ENCODING INDEPENDENT SAMPLE INFORMATION SOURCES

JOHN T. PINKSTON III

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Abstract

The theory of encoding memoryless information sources so that the output can be transmitted with minimum rate and still satisfy a fidelity criterion based on a single letter distortion measure is investigated. This report extends and amplifies the theory developed by Shannon. A general proof of the Source Coding theorem for memoryless sources and single letter distortion measures is presented using variable length codes. It is shown that this proof is more generally applicable than Shannon's previously derived block coding results; moreover, without some additional restrictions, the coding theorem is false if only block codes are permitted. It is also shown that the convergence of encoder rate to R(D) (the minimum rate necessary to achieve average distortion D) with increasing block length n, can be made at least as fast as $(\log n)/n$. Equivalent theories of source coding are developed for cases in which: (i) the fidelity criterion requires every letter to be reproduced with less than a fixed distortion, rather than merely achieving this performance on the average; (ii) there are several fidelity criteria that must be satisfied simultaneously; and (iii) the source outputs are corrupted by a noisy channel before being furnished to the encoder. Means of calculating or estimating R(D) for sources with a difference distortion measure are developed by showing conditions under which R(D) is equal to a more easily calculable lower bound developed by Shannon. Even when equality does not hold, we show that R(D) approaches this bound as $D \to 0$ for all continuous sources, and that for discrete sources, there is always a nonzero region of small D where there is equality. R(D) for a discrete source and distortion measure $d_{ij} = 1 - \delta_{ij}$ is calculated exactly for all D, thereby allowing calculation of the minimum achievable symbol error probability when transmitting over a channel of given capacity. Finally, as an application of the theory, we examine quantizers as a class of source encoders, and show that the rate (output entropy) and distortion of such devices is bounded away from R(D), but is usually quite close.

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I. INTRODUCTION

In the years since the publication of Shannon's classic paper,[1] which marked the beginning of information theory, the bulk of the research effort in this field has been directed toward communication over noisy channels. Some investigators have found tight bounds on the rate at which the probability of error of codes can be made to vanish as a function of code rate and block length; others have attempted to delineate the classes of channels for which coding theorem could be proved; and still others have worked to find practically implementable coding systems that would approach the performance that the coding theorem shows to be possible. The prospect of reliable communication has indeed been a powerful attraction to researchers, and in the past decade many clever schemes toward this end have appeared.

With all of this attention to channel coding, there has been, at least until recently, very little interest in the theory of the related field of source coding. Although there had been previous work on bandwidth compression of certain sources, such as speech (vocoders) and pictures, a mathematical discussion of encoding, or digitizing, an information source so that its outputs can be reconstructed from the code within a desired fidelity limit was first presented by Shannon in his original paper, and again in 1959.[2] "Rate-Distortion Theory," as this branch of information theory is often called, attempts to answer questions about the smallest number of bits or, almost equivalently, the minimum channel capacity, which is needed on the average to adequately describe the outputs of an information source to some user. The assumption here is that a less than perfect reproduction of the source may be adequate, for clearly if only perfect reconstruction is to be tolerated, this number of bits is just equal to H(X), the source entropy, and so there is no problem.

Suppose then, that the user defines "adequate reproduction" by specifying a distortion measure $d(x,y)$ — a function that tells how unhappy he is when the source really produced an output x, and he is given y as the representation. The fidelity criterion is that this distortion be less than some specified amount on the average; that is,

$$E\{d(x,y)\} \leq D.$$

The problem faced by the communications engineer is to design an economical system that will satisfy the user, and since the cost is often proportional to the number of bits to be transmitted, we would like to know the minimum number of bits required per source output to yield the desired fidelity. For example, the source might produce sequences of independent Gaussian random variables, and the user might require the mean-square error (the average of the squared differences between the source output and reconstructed version) to be below a specified value. Equivalently, one might ask for the minimum possible average distortion when a fixed number of bits per source output is available for the representation. This report deals with these kinds of questions.

It does not seem necessary to give very much in the way of motivation for research in this area; the relevance of these questions to telemetry and related fields should be obvious. Unfortunately, we shall see that the present state of the theory is that these questions can be answered only for the most simple of sources and distortion measures, and any application to real sources still lies in the future. One must learn to walk before he can run, and so, with this in mind, let us proceed with the formal development of the theory of source encoding.

By an information source, we mean any device that puts out a stochastic signal of interest to some user. The source will be called time discrete if its output can occur only at regularly spaced intervals, and if this output is a waveform, the source will be termed time continuous. We shall be interested exclusively in time-discrete sources here, and shall symbolize a sequence of n source outputs by a vector $\underline{x} = x_1 x_2 \ldots x_n$ (see Fig. 1). Each of these x's is some element of the source alphabet X, which is the set of all possible outputs that might occur. If X is a finite set, we say that the source is discrete, and if X is the set of real numbers R, then we say that the source is amplitude continuous, or sometimes just continuous. Although X may be a more general space, we shall be concerned primarily with sources having one of the two above-mentioned alphabets.

The behavior of the source is governed by a probability measure which assigns a probability or a probability density to any combination of source outputs determined by whether the source is discrete or continuous. If the probability of any event at a particular time is statistically independent of the outputs at all other times so that

$$\Pr(x_1 \ldots x_n) = \prod_{i=1}^{n} \Pr(x_i),$$

then we say that the source is memoryless. In this case, the source is described by the univariate probability (mass or density) function p(x). We shall deal only with memoryless sources in this report.

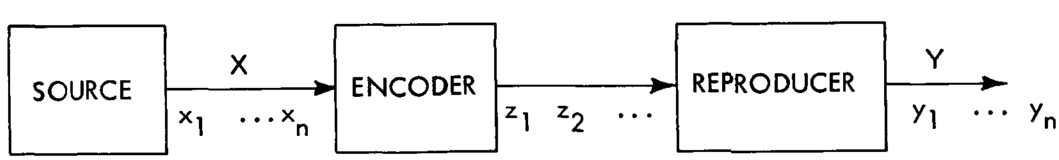

Fig. 1. The source encoding configuration.

A source encoder is any device that maps a sequence of source outputs $x_1 \ldots x_n$ into a sequence of digits $z_1 z_2 \ldots$ taken from some finite alphabet Z. We refer to this number n as the block length of the encoder. We do not assume that all $\underline{x}$ blocks produce z sequences of equal length. This digital stream is assumed to be furnished uncorrupted to the receiver, which uses it to construct a sequence $\underline{y} = y_1 \ldots y_n$ of letters from Y, the reconstruction alphabet, which is then presented to the customer. This set Y is often identical with X, although this is not required. Figure 1 shows this configuration. It can be seen that since the number of encoder output sequences is countable, no generality is lost (or gained) by considering the encoder to consist of two sections, as shown in Fig. 2. The first box takes the n source outputs and produces an integer w, which

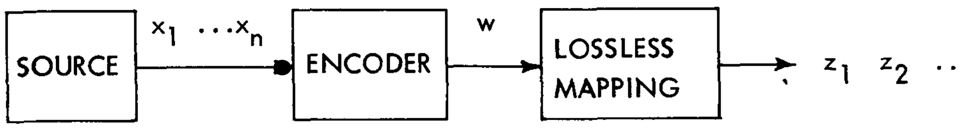

Fig. 2. An equivalent encoding configuration.

is then encoded into the z sequence by a lossless, reversible mapping. The average length of the z sequence is directly related to the entropy of the w distribution, H(W), by the noiseless coding theorem (cf. Abramson[24]).

We suppose that the user specifies a distortion measure $d(x,y)$, $x \in X$, $y \in Y$, which is a non-negative function telling how much distortion is incurred when the receiver puts out y when the source had actually produced x. A distortion measure of this form (a function of only one X letter and one Y letter) is called a single letter distortion measure, and the distortion between blocks is taken to be

$$d(\underline{x},\underline{y}) = \sum_{i=1}^{n} d(x_i, y_i).$$

If X and Y are finite, then $d(x,y)$ is a matrix, and if each is the set of real numbers, then $d(x,y)$ is a function of two real variables. It is clearly possible to formulate much more general distortion measures between blocks, but we restrict ourselves to the single-letter distortion measure because this is the only type for which any significant results have been obtained. We now define the average distortion (per letter) of the encoder to be

$$D = \frac{1}{n} E\{d(\underline{x},\underline{y})\},$$

and its rate (in nats) to be

$$R = \frac{1}{n} E\{\text{length of } Z \text{ sequence}\} \log_e J,$$

where J is the alphabet size of Z. Both expectations are taken with respect to the source distribution, and clearly $\underline{y}$ is determined uniquely by $\underline{x}$.

Finally, a subset of the class of all encoders which will be of interest to us consists of those for which the number of Z digits put out is the same for all input sequences. It can be seen that this is equivalent to the condition that the integer output w (see Fig. 2) can take on only a finite number of values, which number we denote M. This class of encoders will be called block encoders, and we see that for every source sequence $\underline{x}$ of length n, there are $M = e^{nR}$ code words with letters taken from the reproduction alphabet which is available for choosing representation for $\underline{x}$, where R is the rate of the block code. In the following discussion, we shall consider the larger class of encoders unless we specifically state that we are restricting our attention to block ones.

Now since D is a measure of how unhappy the user is on the average, and R is proportional to the effort we must expend in the transmission, we would like to make both of these quantities as small as possible. As one would expect, however, these two goals are not compatible. So given the source p(x) and the distortion measure d(x, y), we would like to know what is the smallest rate consistent with the maintenance of D no greater than some specified level, or equivalently, what is the smallest D that we can achieve if our rate is fixed.

The answers to these questions are given by the rate-distortion function R(D), sometimes called the information rate of the source for a distortion level D. This function has the property that there are encoders having average distortion D, for which the rate can be made arbitrarily close to R(D) by choosing the block length large enough. Conversely, there are no such encoders with rates less than R(D). Recall that by block length, we mean the number of source outputs that the encoder operates on at one time, and this number is not related to the average number of Z digits put out by the encoder.

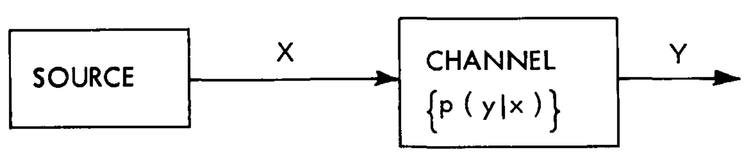

Fig. 3. Source and test channel.

R(D) might have been defined directly as the g.l.b. of the rates of all encoders (of any block length) that have average distortion, at most D. In this case, it would be necessary to give methods by which R(D) might be calculated. The formalism developed differently, however; Shannon, the first writer on this subject, chose to define R(D) in a manner that indicates, at least in principle, how to find this function, and then he proved that the function as he defined it did indeed have the desired properties.

His definition of R(D) for a memoryless time-discrete source, with a difference distortion measure, is as follows: Consider a discrete memoryless channel, characterized by a set of transition probabilities $\{p(y|x)\}$, whose input and output alphabets are the same as the source alphabet X, and the reconstruction alphabet Y, respectively. Suppose that the source is connected directly to the channel, as in Fig. 3. Then the p(x) and $p(y|x)$ functions induce a joint probability measure on the X and Y spaces, and we can define

$$R(D) = \min_{\{p(y|x)\}} I(X;Y),$$

where the minimization over all possible channels is performed, subject to the constraint

$$E_{xy}\{d(x,y)\} \leq D.$$

It is possible that D will be so small that there is no channel that satisfies this constraint, in which case R(D) just does not exist. If this is not the case, then it can be shown that there is always a channel whose mutual information is actually equal to R(D), so that it is valid to write Min instead of g.l.b. Any channel like that shown in Fig. 3 will be called a test channel, and the one that achieves the minimum rate will be referred to as the optimum test channel.

The source coding theorem states that with some mild restrictions on p(x) and d(x, y), R(D) is the minimum achievable encoder rate that is consistent with $E\{d(x,y)\} \leq D$. In other words, for any $\epsilon > 0$, there are encoders with $E\{d(x,y)\} \leq D + \epsilon$ and rate no greater than R(D). Conversely, no encoders exist with average distortion D and rate less than R(D). This result is analogous to the channel-coding theorem which states that the capacity of a channel (the max of a mutual information) is the largest signalling rate that is consistent with the requirement that arbitrarily small probability of error be achievable. Another (unfortunate) analogy is that, like channel capacity, R(D) even as defined above is difficult to calculate.

We shall defer the proof of this theorem to Section II. It is not difficult to show that except at $D = D_{min}$, the smallest value of D for which R(D) is defined, the approach to any point on the R(D) curve need not be along a horizontal line in the R-D plane, as in the statement of the theorem above. Rather, it may be along any path lying above the R(D) curve. For $D = D_{min}$, this trajectory may be anything except vertical, for the same reasons that we cannot have zero probability of error in channel coding, even at rates strictly below capacity. The interested reader who feels unsure of himself (or lazy) may find the proof of these statements in Shannon's paper.[2] Some results concerning the rate at which this approach occurs as a function of encoder block length have also been obtained by Pilc[7] and Goblick.[3] Unlike channel coding, in which the probability of error vanishes exponentially with n, for source coders the approach seems to be something like $n^{-\alpha}$. This subject will be discussed in greater depth in section 2.3.

Let us now look at some examples and properties of rate-distortion functions.

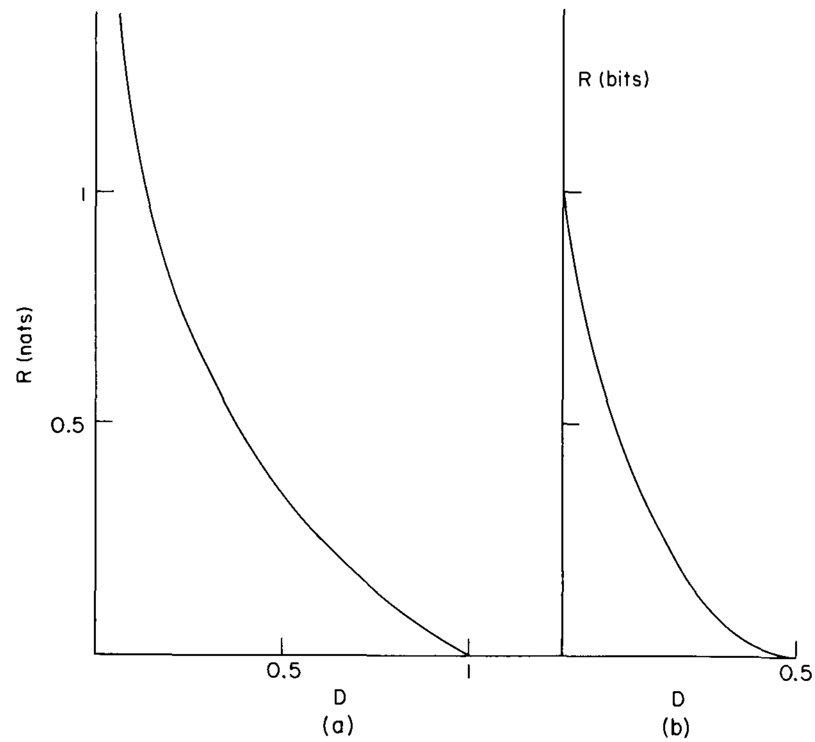

Fig. 4. (a) $R(D) = \frac{1}{2} \log D$.
(b) $R(D) = 1 + D \log D + (1-D) \log (1-D)$.

Figure 4a shows R(D) for a time-discrete, amplitude-continuous memoryless source with a unit variance Gaussian probability density function, and the distortion measure

$$d(x, y) = (x-y)^2.$$

In this case, X and Y are both the real line, and the functional form of R(D) is

$$R(D) = \begin{cases} \frac{1}{2} \log \frac{1}{D} & 0 < D \leq 1 \\ 0 & D > 1 \end{cases}$$

Figure 4b shows R(D) for a binary, equiprobable letter source, with the distortion measure

$$d(x, y) = 1 - \delta_{xy} = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise} \end{cases}$$

so that the distortion between two blocks is just the fraction of places in which they differ. Here we have

$$R(D) = 1 - H(D).$$

We shall defer to Chapter III the discussion of simple methods by which these functions, and others, may be calculated. We merely mention that attempts at finding R(D) by directly attacking the minimization in the defining relation have not been successful. One can proceed up to a point (as many previous researchers have done) as follows: Since I(X;Y) is convex in the $\{p(y|x)\}$, it is necessary and sufficient that

$$\frac{\partial}{\partial p(y|x)}\left[I(X;Y) + \sum_x \mu(x) \sum_y p(y|x) + \lambda \sum_{xy} p(x)\, p(y|x)\, d(x,y)\right] \geq 0$$

with equality unless $p(y|x) = 0$. The undetermined multipliers λ and $\mu(x)$ are to satisfy the constraints $E\{d(x,y)\} = D$ and $\Sigma\ p(y|x) = 1$, respectively. Since

$$\frac{\partial I(X;Y)}{\partial p(y|x)} = \log \frac{p(y|x)}{q(y)},$$

where $q(y) = \sum_x p(x)\, p(y|x)$, the inequality becomes

$$\log \frac{p(y|x)}{q(y)} + \mu(x) + \lambda\, d(x,y) \geq 0$$

or

$$p(y|x) \geq q(y) \frac{e^{-\lambda\, d(x,y)}}{A(\lambda, x)} \qquad \text{for all } x, y,$$

where $\mu(x)$ has been absorbed in the constant $A(\lambda, x) = \sum_y q(y)\, e^{-\lambda\, d(x,y)}$, and equality holds unless $p(y|x) = 0$. But the right-hand side is non-negative, and so equality must also hold, even when $p(y|x) = 0$. This gives us a set of simultaneous equations that presumably could be solved for the $p(y|x)$ in any particular case, but the lack of any simple general expression for the solution renders this approach of little computational use.

Henceforth, we shall assume that all sources are memoryless and time-discrete, unless it is specifically stated otherwise. Note that the curves shown are continuous, monotone decreasing, and convex downward. In fact, it is easy to see that all rate-distortion functions must have these properties. Clearly, R(D) is nonincreasing because increasing D does not delete any element (channel) from the set of permissible channels over which the mutual information is minimized. To see that R(D) is convex downward, suppose that R_1, D_1 and R_2, D_2 are two points on the R(D) curve whose test channels are $p_1(y|x)$ and $p_2(y|x)$, respectively. Then the channel with $p(y|x) = \alpha p_1(y|x) + (1-\alpha)\, p_2(y|x)$, with $0 \leq \alpha \leq 1$, has distortion $\alpha D_1 + (1-\alpha)D_2$ and rate less than or equal to $\alpha R_1 + (1-\alpha)R_2$, since mutual information is convex downward as a function of the transition probabilities. Thus the minimum mutual information to get distortion at most $\alpha D_1 + (1-\alpha)D_2$ can be no greater than $\alpha R_1 + (1-\alpha)R_2$. Finally, convexity implies continuity of the curve, except perhaps at the end points.

Comparing Fig 4a and 4b further, we see that as $D \to 0$, $R(D) \to \infty$ for the Gaussian source, while $R(0) = 1$ bit for the binary source. This behavior is what we would expect because we know that to perfectly reproduce an equiprobable binary source, we need one bit per source output, while exact specification of a real number cannot be done in a finite number of digits. Furthermore, we see that both curves have a finite intercept on the D axis. This point, called D_{max}, is that average distortion that can be obtained even when nothing is transmitted ($R = 0$). For the binary case, this is 1/2 because, with no information at all, we could guess at all of the source letters and be right half of the time on the average. Similarly, for the Gaussian source, we could always put out zero as our reproduction, and the average distortion would be

$$D_{max} = E\left\{(x-0)^2\right\} = \sigma^2 = 1.$$

In general, we have

$$D_{max} = \underset{y \in Y}{\text{Min}}\, E_x\{d(x,y)\},$$

since guessing any letter other than that one which minimizes this expectation can only increase the distortion. It is not necessary that D_{max} be finite, but it will be so in most of the sources and distortions with which we deal.

At this point, we may also note that in both of our examples the smallest possible distortion was zero. This need not be true; in general, the smallest D for which R(D) is defined

$$D_{min} = E_x\left\{\underset{y \in Y}{\text{Min}}\, d(x,y)\right\}.$$

Thus the range of D that is of interest is $D_{min} \leq D \leq D_{max}$, since $R(D) = 0$ for $D > D_{max}$ and is not defined for $D < D_{min}$ because it is impossible to get such distortions. The following theorem shows, however, that $\underset{y \in Y}{\text{Min}}\, d(x,y)$, and therefore D_{min}, may be taken to be zero without loss of generality.

Theorem 1: Suppose $R_1(D)$ and $R_2(D)$ are the rate-distortion functions for the same source, and distortion measures $d_1(x,y)$ and $d_2(x,y)$, respectively. Suppose $d_1(x,y) = \alpha d_2(x,y) + \beta(x)$, where α is a positive constant, and $\beta(x)$ is any function of x alone. Then $R_1(D) = R_2\left(\frac{D-\bar{\beta}}{\alpha}\right)$, where $\bar{\beta} = E_x\{\beta(x)\}$.

Proof: By definition,

$$R_1(D) = \text{Min}\, I(X;Y),$$

where the minimization is subject to the constraint

$$E\{d_1(x,y)\} \leq D.$$

But this constraint can be written

$$E\{\alpha d_2(x,y)+\beta(x)\} \leq D$$

or

$$E\{d_2(x,y)\} \leq \frac{D-\beta}{\alpha},$$

from which the result follows by definition of $R_2(\ )$.

In the discussion up to this point, the "rate" of a source was taken to be the smallest average number of digits that an encoder can put out and still satisfy the fidelity criterion. This is not the only possible interesting definition of this quantity. A second one is the minimum rate necessary for a block encoder to satisfy the fidelity criterion (that is, all of the z sequences are forced to have the same length). In the first case, we are interested in the smallest possible entropy of the distribution of the integer w in Fig. 2, and in the second case, we care only about the number of these integers with nonzero probability. Still a third "source rate" is the smallest channel capacity that will allow the transmission of the source outputs with satisfactory distortion.

The relationship between the source and distortion measure on the one hand, and the strength of statements that can be made about representing and transmitting the source outputs on the other, will be discussed in more detail in Section II. We shall see there that the three rates presented above (the minimum variable length encoder rate, the minimum block encoder rate, and the minimum channel capacity that is necessary to satisfy the fidelity criterion) are not, in general, the same, and that the conditions needed to prove equality with R(D) are most general for the variable length rate and least for the channel capacity.

Starting from the basic source-coding theorem described above, there are many avenues of generalization open, most of which are discussed by Shannon.[2] If the source is not memoryless, one can still prove a coding theorem by defining R(D) as a Min I(X;Y) of blocks of source and reproduction letters.[8] Similarly, the distortion measure might not be a single-letter measure, in which case one must again go to blocks of source outputs to get the desired results. In neither of these cases is the theory entirely satisfactory because the task of calculating R(D) from its definition, difficult enough in the single-letter case, is virtually impossible. The only example of a source with memory for which R(D) is known is a colored Gaussian source with a mean-square error distortion, and the way this problem is solved is by using the properties of the Gaussian distribution to change coordinates so that one gets a new set of random variables that are independent. Such a rotation changes neither the Euclidian metric nor, consequently, the mean-square error. Even the problem of finding R(D) for a binary Markov source that repeats its last output with probability $p \neq 1/2$, with frequency of error as the distortion measure, has resisted substantial efforts at solution.[9] Indeed, this is a special case of a classic unsolved problem (Dobrushin's 22nd).[10] Thus the theory as it stands now is of limited use in these cases.

The potential applications of the theory are obvious. In a given communication problem one can determine what the smallest possible distortion can be, or what is the smallest number of bits that he must use to achieve a specified fidelity. The performance of any proposed system can then be compared with this theoretical minimum. Thus far, there have been a few comparisons of this sort, involving analog modulation[4] and quantization schemes,[5] but these have been restricted to the consideration only of Gaussian sources with a mean-square error fidelity criterion.

The problems of source coding theory fall roughly into three categories: theory, calculation of R(D), and applications. The first involves such things as examination of the conditions necessary for proving a coding theorem and consideration of the rates of encoders with finite block lengths. The second reflects the fact that, like channel capacity, R(D) is not usually easy to calculate, so a body of techniques has been developed for evaluating, or at least approximating, this function in certain interesting cases. Finally, the third area involves the examination of practical encoding schemes in the light of the theory, and the attempt to find ways of improving their performance. Sections II, III, and IV will each be devoted to one of these topics.

II. THEORETICAL DEVELOPMENT

We have presented, for a specified source and distortion measure, definitions of the information rate of an encoder for this source, and of the function R(D). We intimated in Section I that these quantities were related by the fact that no encoder for the source having an average distortion D could have a rate less than R(D), and that encoders existed with rates arbitrarily close to this lower bound. The last existence statement is usually referred to as the "Source Coding Theorem," which in the rest of this work will be shortened to "coding theorem" when no confusion is likely to occur. The fact that R(D) is a lower bound on encoder rates is then the converse to this theorem.

We shall now prove the coding theorem and its converse for discrete sources with arbitrary distortion matrices, and for continuous sources with only a very mild restriction on $d(x, y)$, when the class of encoders is that presented in Section I. Then we examine some possible definitions of information rate other than the minimum rate of all encoders in this class, and present conditions under which these "rates" are also equal to R(D). Furthermore, we show that the approach of encoder rate to R(D) as a function of block length can be made at least as fast as $\frac{\log n}{n}$. For discrete sources, a "zero distortion rate" is then defined which is similar to Shannon's zero-error capacity of noisy channel,[16] and a simple expression for this rate and some of its applications to questions about the capabilities of block codes are given. Lastly, some extensions of the basic theory are presented. These include: (i) the information rate for a fidelity criterion that requires every letter to be encoded with a given accuracy, rather than the weaker condition that this be true merely on the average over a block; (ii) the rate when several fidelity criteria must be satisfied simultaneously; and (iii) the rate when the encoder must operate on a corrupted version of the source outputs.

2.1 CONVERSE STATEMENT

Let us first prove the converse to the source coding theorem, which states that R(D) is the smallest rate that any encoder giving average distortion D may have. It is applicable to all classes of encoders, and is a direct result of the definition of R(D) as a minimum mutual information. The results of this section are essentially due to Shannon,[2] with only slight modifications necessitated by our definition of source encoders.

We shall assume, for the present, that we have a block encoder taking n letters from the source and producing m symbols which are sent over a memoryless channel of capacity C. Our decoder takes the channel outputs and produces n letters from the reproduction alphabet. This configuration is shown in Fig. 5.

Suppose that the average distortion of this scheme is d^*. We show that $C \geq \frac{n}{m} R(d^*)$ by the following string of inequalities:

$$mC \geq I(Z^m; W^m)$$

$$\geq I(X^n; Y^n),$$

Fig. 5. Channel with encoder and decoder.

by the data-processing theorem, a good proof of which is given by Feinstein.[23] Now

$$I(X^n;Y^n) = H(X^n) - H(X^n|Y^n)$$

$$\geq \sum_{i=1}^{n} [H(X_i) - H(X_i|Y_i)]$$

$$= \sum_{i=1}^{n} I(X_i;Y_i)$$

$$> nR(d^*) \qquad \text{by the definition of } R(D).$$

The only non-obvious inequality that was used was

$$H(X^n|y^n) = H(X_1|Y_1 \ldots Y_n) + H(X_2|X_1Y_1 \ldots Y_n) + \ldots + H(X_n|X_1 \ldots X_{n-1}Y_1 \ldots Y_n)$$

$$\leq H(X_1|Y_1) + \ldots + H(X_n|Y_n).$$

It seems clear that this result holds also for channels with memory, as long as capacity can be meaningfully defined. Discussion of such problems is, however, beyond the

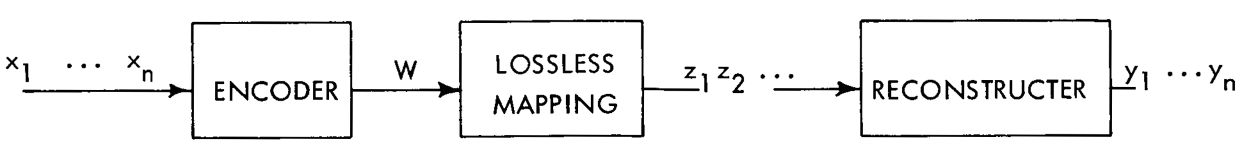

Fig. 6. Variable-length encoder.

scope of this work. This result shows that no matter what kind of processers are used, the capacity of the channel connecting source to sink determines a lower bound on the achievable distortion.

Now, in particular, if the connection between source and user is an encoder such as that shown in Fig. 2 (which, we recall, is equivalent to that of Fig. 1), then we can obtain the desired result that if its average distortion is D, its rate is lower-bounded by R(D) as follows: By the lossless source-coding theorem, the expected number of z digits put out is no less than H(W), the entropy of the integer output distribution (see Fig. 6).

Thus we can write

$$E\{\text{length of } z \text{ sequence}\} \geq H(W)$$
$$= I(X^n;W),$$

since W is completely determined by $x_1 \ldots x_n$. But, by the data-processing theorem,

$$I(X^n; W) \geq I(X^n;Y^n)$$
$$\geq n\, R(D)$$

which completes the proof.

2.2 POSITIVE CODING THEOREMS

There are at least three possible definitions of the rate of a source relative to a fidelity criterion which a user might want to make, and the distinctions among these will be made clear before attempting to prove coding theorems about them.

First, one can define the rate, as was done in Section I, as the minimum rate over the general class of encoders presented there. Since the number of digits put out by the encoder may vary, such encoders will henceforth be called "variable length encoders," and we shall denote the minimum variable-length encoder rate commensurate with a distortion level at most D by $R_v(D)$. Second, one may want to consider only block encoders, and we define the "block coding" rate, $R_b(D)$, to be the minimum rate of all encoders of this class that satisfy the given fidelity criterion. Third, the source rate might be taken to be simply the smallest channel capacity (in bits per source output) such that the source can be block-coded and transmitted over <u>any</u> channel of this capacity with distortion arbitrarily close to some specified amount, and we let this rate be denoted $R_c(D)$.

It is clear that

$$R(D) \leq R_v(D) \leq R_b(D) \leq R_c(D),$$

since we have established that R(D) was a lower bound on all of the other rates, and any source that can be transmitted over any channel of capacity C, as in the third definition, can, a fortiori, be so transmitted over a noiseless channel of the same capacity, which process is identical with the block coding of the second definition. Finally, we have already seen that block encoders are a special case of variable-length encoders.

Furthermore, it is easy to see that these three rates are not the same in general, although they will be for most cases. For example, the four-letter discrete source with probability vector $\underline{p} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}\right)$ and distortion measure

$$d_{ij} = \begin{cases} 0 & \text{if } i = j \\ \infty & \text{otherwise} \end{cases}$$

can be encoded to achieve zero distortion by a variable-length encoder with rate 1.75 bits (by a Huffman code), but to get this fidelity with a block code requires 2 bits per source, since in a block of n outputs, there are 4^n sequences of nonzero probability each of which must have its own code word to avoid having an infinite distortion. Finally, R_c is infinite, since no channel capacity is sufficient to guarantee that the source can be transmitted with zero distortion because the probability of error cannot be made exactly equal to zero in general, but merely to approach this value. This still leaves a positive probability of an infinite distortion if an error occurs.

Let us now turn to the consideration of coding theorems for each of these three definitions of rate. What we are after are conditions on the source and distortion under which each of these rates is equal to R(D), and proofs to this effect. It is convenient to start with the block encoder, and state the following theorem.

Theorem 2: If the distortion matrix $\{d_{ij}\}$ for a discrete source satisfies the condition that at least one column has no infinite entries (that is, there is an output letter j such that d_{ij} is finite for all source letters), then for any D, $D_{min} < D \leq D_{max}$, there exist block encoders with rates arbitrarily close to R(D). Similarly, if the distortion function d(x, y) for a continuous source satisfies the condition that there exists a $y \in Y$ such that $E_x\{d(x,y)\}$ is finite, then the same conclusion holds.

This theorem was first stated and proved by Shannon,[2] and later Goblick[3] obtained the same result using different techniques. The simplest proof known to the author is one constructed by Gallager,[11] based on contributions by Shannon, Goblick, and Stiglitz, and this is reproduced in Appendix B. This proof involves a random-coding argument, in which $M = e^{nR^*}$ code words of length n from the reproduction alphabet are chosen randomly, and the probability of source sequences for which there is no code word resulting in distortion at most D^*, averaged over the ensemble of codes, is shown to approach zero as n becomes large if $R^* > R(D)$. This establishes the existence of codes of rate R^* with vanishing probability of exceeding a distortion of D^*. All that is left is to show that the rare bad $\underline{x}$ sequences can be encoded with finite distortion D_m, so that we can write

$$D \leq D^*(1-p_o) + D_m p_o,$$

where p_o is the probability of $\underline{x}$ sequences unencodable with distortion less than D^*.

Since p_o approaches zero as $n \to \infty$, the right-hand side approaches D^*. But the existence of a finite D_m is established by the assumptions on the distortion in the statement of the theorem, which insure that there is some output letter such that a code word with this letter in all positions has finite expected distortion with all source sequences. This code word can be added to the code with negligible effect on the rate, which establishes the result.

The argument above establishes the sufficiency of the stated requirements on source and distortion measure for a block-coding theorem to hold. It turns out that these conditions are necessary for $R_b(D)$ to equal $R(D)$ for all D, but that they are not necessary for equality to hold merely over a range of D. We shall defer the proof of these facts to section 2.5, where the capabilities of block coders will be examined in depth. The necessity of some kind of requirements on the source, distortion measure, and distortion level for $R_b(D)$ to equal R(D) sets the class of block encoders apart from variable length, for which we will find that no such assumptions are needed. Basically, when we are limited to $M = e^{nR}$ code words, we need some finite distortion word or words to fall back on in the rate event that all other code words have too much distortion. It should be pointed out that these requirements are not very restrictive, and are satisfied by most interesting sources and distortions. Obviously, any time that D_{max}, the D at which R(D) becomes equal to zero, is finite, then the conditions for the theorem to be true are satisfied; however, we shall see in section 2.5 that the converse of this statement is not true.

Next, we turn to the third definition of rate – that of the smallest channel capacity needed to achieve the desired fidelity. It is easy to see that in the discrete case, a necessary and sufficient condition on the distortion matrix for this rate, $R_c(D)$, to be equal to R(D) is that the distortion between each source letter and each output letter that must be used to achieve R(D), must be finite. This rather awkward statement is necessary to allow distortions such as

$$\{d_{ij}\} = \begin{bmatrix} 0 & 0 & \infty \\ \infty & 0 & 0 \end{bmatrix},$$

which can clearly be transmitted over channels of zero capacity, in spite of the fact that some distortions are infinite. The catch is that since only the second output letter is ever used, the channel decoder knows that no matter what is received from the channel, it must be mapped into this letter, and no infinite-distortion reproduction is ever made. For continuous sources, the analogous condition is that $E_x\{d(x,y)\}$ be finite for all y that must be used.

For a discrete source, the necessity of these conditions follows from the fact that all transitions from source to output letters must be considered to have nonzero probability, and so cannot have infinite distortion associated with them. The sufficiency can be proved as follows: Let D_m be the largest d_{ij}. We construct a block code of length n with $M = e^{nR}$ code words, so that the rate is R. These code words are then encoded for transmission over our noisy channel of capacity C. If we let p_c be the probability that our encoder fails to produce a code word with distortion less than D^*, and p_e be the probability of a channel error, then we can write

$$D \leq D^* + D_m(p_e+p_c).$$

But, by Theorem 2, if $R > R(D^*)$, then p_c can be made to vanish as $n \to \infty$. Similarly, by the channel-coding theorem, if $C > R$, then p_e can be made arbitrarily small as the block length of the channel code (not the same as n) gets large. Thus, if $C > R(D^*)$, the upper bound on D approaches D^*, which is the desired result.

Essentially the same argument holds for continuous sources, for which a necessary and sufficient condition for transmission over a channel is that $E\{d(\underline{x}, \underline{y}_j) | \underline{y}(\underline{x}) \neq \underline{y}_j\}$ be finite for all code words $\underline{y}_j$. In this case, D_m may be taken as the largest of all such expectations. We see that this condition is equivalent to $E\{d(x, y)\}$ being finite for all y that must be used (that is, for which q(y), the output density of the optimum test channel, is nonzero), since

$$p(x | \underline{y}_j \text{ received in error}) = \begin{cases} 0 & \text{if } \underline{y}(\underline{x}) = \underline{y}_j \\ \dfrac{p(x)}{A} & \text{otherwise} \end{cases}$$

where A is simply a normalizing constant.

We can now turn to the remaining definition of a source's information rate — that given in Section I. We have already shown that this is the most comprehensive and general of the three, and we shall now give the proof that, in fact, it is always equal to R(D). First, for a discrete source, we have the following theorem.

Theorem 3: Let $\underline{p} = (p_1 \dots p_m)$ be the vector of letter probabilities of a discrete source having m letters, and $\{d_{ij}\}$ be its associated distortion matrix. Then for any $D^* > D_{min}$, and any $\epsilon > 0$, there exists an encoder with rate less than $R(D^*) + \epsilon$ whose average distortion is less than D^*.

Proof: Let $\underline{x} = x_1 \dots x_n$ be a sequence of n outputs from the source, and $\underline{\ell} = \ell_1 \dots \ell_m$ be the composition of $\underline{x}$ (that is, ℓ_j is the number of times the j^{th} source letter appears in $\underline{x}$). To demonstrate the existence of encoders with rate and distortion arbitrarily near R(D), we consider a class of such devices that operate as follows: The source sequence is taken and compared in a preset order with an arbitrarily long list of code words, $\underline{y}_1$, $\underline{y}_2$, ... until a $\underline{y}_k$ is found such that $d(\underline{x}, \underline{y}_k) \leq nd_{\underline{x}}$, where the $d_{\underline{x}}$ are distortion thresholds actually dependent only on the composition of $\underline{x}$, and will soon be specified. The integer k is then put out by the encoder. Clearly, the distortion introduced by the encoder when the sequence $\underline{x}$ is given to it is no greater than $d_{\underline{x}}$, and thus the average distortion satisfies

$$D \leq \sum_{\underline{x}} p(\underline{x})\, d_{\underline{x}}.$$

Such an encoder is not of the form described in Section I and shown in Fig. 1, for which the theorem is stated, but it can be put into this form by adding a further coding step that maps integers into variable-length sequences from a finite alphabet. The proof that it is possible to do this step and have the average output sequence length arbitrarily close to H(K), the entropy of the integer distribution, is given in Appendix A. Thus we need concern ourselves only with the entropy of this distribution, and if R is the encoder's rate per source output, we can write

$$n R = H(K).$$

This quantity can be upper-bounded by

$$H(K) \leq H(K, L) = H(L) + H(K|L),$$

where L is the ensemble of $\underline{x}$ compositions. But

$$H(K|L) = \sum_{\underline{\ell}} p(\underline{\ell})\, H(K|\underline{\ell})$$

$$= \sum_{\underline{x}} p(\underline{x})\, H(K|\underline{x})$$

$$= H(K|X).$$

Thus

$$n R \leq H(L) + \sum_{\underline{x}} p(\underline{x})\, H(K|\underline{x}). \tag{1}$$

Now to make analysis possible, we resort to random-coding techniques, assuming that each letter of each code word $\underline{y}_j$ is chosen randomly with univariate distribution $p_c(y)$ independently of all others. We shall find the expected encoder rate, averaged over this ensemble of codes, and then, by the usual argument, we can assert that there must exist at least one set of $\underline{y}$'s that gives performance as good as the average. In other words, there is a code (a set of $\underline{y}$'s) such that when it is used, the resulting $H(K|\underline{x})$ is no greater than $E_y\{H(K|\underline{x})\}$.

We now define

$$q_{\underline{x}} \triangleq \Pr[d(\underline{x}, \underline{y}) \leq n\, d_{\underline{x}} | \underline{x}]$$

and since all of the code words are statistically independent, the k distribution for a fixed $\underline{x}$ is just

$$p(k|\underline{x}) = q_{\underline{x}}(1-q_{\underline{x}})^{k-1},$$

which is a geometric distribution. Appendix C shows that the entropy of such a distribution with parameter q is

$$H(K) = \frac{H(q)}{q},$$

and that this quantity can be upper-bounded by

$$\frac{H(q)}{q} \leq -\log q + 1.$$

Thus

$$E_y\{H(K|x)\} = -\sum_{k=1}^{\infty} p(\underline{k}|x) \log p(k|\underline{x})$$

$$= \frac{H(q_{\underline{x}})}{q_{\underline{x}}}$$

$$\leq -\log q_{\underline{x}} + 1.$$

Substituting this result in Eq. 1, we find

$$n R \leq H(L) + \sum_{\underline{x}} p(\underline{x}) \left[-\log q_{\underline{x}}\right] + 1. \tag{2}$$

This expression can be further upper-bounded by lower-bounding $q_{\underline{x}}$, which we do as follows: By definition, $d(\underline{x},\underline{y}) = \sum_{i=1}^{n} d(x_i, y_i)$ which is, for fixed $\underline{x}$, a sum of independent random variables (not necessary identically distributed). For such cases, a slight modification of techniques developed by Gallager[12] enables us to write

$$\Pr\left[\sum d(x_i, y_i) \leq n\, d_{\underline{x}} | \underline{x}\right] = q_{\underline{x}} \geq B\, e^{\mu_{\underline{x}}(s) - s\mu'_{\underline{x}}(s)}, \tag{3}$$

where s is a real negative parameter,

$$\mu_{\underline{x}}(s) = \log E_{\underline{y}}\{e^{s\, d(\underline{x},\underline{y})}\}$$

$$= \sum_{i=1}^{n} \log \sum_{y_i} p_c(y_i)\, e^{s\, d(x_i, y_i)},$$

and s is determined from the relation

$$\mu'_{\underline{x}}(s) = n\, d_{\underline{x}}.$$

Finally, B is a constant, independent of $\underline{x}$, and for large enough n can be lower-bounded by $B_o n^{-1/2}$, where B_o is a constant. The details of the derivation of this bound are given in Appendix D.

Substituting relation (3) in (2), we obtain

$$n\, R \leq H(L) + \sum_{\underline{x}} p(\underline{x}) \left[s\mu'_{\underline{x}}(s) - \mu_{\underline{x}}(s)\right] - \log B + 1, \tag{4}$$

and d_o, the average distortion threshold over all $\underline{x}$ sequences satisfies

$$n\, d_o = \sum_{\underline{x}} p(\underline{x})\, n\, d_{\underline{x}}$$

$$= \sum_{\underline{x}} p(\underline{x})\, \mu_{\underline{x}}(s), \tag{5}$$

which upper-bounds the actual encoder distortion, as was shown before.

We now define

$$\mu(s) = \sum_{x \in X} p(x) \left(\log \sum_{y \in Y} p_c(y)\, e^{s\, d(x,y)}\right)$$

so that

$$\sum_{\underline{x} \in X^n} p(\underline{x})\, \mu_{\underline{x}}(s) = \sum_{\underline{x}} p(\underline{x}) \sum_{i=1}^{n} \log \sum_{y_i \in Y} p_c(y_i)\, e^{s\, d(x_i, y_i)}$$

$$= \sum_{i=1}^{n} \sum_{\underline{x}} p(\underline{x}) \log \sum_{y_i} p_c(y_i)\, e^{s\, d(x_i, y_i)}$$

$$= n\, \mu(s).$$

Then Eq. 4 becomes

$$n\, R \leq n\left[s\mu'(s) - \mu(s)\right] + H(L) - \log B + 1. \tag{6}$$

Similary, (5) can be written

$$d_o = \mu'(s).$$

Finally, H(L) can be bounded by

$$H(L) \leq M \log (n+1),$$

since the composition vector has M components, each of which may take on no more than n+1 different values.

Bounding B by $B_o n^{-1/2}$, and dividing (6) through by n, we then have

$$R \leq s\mu'(s) - \mu(s) + \frac{(M+1/2) \log (n+1)}{n} + \frac{1 - \log B_o}{n},$$

in which the third term is a combination of the bound on H(L) and the $n^{-1/2}$ part of the bound on B, with n upper-bounded by n+1 in the last case.

Clearly, the last two terms of this expression can be made arbitrarily small by choosing n large enough, so we have established that for any $\epsilon > 0$, there are coding schemes for which

$$R \leq s\mu'(s) - \mu(s) + \epsilon$$

and (7)

$$d_o = \mu'(s).$$

These expressions are still dependent on $p_c(y)$, the distribution from which the code words were selected. Therefore it remains for us to show that by the proper choice of this distribution, the upper bound of (7) is actually equal to R(D). Suppose, then, that $\{p(y|x)\}$ are the transition probabilities of the test channel between X and Y that minimized I(X;Y) for a fixed distortion level D^* (which means that the I(X;Y) induced by this conditional distribution is equal to $R(D^*)$). It has been shown in Section I that $p(y|x)$ must satisfy

$$\frac{p(y|x)}{q(y)} = \frac{e^{s\, d(x,y)}}{\sum_{y'} q(y')\, e^{s\, d(x,y')}} \qquad \text{(all } x \text{ and } y\text{)},$$

where

$$q(y) = \sum_{x \in X} p(x)\, p(y|x),$$

and s is determined from the condition that the expected distortion between x and y be D^*, or

$$\sum_{x,y} p(x)\, p(y|x)\, d(x,y) = D^*.$$

Let us select $p_c(y)$ to be the output distribution of this test channel, $q(y)$. Then

$$R(D^*) = I(X;Y)$$

$$= \sum_{x,y} p(x)\, p(y|x) \log \frac{p(y|x)}{p_c(y)}$$

$$= \sum_{x,y} p(x) \frac{p_c(y)\, e^{s\, d(x,y)}}{\sum_{y'} p_c(y')\, e^{s\, d(x,y')}} \left[s\, d(x,y) - \log \sum_{y'} p_c(y')\, e^{s\, d(x,y')} \right]$$

$$= s\mu'(s) - \mu(s),$$

since the condition on the distortion becomes

$$D^* = \sum_{x,y} p(x)\, p(y|x)\, d(x,y)$$

$$= \sum_{x,y} p(x) \frac{d(x,y)\, p_c(y)\, e^{s\, d(x,y)}}{\sum_{y'} p_c(y')\, e^{s\, d(x,y')}}$$

$$= \mu'(s).$$

Thus this selection of $p_c(y)$ causes the upper bound on achievable coder rate and distortion of Eq. 7 to be equal to R(D), which completes the proof. It is interesting to note that the coding theorem for block codes can be proved by using the machinery developed above.

It is easy to see that in their present form, the preceding derivations are applicable only to discrete sources, because of the necessity of the composition argument that introduced the H(L) term. Thus, to get the same theorem for continuous sources, we must employ different tactics. Our encoders in this case will be built around a block encoder, which is shown in Appendix B to be capable of satisfactorily encoding the source outputs with probability approaching one. From this fact, it will follow that all that we have to do is add provisions for encoding those rare bad $\underline{x}$ sequences with finite distortion and rate, since the over-all distortion and rate can then be made arbitrarily near those of the block code. To do this finite-distortion encoding, we simply quantize each component of $\underline{x}$ with an infinite-level, uniform quantizer, and show that the rate of this device, which is just the entropy of its output, is finite. To this end, we state the following lemma.

Lemma: Suppose that the source density $p(x)$ is such that $p(x) \log p(x)$ has a finite Riemann integral, and that there exist constants ϵ and A such that $0 < \epsilon$, $A < \infty$ and $d(x, y) \leq A$ whenever $|x-y| < \epsilon$. Then there is an infinite-level quantizer that has a finite rate and distortion when operating on the given source.

Proof: Let the line be quantized into intervals of length Δ, and let q_j be the probability of I_j, the j^{th} interval. Let

$$p_j = \operatorname*{Max}_{x \in I_j} p(x).$$

Then the entropy of the discrete distribution $\{q_j\}$ is

$$H(Y) = -\sum q_j \log q_j$$

$$\leq -\sum \Delta\, p_j \log \Delta\, p_j,$$

since the function p log p is monotone increasing on $0 \leq p \geq \frac{1}{e}$, and Δ can be chosen small enough so that $q_j < \frac{1}{e}$ for all j. Thus

$$H(Y) \leq \Delta\left(-\sum p_j \log p_j\right) - \left(\Delta \sum p_j\right) \log \Delta.$$

But as $\Delta \to 0$ (8)

$$-\Delta \sum p_j \log p_j \to -\int p(x) \log p(x)\, dx = H(X)$$

and

$$\Delta \sum p_j \to \int p(x)\, dx = 1.$$

Since H(X) is finite by assumption, there is some finite Δ_1 for which the sums in Eq. 8 are finite. If we then choose Δ, the quantization interval spacing, equal to Min (Δ_1, ϵ), we are guaranteed that H(Y) is finite and that the distortion never exceeds A. Q.E.D.

We are now prepared to demonstrate an encoder for continuous sources without the requirement that there be an output letter guaranteeing finite distortion, that was necessary for the proof of the block-coding theorem. The scheme that we shall use consists basically of a block code with $M = e^{nR^*}$ code words. The source sequence $\underline{x}$ to be encoded is compared with each of these code words in turn, and if one is found for which $d(\underline{x}, \underline{y}) \leq nD^*$, this word is transmitted. We know from Appendix B that if $R^* < R(D^*)$, then codes exist for which the probability of failing to find such a code word, which we denote p_o, is arbitrarily small. In the rare event that no good code word (average distortion less than D^*) is found, then the encoder simply quantizes each component of $\underline{x}$ independently, and sends the quantizer outputs.

If we make the same assumptions about the source and distortion measure as in the statement of the lemma above, then we know that such a quantizer exists with finite distortion and output entropy, which we denote D_q and R_q, respectively. Then the over-all rate and distortion of the encoder satisfy

$$D \leq (1-p_o) D^* + p_o D_q$$

and

$$R \leq (1-p_o) R^* + p_o R_q + H(p_o),$$

which approach D^* and R^*, respectively, as $p_o \to 0$. It is shown in Appendix A that the output of the infinite-level quantizer can be encoded by a variable-length code, so that our encoder can be put into the form of those described in Section I. Since p_o can be made to approach zero if R^* is chosen greater than R(D), we have proved the source-coding theorem for continuous sources, which we summarize in the following theorem.

Theorem 4: Suppose that the source entropy H(X) is finite (with the integral taken to be a Riemann integral), and the distortion measure satisfies the condition stated in the lemma above. Then source encoders exist (variable-length) with rate and distortion arbitrarily close to R(D).

2.3 RATE OF APPROACH TO R(D)

It has been shown that by allowing encoders to have arbitrarily large block lengths, it is possible to get rates and distortions approaching the R(D) function. But it is of practical and theoretical interest to know more than this; we would like to say not only what the limiting performance is but also how large a block length is needed to get close to this limit. Thus researchers have been led to investigate the rates and distortions of encoders as a function of their block length, just as channel-coding theorists have sought the minimum probability of error for specified code length.

Since this problem has been extensively studied by Pilc,[7] we shall spend little time on it. Our contribution is merely to note that an upper bound to the possible rate for a given distortion as a function of the encoder block length can be obtained easily as a by-product of the results of section 2.2. This result agrees with some obtained by Pilc, and its derivation seems simpler. In fact, we see directly from Eq. 6 that there exist devices acting on n source letters and putting out an integer as output which guarantee distortion, at most, D, and satisfy

$$H(K) \leq n\,R(D) + c_1 \log n + c_2,$$

where H(X) is the entropy of the output distribution and the c's are constants. Appendix A tells us that this distribution can then be encoded into code words with letters in a finite alphabet for which $\bar{n}$, the average number of digits put out, satisfies

$$\bar{n} \leq H(K) + 1.$$

Therefore, since R, the rate per source output, satisfies $n\,R = \bar{n}$, we have

$$R \leq R(D) + \frac{c_1 \log n}{n} + \frac{c_2 + 1}{n}.$$

Thus, in general, we see that the difference between R(D) and the encoder rate can be made to decrease as fast as $\frac{\log n}{n}$. This algebraic rate of approach corroborates the results of Pilc, and is in contrast to the exponential decay of error probability for channel codes.

There are two sources for this $\frac{\log n}{n}$ term, one being the bound on the probability of a randomly chosen code word having acceptable distortion, and the other being the bound on the entropy of the composition of the $\underline{x}$ sequences. It is the latter term that makes the derivation above valid only for discrete sources. It seems unlikely that tighter results can be obtained for general sources and distortions, since the probability bound that is used is the one that is best known.

We shall see in some examples of special cases in which the symmetry of the distortion measure and the q(y) distribution cause the $d(x_i,y_i)$ random variables to be identically distributed. This property permits simpler derivations, although no tighter results, than in the general case. It also permits us to dispense with the composition argument, and thus the H(L) term, in which case continuous sources may be treated no differently from discrete ones.

2.4 SOME EXAMPLES

We shall examine two special cases of sources and distortion measures which have some interesting features. It is hoped that the repeating of the coding theorem proofs for these cases will not be too repetitious, but will add insight into the general problems, as well as point out spots where special properties allow us to obtain simpler bounds than those discussed above.

First, consider encoding an equiprobable binary source with Hamming distance as the distortion measure. This means that the distortion matrix is

$$\{d\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and the distortion between two blocks of length n is just 1/n times the number of positions in which they differ. As in section 2.2, we consider an encoder that has an arbitrarily long list of code words $\underline{y}_j$ which it compares in a preset order with $\underline{x}$, the source output sequence, until a $\underline{y}_j$ is found for which $d(\underline{x},\underline{y}_j) \leq nD$. Note that D does not depend on $\underline{x}$ in this special case. The encoder puts out an integer to specify which code word first satisfactorily matches $\underline{x}$, and from the results in Appendix A we know that we may consider the rate to be H(K), the entropy of this integer distribution. Of course, the average distortion must be less than D.

As usual, we consider the $\underline{y}_j$ to have each letter chosen randomly and independently of all other letters with equal probability of being a one or a zero and, just as in section 2.2, we can assert that there exists a set of code words for which

$$H(K) \leq \frac{H(q)}{q},$$

where

$$q = \Pr[d(\underline{x}, \underline{y}) \leq n\,D],$$

and is independent of $\underline{x}$, because of the symmetries of the distortion measure and the distribution from which the code words are chosen. It is shown in Appendix C that

$$\frac{H(q)}{q} \leq -\log q + 1,$$

and to further upper-bound this quantity, we must lower-bound q. Since $d(\underline{x}, \underline{y})$ is the sum of independent equiprobable binary random variables,

$$q = \frac{1}{2^n}\left[1 + \binom{n}{1} + \dots + \binom{n}{nD}\right]$$

$$\geq 2^{-n}\binom{n}{nD}.$$

We now use a bound developed by Shannon for this type of problem, which is

$$\binom{n}{nD} > \frac{2^{nH(D)}}{\sqrt{2\pi\, n D(1-D)}} \exp\left[-\left(\frac{1}{12\,nD} + \frac{1}{12\,n\,(1-D)}\right)\right].$$

The derivation of this bound from Stirling's approximation to the factorials in the binomial coefficient may be found in Peterson.[25] From this result, we have

$$q \geq \frac{2^{-n(1-H(D))}}{\sqrt{2\pi\, n D(1-D)}} \exp\left[-\left(\frac{1}{12\,nD} + \frac{1}{12\,n\,(1-D)}\right)\right].$$

Since the rate per source output satisfies n R = H(K),

$$n\,R \leq -\log q + 1$$

$$\leq n(1-H(D)) + \left(\frac{1}{12\,nD} + \frac{1}{12\,n\,(1-D)}\right)\log_2 e + \frac{1}{2}\log n + \frac{1}{2}\log 2\pi\, D(1-D), \qquad (9)$$

which, upon dividing through by n, becomes

$$R \leq 1 - H(D) + \frac{c_1 \log n}{n} + \frac{c_2}{n} + \frac{c_3}{n^2}.$$

The 1 - H(D) part of expression (9) can easily be shown directly to be R(D) for the binary source and Hamming distance distortion measure. We shall defer this to section 3.3, where we develop more general machinery from which this result follows as a special case. Thus the coding theorem is proved for this case, and the rate of approach is seen to be $\frac{\log n}{n}$, as expected. Finally, we merely note that in Appendix A it is shown that encoding the integers into a variable length code only adds a 1/n term to the bound on R, so that the $\frac{\log n}{n}$ term continues to be the dominant one.

Our second example is a continuous, "modular" source (that is, one that produces as outputs points on a circle). The probability density that we shall consider is one that is uniform over the entire circle, and the distortion measure will be assumed to be a function only of the angle (<180°) between the two points. This is the modular analog of a difference distortion measure.

As usual, we encode by selecting an arbitrarily long list of code words and search this list until a good one is found. The integer k is put out by the encoder if $\underline{y}_k$ is the first code word on the list satisfying $d(\underline{x}, \underline{y}_k) \leq nD$. As before, we calculate for each $\underline{x}$ the entropy of this integer output when the code words are chosen randomly, with the standard argument that there must be a set of code words giving an entropy, at most, as large as this expectation.

We choose each letter of each code word independently from a distribution that is uniform around the circle, and so for a given $\underline{x}$, it is clear that the k distribution is geometric, with

$$p(k) = q_{\underline{x}}(1-q_{\underline{x}})^{k-1},$$

where

$$q_{\underline{x}} = \Pr[d(\underline{x}, \underline{y}) \leq n D | \underline{x}].$$

Now because of our assumptions about the distortion measure,

$$d(\underline{x}, \underline{y}) = \sum_{i=1}^{n} d(x_i, y_i)$$

$$= \sum d(x_i - y_i),$$

where $x_i - y_i$ should be read as the angle between these two points. But y_i is uniformly distributed, and therefore $x_i - y_i$ is also, independently of x_i. Thus $d(\underline{x},\underline{y})$ is a sum of identically distributed random variables, $d(u_i)$, where the u_i are uniformly distributed, and are independent of $\underline{x}$. Thus $q_{\underline{x}} = q$, independent of $\underline{x}$. The argument that the rate satisfies

$$n R \leq -\log q + 1$$

then goes through exactly as before, with the minor difference that one single q is good for all $\underline{x}$, and thus the composition argument that is necessary in the general case can be dispensed with.

Still following the usual procedure, we seek a lower bound to q, but now, by virtue of the fact that the d's are identically distributed, a slightly different bound is possible. We rewrite

$$q = \Pr\left[\sum d_i \leq n D\right],$$

where we have treated the $d_i = d(u_i)$ as random variables with their own distributions, and use the lower bound given by Gallager,[26] which gives

$$q \geq \frac{1 - o(n)}{\sqrt{2\pi n s^2 \mu''(s)}} \exp[\mu(s) - s\mu'(s)],$$

where s is determined from

$$\mu'(s) = n D$$

and

$$\mu(s) \triangleq \log E\left\{e^{s \Sigma d_i}\right\}$$

$$= n \log E\left\{e^{s d}\right\},$$

where d is a random variable with the same distribution as the d_i. It is simple to show that

$$s\mu'(s) - \mu(s) = n R(D),$$

by showing that the optimum test channel must have a uniform output and using the procedure of section 2.2. Thus

$$n R \leq -\log q + 1$$

$$= n R(D) + \frac{1}{2} \log (2\pi n s^2 \mu''(s)) + 1,$$

and dividing through by n shows that the convergence is again $\frac{\log n}{n}$.

Note again that the list-encoding argument works for this continuous source because the entropy of the integer output did not depend on $\underline{x}$, which allows us to dispense with the composition argument.

2.5 CAPABILITIES OF BLOCK CODES

In section 2.2, we discussed conditions under which block coders could approach R(D) for all values of D. Now we shall investigate this class of encoders in greater detail, and it will be seen that, even when they are not capable of achieving the performance guaranteed by the coding theorem for all D, often there is still a range of D for which they are optimal. In order to get at this result and to find this range of distortions (or equivalently, a range of rates), we first investigate the encoding by means of block codes of discrete sources with a distortion measure whose values are either zero or infinity. We derive an expression for the smallest number of code words necessary to give zero distortion, and knowledge of this minimum block-code rate for such a distortion measure will then allow us to solve several related problems involving block codes. For example, not only can we specify the range of rates for which such codes can approach the Rate-Distortion function in performance but also we can calculate the rate of a discrete source relative to a fidelity criterion that requires every letter to be encoded with distortion no more than a specified amount, rather than merely achieving this performance on the average.

The criterion on average distortion with letter distortions either zero or infinite is equivalent to requiring that for every possible source sequence there be a code word whose distortion with this sequence is zero. Thus, for each source letter x, there is a set of "allowable" output letters, and one of these must be used to adequately represent x. These sets can be represented simply by an adjacency diagram like those shown in Fig. 7, which have the interpretation that any output letter that is connected by a line to an input letter may be used to represent that input letter. In both of these examples, the

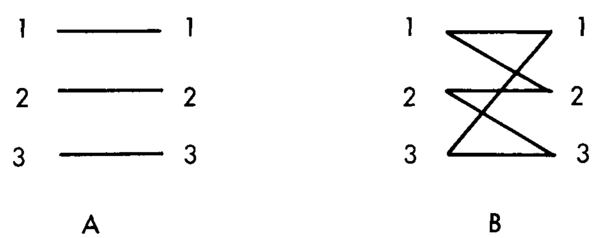

Fig. 7. Typical adjacency diagrams.

input and output are 3-letter alphabets, but in A only the corresponding letter may be used to represent a source output, while in B the first source letter may be coded into (represented by) either of the first two output letters, etc. For the first case, it is easy to see that we need one code word for each possible source sequence, and so we need 3^n words to insure that blocks of n will be adequately represented. In the second case, however, the set $\{1, 2\}$ is a satisfactory set of code words for $n = 1$, and $\{11, 22, 33\}$ is such a set for $n = 2$. We shall see that the rate of this second code, $R = \frac{1}{2} \log 3$, is the smallest that any satisfactory code may have.

Let us now turn to the formal development of our results, and for a block length of n, define $M_{o,n}$ as the smallest number of code words that insures that every source sequence can be represented by at least one code word. Then we define the rate, R_o, by

$$R_o \stackrel{\Delta}{=} \inf_n \frac{1}{n} \log M_{o,n}.$$

Clearly, $R_o \geq R(0)$, the rate-distortion function for the given source and distortion measure evaluated at $D = 0$, since this last function gives the smallest possible rate of any coding scheme, which includes block codes as a subset. Furthermore, if we require all of the source-letter probabilities to be nonzero, which merely states that all letters are really there, then it is easy to see that $M_{o,n}$ is independent of these probabilities. This is true because the probability of each source sequence will then be nonzero, and so must have a representative code word. Thus the inequality $R_o \geq R(0)$ must hold for all source distributions, and so we can write

$$R_o \geq \sup_{\underline{p}} R(0),$$

where $\underline{p}$ is the source probability vector, and the sup is over the open set described by the conditions $p_i > 0$ for all i, and $\sum_i p_i = 1$. Now since I(X;Y) is a continuous function of the probability distributions, so is R(D), and, therefore, we can include the boundary of the region and write

$$R_o \geq \operatorname{Max}_{\underline{p}} R(0),$$

where now the Max is over the set $p_i \geq 0$ and $\sum_i p_i = 1$. Finally, inserting the defining relation for R(0), we have

$$R_o \geq \operatorname{Max}_{\underline{p}} \operatorname{Min}_{p(j|i)} I(X;Y), \tag{10}$$

where the transition probabilities $p(j|i)$ of the test channel are restricted to be nonzero only for those transitions that result in zero distortion (that is, values of i and j for which the distortion between i and j is zero).

We shall now show that the inequality of Eq. 10 is actually an equality by demonstrating the existence of satisfactory codes with rates arbitrarily close to the right-hand side. Since the extrema of functions over closed sets are always attained for some member of the set, there must exist p^* and $p^*(j|i)$ such that the mutual information that they determine is the desired Max Min. Since I(X;Y) is a differentiable function of the probability distributions, it must be stationary with respect to variations in the

$p(j|i)$ distribution at the saddle point. If this were not so, then this point could not be a minimum. Furthermore, since $I(X;Y)$ is convex $\cap$ in $\underline{p}$ and convex $\cup$ in the $p(j|i)$ distribution, the order of the Max and Min can be interchanged without affecting the result.[15] Then by the same reasoning that was used above, $I(X;Y)$ must also be stationary with respect to variations in the nonzero p_i when $p_i = p_i^*$.

Thus, including Lagrange multipliers to satisfy the constraints that probability distributions sum to one, we can write

$$\frac{\partial}{\partial p(j|i)}\left[I(X;Y) + \sum_{i,j} \lambda_i\, p(j|i) + \mu \sum_i p_i\right] = 0 \tag{11}$$

and

$$\frac{\partial}{\partial p_i}\left[I(X;Y) + \sum_{i,j} \lambda_i\, p(j|i) + \mu \sum_i p_i\right] = 0 \tag{12}$$

for all i and j such that p_i^* and $p^*(j|i)$ are nonzero. Since we know

$$\frac{\partial I}{\partial p(j|i)} = p_i \log \frac{p(j|i)}{q_i}$$

and

$$\frac{\partial I}{\partial p_i} = \sum_j p(j|i) \log \frac{p(j|i)}{q_j} - 1,$$

where $q_j = \sum_i p_i\, p(j|i)$, then Eq. 11 becomes

$$p_i \log \frac{p(j|i)}{q_j} + \lambda_i = 0$$

or

$$\log \frac{p(j|i)}{q_j} = r_i,$$

which is independent of j. Then by substituting r_i for $\log \frac{p(j|i)}{q_j}$, Eq. 12 becomes

$$\sum_j p(j|i)\, r_i - 1 + \mu = 0,$$

and by taking r_i outside the summation,

$$r_i = 1 - \mu,$$

and thus is independent of i also. Therefore for maximizing the minimizing distributions, it must be true that

$$\frac{p(j|i)}{q_j} = h^{-1}$$

a constant, for all i and j such that $p_i\, p(j|i) > 0$. It then follows that

$$I(X;Y) = \log \frac{1}{h},$$

and if we define the sets $S_i = \{j | d_{ij} = 0\}$, we can write

$$\sum_{j \in S_i} q_j = \sum_{j \in S_i} h\, p(j|i) = h \qquad \text{(all } i \text{ such that } p_i > 0\text{)},$$

since $p(j|i)$ must be zero for $j \notin S_i$.

For those source letters whose maximizing probabilities turn out to be zero, an expression similar to Eq. 12 must hold, namely

$$\frac{\partial}{\partial p_i}\left[I(X;Y) + \sum_{i\,j} \lambda_i\, p(j|i) + \mu \sum_i p_i\right] \leqslant 0 \tag{13}$$

for all i such that $p_i = 0$. By taking the derivative, this becomes

$$\sum_j p(j|i) \log \frac{p(j|i)}{q_j} - 1 + \mu \leqslant 0$$

or

$$\sum_j p(j|i) \log \frac{p(j|i)}{q_j} \leqslant 1 - \mu = \log \frac{1}{h}. \tag{14}$$

Since the $p(j|i)$ do not affect $I(X;Y)$ when p_i is zero, this inequality must be true for all choices of the $p(j|i)$ values, as long as these are zero for $j \notin S_i$. So, in particular, it must hold when

$$p(j|i) = \begin{cases} \dfrac{q_j}{\sigma}, & j \in S_i \\ 0 & \text{otherwise} \end{cases}$$

where σ is a normalizing constant equal to $\sum_{j \in S_i} q_j$. But with this choice of $p(j|i)$, Eq. 14 becomes

$$\log \frac{1}{\sigma} \leq \log \frac{1}{h}$$

or

$$\sigma = \sum_{j \in S_i} q_j \geq h.$$

Now consider constructing a code by picking each letter of each code word randomly and independently with probability distribution q_j. With this method, the probability that a randomly selected letter will be an acceptable representation of the i^{th} source letter is just $\sum_{j \in S_i} q_j$, which has been shown to be greater than or equal to h, independent of i.

Thus the probability that a randomly chosen code word $\underline{y}$ of block length n will be acceptable for a given source sequence $\underline{x}$ is

$$\Pr[\underline{y} \sim \underline{x} | \underline{x}] \geq h^n$$

for all x.

If $M = e^{nR}$ code words are so chosen, then the code rate is R, and the probability (averaged over all codes) that a given $\underline{x}$ sequence is not covered is

$$\begin{aligned} \Pr[\text{no word for } \underline{x} | \underline{x}] &\leq (1-h^n)^M \\ &\leq e^{-Mh^n} \\ &= e^{-e^{n\left(R - \log \frac{1}{h}\right)}}. \end{aligned}$$

Now setting $R = \log \frac{1}{h} + \delta$, where δ is an arbitrily small positive number, we can bound the probability of all $\underline{x}$ sequences for which there is no code word by

$$\begin{aligned} \Pr[\underline{x} | \nexists \text{ no code word for } \underline{x}] &= \sum_{\underline{x}} p(\underline{x}) \Pr[\text{no word for } \underline{x} | \underline{x}] \\ &\leq e^{-e^{n\delta}}. \end{aligned}$$

By the usual random-coding argument, there must exist a code with performance at least as good as this average, and we can choose n large enough that

$$e^{-e^{n\delta}} < p_{min}^n,$$

where p_{min} is the smallest source-letter probability. Thus for large enough n, there is a code for which

$$\Pr[\underline{x} \mid \exists \text{ no code word for } \underline{x}] \leq e^{-e^{n\delta}} < (p_{min})^n.$$

But this probability must then be zero, since the smallest nonzero value that it could take on is p_{min}^n.

Thus we have shown that for code rates arbitrarily close to $\log \frac{1}{h} = \text{Max Min } I(X;Y)$, there exist codes that will give zero distortion when used with the given source. Since it was shown earlier that such performance was not possible for rates below this value, we have established the block-coding rate of a discrete source with letter distortions that are either zero or infinite to be

$$R_o = \underset{p}{\text{Max}} \underset{p(j|i)}{\text{Min}} I(X;Y).$$

Furthermore, the conditions on stationarity of I given by Eqs. 11, 12, and 13 are sufficient as well as necessary. Thus we know that if we can find distributions satisfying the condition that for some number h, $\frac{p(j|i)}{q_i} = h^{-1}$ for all i and j such that $p_i\, p(j|i) > 0$, then the I(X;Y) determined by these distributions is the desired R_o. We have seen that this condition is equivalent to having all transitions leading to a given output letter have the same probability, and that for each source letter with positive probability, the sum of the probabilities of these output letters that can be reached from this input be the same.

It is interesting to compare these results with those of Shannon on the zero-error capacity of a noise channel.[16] His upper bound on this capacity is identical with our R_o. There are, however, cases for which the zero-error capacity is not known, unlike the zero-distortion rate.

As an example of the calculation of R_o, consider the distortion shown in Fig. 7b. It is easy to see that a source whose letter probabilities are all 1/3, and a channel whose transition probabilities are 1/2 everywhere that there is a line in the diagram, and zero elsewhere, satisfy the conditions of stationarity given above. Thus R_o is I(X;Y) for this source and channel, which comes out to be $\frac{1}{2}\log 3$.

The techniques just developed allow us to solve several other problems concerning block codes for discrete sources. If we want to know the rate needed for zero-average distortion for an arbitrary distortion matrix, it is easy to see that this rate is just R_o for a distortion that is zero everywhere that the given one is, and infinite elsewhere. A second problem is finding the block-code rate necessary for achieving a finite-average distortion when some of the d_{ij} are infinite. In this case, the answer is R_o for the distortion that is infinite everywhere that the given one is, and is zero elsewhere. We shall refer to this distortion as the modified distortion measure. This result allows us to specify when block coders approach R(D), as advertised in section 2.2, as follows:

Theorem 5: Block encoders exist with rate and distortion approaching R(D) if and only if R(D) is greater than the smallest block-code rate that can yield a finite-average distortion for the given source and distortion measure (which rate has been shown to be equal to R_o for the source and modified distortion described above).

Proof: First, the "only if" part of the theorem must hold because otherwise there would be block codes with rates less than R_o, giving zero distortion for the given source and modified distortion measure. Conversely, suppose $R(D) > R_o$, so that we can pick a number R_b such that $R(D) > R_b > R_o$. Then for sufficiently large n, there exist block codes of rate R_b (having e^{nR_b} code words) with finite distortion (say D_m). Then to encode the source, we pick a block code of rate $R^* > R(D)$, which therefore has e^{nR^*} code words, and add the e^{nR_b} code words to this set. If $R^* > R_b$, this addition has a negligible effect on the rate for large n. The distortion of this code can then be bounded by

$$D \leq D^* + P_o D_m,$$

which can be made arbitrarily close to D^* for large enough block length n. Q.E.D. Another statement of this theorem is that $R_b(D) = R(D)$ if and only if $R(D) \geq R_o$.

Finally, the fidelity criterion might be that every letter be encoded with a distortion no greater than D (as opposed to the more usual constraint on the average over a block, which we have been considering up to this point). In this case, the "every letter" block-coding rate $R_{e\ell}(D)$ is just R_o for the distortion that is zero if the original d_{ij} is D or less, and infinite otherwise.

2.6 SOME EXTENSIONS OF THE THEORY

We shall consider two extensions of the basic theory developed in section 2.2. In the first of these, we suppose that the outputs of the source are corrupted by being passed through some noisy memoryless channel (without coding) before the encoder is permitted to look at them. The configuration is shown in Fig. 8a. We are still interested in the distortion between the x's and the y's and, as before, we ask for the relation between this distortion and the necessary rate of the encoder. This problem may be termed the "Noisy Source" problem, and we shall denote the analog of the rate-distortion function for this case by $R_n(D)$.

This "noisy source rate" may be found by reducing the problem as stated to an equivalent one involving ordinary (noisefree) source coding. We assume that we are given the probability distribution of the source, p(x), and also the transition probabilities of the channel, $p(w|x)$. We further assume that a distortion measure d(x, y) between x and y is specified. We define another distortion function, $\hat{d}(w, y)$, between w and y by

$$\hat{d}(w, y) = \int_X p(x|w)\, d(x, y)\, dx, \tag{15}$$

where $p(x|w)$ is the conditional distribution given by

$$p(x|w) = \frac{p(w|x)\, p(x)}{\int_{X'} p(w|x')\, p(x')\, dx'}.$$

If these sets are discrete, merely replace integrals by sums.

Now let $\hat{R}(D)$ be the ordinary rate-distortion function corresponding to a source with output distribution

$$p(w) = \int_X p(w|x)\, p(x)\, dx,$$

and distortion measure $\hat{d}(w, y)$. We claim that $R_n(D) = \hat{R}(D)$. To prove this fact, it suffices to show that all encoders satisfying one of the fidelity criteria must satisfy the other. For any encoder with w and y as input and output alphabets, respectively, there are defined distortions $E\{d(x, y)\}$ and $E\{\hat{d}(w, y)\}$ when connection is made as in parts (a) and (b), respectively, of Fig. 8. Now we can write

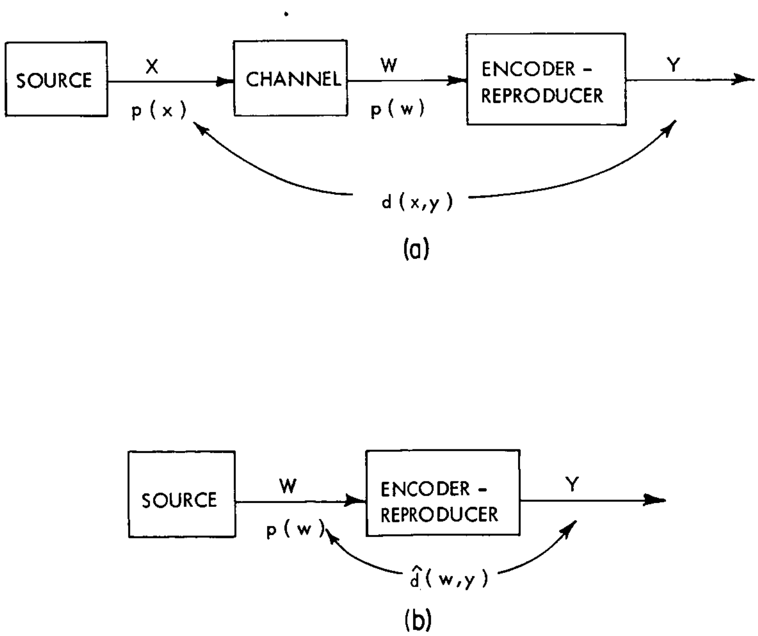

Fig. 8. (a) "Noisy Source" configuration.
(b) Noiseless analog.

$$E\{d(x,y)\} = \int_x \int_y p(x,y)\ d(x,y)\ dxdy$$

$$= \int_x \int_y \int_w p(x,y,w)\ d(x,y)\ dxdydw$$

$$= \int_x \int_y \int_w p(w,y)\ p(x|w,y)\ d(x,y)\ dxdydw.$$

But $p(x|w,y) = p(x|w)$, since the configuration is cascade, so

$$E\{d(x,y)\} = \int_w \int_y p(w,y) \int_x p(x|w)\ d(x,y)\ dxdwdy$$

$$= \int_w \int_y p(w,y)\ \hat{d}(w,y)\ dwdy$$

$$= E\{\hat{d}(w,y)\}.$$

Thus the average distortions that any particular encoder gives rise to in each of the two cases of Fig. 8 are identical. Therefore, the class of all encoders that satisfy the fidelity criterion

$$E\{d(x,y)\} \leqslant D,$$

in case (a), is identical with the class that satisfies the criterion

$$E\{\hat{d}(w,y)\} \leqslant D$$

in case (b). Since the encoder rate obviously does not depend on which configuration the encoder is in, the result is that

$$R_n(D) = \hat{R}(D).$$

To conclude this discussion of the noisy source problem, we can give two examples of calculation of $R_n(D)$, using the result derived above. First, consider the binary symmetric source and channel of Fig. 9, where the source letters are equiprobable, the channel crossover probability is ϵ, and the distortion measure is Hamming distance, that is, $d_{ij} = 1 - \delta_{ij}$. Then the d matrix can be found by using Eq. 15, and turns out to be

$$\{\hat{d}\} = \begin{bmatrix} \epsilon & 1-\epsilon \\ 1-\epsilon & \epsilon \end{bmatrix},$$

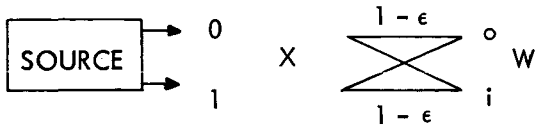

Fig. 9. Binary source corrupted by a BSC.

since $p(x|w) = p(w|x)$ in this case. The p(w) distribution is, of course, the same as p(x), since the channel is symmetric. Thus we need only find R(D) for an equiprobable binary source with $\{\hat{d}\}$ as the distortion. We note that R(D) = 1 - H(D) when the distortion is $d_{ij} = 1 - \delta_{ij}$, and

$$\hat{d}_{ij} = (1-2\epsilon)(1-\delta_{ij}) + \epsilon,$$

and we recall that, by Theorem 1, if

$$d_1 = \alpha d_2 + \beta,$$

then

$$R_1(D) = R_2\left(\frac{D-\beta}{\alpha}\right),$$

where R_1 and R_2 correspond to the distortion measures d_1 and d_2, respectively. If we set $\alpha = 1 - 2\epsilon$ and $\beta = \epsilon$, it follows that

$$R_n(D) = 1 - H\left(\frac{D-\epsilon}{1-2\epsilon}\right)$$

which, we note, behaves as we would expect, with D_{min}, the smallest possible distortion equal to ϵ, and $D_{max} = 1/2$.

The second example is a zero-mean Gaussian source of variance σ_x^2, corrupted by additive Gaussian noise with variance σ_z^2 (see Fig. 10). The distortion measure is

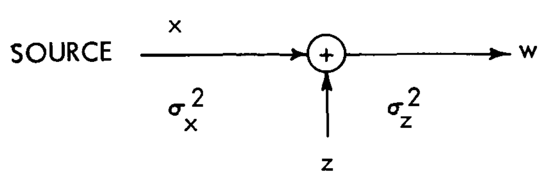

Fig. 10. Source corrupted by additive Gaussian noise.

taken to be mean-square error, $d(x, y) = (x-y)^2$. Clearly, p(w) is Gaussian with variance $\sigma_w^2 = \sigma_x^2 + \sigma_z^2$, and

$$p(x|w) = \frac{p(w|x)\, p(x)}{p(w)}$$

is Gaussian with mean Aw, where

$$A = \frac{\sigma_x^2}{\sigma_z^2 + \sigma_x^2}$$

and with variance $A\,\sigma_z^2$. Then we can calculate

$$d(w, y) = \int p(x|w)(x-y)^2$$

$$= \int x^2\, p(x|w)\, dx - 2y \int x\, p(x|w)\, dx + y^2$$

$$= A\,\sigma_z^2 + A^2 w^2 - 2y\, Aw + y^2$$

$$= A\,\sigma_z^2 + (y-Aw)^2.$$

Letting v = Aw, we see that we want R(D) for a zero-mean Gaussian source p(v) with variance

$$\sigma_v^2 = A^2\,\sigma_w^2 = A^2\left(\sigma_x^2 + \sigma_z^2\right)$$

and distortion

$$\partial(v, y) = A\,\sigma_z^2 + (y-v)^2.$$

Using the result of Theorem 1 and the fact that for a Gaussian source with mean-square distortion, $R(D) = \frac{1}{2}\log\frac{\sigma^2}{D}$, we have

$$R_n(D) = \frac{1}{2}\log\left(\frac{A^2\left(\sigma_x^2 + \sigma_z^2\right)}{D - A\,\sigma_z^2}\right)$$

which, upon substitution for A, becomes

$$R_n(D) = \frac{1}{2}\log\left(\frac{\sigma_x^4}{D\left(\sigma_x^2 + \sigma_z^2\right) - \sigma_x^2\sigma_z^2}\right).$$

Again, we note that this form is what we expected, with the smallest D being just the minimum variance estimate of x made on the basis of w, and the rate becoming infinite as this D is approached. Furthermore, D_{max} is just that D for which R = 0, or for which the argument of the log is 1. This turns out to be σ_x^2, also as expected.

The second extension of the theory that we shall discuss comes about when the source outputs are required to be encoded to simultaneously satisfy several fidelity criteria, rather than just one. The user specifies distortion measures $d_1(x,y) \dots d_n(x,y)$, and tolerance levels $D, \dots D_n$, and insists that $E\{D_i(x,y)\} \leq D_i$ for all i simultaneously.

For this case, the rate-distortion function becomes a function of the n variable $D_1 \dots D_n$, and is defined by

$$R(D_1 \dots D_n) = \min_{\{p(y|x)\}} I(X;Y),$$

with the constraints

$$E\{d_i(x,y)\} \leq D_i \qquad 1 \leq i \leq n.$$

The proof of the coding theorem for this case is so similar to the proof in the single-distortion case that we merely indicate the few places where the two differ, and leave the filling in of the details as an exercise.

The fact that no encoder can have rate below $R(D_1, \dots D_n)$ while still satisfying the fidelity criteria is proved exactly as in section 2.2. The positive-coding theorem parallels the proof in Appendix B if all of the distortions satisfy the condition on the single-distortion measure there, that there be an output letter guaranteeing finite distortion. The selection of code words and definitions of R^* and $D_1^* \dots D_n^*$ are exactly as for the ordinary case, and P_o is defined by

$$P_o = 1 - \Pr\left[d_i(\underline{x}, \underline{y}(\underline{x})) \leq n\left(D_i^* + \frac{\epsilon}{2}\right) \text{ for all } i\right]$$

so that similarly, the i^{th} distortion satisfies

$$D_i \leq D_i^* + \frac{\epsilon}{2} + P_o D_{max}.$$

The definition of the set A must be modified to

$$A = \left\{(\underline{x},\underline{y}) \,\middle|\, I(\underline{x};\underline{y}) > N\left(R^* + \frac{\epsilon}{2}\right) \text{ or } d_1(\underline{x},\underline{y}) > N\left(D_1^* + \frac{\epsilon}{2}\right) \text{ or} \dots \text{or } d_n(\underline{x},\underline{y}) > N\left(D_n^* + \frac{\epsilon}{2}\right)\right\},$$

where N is now the block length of the encoder. Then the same argument as that in Appendix B carries through for the rest of the proof.

III. EVALUATION OF R(D)

We now turn to consideration of methods by which the rate-distortion function R(D) may be evaluated, at least for some interesting special cases. The desirability of knowing this function has already been discussed in Section I, and thus needs no additional motivation. Unfortunately, like channel capacity, R(D) is not easily calculated in general; direct attacks using the definition as a Min I(X;Y) can be made, but they cannot be carried through to a general closed form answer. Various researchers have obtained sets of linear equations to be solved for the transition probabilities of the test channel,[1,3,17] but the fact that the general solution is unwieldly and cannot even be guaranteed to produce non-negative probabilities severely limits the usefulness of such results. Indeed, the R(D) function has been found explicitly only for a very few special cases: Shannon[2] solved the case of a uniform discrete source and a distortion matrix with the same set of entries in each row and column, and the case of a time-discrete Gaussian source with mean-square distortion; Kolmogorov,[6] Jordan, and Holsinger all independently extended the last solution to time-continuous Gaussian sources. This is the only known result for the time-continuous case.

Since the task of finding R(D) exactly is so difficult, we are led to investigate means of approximating it. We shall give ways of estimating the rate of a source relative to a class of distortion measure called difference distortion measures. In the continuous case, a difference distortion measure is one that is a function only of the magnitude of the difference between input and reproduction, $d(x,y) = d(|x-y|)$. A common example is mean-square error, where $d(x,y) = (x-y)^2$. For discrete sources the condition is that the distortion matrix have the same set of entries, although permuted, perhaps, in each row and column.

Most of the results given here will stem from a lower bound to R(D) for continuous sources and difference distortion measures which was developed by Shannon. We shall extend this result to include discrete sources, and give a means of calculating the bound in both cases. Then this bound will be related to R(D) by deriving necessary and sufficient conditions under which equality holds, and we shall show that even when R(D) is not identically equal to the lower bound, it still provides a good approximation at small distortion levels. Finally, R(D) for the case of a discrete source and distortion $d_{ij} = 1 - \delta_{ij}$ will be calculated for all D.

3.1 LOWER BOUND ON R(D)

In his 1959 paper, Shannon derived a lower bound to R(D) for a continuous source with a difference distortion measure, which has the form

$$R(D) \geq H(X) - \phi(D), \tag{16}$$

where H(X) is the (continuous) entropy of the source distribution p(x), and $\phi(D)$ is defined by

$$\phi(D) \triangleq \underset{p_z(z)}{\mathrm{Max}}\ H(Z), \tag{17}$$

where p(z) is any probability density, and the maximization is subject to the constraint

$$E_z\{d(z)\} \triangleq \int d(z)\, p(z)\, dz \leq D.$$

Note that d can be written as a function of one argument only because it has been assumed to be a difference distortion. At this point, let us state the convention that when the limits of integration are unspecified, they are $-\infty$ and ∞.

Now suppose that we have found the test channel for the given source, distortion, and distortion level, so that

$$\begin{aligned} R(D) &= I(X;Y) \\ &= H(X) - H(X|Y) \\ &= H(X) - \int p(y)\, H(X|y)\, dy. \end{aligned}$$

Denoting the expected distortion between the random variable x and the point in the output space y by D_y, we can write

$$H(X|y) \leq \phi(D_y),$$

by the definition of ϕ as the maximum of all such entropies. Therefore

$$R(D) \geq H(X) - \int \phi(D_y)\, p(y)\, dy.$$

Now $\phi(D)$ is convex $\cap$ as a function of its argument, which can be seen as follows: Suppose $p_1(z)$ and $p_2(z)$ are the maximizing distributions that give $\phi(D)$ and $\phi(D_2)$, respectively. Then

$$\int (\alpha\, p_1(z) + \beta\, p_2(z))\, d(z)\, dz = \alpha\, D_1 + \beta\, D_2$$

so

$$\begin{aligned} \phi(\alpha\, D_1 + \beta D_2) &\geq H(\alpha\, p_1 + \beta\, p_2) \\ &\geq \alpha\, H(p_1) + \beta\, H(p_2) \\ &= \alpha\, \phi(D_1) + \beta\, \phi(D_2). \qquad \text{Q.E.D.} \end{aligned}$$

As a result of this convexity,

$$\overline{\phi(D)_y} \leq \phi(\overline{D}_y) = \phi(D)$$

where the bar denotes expectation with respect to the p(y) distribution; that is,

$$D = \overline{D_y} = \int D_y \, p(y) \, dy.$$

Thus it follows that

$$R(D) \geq H(X) - \phi(D).$$

Up to this point, the derivation has followed Shannon.[2] Hereafter, the results presented are original. A few of these have appeared elsewhere,[19] and are repeated here for convenience.

The function $\phi(D)$ can be easily calculated by using calculus of variations techniques. Introducing Lagrange multipliers s and t to allow for the constraints

$$\int p(z) \, d(z) \, dz = D$$

and

$$\int p(z) = 1,$$

respectively, we have

$$\phi(D) = \underset{p(z)}{\text{Max}} \int \left[-p(z) \log p(z) + s d(z) \, p(z) + t p(z)\right] dz$$

$$= \underset{p(z)}{\text{Min}} \int p(z) \left[\log p(z) - s d(z) - t\right] dz.$$

The condition on the extremal p(z) is that the derivative of the integrand with respect to p be zero, or

$$\log p(z) - s \, d(z) - t + 1 = 0.$$

The solution to this equation is

$$p(z) = C \, e^{sd(z)},$$

where t has been absorbed into the constant C, which is adjusted so that p(z) integrates to 1. It is convenient to define a function

$$A(s) = \int e^{sd(z)}\, dz \tag{18}$$

so that

$$C = \frac{1}{A(s)},$$

and the parameter s is still to be determined from the constraint

$$\int p(z)\, d(z)\, dz = D$$

or

$$\int d(z)\, e^{sd(z)}\, dz = D\, A(s). \tag{19}$$

The expression for $\phi(D)$ then becomes

$$\phi(D) = -\int p(z) \log p(z)\, dz$$

$$= \int \frac{e^{sd(z)}}{A(s)} [\log A(s) - s\, d(z)]\, dz$$

$$= \log A(s) - s\, D, \tag{20}$$

with s still determined by Eq. 19. $\phi(D)$ can be calculated for some special cases by finding A(s) from Eq. 18, solving Eq. 19 for s, and plugging the result in Eq. 20. In general, however, it is convenient to express ϕ and D parametrically in s. From Eq. 19, it can be seen that

$$D = \frac{A'(s)}{A(s)},$$

and substituting this in Eq. 20 yields

$$\phi = \log A(s) - \frac{s\, A'(s)}{A(s)}. \tag{21}$$

Clearly, A(s) and A'(s) are well-behaved for $s < 0$, as long as d(z) increases faster than logarithmically, but not faster than exponentially in $|z|$. The range $s \geq 0$ is not of interest to us, since as $s \to 0$, $D \to \infty$, and as $s \to -\infty$, $D \to \underset{z}{\text{Min}}\, d(z)$, which is usually assumed to be zero and to occur at $z = 0$.

Clearly, D is continuous in s, and can be seen to be monotone increasing in this parameter by

$$\frac{dD}{ds} = \frac{d}{ds}\left(\frac{A'(s)}{A(s)}\right)$$

$$= \frac{A''(s)}{A(s)} - \left(\frac{A'(s)}{A(s)}\right)^2$$

$$= \int (d(z)-D)^2 \; p(z) \, dz > 0.$$

Similarly, ϕ is monotone increasing in s, since

$$\frac{d\phi}{ds} = -s \frac{d}{ds}\left(\frac{A'(s)}{A(s)}\right),$$

which is positive for $s < 0$, the range of interest. It is perhaps interesting to note that the parameter s is the slope of the lower-bound curve, since

$$\frac{d\phi}{dD} = \frac{\frac{d\phi}{ds}}{\frac{dD}{ds}} = -s.$$

This derivation has applied only to continuous sources, but the discrete case can be treated in an almost identical manner. Here, the analog of a difference distortion measure is a matrix containing the same set of entries in each row and column. A lower bound on R(D) that is analogous to the one presented above for the continuous case can easily be derived for this case. If the source and reproduction alphabets each have M letters, we define

$$\phi(D) = \operatorname*{Max}_{\underline{z}} H(Z),$$

where $\underline{z}$ is an M-component probability vector, and the Max is taken subject to the constraint $\Sigma z_i d_i = D$, where $\{d_i\}$ is the set of column entries of the distortion matrix. By proceeding exactly as in the continuous case, it is easy to show that

$$I(X;Y) \geqslant H(X) - \phi(D)$$

for all channels such that $E\{d_{ij}\} \leqslant D$, and therefore, in particular, this relation must hold for the channel of this class with the smallest mutual information, which is R(D).

The function $\phi(D)$ can also be calculated exactly as before, with the result

$$\phi = \log A(s) - s \frac{A'(s)}{A(s)}$$

$$D = \frac{A'(s)}{A(s)},$$

where now

$$A(s) = \sum_i e^{sd_i},$$

and again we are only concerned with negative values of s.

At this point, it should be pointed out that the assumption of a difference distortion (which term includes both discrete and continuous cases) was slightly stronger than we needed to derive these lower bounds. All that was really needed was that, for every output letter, the set of possible distortions between it and the source be the same, which is equivalent to merely requiring each column of the distortion matrix to have the same set of elements, rather than to place such a requirement on both rows and columns. Non-difference distortions satisfying this condition are rather artificial and uninteresting, and so in the interest of ease of presentation the difference between the classes of difference distortions and those for which the lower bound applies has not been stressed.

We shall conclude with some examples of lower bounds of the type derived above. For the case of a zero-mean Gaussian source with variance σ^2, and distortion measure $d(x,y) = (x-y)^2$, we calculate

$$A(s) = \int_{-\infty}^{\infty} e^{sz^2}\, dz$$

$$= \sqrt{\frac{-\pi}{s}} \int \frac{e^{\frac{z^2}{2\left(\frac{1}{2s}\right)}}}{\sqrt{\frac{-\pi}{s}}}\, dz$$

$$= \pi(-s)^{-1/2}$$

so

$$A'(s) = \frac{\sqrt{\pi}}{2}(-s)^{-3/2}$$

and

$$D = \frac{A'(s)}{A(s)} = \frac{-1}{2s}.$$

Then

$$\phi = \log A(s) - s\,D$$

$$= \frac{1}{2}\log \pi - \frac{1}{2}\log(-s) + \frac{1}{2}$$

$$= \frac{1}{2}\log\left(\frac{-\pi e}{s}\right).$$

Substituting for s in terms of D, we have

$$\phi(D) = \frac{1}{2} \log (2\pi e D),$$

and since the entropy of a Gaussian distribution of variance σ^2 is known to be $\frac{1}{2} \log (2\pi e\sigma^2)$, the lower bound is

$$R(D) \geq H(X) - \phi(D)$$

$$= \frac{1}{2} \log \frac{\sigma^2}{D}. \tag{22}$$

A second example is a binary source with distortion $d_{ij} = 1 - \delta_{ij}$, or

$$\{d\} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Here,

$$A(s) = \sum e^{sd_i} = 1 + e^s$$

and

$$A'(s) = e^s.$$

Therefore

$$D = \frac{e^s}{1 + e^s},$$

which implies

$$e^s = \frac{D}{1 - D}.$$

Then

$$\phi = \log (1+e^s) - s\,D$$

$$= \log \left(1 + \frac{D}{1 + D}\right) - D \log \frac{D}{1 - D}$$

$$= \log \left(\frac{1}{1 - D}\right) - D \log \frac{D}{1 - D}$$

which becomes

$$\phi = H(D).$$

Therefore

$$R(D) \geq H(X) - H(D),$$

which is valid for any probability assignment on the two source letters. We shall see that both of these lower bounds are, in fact, equal to R(D).

3.2 RELATION OF THE LOWER BOUND TO R(D)

We shall now show that the lower bound to R(D) just presented is more than just a lower bound. In fact, it is intimately related to R(D). We shall show that, in general, for continuous sources, $R(D) \to H(X) - \phi(D)$ as $D \to 0$, thereby making this bound a useful tool for approximating R(D) for small D. Furthermore, under certain conditions, these two functions are actually identical. For discrete sources, similar results hold, except that instead of merely approaching R(D), as $D \to 0$, there is always a nonzero region of D for which equality holds.

The conditions under which R(D) is actually equal to $H(X) - \phi(D)$ are stated in the following theorem.

Theorem 6: Let $p(z) = \frac{e^{sd(z)}}{A(s)}$, where d( ) is the difference distortion measure

$$A(s) = \int e^{sd(z)}\, dz,$$

and the parameter s is related to the allowable distortion level D by

$$D = \frac{A'(s)}{A(s)}.$$

Thus $H(Z) = \phi(D)$ for this distortion measure. Furthermore, suppose there exists a probability density p(y) such that p(y) convolved with p(z) yields p(x), the source density. Then under these conditions $R(D) \equiv H(X) - \phi(D)$, where $\phi(D)$ is defined by Eq. 17.

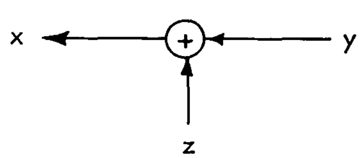

Fig. 11. "Backward" additive noise channel.

Proof: We note that this condition implies the existence of a "backward additive noise channel" as shown in Fig. 11, where $x = y + z$, with y and z independent. The average distortion for this channel is

$$E\{d(x,y)\} = E\{d(x-y)\}$$
$$= E\{d(z)\}$$
$$= \int d(z) \frac{e^{sd(z)}}{A(s)}$$
$$= \frac{A'(s)}{A(s)}$$
$$= D,$$

and the mutual information is

$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(X) - H(Z).$$

Because of our choice of p(z), we then have

$$I(X;Y) = H(X) - \phi(D).$$

Since we have demonstrated a channel with rate and distortion equal to the lower bound, it follows that this bound must be identical with R(D).

This procedure is important because almost all of the known R(D) functions are found by this method. For example, if p(x) is Gaussian with zero mean and variance σ^2, and $d(u) = u^2$, then for distortion level D, it can be seen that p(z) is Gaussian with variance D, and as long as $D \leq \sigma_x^2$, we can choose p(y) to be Gaussian with variance $\sigma_x^2 - D$. Thus for $D < \sigma_x^2 = D_{max}$, the rate-distortion function for a Gaussian source with mean-square error as the distortion is

$$R(D) = H(X) - \phi(D)$$
$$= \frac{1}{2} \log \frac{\sigma_x^2}{D}$$

by Eq. 22. This result was first derived by Shannon. It can be shown similarly, for a source with

$$p(x) = \frac{c}{2} e^{-c|x|}$$

and distortion

$$d(u) = |u|,$$

that

$$p(z) = \frac{e^{\frac{-|z|}{D}}}{2D}$$

and

$$p(y) = c^2 D^2 \delta(y) + (1-D^2 c^2) \frac{c}{2} e^{-c|y|}$$

satisfy the condition $p(x) = p(y) * p(z)$, and since $\phi(D) = \log 2eD$, then

$$R(D) = \log \frac{1}{cD}.$$

Finally, one last class of source-distortion measure pairs for which $R(D) = H(X) - \phi(D)$ is a modular source (which was defined in section 2.4) with a uniform probability distribution, and any difference distortion measure. We simply select p(y) to be uniform, and note that anything convolved with a uniform distribution on a circle still yields a uniform distribution.

In general, even if the source does not satisfy the conditions for equality given in Theorem 3, the lower bound is still important because for quite general continuous sources and difference distortion measure (see Pinkston[19] for precise conditions on p(x) and d(x, y)), it is true that

$$\lim_{D \to 0} (R(D) - H(X) + \phi(D)) = 0.$$

This fact is proved by finding an upper bound to R(D) and showing that if d( ) is bounded away from zero except in a neighborhood of the origin, then this upper bound approaches

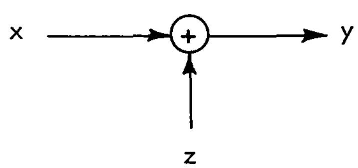

Fig. 12. Additive noise channel.

the lower bound. To see this, consider the additive noise channel of Fig. 12, where

$$p(z) = \frac{e^{sd(z)}}{A(s)}$$

and

$$\frac{A'(s)}{A(s)} = D,$$

which is already familiar. Clearly, for this channel

$$E\{d(x,y)\} = E\{d(x-y)\}$$

$$= E\{d(z)\}$$

$$= \frac{A'(s)}{A(s)} = D.$$

Since the channel's distortion is D, by definition, R(D) must be no greater than its mutual information, so

$$R(D) \leqslant I(X;Y) = H(Y) - H(Y|X)$$

$$= H(Y) - H(Z)$$

$$= H(Y) - \phi(D).$$

But now the difference between the upper and lower bounds is just $H(Y) - H(X)$, and as $D \to 0$, it can be shown that $s \to -\infty$, so that $p(z) = \frac{e^{sd(z)}}{A(s)}$ approaches an impulse, since we assumed that d(0) was the unique minimum value of d(z). Thus the convolution of p(z) with p(x) approaches an identity transformation, so that $p(y) \to p(x)$, and $H(Y) \to H(X)$. Although none of these steps is difficult to show, we shall omit the proofs because they are tedious and have already been written up elsewhere.[19]

Thus we have established that $R(D) \to H(X) - \phi(D)$ as $D \to 0$, and so the latter function provides a good approximation to R(D) for small values of D.

For discrete sources, the condition for equality of R(D) with the lower bound can be phrased, analogously to the continuous case, that there must exist a p(y) such that p(x) is the output distribution of an algebraically additive channel with input p(y). But this statement is not nearly as graphic as its continuous analog, and for our purposes, it is simpler to make the condition as follows:

Theorem 7: Suppose there exists a channel for which p(x) is the input distribution, and

$$p(x|y) = \frac{e^{sd(x,y)}}{A(s)}.$$

Then $R(D) = H(X) - \phi(D)$.

The proof is so similar to that of Theorem 6 that it will be left as an exercise. It is clear that such a channel will exist if there is an output distribution q(y) such that

$$p(x) = \sum_y q(y)\, p(x|y).$$

Since we are dealing with discrete variables, we shall find it convenient to write p(x) and q(y) as vectors p and q, respectively, each of dimension M. Similarly, we will write d(x, y) as an M × M matrix, with entries d_{ij}, i corresponding to the source letter, and j to the output letter. Then the condition for equality of R(D) with the lower bound is that the equation

$$\Omega \underline{q} = \underline{p} \tag{23}$$

must have a solution $\underline{q} = \Omega^{-1} \underline{p}$, with all elements of $\underline{q}$ non-negative, where Ω is an M × M matrix with $\Omega_{ij} = \frac{e^{sd_{ij}}}{A(s)}$.

If the distortion matrix has $d_{ij} = 0$ and $d_{ij} > 0$ if $i \neq j$, then Ω approaches the identity matrix as $s \to -\infty$, and clearly $\underline{q} = \underline{p}$ is a valid solution in this case. The condition $s = -\infty$ has been shown (in section 3.1) to correspond to minimum distortion, and so we have R(D) equal to the lower bound at $D = D_{min}$, which may be taken equal to zero with no loss of generality. Furthermore, since the elements of Ω are continuous functions of s, the elements of the inverse will be likewise, and so $\underline{q}$ will vary continously with s. Thus there must be a range of values of s, $-\infty < s < s_1$ for which Eq. 23 has a valid solution. It then follows that there is a $D_1 > 0$ such that $R(D) = H(X) - \phi(D)$ for all $0 \leq D \leq D_1$, and

$$D_1 = \frac{A'(s_1)}{A(s_1)}.$$

It is interesting to note that if the distortion matrix satisfies the condition that all rows, as well as all columns, have the same entries, and if the source letters are equiprobable, then $\underline{q} = \underline{p} = \left(\frac{1}{M}, \ldots, \frac{1}{M}\right)^T$ is always a solution, and R(D) is equal to the lower bound for all $D \leq D_{max}$. This situation is analogous to the uniform modular source dicussed earlier.

We have seen that the lower bound $H(X) - \phi(D)$ is a powerful tool for obtaining information about the rate-distortion function when d(x, y) is a difference-distortion measure. In fact, virtually every R(D) function that is known exactly is equal to this bound. The only nontrivial exception to this statement known to the author is R(D) for a discrete source with Hamming distance as the distortion measure. We shall see how this function can be found for all D.

3.3 APPLICATION TO THE CONVERSE OF THE CODING THEOREM

Suppose that our source is discrete, and our distortion matrix is $d_{ij} = 1 - \delta_{ij}$, where δ_{ij} is the Kronecker delta. The average distortion is then just the probability that a source letter is not reproduced correctly. Knowing R(D) for this matrix and some source, we can find the minimum achievable probability of error when transmitting this source over a channel of capacity C simply by solving R(D) = C for D, the desired minimum probability. Since R(0) = H(X) and R(D) is monotone decreasing, this minimum per letter probability of error will turn out to be strictly positive if $C < H(X)$, which establishes the coding theorem converse. The present derivation goes beyond this, and actually gives a method of finding this minimum error probability, rather than merely showing it to be positive. It will be convenient to consider the source probabilities ordered, so that $p_1 \leq p_2 \leq \ldots \leq p_M$, which involves no loss of generality.

We shall now find the lower bound to R(D) for this case and see where equality holds. First,

$$A(s) = \sum e^{sd_i} = 1 + (M-1)e^s;$$

therefore,

$$D = \frac{A'(s)}{A(s)} = \frac{(M-1)e^s}{1 + (M-1)e^s}.$$

Solving for s, we get

$$e^s = \frac{D}{(M-1)(1-D)}$$

or

$$s = \log D - \log (M-1) - \log (1-D),$$

so that

$$A(s) = \frac{D}{1 - D}.$$

Now

$$\phi = \log A(s) - sD$$

$$= -\log (1-D) - D[\log D - \log (M-1) - \log (1-D)]$$

$$= H_2(D) + D \log (M-1),$$

where $H_2(D) = -D \log D - (1-D) \log (1-D)$. So the bound is

$$R(D) \geq H(X) - H_2(D) - D \log (M-1).$$

Clearly, if we set the right-hand side of this inequality equal to C, the channel capacity, and solve for D, this will be a lower bound to the minimum possible probability of error. This result is identical with that obtained by Gallager.[20]

We now want to find the region of equality. Writing

$$\Omega = \frac{1}{A(s)} \begin{bmatrix} 1 & \alpha & & \cdots & \alpha \\ \alpha & 1 & \alpha & \cdots & \vdots \\ \vdots & & & & \alpha \\ \alpha & \cdots & & \alpha & 1 \end{bmatrix}$$

where $\alpha = e^s$, and $\alpha < 1$, since $s < 0$, we see that the equations to be solved are

$$\frac{1}{A(s)}\left[q_j + \alpha(1-q_j)\right] = p_j; \qquad \text{for all } j.$$

The solution is

$$q_j = \frac{A(s)\, p_j - \alpha}{1-\alpha}$$

$$= \frac{\left[1+(M-1)\alpha\right] p_j - \alpha}{1-\alpha}. \tag{24}$$

Now α is monotonic as a function of s, and therefore also of D, and $\alpha \to 0$ as $s \to -\infty$ and $D \to 0$. It is clear that for small enough α, all of the q_j are positive, and as α increases, the first one to go negative will be the one corresponding to p_1, the smallest p. We therefore solve the equation

$$q_1(\alpha) = 0$$

for the α below which all of the q's are positive. This equation becomes

$$p_1 + \alpha(M-1)\, p_1 - \alpha = 0$$

or

$$\alpha = \frac{p_1}{1-(M-1)p_1},$$

from which

$$D_1 = \frac{(M-1)\alpha}{1+(M-1)\alpha} = (M-1)p_1$$

is the value of D below which $R(D) = H(X) - H_2(D) - D \log(M-1)$. Clearly, $D_1 \leq D_{max} = 1 - p_{max} = \sum_{i=1}^{M-1} p_i$, with equality if and only if the $M-1$ smallest p_i are equal. As we have seen, equality holds if all p_i are equal, and now we can relax this condition to only the $M-1$ smallest. For example, for $p = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$,

$$R(D) = H(X) - H_2(D) - D$$

$$= 1.5 - H_2(D) - D \text{ bits}; \qquad 0 \leq D \leq \frac{1}{2}.$$

In general, we now have $R(D)$ for $D \leq D_1 = (M-1)p_{min}$. Before we proceed to the task of finding $R(D)$ for $D > D_1$, let us remark that for a binary source, our development indicates that $D_1 = D_{max}$. Therefore

$$R(D) = H(X) - H_2(D)$$

for all binary sources and the distortion $d_{ij} = 1 - \delta_{ij}$.

We now turn to the calculation of $R(D)$ for all D. For this task, we shall need two lemmas, and it is convenient to present them at this time.

Lemma 1: Suppose $R(D)$ for some discrete source p and distortion matrix $\{d_{ij}\}$ is known, and suppose we form the new distortion

$$\hat{d}_{ij} = d_{ij} + w_i$$

(that is, every element of each row has a constant added to it which may be different for different rows). Then

$$\hat{R}(D) = R(D-\overline{w}),$$

where $\overline{w} = \sum_i p_i w_i$, and $\hat{R}$ corresponds to the source $\underline{p}$ and the distortion $\hat{d}$.

Proof: This result is a special case of Theorem 1.

Lemma 2: Suppose a distortion matrix has a row that is all zeros, and p_1 is the probability of the source letter corresponding to that row. Then

$$R(D) = (1-p_1)\hat{R}\left(\frac{D}{1-p_1}\right),$$

where $\hat{R}(\ )$ corresponds to the distortion matrix with the row of zeros deleted, and the source with letter 1 deleted, and probability vector

$$\underline{p}' = \left(\frac{p_2}{1-p_1}, \ldots, \frac{p_M}{1-p_1}\right).$$

Proof: An all-zero row means that we do not care how that source letter is reproduced. Suppose x_1 is this letter. Then we can choose $p_{y|x}(j|1)$ so that $I(X_1;Y) = 0$. With this choice,

$$R(D) = \text{Min } I(X:Y)$$

$$= \text{Min} \left[p_1 I(x_1;Y) + \sum_{i=2}^{M} p_i I(x_i;Y) \right]$$

$$= \text{Min} \left[(1-p_1) \sum_{i=2}^{M} \frac{p_i}{1-p_1} I(x_i;Y) \right],$$

and the constraint is

$$E_{xy}\{d_{ij}\} \leq D$$

or

$$\sum_i p_i E_y\{d_{ij} | x=i\} \leq D;$$

but $E_y\{d_{ij} | x=1\} = 0$, so the constraint becomes

$$\sum_{i=2}^{M} \frac{p_i}{1-p_1} E_y\{d_{ij} | x=i\} \leq \frac{D}{1-p_1}.$$

Now, by definition of R(D), the assertion follows.

This lemma roughly states that if a row is all zeros, we need not expend any information on the transmission of the corresponding source letter.

We are finally prepared to find R(D) for the distortion $d_{ij} = 1 - \delta_{ij}$ for all values of D. We simply note that the solutions for q_j of Eq. 24 are monotonic in α, and therefore in D. Thus once a q_j goes to zero, it never becomes positive for any larger D. Since p_1 is the smallest p_i, q_1 is the first output probability to become 0, which occurs at $D = D_1 = (M-1) \log p_1$. Then we know that for $D > D_1$, output 1 will never be used, and we can, therefore, remove it from the output alphabet and delete the corresponding column from $\{d_{ij}\}$ without affecting R(D). Thus for $D > D_1$, we might as well have the distortion matrix

$$d_{ij} = \begin{bmatrix} 1 & 1 & 1 & \cdots & & 1 \\ 0 & 1 & 1 & \cdots & & 1 \\ 1 & 0 & 1 & & \cdots & 1 \\ \vdots & & & & & \vdots \\ 1 & \cdots & & & 1 & 0 \end{bmatrix}$$

with M rows and M - 1 columns, and the first row is all 1's. Now by Lemma 1, we can subtract 1 from the top row, and write

$$R(D) = R^{(1)}(D-p_1),$$

where $R^{(1)}$ corresponds to the matrix

$$d_{ij}^{(1)} = \begin{bmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 1 & \cdots & & 0 \\ 1 & 0 & 1 & & \cdots & 1 \\ \vdots & & & & & \vdots \\ 1 & \cdots & & & 1 & 0 \end{bmatrix}$$

Now we have a matrix with an all-zero row, so by Lemma 2

$$R^{(1)}(D) = (1-p_1)\, R^{(2)}\left(\frac{D}{1-p_1}\right).$$

Combining these results, we have

$$R(D) = (1-p_1)\, R^{(2)} \frac{D-p_1}{1-p_1}, \qquad \text{for } D > D_1,$$

where $R^{(2)}$ corresponds to the $M - 1 \times M - 1$ matrix with $d_{ij} = 1 - \delta_{ij}$, and the source $p^{(2)} = \left(\frac{p_2}{1-p_1}, \ldots, \frac{p_M}{1-p_M}\right)$.

$R^{(2)}(D)$ can now be lower-bounded exactly as before, and this lower bound is valid for values of the argument up to $\frac{(M-2)p_2}{1-p_1}$, where p_2 is the second smallest source probability. Thus the second break point comes at D_2, where

$$\frac{D_2 - p_1}{1-p_1} = \frac{(M-2)p_2}{1-p_1}$$

or

$$D_2 = p_1 + (M-2)p_2 \geq D_1.$$

So we have

$$R(D) = H(X) - H(D) - D \log (M-1) \qquad \text{for } 0 \leq D \leq (M-1)p_1$$

and

$$R(D) = (1-p_1)\left[H_{M-1}(X) - H_2\left(\frac{D-p_1}{1-p_1}\right) - \left(\frac{D-p_1}{1-p_1}\right)\log(M-2)\right]$$

$$\text{for } (M-1)p_1 < D \leq p_1 + (M-2)p_2,$$

where

$$H_{M-1}(X) = -\sum_{i=2}^{M} \frac{p_i}{1-p_1}\log\frac{p_i}{1-p_1}.$$

It is clear that this process can be continued at the expense of some algebraic complexity. The result is

$$R(D) = (1-S_k)\left[H_{M-k}(X) - H_2\left(\frac{D-S_k}{1-S_k}\right) - \left(\frac{D-S_k}{1-S_k}\right)\log(M-k-1)\right],$$

$$\text{for } D_k \leq D \leq D_{k+1},$$

where k runs from 0 up to $M-2$, and

$$S_k = \sum_{i=1}^{k} p_i,$$

$$D_k = S_{k-1} + (M-k)p_k,$$

and $H_{M-k}(X)$ is the entropy of the distribution $\left(\frac{p_{k+1}}{1-S_k}, \ldots, \frac{p_M}{1-S_k}\right)$. By inverting this function to get D as a function of R, one can find the minimum achievable symbol error probability at any signalling rate. It is interesting to note (and straightforward to verify directly) that the slope of this $R(D)$ curve varies continuously over the entire range of D.

As an example, if $\underline{p} = \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{2}\right)$, then $D_1 = 1/4$ and $D_2 = D_{max} = 1/2$, and the rate (in bits) is

$$R(D) = \begin{cases} 1.41 - H_2(D) - D & 0 \leq D \leq \frac{1}{4} \\ \frac{7}{8}\left[0.985 - H_2\left(\frac{8D-1}{7}\right)\right] & \frac{1}{4} \leq D \leq \frac{1}{2} \end{cases}$$

3.4 DISCUSSION

We have presented a lower bound to $R(D)$ for difference distortion measures, which is due to Shannon, and have shown that under certain conditions, this bound is actually an equality. Furthermore, we have shown that even when equality does not hold, the bound provides a good approximation to R(D) for small values of D, since the two approach as $D \to 0$ for continuous sources, and, in general, are equal over some nonzero range of D for discrete ones. This shows that for small D, the only parameter of the source that matters is H(X), its entropy, and this merely affects the vertical placement of the R(D) curve. The shape of this curve is determined solely by the distortion measure.

Using these results, we were then able to find R(D) for all D, when $d_{ij} = 1 - \delta_{ij}$. This function had to be calculated in pieces, and the algebraic complexity that ensued perhaps gives some indication of the difficulties involved in the exact computation of R(D) curves.

IV. APPLICATION TO QUANTIZERS

An obvious use for the theory of source encoding that has been developed in Section II and III is to provide a yardstick for evaluating an actual source coder. Knowing what the best possible performance is, the communications engineer can decide whether a proposed system comes sufficiently close to the optimum, or whether there is enough room for improvement to justify the search for better schemes. The main difficulty with such comparisons has been the lack of knowledge of the R(D) function itself, and, to the author's knowledge, the only attempts in this direction have been made by Goblick and Holsinger[5] and by Goblick,[4] who restrict themselves to consideration only of Gaussian sources with mean-square error as the distortion measure.

With the machinery developed in Section III for estimating R(D), we are finally in a position to evaluate some practical source-encoding schemes for fairly broad classes of source distributions and difference distortion measures. If one is faced with a continuous source to be digitized, the first scheme that comes to mind is a simple quantizer. In evaluating such a device, an obvious quantity of interest is the amount of distortion that it introduces, on the average, between source and user. Furthermore, if one is willing to use noiseless source coding techniques (cf. Fano[31]) to encode the quantizer output without further distortion, then the other important parameter is the entropy of the output levels, since this determines the channel capacity needed to transmit the quantized values. Assuming that we are willing to do this coding, we shall investigate the relationship between achievable quantizer rates (output entropy) and distortion. We shall show that this output entropy is also the mutual information between input and output, from which it follows that R(D) is a lower bound on quantizer rates for a distortion of D. Furthermore, for difference distortion measures of the form $d(u) = |u|^{\nu}$ $(\nu > 0)$, and small values of D, this bound cannot be approached, but rather it turns out that the lowest possible rate is some fixed amount above R(D) independent of D. This result shows that as a class, quantizers are suboptimal in a rate-distortion sense.

Quantizers have been studied by many investigators, and for a complete bibliography, the reader is referred to Bruce.[18] Most of this work, however, has concentrated on the relationship between the distortion and the number of output levels, rather than the entropy of these levels. A typical example is the paper by Max,[21] which discusses the minimization of the mean-square error of quantizers with a fixed number of levels. This comparison of distortion with the number of outputs is appropriate if the designer is concerned with the encoder complexity, but the output entropy is more appropriate if it is desired to transmit the source as accurately as possible over a fixed capacity channel.

4.1 SUBOPTIMALITY OF QUANTIZERS

A quantizer is a memoryless device that takes a real number x as its input, and produces as output y(x), which takes on only a discrete number of values. The quantizer is determined by a number of adjacent intervals $I_1 \dots I_n$, with division points

$\xi_o \ldots \xi_n$, such that

$$I_j = [\xi_{j-1}, \xi_j],$$

and by representative points for each interval $\eta_j \in I_j$. Then $y(x) = \eta_j$ if and only if $x \in I_j$ (see Fig. 13).

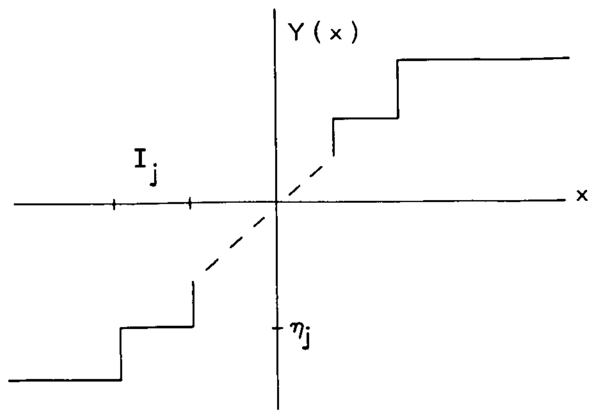

Fig. 13. Transfer function of a quantizer.

It is interesting to note that in the language of Section II a quantizer is a source encoder with a block length of one. Furthermore, emphasis on the output entropy corresponds to considering the device as a variable-length encoder, while emphasis on the number of output levels corresponds to block encoding.

The distortion introduced by such a device is

$$E\{d(x,y)\} = \int_X p(x)\, d(x, y(x))\, dx$$

$$= \sum_{j=1}^{n} \int_{x \in I_j} p(x)\, d(x, \eta_j)\, dx, \tag{25}$$

where p(x) is the probability density function of the source, and d(x, y) is the distortion measure. Let us define q_j ($j = 1, \ldots n$) to be the probability that $y = \eta_j$, so that

$$q_j = \int_{I_j} p(x)\, dx = \int_{\xi_{j-1}}^{\xi_j} p(x)\, dx.$$

The entropy of the output is then

$$H(Y) = -\sum q_j \log q_j,$$

and since the mutual information I(X; Y) is equal to H(Y) (by virtue of the fact that

$H(Y|X) = 0)$, it follows that for a quantizer with distortion D

$$H(Y) \geq R(D), \tag{26}$$

the rate distortion function for the given source and distortion measure.

We define Q(D) to be the smallest possible H(Y) of any quantizer giving distortion D or less. Obviously, since Eq. 26 holds for all quantizers,

$$Q(D) \geq R(D).$$

We shall now show that, for small D, Q(D) is, in fact, strictly greater than R(D) when the distortion measure is

$$d(x, y) = |x-y|^{\nu}, \qquad \nu > 0.$$

To do this, we shall derive a lower bound to H(Y) for any quantizer with distortion D, which, since it is universally valid, must also then be a lower bound to Q(D). Since $H(Y) = I(X; Y)$ has already been shown, we can write

$$H(Y) = H(X) - H(X|Y)$$

$$= H(X) - \sum_j q_j H(X|y=\eta_j).$$

Let the length of the interval I_j be $2T_j$, so that

$$H(X|y=\eta_j) \leq \log 2T_j.$$

At this point, we assume temporarily that the source is strictly limited to a finite range of real numbers, so that all of the T_j are finite. Then

$$H(Y) \geq H(X) - \sum_j q_j \log qT_j$$

$$= H(X) - \frac{1}{\nu} \sum_j q_j \log (2T_j)^{\nu}$$

$$\geq H(X) - \frac{1}{\nu} \log \sum_j q_j (2T_j)^{\nu}, \tag{27}$$

by the convexity of the log. The usefulness of the last two steps, which may seem rather unmotivated now, will soon become clear. We now want to relate the argument of the logarithm in Eq. 27 with the average distortion. To express D in terms of the T_j, we write

$$D = \sum_j D_j,$$

where

$$D_j = \int_{I_j} p(x)\; d(x, \eta_j)\; dx$$

$$= \int_{I_j} p(x)\, |x - \eta_j|^{\nu}\; dx. \tag{28}$$

Now if $p(x)$ is convex $\cup$ over I_j, replacing it by its average value, $\frac{q_j}{2T_j}$, can only decrease the integral in Eq. 28, and then moving η_j to the center of the interval results in a further decrease. Symbolically, these steps are

$$D_j \geq \int_{I_j} \frac{q_j}{2T_j} |x - \eta_j|^{\nu}\; dx$$

$$\geq \frac{q_j}{2T_j} \int_{-T_j}^{T_j} |u|^{\nu}\; du$$

$$= \frac{q_j T_j^{\nu}}{\nu + 1}. \tag{29}$$

If the density $p(x)$ is not convex $\cup$ over I_j, we can still obtain a bound on D_j, by defining

$$p_{min,j} \triangleq \underset{x \in I_j}{\text{Min}}\; p(x)$$

so that, from Eq. 28,

$$D_j \geq p_{min,j} \int_{I_j} |x - \eta_j|^{\nu}\; dx$$

$$\geq \frac{2T_j^{\nu+1}}{\nu + 1}\, p_{min,j}, \tag{30}$$

by moving η_j to the center of I_j exactly as we did in obtaining Eq. 29. Now define

$$\epsilon_j = \begin{cases} 0 & \text{if } p(x) \text{ is convex } \cup \text{ over } I_j \\ \dfrac{q_j}{2T_j} - p_{min,j} & \text{otherwise} \end{cases}$$

which is just the difference between the average and minimum values of p(x) over I_j if this density is not convex ∪ there, and will become small as the quantization intervals become fine if p(x) is reasonable smooth. Then Eqs. 29 and 30 both become

$$D_j \geq \frac{1}{\nu + 1}\left[q_j T_j^{\nu} - 2\epsilon_j T_j^{\nu+1}\right],$$

and so

$$D \geq \frac{1}{\nu + 1}\left[\sum_j q_j T_j^{\nu} - 2\sum_j \epsilon_j T_j^{\nu+1}\right]. \tag{31}$$

Equations 27 and 31 thus give a lower bound on the rate and distortion of any quantizer.

We now restrict our attention only to the case of small distortion, which is usually the region of practical interest. This assumption on D is equivalent to all of the quantization intervals being small (or at least all those for which p(x) is not convex ∪). In this case, the second term of Eq. 31 becomes negligible relative to the first, since $\epsilon_j \to 0$ as the $T_j \to 0$, and q_j is of the order of T_j (see Appendix E for a rigorous treatment of this approximation). Thus for small D, we can make the approximation

$$D_j \approx \frac{q_j T_j^{\nu}}{\nu + 1},$$

and so

$$\sum_j q_j (2T_j)^{\nu} = 2^{\nu}(\nu+1)\, D.$$

Substituting this result in Eq. 27, we have

$$Q(D) \geq H(X) - \frac{1}{\nu}\log 2^{\nu}(\nu+1)\, D, \tag{32}$$

which is really an approximation, valid for small D.

Now if the random variable x is not bounded, the same result may still be obtained if the contributions of the extremal intervals to the entropy and distortion become

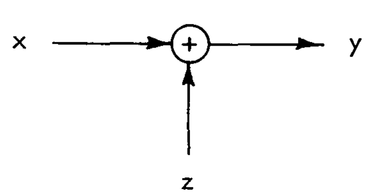

Fig. 14. Additive noise channel.

negligible as the boundaries of these intervals are moved outward.

We can give a plausability argument for the assertion that under our assumption that D is small, Eq. 32 is actually an equality. Consider a uniform quantizer the length of whose intervals must be small, since D is small. In this case, the output y can be written

$$y = x + z,$$

where z is a random variable that is approximately independent of x and uniformly distributed on $(-T, T)$, where 2T is the quantizer interval length. Thus the quantizer looks approximately like an additive noise channel as shown in Fig. 14.

The rate and distortion are then found to be

$$\begin{aligned} I(X;Y) &= H(Y) - H(Y|X) \\ &= H(Y) - H(Z) \\ &= H(Y) - \log 2T \end{aligned}$$

and

$$\begin{aligned} D &= E\{d(y-x)\} = E\{d(z)\} \\ &= \frac{1}{2T}\int_{-T}^{T} |u|^{\nu}\, du \\ &= \frac{T^{\nu}}{\nu+1}. \end{aligned}$$

Thus

$$T = [D(\nu+1)]^{1/\nu}$$

and

$$I(X;Y) = H(Y) - \frac{1}{\nu}\log 2^{\nu}(\nu+1)\,D.$$

Since T will be small if D is small, p(y) will be nearly equal to p(x), and so

$$H(Y) \approx H(X)$$

by the same argument as that in section 3.2. Replacing H(Y) with H(X) causes the expression above to become identical with the lower bound of Eq. 26. Although the derivation of this result was not rigorous, the calculations of actual quantizers presented below leave little doubt as to its validity. If the reader is still unconvinced, he may perhaps find some comfort in the fact that this result will not be needed in the sequel.

The next step is to find R(D), so that it can be compared with the lower bound on Q(D). We shall use the lower bound $H(X) - \phi(D)$ as our estimate of R(D), since this has been shown to be a close approximation for small values of D. For the distortion measure $d(u) = |u|^{\nu}$, it has been shown that

$$A(s) = \frac{2}{\nu(-s)^{1/\nu}} \Gamma\left(\frac{1}{\nu}\right)$$

from which

$$\phi(D) = \log \frac{2\Gamma\left(\frac{1}{\nu}\right)}{\nu} + \frac{1}{\nu} \log D + \frac{1}{\nu} (\log \nu + 1).$$

The detailed derivation of this result has been given elsewhere.[32]

Thus we see that, for small D,

$$R(D) \approx H(X) - \frac{1}{\nu} \log D - \log \frac{2\Gamma\left(\frac{1}{\nu}\right)}{\nu} - \frac{1}{\nu} (\log \nu + 1).$$

We see that the difference between this expression and Eq. 32 is

$$Q(D) - R(D) \geq \log \frac{2\Gamma\left(\frac{1}{\nu}\right)}{\nu} + \frac{1}{\nu} (\log \nu + 1) - \frac{1}{\nu} \log 2^{\nu}(\nu+1), \tag{33}$$

which is a constant, independent of D. Thus we have shown that at small distortion levels, for a distortion of the form $d(u) = |u|^{\nu}$, quantizers are suboptimal as a class of source encoders. The extent of this suboptimality is given in Eq. 33.

It is perhaps of interest to note that any attempt to derive a result like the above for all D is doomed to failure because, clearly, at $D = D_{max}$, $R(D) = Q(D) = 0$. We should also note that although quantizer rates are strictly bounded away from R(D) at small D, for practical purposes this difference as expressed in Eq. 33 is not very large. For example, if $\nu = 2$, it comes out to be very nearly 1/4 bit. It is then an engineering decision whether that extra 1/4 bit is worth the added complexity of the encoding equipment.

Finally, the assumption of small D is quite reasonable from an engineering point of view. It seems very unlikely that a user would ever be willing to tolerate a distortion level more than from .01 to .1 of D_{max}. This would correspond to two-decimal digit accuracy in reproducing a unit variance source with mean-square error distortion. Experimental results seem to indicate that $D = .01\, D_{max}$ is easily small enough for the approximations made above to be quite good. Finally, we note that for comparison purposes, we only know R(D) accurately for small D, in any case.

4.2 EXAMPLES

The first example that we shall consider is that of a unifrom modular source (defined in section 2.4) with the difference distortion $d(u) = |u|$, where u is the distance around the circle from x to y. For this case, both R(D) and the lower bound to Q(D) can be found exactly for all D, without resorting to any approximations. First, we can calculate R(D) by noting, as in section 3.2, that for all D a uniform p(y) distribution satisfies the condition that it convolves with $\frac{e^{sd(z)}}{A(s)}$ to give p(x). Therefore

$$R(D) = H(X) - \phi(D),$$

and clearly

$$H(X) = \log 2T,$$

where 2T is the circumference of the circle that is the X space. From Eq. 21,

$$\phi(D) = \log A(s) - s\frac{A'(s)}{A(s)}$$

$$D = \frac{A'(s)}{A(s)},$$

where

$$A(s) = \int e^{sd(u)}\,du$$

$$= 2\int_0^T e^{su}\,du$$

$$= \frac{2}{s}\left[e^{sT}-1\right].$$

These expressions are easy to evaluate by computer, and the resulting R(D)

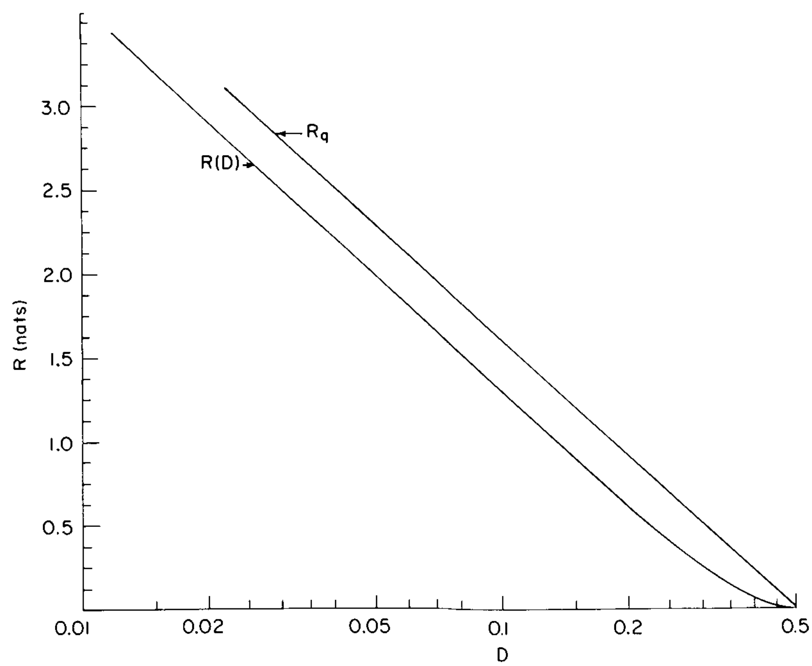

Fig. 15. Uniform modular source d(u) = |u|.

for T = 1 is shown in Fig. 15. Note that as s becomes large negative,

$$A(s) \approx -\frac{2}{s},$$

and so

$$D \approx -\frac{1}{s}$$

and

$$\phi(D) = \log 2D + 1.$$

Thus, since H(X) = log 2,

$$R(D) \approx \log \frac{1}{eD}.$$

This approximation is seen from Fig. 15 to be quite close for D below approximately .15.

The derivation of a lower bound to Q(D) for modular sources is almost identical with that of the previous section. In the case of a uniform source, however, it should be pointed out that $\epsilon_j = 0$ for all j, so no approximations are needed to obtain an expression like Eq. 32. Since $H(X|y=\eta_j) = \log 2T_j$, where $2T_j$ is the length of the j^{th} interval,

$$H(Y) = H(X) - \sum_j q_j \log 2T_j$$

$$\geq H(X) - \log \left(2 \sum_j q_j T_j\right),$$

where

$$q_j = \Pr[y=\eta_j] = \frac{T_j}{T}$$

and

$$D = \sum_j \int_{I_j} \frac{|x-\eta_j|}{2T}$$

$$\geq \frac{1}{2T} \sum_j \int_{-T_j}^{T_j} |u| \, du$$

$$= \frac{1}{2} \sum_j q_j T_j$$

so

$$Q(D) \geq H(X) - \log 4D$$

$$= \log \frac{T}{2D}. \tag{35}$$

This bound is also plotted in Fig. 15 for T = 1. From this, we see that except at R = 0, the minimum quantizer rate is strictly above R(D), and that for D below approximately .15, the rate difference is $1 - \log_e 2$ nats, which corresponds to a factor of 1.35 in distortion. Finally, with reference to our assertion that this lower bound is actually equal to Q(D), we can see that a uniform quantizer with n levels has rate

$$H(Y) = \log n$$

and, from Eq. 34, the distortion is

$$D = \frac{T}{2n},$$

since $q_j = \frac{1}{n}$ for all j. Setting T = 1 as before, we see that

$$H(Y) = -\log 2D,$$

which shows that uniform quantizer rates and distortions fall on the lower-bound curve.

For our second example, we consider a unit variance Gaussian source with distortion measure $d(u) = u^2$, or mean-square error. For this case, we know from section 3.2 that

$$R(D) = \frac{1}{2} \log \frac{1}{D}$$

and, from Eq. 32, for small D,

$$Q(D) \geq H(X) - \frac{1}{2} \log 12D.$$

Since for a Gaussian source,

$$H(X) = \frac{1}{2} \log 2\pi e\sigma^2,$$

we have

$$Q(D) \geq \frac{1}{2} \log \frac{\pi e}{6D}.$$

Both of these functions are plotted in Fig. 16. The vertical difference between the curves is

$$Q(D) - R(D) \geq \frac{1}{2} \log \frac{\pi e}{6} = .175 \text{ nat}$$

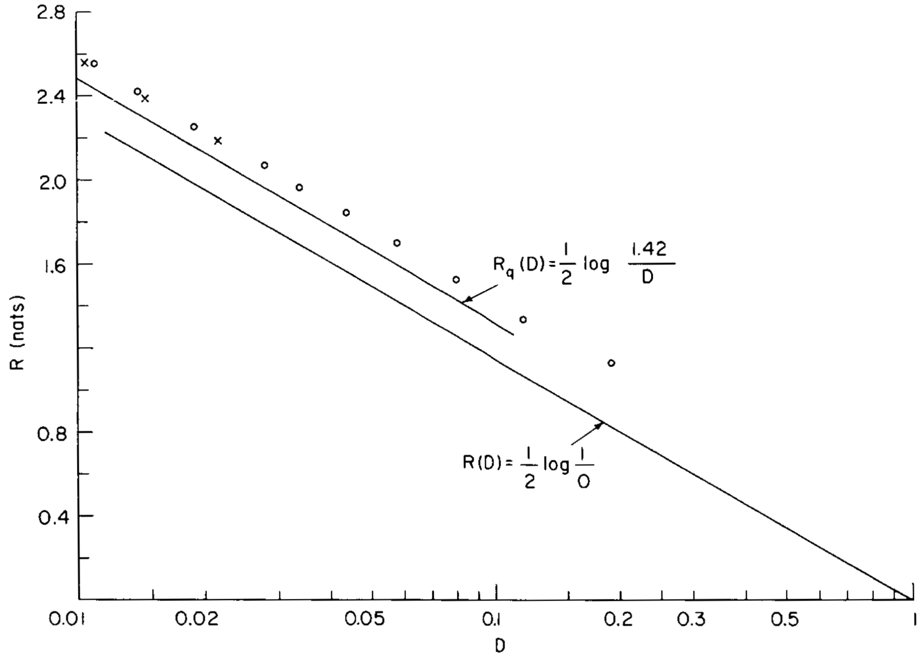

Fig. 16. Rate and distortion of some minimum distortion quantizers.
× = uniform quantization intervals.
O = unrestricted interval sizes.

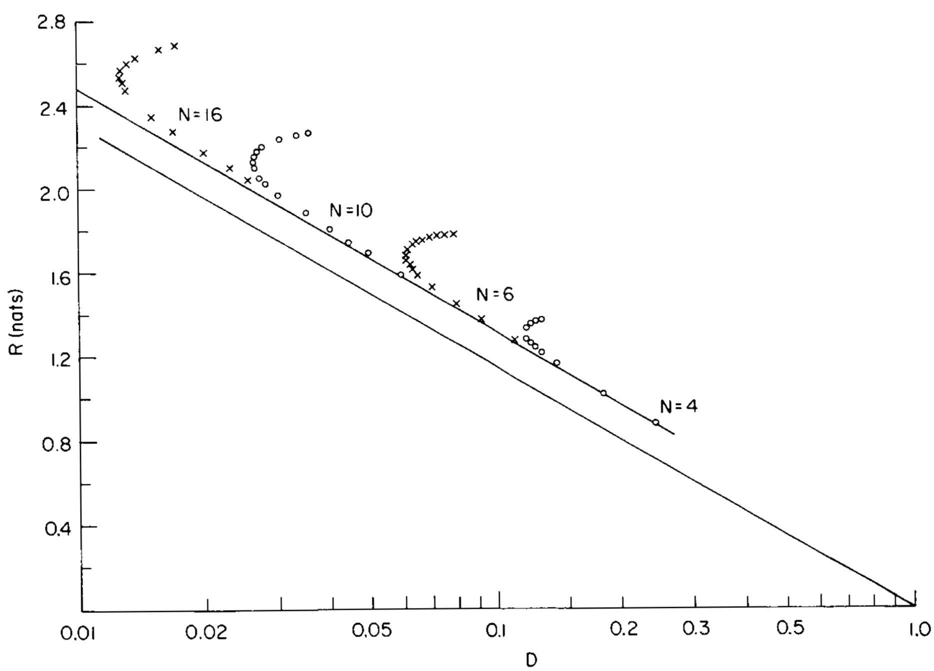

Fig. 17. R-D trajectories for uniform n-level quantizers.

which comes out to be very nearly 1/4 bit.

Also shown in Fig. 16 are the points corresponding to quantizers of minimum distortion for a fixed number of levels, which have been computed by Max[21] for both uniform and arbitrary spacing of the levels. Figure 17 shows the H-D trajectory as the length of the intervals of an equal-interval quantizer is varied, for n, the number of levels, equal to 4, 6, 10, and 16. This computation has been carried out previously by Goblick.[5] One interval boundary was kept fixed at the origin, and the others were at ±t, ±2t, and so forth, where the parameter t was variable. The point of smallest distortion on these trajectories is the same as Max's minimum distortion point for equal spacings. Large values of t correspond to the top (high-rate) portion of the trajectory shown here, and small values to the lower section. Note that in a rate-distortion sense, this point is not at all optimal, lying farther from the R(D) curve than other points on the trajectory, and one can get the same distortion with smaller rate by using more quantization levels.

4.3 DISCUSSION

We have seen, at least for distortion measures of the form $|u|^\nu$, that all quantizers are suboptimal in a rate-distortion sense. The discrepancy between the best possible encoder performance and that of the best quantizer (indeed, that of most reasonable quantizers) is not great, however, being approximately 1/4 bit in the case of mean-square error. Furthermore, it seems clear that good quantizers are not difficult to find, judging from the results shown in Fig. 17, which shows that even uniform intervals come very close to the lower bound.

The problem of finding practical schemes that perform better than quantizers is still open. One possibility is to increase the block length by going to multidimensional quantizers, which treat a sequence of source outputs as an n-vector, and whose output

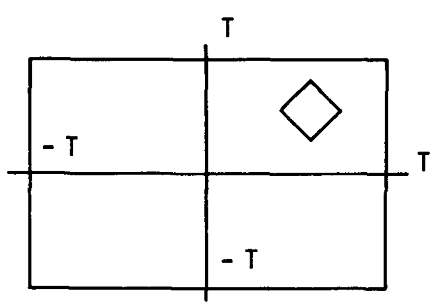

Fig. 18. Quantizer space and a typical quantization region.

specifies the region of n-space in which this vector falls. How to choose these regions in general is not at all clear, nor is such a device necessarily easy to implement. One example has been worked out — for a uniform modular source with $d(u) = |u|$, and n, the number of dimensions, equal to 2. The space to be quantized is thus a square with sides of length 2T, with opposite edges identified with each other. This is equivalent to a torus. The quantization regions are also chosen to be squares, oriented at a 45° angle to the axes, as shown in Fig. 18. The sides of these small squares must be $\frac{2T}{\sqrt{n}}$, where n is an integer, in order for them to pack properly into the space. The reader can easily see that in this case, the rate per source output is

$$R = \frac{1}{2} \log n$$

and the average distortion is

$$D = \frac{\sqrt{2}\, T}{2\sqrt{n}},$$

from which we obtain

$$R = \log \frac{\sqrt{2}\, T}{3D}.$$

Comparing this rate with the lower bound for one-dimensional quantizers, given by Eq. 35,

$$Q(D) \geq \log \frac{T}{2D},$$

we see that there has indeed been an improvement, but only $\log \frac{3}{2\sqrt{2}} = .06$ nat, out of the approximately .3 that separates Q(D) from R(D). This could hardly be called a substantial improvement. Also note that even this small gain is intimately connected to the particular distortion measure, $d(u) = |u|$. If instead $d(u) = u^2$, it is easy to see that there is no improvement because a rotation preserves the Euclidian metric.

Another possibility for an encoder would be one that finely quantizes each source output, and then performs further encoding on blocks of these outputs as though they were taken directly from a large-alphabet discrete source. This method would have the advantage that real numbers would not have to be stored, as was the case in the previous scheme, but since the problem of practically encoding discrete sources is far from solved, it still leaves much to be desired.

All in all, the engineer will have to decide whether the fraction of a bit of improvement over the simple quantizer's performance is worth the increase (probably great) in the complexity of the encoding equipment.

V. SUMMARY AND CONCLUSIONS

We have presented some contributions to the theory of encoding memoryless information sources with single-letter distortion measures, and some applications of this theory to practical systems. Although most of the material discussed in Section I is not original, we have given a fairly thorough background and summary of this field because such a development does not seem to have been made elsewhere.

In Section II a proof was presented of a general form of the source-coding theorem and a discussion was given of the capabilities of various classes of source encoders. The main results, besides the proof of the coding theorem, were: a demonstration that the difference between R(D) and the rate of an encoder can be made to decrease at least as fast as $\frac{\log n}{n}$, where n, the encoder block length, is the number of source outputs operated on at one time; a demonstration that variable-length codes are more powerful that block ones, and a statement of necessary and sufficient conditions under which the two classes are equivalent; calculation of the rate of a source when the fidelity criterion is that every letter be reproduced with less than a specified distortion; and extension of the theory to allow several fidelity criteria, and the encoding of source outputs after they have been corrupted by a noisy channel.

Section III prepared the way for the application of the theory by developing machinery by means of which R(D) can be calculated for several interesting classes of sources and distortion measures. By using these methods, R(D) was calculated for discrete sources in the special case for which $d_{ij} = 1 \cdot \delta_{ij}$, which allows us to find the minimum achievable probability of error when signalling over a channel of given capacity.

Using the techniques developed in Section III, we were able, in Section IV, to compare the performance (in a rate-distortion sense) of simple quantizers with R(D). We found that at small D, quantizers could not approach R(D), but that the difference between the best quantizer rate and the minimum achievable was usually a fraction of a bit.

As far as further research in this field goes, the author is not optimistic. Real sources are usually very difficult to handle analytically, since they are likely to have dependencies between samples and, worse yet, the source may not even be stationary. Indeed, the actual statistics of a source are often not known at all. Furthermore, it is not always clear what the proper distortion measure should be.

The current situation in source encoding is that at the theoretical end, analytical results exist only for the simplest of sources and distortion measures and, at the other end of the scale, encoders are being built for complex sources by exploiting the peculiar characteristics of the signal to be represented. Source coding theory can still offer no guidance in these cases. Even for simple sources, practical (constructive) encoding schemes are nonexistent, unless, of course, one wants to put such a simple-minded device as a quantizer in this class. This lack is probably related to the fact that, unlike channel coding, the small payoff in source coding (1/4 bit better than quantizers for mean-square error) has not motivated people to search for such schemes.

Probably the most promising line of further investigation lies not in additional purely theoretical development, but in attempts to bridge the gulf between theory and practice. It would be of great interest to know R(D) for sources with memory, since these usually approximate reality more closely than memoryless ones, but, as we have mentioned, the binary Markov source with Hamming distance distortion has thus far resisted substantial efforts, and it is hard to believe that there are many simpler cases than this one. The solution of such problems must therefore await the development of some new and more powerful mathematical techniques. It is possible, however, that examination of a particular source in the light of the theory will lead to some improvements in the methods of encoding it.

It would also be interesting to know how sensitive the results of this theory are to variations in the source model or the distortion measure. For example, how well does a code designed for a memoryless source work when the source really has memory. Finally, it would be desirable to have the results on quantizers in a neater form, and even to have some more general results about encoders with very small block lengths.

In conclusion, though, it is the author's opinion that this field of source coding, especially the theoretical aspects, will provide poor pickings, even for that most omniverous and desperate of researchers, the graduate student in search of a thesis.

APPENDIX A

Encoding an Infinite-Level Discrete Source

We shall show that an infinite-level discrete source with entropy H(X) can be encoded by a variable-length code with letters taken from an L-letter alphabet with average code-word length satisfying

$$H(X) \leq \bar{n} \leq H(X) + 1,$$

where H(X) is expressed in base L units. The first step in establishing this result is to show that the Kraft inequality holds for infinite alphabet sources. For finite, M-letter sources, this result states that the inequality

$$\sum_{i=1}^{M} L^{-n_i} \leq 1$$

is a necessary and sufficient condition for the existence of a prefix code with lengths $\{n_i\}$, where L is the size of the encoding alphabet. A proof of this has been given by Fano.[27] This result follows from the fact that a code word of n_i letters (corresponding to a node of order n_i) removes the fraction L^{-n_i} of all possible nodes from further consideration. A code clearly exists if and only if the fraction of nodes thus eliminated by all code words does not exceed one.

Now let us choose n_i to be that integer such that

$$\log_L \frac{1}{p_i} \leq n_i < \log_L \frac{1}{p_i} + 1,$$

where p_i is the probability of the i^{th} source letter. It follows that

$$p_i \geq L^{-n_i}$$

and so

$$\sum_{i=1}^{\infty} L^{-n_i} \leq \sum_{i=1}^{\infty} p_i = 1,$$

so that the Kraft inequality is satisfied. Thus there exists a code with these $\{n_i\}$, and since

$$p_i \log_2 \frac{1}{p_i} \leq n_i p_i \leq p_i \left[\log_L \frac{1}{p_i} + 1\right],$$

the average length, $\bar{n} = \Sigma\, n_i p_i$, satisfies

$$H(X) \leq \bar{n} \leq H(X) + 1,$$

which was to be shown.

APPENDIX B

Source Coding Theorem for Block Codes

In this appendix, X and Y are the input and reproduction alphabets respectively, p(x) is the source distribution, and d(x, y) is the single-letter distortion measure. Sequences of source and reproduction letters will be denoted $\underline{x} = x_1 \ldots x_n$ and $\underline{y} = y_1 \ldots y_n$, respectively. The distortion between blocks is taken to be the sum

$$d(\underline{x}, \underline{y}) = \sum_{i=1}^{n} d(x_i, y_i).$$

Subsequently, we assume that p(x) and d(x, y) are such that there exists $\hat{y} \in Y$ with the property

$$E\{d(x, \hat{y})\} \triangleq D_{max} < \infty.$$

The proof of the source-coding theorem follows from the following lemma.

Lemma: Suppose there is a memoryless channel with input probabilities p(x) and transition probabilities $p(y|x)$, $x \in X$, $y \in Y$, and there exists a $\hat{y}$ as described above.

Let

$$R^* = I(X; Y) = E\{I(x; y)\}$$

and

$$D^* = E\{d(x, y)\},$$

where both expectations are with respect to the joint probability induced on X and Y by the channel. Then for any $\epsilon > 0$, there exists a block code of some length n with $M = e^{n(R^*+\epsilon)}$ code words and distortion less than $D^* + \epsilon$.

Proof: With the usual random-coding argument, we choose each letter of M − 1 code words independently with probabilities q(y), where

$$q(y) = \sum_{X} p(x)\ p(y|x)$$

is the output distribution of the channel. The last code word has $\hat{y}$ in each position, which guarantees a finite distortion representation for all $\underline{x}$. Define $\underline{y}(\underline{x})$ to be the smallest distortion code word when $\underline{x} = x_1 \ldots x_n$ is the source output, and let

$$P_o = \Pr\left[d(x, y(x)) > n\left(D^* + \frac{\epsilon}{2}\right)\right].$$

Then clearly the distortion satisfies

$$D \leq D^* + \frac{\epsilon}{2} + P_o D_{max}. \qquad \text{(B. 1)}$$

We shall now find the expected value of P_o over the ensemble of codes.

Define the set A of pairs of source and output sequences:

$$A = \left\{ (\underline{x}, \underline{y}) : I(\underline{x}; \underline{y}) > n\left(R^* + \frac{\epsilon}{2}\right) \text{ or } d(\underline{x}, \underline{y}) > n\left(D^* + \frac{\epsilon}{2}\right) \right\},$$

and let

$$A_{\underline{x}} = \{\underline{y} : (\underline{x}, \underline{y}) \in A\}.$$

Let

$$P_{\underline{x}} = \Pr[d(\underline{x}, \underline{y}(\underline{x})) \geq n(D^* + \epsilon) \,|\, \underline{x}].$$

Then

$$P_{\underline{x}} \leq \left[1 - \sum_{\underline{y} \in A^c_{\underline{x}}} q(y)\right]^{M-1}$$

$$\leq e^{-(M-1) \sum_{\underline{y} \in A^c_{\underline{x}}} q(y)}$$

But for $\underline{y} \in A^c_{\underline{x}}$, $I(\underline{x}; \underline{y}) \leq n\left(R^* + \frac{\epsilon}{2}\right)$, or

$$q(\underline{y}) \geq p(\underline{y} | \underline{x})\, e^{-n(R^* + \epsilon/2)}$$

so

$$P_{\underline{x}} \leq \exp\left[-(M-1)\, e^{-n(R^* + \epsilon/2)} \Pr\left(A^c_{\underline{x}} | \underline{x}\right)\right].$$

Since the right-hand side of this inequality is a convex ∪ function of $\Pr\left(A^c_{\underline{x}} | \underline{x}\right)$, which we write as $f(\cdot)$, then

$$P_{\underline{x}} \leq \left[1 - \Pr\left(A^c_{\underline{x}} | \underline{x}\right)\right] f(0) + P\left(A^c_{\underline{x}} | \underline{x}\right) f(1)$$

$$\leq P(A_{\underline{x}} | \underline{x}) + \exp\left[-(M-1)\, e^{-n(R^* + \epsilon/2)}\right],$$

and averaging over all $\underline{x}$,

$$P_o \leq P(A) + \exp\left[-(M-1)\, e^{-n(R^* + \epsilon/2)}\right].$$

By the law of large numbers, we can find an n large enough so that P(A) is arbitrarily small, and since $M = e^{n(R^* + \epsilon)}$, the argument of the exponential function $\to -\infty$ as $n \to \infty$, so we can find an n for which

$$P_o \leq \frac{\epsilon}{2D_{max}},$$

which, when Substituted in Eq. B. 1, gives

$$D \leq D^* + \epsilon,$$

and the rate, $\frac{1}{n} \log M$, is $R^* + \epsilon$.

By the usual argument, there must be a code in the ensemble with P_o as small as the average calculated above, which completes the proof of the lemma.

It is now merely necessary to note that if $\{p(y|x)\}$ are the transition probabilities of the optimum test channel for the given source, distortion measure, and tolerance level, then

$$E\{d(x,y)\} = D$$

and

$$I(X;Y) = R(D),$$

so by the lemma, a block code exists with rate and distortion arbitrarily close to the R(D) and D above.

APPENDIX C

Entropy of a Geometric Source

We want to find the entropy H(X) for an infinite-level discrete source for which the k^{th} letter appears with probability

$$p(k) = q(1-q)^{k-1}.$$

Then

$$H(X) = -\sum_{k=1}^{\infty} p(k) \log p(k)$$

$$= -\sum q(1-q)^{k-1} [\log q + (k-1) \log (1-q)]$$

$$= -\log q - q(1-q) \log (1-q) \sum_{k=1}^{\infty} (k-1)(1-q)^{k-2}.$$

To do the remaining summation, we note that

$$\sum_{n=0}^{\infty} x^{n-1} = \frac{d}{dx} \sum_{n=0}^{\infty} x^n$$

$$= \frac{d}{dx}\left(\frac{1}{1-x}\right)$$

$$= \frac{1}{(1-x)^2}.$$

Replacing x with 1 - q, we have

$$\sum_{k=1}^{\infty} (k-1)(1-q)^{k-2} = \frac{1}{q^2},$$

and so

$$H(X) = -\log q - \frac{(1-q)}{q} \log (1-q) \qquad \text{(C. 1)}$$

$$= \frac{H(q)}{q}.$$

Finally, to obtain a useful upper bound on this entropy, we can expand log (1-q) in (C. 1) in powers of q (since $q < 1$), and write

$$H(X) = -\log q - \frac{(1-q)}{q}\left[-q - \frac{q^2}{2} - \dots\right]$$

$$\leqslant -\log q + \frac{1-q}{q}[q+q^2+\dots]$$

$$= -\log q + 1.$$

APPENDIX D

An Exponential Bound

Gallager[12] has shown that if $\xi_j (1 \leq j \leq n)$ are independent random variables, then

$$\Pr\left[\sum \xi_j \leq \mu'(s)\right] \geq B\, e^{\mu(s)-s\mu'(s)},$$

where $\mu(s)$ is the semi-invariant moment-generating function of $\Sigma\, \xi_j$, and B is a complicated expression given by Gallager.[28] This result is derived by using a form of the Central Limit theorem which is due to Berry. It states

$$|G(z)-\phi(z)| \leq \frac{C\rho_{3,n}}{n},$$

where G(z) is the c.d.f. of the normalized sum, $\phi(z)$ is the zero-mean, unit-variance Gaussian c.d.f., C is a constant, and $\rho_{3,n}$ has been defined by Gallager.[12] We use the fact (based on a theorem of Feller[29]), that the numerator may be taken to be

$$C\rho_{3,n} = \frac{33\lambda}{4\left(\frac{1}{n}\Sigma\ \mathrm{var}\,(\xi_j)\right)^{1/2}},$$

where

$$\lambda = \underset{j}{\mathrm{Max}} \frac{E\{|\xi_j-\xi_j|^3\}}{\mathrm{Var}\,(\xi_j)}.$$

Writing these results in our notation, we have

$$\Pr \sum d(x_i, y_i) \leq \mu_x'(s)\,|x \;\; \geq B(x)\, e^{\mu_x(s)-s\mu_x'(s)}$$

and

$$B(x) = \frac{\exp\left[z_o s\sqrt{\mu_x''(s)} - \frac{z_o^2}{2}\right]}{\sqrt{2\pi}\left[z_o - s\sqrt{\mu_s''(s)}\right]}\, 1 - \frac{1}{\left[z_o - s\sqrt{\mu_x''(s)}\right]^2},$$

where z_o is specified by

$$\phi(z_o) = \frac{1}{2} + \frac{2C\rho_{3,n}}{n}.$$

Now since X is finite, there are numbers $\mu_{max}(s)$ and $\mu_{min}(s)$ such that

$$n\mu_{min}(s) \leq \mu_x(s) \leq n\mu_{max}(s).$$

We then have

$$B(x) \geq B = \frac{\exp\left[z_o s\sqrt{n\mu_{max}}\ \rho\ \frac{z_o^2}{2}\right]}{\sqrt{2\pi}\left[z_o - s\sqrt{n\mu_{max}}\right]}\left\{1 - \frac{1}{\left[z_o - s\sqrt{n\mu_{min}}\right]^2}\right\}.$$

Finally, using the fact that for large n,

$$z_o = \frac{2\sqrt{2\pi}\ C\rho_{3,n}}{\sqrt{n}} + o_1\left(\frac{1}{\sqrt{n}}\right),$$

we see that

$$B \geq \frac{B_1}{\sqrt{n}} + o_2\left(\frac{1}{\sqrt{n}}\right),$$

where B_1 is a constant (cf. Gallager[30]) and thus, for large enough n, we can find a constant B_o such that

$$B \geq \frac{B_o}{\sqrt{n}}.$$

APPENDIX E

On the Approximation in Section IV

In section 4.1, a lower bound on the minimum quantizer entropy Q(D) was derived. This expression could be put into a convenient and simple form by making the approximation that for all intervals for which p(x) was convex ∩, the second term of the inequality

$$D_j \geq \frac{q_j T_j^{\nu}}{\nu + 1} - \frac{2\epsilon_j T_j^{\nu+1}}{\nu + 1}$$

becomes negligible relative to the first, or equivalently that the ratio of the two approaches zero as $T_j \to 0$. Sufficient conditions for this to hold are given in the following theorem.

Theorem: let p(x) be bounded away from zero and have a bounded derivative over the region where it is convex ∩ (that is, there exist constants B and C such that $p(x) > B$ and $|p'(x)| < C$ for all x in the region). Then $\epsilon_j T_j / q_j \to 0$ as $T_j \to 0$.

Proof: Since the slope of p(x) is bounded by C,

$$\epsilon_j \leq p_{max,j} - p_{min,j}$$

$$\leq 2\, C\, T_j,$$

and

$$q_j \geq 2\, T_j\, p_{min,j}$$

$$\geq 2\, B\, T_j,$$

we have

$$\frac{\epsilon_j T_j}{q_j} \leq \frac{C}{B} T_j,$$

which vanishes as $T_j \to 0$. Q.E.D.

It is easy to see that as $D \to 0$, all T_j for intervals over which p(x) is nonzero must approach zero, since every such interval makes a positive contribution to the over-all distortion. Thus the approximation made in section 4.1 is justified.

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13. ABSTRACT

The theory of encoding memoryless information sources so that the output can be transmitted with minimum rate and still satisfy a fidelity criterion based on a single letter distortion measure is investigated. This report extends and amplifies the theory developed by Shannon. A general proof of the Source Coding theorem for memoryless sources and single letter distortion measures is presented using variable length codes. It is shown that this proof is more generally applicable than Shannon's previously derived block coding results; moreover, without some additional restrictions, the coding theorem is false if only block codes are permitted. It is also shown that the convergence of encoder rate to R(D) (the minimum rate necessary to achieve average distortion D) with increasing block length n, can be made at least as fast as (log n)/n. Equivalent theories of source coding are developed for cases in which: (i) the fidelity criterion requires every letter to be reproduced with less than a fixed distortion, rather than merely achieving this performance on the average; (ii) there are several fidelity criteria that must be satisfied simultaneously; and (iii) the source outputs are corrupted by a noisy channel before being furnished to the encoder. Means of calculating or estimating R(D) for sources with a difference distortion measure are developed by showing conditions under which R(D) is equal to a more easily calculable lower bound developed by Shannon. Even when equality does not hold, we show that R(D) approaches this bound as $D \to 0$ for all continuous sources, and that for discrete sources, there is always a nonzero region of small D where there is equality. R(D) for a discrete source and distortion measure $d_{ij} = 1 - \delta_{ij}$ is calculated exactly for all D, thereby allowing calculation of the minimum achievable symbol error probability when transmitting over a channel of given capacity. Finally, as an application of the theory, we examine quantizers as a class of source encoders, and show that the rate (output entropy) and distortion of such devices is bounded away from R(D), but is usually quite close.

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