-1} is defined in terms of z_N. Substitution of this relationship into the boundary condition at i = N eliminates z_N and results in the homogeneous solution of the form

$$\left[S \right] \left[z_{N-1} \right] = 0 \qquad 1.37$$

where

$$\left[S \right] = \left[C_N - B_N P_{N-1}^{-1} \right]$$

(z_{N-1} could be eliminated also; however, z_N is a singular matrix for hinge support boundaries.) In order for a nontrivial solution to exist, $z_{N-1} \neq 0$ and thus the determinant of the (4 x 4) matrix S must vanish, i.e.,

$$|S| = 0 \qquad 1.38$$

$|S|$ is described as the characteristic determinant for natural frequencies of the shell. The natural frequencies are the roots of the determinantal equation above.

In its most simplified form, the numerical procedure requires the selection of a trial frequency Ω and Fourier index n to start the process. The B_i^* matrix in Eq. (1.22) can now be determined and, using the outlined elimination procedure, the characteristic determinant can be evaluated. Repeated trials for incremental values of Ω_i can be made until the determinant value changes sign. Using an inverse interpolation procedure, the particular natural frequencies Ω_i are obtained at which $|S| = o$. These frequencies correspond to the particular Fourier index n selected.

Once the characteristic values are obtained, the corresponding solution for $z_{N-1}^{(j)}$ is obtained from

$$z_{N-1}^{(j)} = d\,(-1)^{i+1}\,|M_i| \tag{1.39}$$

where $z_{N-1}^{(j)}$ denotes the j-th element of z_{N-1}, d is an arbitrary constant, and $|M_i|$ is the determinant obtained from any submatrix of rank 3 contained in $|S|$ by deleting the j-th column. After $z_{N-1}^{(j)}$ is calculated from the above, the corresponding mode shapes may be found for a particular natural frequency using the modified form of the recursion relationships Eq. (1.36) previously developed.

2.0 FREE VIBRATION OF SHELLS OF REVOLUTION HAVING ARBITRARY STIFFNESS DISTRIBUTIONS

2.1 INTRODUCTION

An extension of the numerical procedure given in the previous section is presented. The procedure is modified to treat the problem of the free vibrations of shells of revolution having arbitrary stiffness distributions. The formulation presented here is based on the Fourier approach developed in Reference 6 for the static analysis of this class of shells and the numerical procedure developed in Section 1 for studying the free vibration of shells of revolution.

2.2 ANALYTICAL FORMULATION

2.2.1 Equations of Motion

The general equilibrium equations of Sanders' theory including the effects of transverse shear distortion are given by Equations 8a-e in Reference 6. Extending these equations to the dynamical form using D'Alembert's principle results in the following equations of motion for the shell.

$$a\left[\frac{\partial \rho}{\partial \xi}N_\xi + \rho\frac{\partial N_\xi}{\partial \xi} + \frac{\partial \overline{N}_{\xi\theta}}{\partial \theta} - \frac{\partial \rho}{\partial \xi}N_\theta\right] + a\rho\,\omega_\xi\,Q_\xi + \frac{1}{2}\left(\omega_\xi - \omega_\theta\right)\frac{\partial \overline{M}_{\xi\theta}}{\partial \theta} + a^2\rho\mu\frac{\partial^2 U_\xi}{\partial t^2} = 0 \tag{2.1}$$

$$a\left[\frac{\partial N_\theta}{\partial \theta} + 2\frac{\partial \rho}{\partial \xi}\overline{N}_{\xi\theta} + \rho\frac{\partial \overline{N}_{\xi\theta}}{\partial \xi}\right] + a\rho\,\omega_\theta Q_\theta + \frac{\rho}{2}\frac{\partial}{\partial \xi}\left[\omega_\theta - \omega_\xi\,\overline{M}_{\xi\theta}\right] + a^2\rho\mu\frac{\partial^2 U_\theta}{\partial t^2} = 0$$

$$a\left[\frac{\partial \rho}{\partial \xi}Q_\xi + \rho\frac{\partial Q_\xi}{\partial \xi} + \frac{\partial Q_\theta}{\partial \theta} - \rho\left(\omega_\xi N_\xi + \omega_\theta N_\theta\right)\right] + a^2\rho\mu\frac{\partial^2 W}{\partial t^2} = 0$$

$$\frac{\partial \rho}{\partial \xi} M_\xi + \rho \frac{\partial M}{\partial \xi}\xi + \frac{\partial \overline{M}}{\partial \theta}\xi\theta - \frac{\partial \rho}{\partial \xi} M_\theta - a\rho \left[Q_\xi + \mu^* \frac{\partial^2 \Phi}{\partial t^2}\xi \right] = 0$$

$$\frac{\partial M}{\partial \theta}\theta + 2\frac{\partial \rho}{\partial \xi}\overline{M}_{\xi\theta} + \rho \frac{\partial \overline{M}}{\partial \xi}\xi\theta - a\rho \left[Q_\theta + \mu^* \frac{\partial^2 \Phi}{\partial t^2}\theta \right] = 0$$

where

$\mu(\xi, \theta)$ = mass per unit area of the shell

$\mu^*(\xi, \theta)$ = mass moment per unit area of the shell

t = time coordinate

and the components of membrane force $N_{()}$ transverse force $Q_{()}$ and moment $M_{()}$ are as shown in Figure 2 of Reference 1. The stress resultant relationships in terms of the displacements U_ξ, U_θ, W and rotations Φ_ξ, Φ_θ were developed in Reference 1 and are given with thermal effects neglected by

$$N_\xi = \frac{1}{a}\left\{ B_1 \frac{\partial U_\xi}{\partial \xi} + B_3 \gamma U_\xi + B_3 \frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + \left(B_1\ \omega_\xi + B_3\ \omega_\theta \right) W \right.$$

$$\left. + C_1 \frac{\partial \Phi}{\partial \xi}\xi + C_3 \gamma \Phi_\xi + C_3 \frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}\theta \right\}$$

$$N_\theta = \frac{1}{a}\left\{ B_3 \frac{\partial U_\xi}{\partial \xi} + B_2 \gamma U_\xi + B_2 \frac{1}{\rho}\frac{\partial U_\theta}{\partial \theta} + \left(B_3\ \omega_\xi + B_2\ \omega_\theta \right) W \right. \qquad (2.2)$$

$$\left. + C_3 \frac{\partial \Phi}{\partial \xi}\xi + C_2 \gamma \Phi_\xi + C_2 \frac{1}{\rho}\frac{\partial \Phi}{\partial \theta}\theta \right\}$$

$$Q_\xi = G_2 \left\{ -\frac{1}{a}\ \omega_\xi U_\xi + \frac{1}{a}\frac{\partial W}{\partial \xi} + \Phi_\xi \right\}$$

$$\overline{N}_{\xi\theta} = \frac{1}{2a}\left\{\frac{1}{\rho}\left[G_1 + G_{12}\frac{1}{2a}\left(\omega_\xi - \omega_\theta\right)\right]\frac{\partial U_\xi}{\partial\theta}\right.$$

$$+ \left[G_1 - G_{12}\frac{1}{2a}\left(\omega_\xi - \omega_\theta\right)\right]\frac{\partial U_\theta}{\partial\xi} - \gamma\left[G_1 + G_{12}\frac{1}{2a}\left(\omega_\xi - \omega_\theta\right)\right]U_\theta$$

$$\left. + G_{12}\frac{1}{\rho}\frac{\partial\Phi_\xi}{\partial\theta} + G_{12}\frac{\partial\Phi_\theta}{\partial\xi} - G_{12}\gamma\Phi_\theta\right\}$$

$$M_\xi = \frac{1}{a}\left\{C_1\frac{\partial U_\xi}{\partial\xi} + C_3\gamma U_\xi + C_3\frac{1}{\rho}\frac{\partial U_\theta}{\partial\theta} + \left(C_1\omega_\xi + C_3\omega_\theta\right)W\right.$$

$$\left. + D_1\frac{\partial\Phi_\xi}{\partial\xi} + D_3\gamma\Phi_\xi + D_3\frac{1}{\rho}\frac{\partial\Phi_\theta}{\partial\theta}\right\}$$

$$M_\theta = \frac{1}{a}\left\{C_3\frac{\partial U_\xi}{\partial\xi} + C_2\gamma U_\xi + C_2\frac{1}{\rho}\frac{\partial U_\theta}{\partial\theta} + \left(C_3\omega_\xi + C_2\omega_\theta\right)W + D_3\frac{\partial\Phi_\xi}{\partial\xi}\right.$$

$$\left. + D_2\gamma\Phi_\xi + D_2\frac{1}{\rho}\frac{\partial\Phi_\theta}{\partial\theta}\right\}$$

$$\overline{M}_{\xi\theta} = \frac{1}{2a}\left\{\frac{1}{\rho}\left[G_{12} + G_{13}\left(\frac{1}{2a}\right)\left(\omega_\xi - \omega_\theta\right)\right]\frac{\partial U_\xi}{\partial\theta}\right.$$

$$+ \left[G_{12} - G_{13}\frac{1}{2a}\left(\omega_\xi - \omega_\theta\right)\right]\frac{\partial U_\theta}{\partial\xi} - \gamma\left[G_{12} + G_{13}\frac{1}{2a}\left(\omega_\xi - \omega_\theta\right)\right]U_\theta$$

$$\left. + G_{13}\frac{1}{\rho}\frac{\partial\Phi_\xi}{\partial\theta} + G_{13}\frac{\partial\Phi_\theta}{\partial\xi} - G_{13}\gamma\Phi_\theta\right\}$$

$$Q_\theta = G_3\left\{-\frac{1}{a}\omega_\theta U_\theta + \frac{1}{a\rho}\frac{\partial W}{\partial\theta} + \Phi_\theta\right\}$$

The shell stiffness parameters $B_{()}$, $C_{()}$, $D_{()}$, and $G_{()}$ are described by the integral relationships Equation 16 of Reference 1. Substitution of Equations 2.2 into Equations 2.1 yield the shell equations of motion in terms of the variables U_ξ, U_θ, W, Φ_ξ, and Φ_θ.

2.2.2 Assumed Solution

The solution to the resulting equations of motion are assumed to be in the following Fourier series form:

$$U_\xi = e^{i\alpha t} \frac{a\sigma_o}{E_o} \sum_{n=o}^{\infty} u_n(\xi) \text{ Cos } n\theta$$

$$U_\theta = e^{i\alpha t} \frac{a\sigma_o}{E_o} \sum_{n=1}^{\infty} v_n(\xi) \text{ Sin } n\theta$$

$$W = e^{i\alpha t} \frac{a\sigma_o}{E_o} \sum_{n=0}^{\infty} w_n(\xi) \text{ Cos } n\theta \qquad (2.3)$$

$$\Phi_\xi = e^{i\alpha t} \frac{\sigma_o}{E_o} \sum_{n=0}^{\infty} u\phi_{\xi n}(\xi) \text{ Cos } n\theta$$

$$\Phi_\theta = e^{i\alpha t} \frac{\sigma_o}{E_o} \sum_{n=1}^{\infty} u\phi_{\theta n}(\xi) \text{ Sin } n\theta$$

where σ_o, E_o, h_o are reference stress, modulus of elasticity, and thickness levels selected to provide nondimensional forms. The circular frequency (rad/sec) is described by α.

Since shell configuration considered may have an arbitrary stiffness distribution, the stiffness parameters and mass quantities μ, μ^* will also be expanded in Fourier series in the circumferential variable given by

$$\mu = \frac{E_o h_o}{a^2} \sum_{j=0}^{\infty} m_j \text{ Cos } j\theta$$

$$\mu^* = E_o h_o \sum_{j=0}^{\infty} m^*_j \text{ Cos } j\theta \qquad (2.4)$$

$$B_m = E_o h_o \sum_{j=0}^{\infty} b_{mj}(\xi) \text{ Cos } j\theta, \quad G_m = E_o h_o \sum_{j=0}^{\infty} g_{mj}(\xi) \text{ Cos } j\theta$$

$$C_m = E_o h_o^2 \sum_{j=0}^{\infty} c_{mj}(\xi) \text{ Cos } j\theta, \quad G_{12} = E_o h_o^2 \sum_{j=0}^{\infty} g_{12j}(\xi) \text{ Cos } j\theta$$

$$D_m = E_o h_o^3 \sum_{j=0}^{\infty} d_{mj}(\xi) \text{ Cos } j\theta, \quad G_{13} = E_o h_o^3 \sum_{j=0}^{\infty} g_{13j}(\xi) \text{ Cos } j\theta$$

Then, expansions assume a plane of symmetry with respect to planform geometry (see discussion of Page 22 of Reference 1).

2.2.3 Eigenvalue Problem

Substitution of assumed solutions given by Equations 2.4 into the equations of motion yield five series expressions in the circumferential coordinate relating the Fourier coefficients u_j, v_j, w_j, $\phi_{\xi j}$, and $\phi_{\theta j}$ and the circular frequency α. By employing the appropriate trigonometric identities and orthogonality relationships on the Fourier series the circumferential variable θ can be eliminated yielding a homogeneous system of 5K ordinary differential equations relating the 5K unknown Fourier coefficients and the circular frequency α. (K is the assumed number of coefficients retained in the series expressions.) The resulting equations can be described in the following matrix form

$$Fz'' + Gz' + \left(Hz + M_o\Omega^2\right) = P \qquad (2.5)$$

where

$$z = \begin{bmatrix} u_o \\ v_o \\ w_o \\ \phi_{\xi o} \\ \phi_{\theta o} \\ u_i \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \phi_{\theta K-1} \end{bmatrix}_{(5K \times 1)} \qquad F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \cdots\cdots & f_{1,5K} \\ f_{21} & f_{22} \cdots\cdots & & \cdot \\ f_{31} & \cdot & & \cdot \\ \cdot & & & \cdot \\ \vdots & & & \vdots \\ \cdot & & & \cdot \\ f_{5K,1} & \cdots\cdots\cdots & & f_{5K,5K} \end{bmatrix}_{(5K \times 5K)}$$

$$G = \begin{bmatrix} g_{11} & g_{12} \cdots & g_{1,\ 5K} \\ g_{21} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ g_{5K,\ 1} & \cdots & g_{5K,\ 5K} \end{bmatrix}_{(5K \times 5K)} \qquad H = \begin{bmatrix} h_{11} & h_{12} \cdots & h_{1,\ 5K} \\ h_{21} & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ h_{5K,\ 1} & \cdots & h_{5K,\ 5K} \end{bmatrix}_{(5K \times 5K)}$$

and

$$\Omega^2 = \frac{\alpha^2 a^2}{h_o E_o}$$

The elements of matrices F, G, H are given in Appendix A of Reference 1.

The matrix M_o is a 5K x 5K array defined as follows:

$$M_o = \begin{bmatrix} \hat{m}_{11} & \hat{m}_{12} & \hat{m}_{13} \cdots & \hat{m}_{1,\ 5K} \\ \hat{m}_{21} & \hat{m}_{22} \cdots & & \cdot \\ \hat{m}_{31} & \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \hat{m}_{5K,\ 1} & \cdots & \cdots & \hat{m}_{5K,\ 5K} \end{bmatrix}$$

where

$$\begin{aligned} \hat{m}_{5k+1,\ 5j+1} &= M^{kj} \\ \hat{m}_{5k+2,\ 5j+2} &= \bar{M}^{kj} \\ \hat{m}_{5k+3,\ 5j+3} &= M^{kj} \qquad \left| k,\ j = 0,\ 1,\ 2 \ldots K \right| \\ \hat{m}_{5k+4,\ 5j+4} &= M^{*kj} \\ \hat{m}_{5k+5,\ 5j+5} &= \bar{M}^{*kj} \\ \hat{m}_{others} &= O \end{aligned}$$

and

$$M^{kj} = \frac{1}{2} m_{(k+j)} + \left\{1 - \delta^2(j-k) + \delta(k)\right\} m_{\left|k-j\right|}$$

$$\overline{M}^{kj} = \frac{1}{2} -m_{(k+j)} + \left\{1 - \delta^2(j-k) + \delta(k)\right\} m_{\left|k-j\right|}$$

$$M^{*kj} = \frac{1}{2} m^*_{(k+j)} + \left\{1 - \delta^2(j-k) + \delta(k)\right\} m^*_{\left|k-j\right|}$$

$$\overline{M}^{*kj} = \frac{1}{2} -m^*_{(k+j)} + \left\{1 - \delta^2(j-k) + \delta(k)\right\} \overline{m}^*_{\left|k-j\right|}$$

$$\delta(\ell) = \begin{cases} 1 & \ell > 0 \\ 0 & \ell = 0 \\ -1 & \ell < 0 \end{cases}$$

It should be noted for the general case of shells having arbitrary stiffness distributions, it is not possible to uncouple the shell equations into separate sets for each Fourier index n as was accomplished in Reference 2. Equations 2.5 together with the appropriate boundary conditions (Equation 45, Reference 1) define the appropriate equations for determining the free vibrations of shells of revolution having arbitrary stiffness distribution.

2.3 Numerical Analysis

The form of Equations 2.5 and boundary condition (Equation 45, Reference 1) are similar to those considered in studying the free vibrations of shells of revolution (Equation 1.22). The identical numerical procedure described in Section 1.0 can be utilized in obtaining the matrix solution of the characteristic frequency equation. For this reason, the described numerical procedure will not be repeated here.

3.0 SUMMARY

A general numerical procedure has been presented for the determination of natural modes and frequencies of vibrations of rotationally symmetric shells of revolution with any combination of homogeneous boundary conditions. An advantage of the numerical procedure presented is that the solution revolves around the operation on 4 x 4 matrices to obtain a 4 x 4 determinant for evaluation of the characteristic values.

The procedure is also modified to handle an extended class of shells of revolution; that is, shells of revolutions with both meridional and circumferential variations of the stiffness parameters.

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