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THE EFFECT OF 3-DIMENSIONAL NONLINEAR RESONANCES ON THE MOTION OF A PARTICLE NEAR THE EARTH-MOON EQUILATERAL LIBRATION POINTS

by

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ABSTRACT

In the uniformly rotating reference frame of the restricted 3-body problem (in which Earth and Moon occupy fixed positions on the abscissa), the equilateral libration points L_4 and L_5 are known to be points of equilibrium. A particle placed at rest at one of these points will remain at rest for all times. According to linear theory, for very small disturbances from equilibrium the particle will tend to move along bounded trajectories in the immediate vicinity of these points.

When the force field near L_4 and L_5 is not assumed to be linear, and in addition other perturbing effects are included, the particle's motion might be excited sufficiently and lead to unstable divergent trajectories.

This dissertation presents the results of an analytic study of the 3-dimensional stability of motion of a particle near L_4 in a nonlinear Earth-Moon force field, upon which is superimposed a linear solar gravitational field distribution. In particular, the long period features of the particle's motion are studied, which stem from the excitation at or close to the particle's natural frequencies, and are introduced by the presence of resonance terms in the internal (Earth and Moon) and external (solar) force fields.

The results show that in the presence of the internal nonlinearities the stability of motion predicted by the linear theory is valid for only a very restricted region of initial displacement and velocity disturbances. Disturbances outside this region would lead to divergence of the solution. The nonlinear coupling of the out-of-plane terms with the in-plane terms was found to be of minor importance and did not contribute to an appreciable transfer of energy from one mode of motion to the other.

The inclusion of the external force terms was found to admit some equilibrium solutions of the variational equations. Of those, the one stable equilibrium solution found was characterized by a coplanar elliptic particle orbit around L_4 which had its major axis (of magnitude roughly 120,000 mi) oriented at right angles to the line joining Earth to L_4. This orbit was traversed in a clockwise sense at mean angular rate equal to that of the Sun, as seen in the rotating coordinate frame, and very close to the particle's faster coplanar natural frequency. The particle's motion thereby became synchronized with that of the Sun.

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C_J = Jacobi constant introduced in Eq. (43)

D = mean Earth-Moon distance defined by Eq. (8)

D_1 = integration constant introduced in Eq. (65)

$e \cong .055$ = eccentricity of lunar orbit

f, η = functions of ξ defined in Eqs. (69) and (70)

$f_x, f_y \cdots$ = forcing functions introduced in Eqs. (37)

G = Universal gravitational constant

$G(\beta, \alpha'), S(\beta, \alpha')$ = generating functions introduced in Eq. (45)

H = Hamiltonian = $H^{(o)} + H'$

H' = contains the higher order nonlinearities

$\overline{\tilde{H}}_3, \overline{\tilde{H}}_4, \overline{H}_4$ = slowly varying Hamiltonians used in Eq. (48)

H_r, H_{p_r} = partial derivatives of H as defined in Eqs. (22)

H_s = Hamiltonian resulting from solar effects

$H^{(o)}$ = contains the linear and quadratic terms

$H^{(o)}_{\alpha_i}, H^{(o)}_{\beta_i}$ = partial derivatives of $H^{(o)}$ as defined in Eqs. (41)

i = inclination of E-M plane with ecliptic

J = transformation matrix defined in Eq. (32)

$J(\beta', \alpha', t)$ = $J_1 + J_2 + J_3$ = generating function introduced in Section IX.

K' = Hamiltonian containing only secular and slowly varying terms

K^* = time independent Hamiltonian = $K_i^* + K_e^*$

K_{i2}^* = coplanar part of K^* defined in Eq. (63)

L = Lagrangian

$m = n_s/n = .074801$ = dimensionless Earth orbital angular velocity

m_i = mass of ith body

n = mean angular velocity of E-M system

n_s = mean angular velocity of Earth around Sun

P_x, P_y, P_z
p_x, p_y, p_z = momenta defined by Eqs. (19)

Q, P = normal canonical coordinates introduced via Eq. (30)

$\bar{R}$ = displacement vector in inertial space

r_{ij} = distance between masses

$\bar{r}$ = position vector measured from L_4

$\bar{r}_{1L}$ = position vector from Earth to point L_4

T = kinetic energy

$T_\alpha = 2\pi/\omega_\alpha$ = period of slow oscillation in α_1^* and α_2^* obtained from Eq. (84)

V = potential energy

W = normal out-of-plane solar acceleration at Moon's position, introduced in Eqs. (121)

$\bar{x}, \bar{y}, \bar{z}$ = solutions to homogeneous linear equations

$\tilde{x}, \tilde{y}, \tilde{z}$ = forced response of linearized system

x_s, y_s, z_s = solar coordinates in xyz system, defined in Eq. (17)

α', β' = set of "slowly varying integration constants"

α^*, β^* = set of variables canonical with respect to K^*

α_i, β_i = set of integration constants

$\beta_i^{\neq}$ = $\left[t - (-1)^i \beta_i\right]$ where $(i = 1,2,3)$

$\Delta^* = \beta_1^* + \beta_2^*$ = angular variable

Δ_{13} = angular variable defined in Eq. (51)

$\Delta\omega_1, \Delta\omega_2$ = frequency shifts in the coplanar natural frequencies, defined in Eq. (79)

$\xi, \phi, \eta_o, \Omega, \upsilon'$ = angular variables introduced in Eqs. (15), (16), and (17) and defined in Appendix B

$\epsilon_{12}, \epsilon_{13}, \epsilon_{\xi 1}$ = detuning frequencies

η = constant defined in connection with Eq. (26)

μ = dimensionless quantity defined by Eq. (12) = $m_M/(m_E + m_M) \cong 1/82.45$

$\xi = \alpha_2^*/D_1$ = variable introduced by Eq. (67)

ρ, υ = perturbation quantities defined by Eqs. (15) and (16)

Φ_o = 6 × 6 matrix defined by Eq. (24)

$\dot{\Omega}, i$ = angular velocities of line of nodes, and of E-M plane inclination, respectively

ϖ = angular velocity vector of xyz system

$\omega_1, \omega_2, \omega_3$ = eigenvalues of the linear homogeneous set of differential equations (i.e., the natural frequencies about L_4)

ω_M = angular velocity of a hypothetical isolated E-M system (i.e., no solar perturbations present)

$(\ )^T$ = denotes transpose of ( )

$(\dot{\ })$ = denotes total time derivative

I. INTRODUCTION

The subject of the Earth-Moon libration points has aroused in recent years the curiosity and interest of a great many researchers in the field of celestial mechanics and analytical dynamics. This renewed interest by modern day investigators in this classical problem has been stimulated by the recent telescopic sightings by K. Kordylewski(1,2) of two faint cloud-like objects or shapes in the vicinity of the L_4 and L_5 Earth-Moon libration points. These findings have led to a great amount of speculation regarding the origin and stability of motion of such clouds, believed by many to be composed of minute dust particles.

Although a number of more recent naked eye sightings from high flying aircraft have since been reported by a few investigators in this country, the issue of the existence or nonexistence of these libration dust clouds has not yet been resolved to everyone's satisfaction by any of the current studies, and is still the subject of debate between proponents and detractors of this hypothesis. While the definitive answer to this question might not be obtained until concrete evidence and data will be gathered near these points from a space vehicle, the quest so far has not been all in vain. In the process a great many areas for further research of both a theoretical and a practical, mission oriented, nature have been exposed and tackled, which will keep many researchers busy for quite a while.

In the present dissertation we shall not attempt to shed new light on the question of the existence of dust clouds, but shall confine instead our attention to the study of the interesting underlying theoretical problem in nonlinear analytical dynamics of a particle. This particle may be associated, if one desires to do so, with the center of mass of a hypothetical dust cloud. It should be pointed out however that the uncritical application of some of the results and conclusions of the present study to the dust cloud problem might lead to misleading conclusions, since such important destabilizing effects as solar radiation pressure and particle collisions have not been considered here.

II. LIBRATION POINT GEOMETRY

Some of the geometrical features of libration points are briefly indicated below for the purpose of orientation.

The five libration points (also known as Lagrangian points) of the classical restricted 3-body problem (i.e., Sun is neglected, and Earth and Moon revolve in circular orbits about their common center of mass) are indicated in Fig. 1. They are points of equilibrium in the coordinate frame XYZ, rotating around the Z axis with the mean angular velocity n of the Earth-Moon system, in the sense that no net accelerations are experienced by particles at rest at these points.

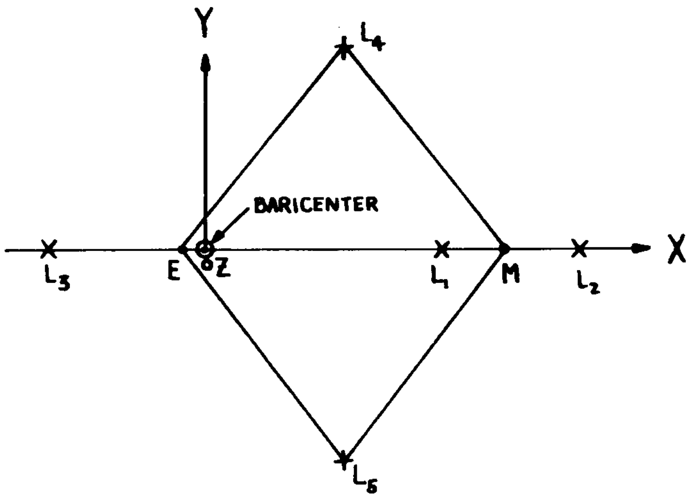


Fig. 1: Libration points of the restricted 3-body problem.

By means of linear small perturbation analysis the collinear points L_1, L_2, L_3 were found to be unstable to small initial disturbances, while the equilateral points L_4 and L_5 were found to be points of stable equilibrium around which small amplitude conditionally periodic (i.e., in this case doubly periodic but not necessarily simply periodic) motions resulted for small initial disturbances.

The more realistic physical model used in the present analysis is shown in Fig. 2. The Sun, lunar orbital eccentricity e ($\approx$ .055) and inclination i of the Earth-Moon plane with the ecliptic ($i \approx 5^o$) are included. The Earth is assumed to move in a circular orbit around the Sun.

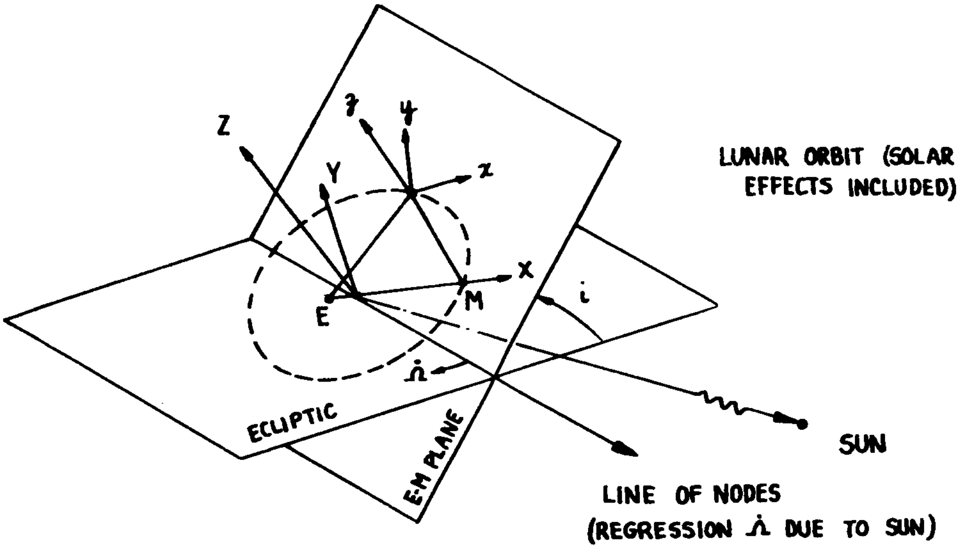


Fig. 2: Three dimensional geometry of the 4-body problem.

III. BRIEF REVIEW OF PAST WORK ON THE SUBJECT

Most of the basic work on the restricted 3-body problem stems back to some of the classical studies in analytical dynamics of Lagrange, Jacobi, Poincaré, etc. which are discussed in most of the standard textbooks on Celestial Mechanics. Some of the main features and results are briefly summarized in the following sections.

More recent analytic work on the 3-body problem concerned itself with such questions as the existence of periodic orbits both in the vicinity of the libration points, as well as periodic orbits which fill the whole Earth-Moon space and possibly loop a number of times around both primary bodies.

Studies which included the solar force field are of a more recent vintage and are predominantly of a numerical nature, in that they tackle the problem by direct integration of the full set of differential equations of motion for various periods of time t, and usually for a very restricted set of initial conditions(3-5) (i.e., zero particle displacements and velocities, and collinear position of the major bodies in the order Earth-Moon-Sun). The application of Hamiltonian techniques to the 2-dimensional libration point problem was suggested in an analytic study by Breakwell and Pringle.(6) These techniques are extended in the present thesis to the 3-dimensional problem which also includes the effects of lunar orbital eccentricity.

1. THE CLASSICAL RESTRICTED 3-BODY PROBLEM: PAST RESULTS AND THEIR LIMITATIONS

Some of the basic results of the 3-body theory, as related to the libration points, and some of the questions left unanswered by the theory are mentioned in A and B, respectively.

A. 1. The existence of the five Lagrangian equilibrium points shown in Fig. 1 was discovered.

2. The stability of motion near these points was investigated by linearizing the equations of motion near these points.

3. For small deviations from equilibrium the coplanar homogeneous set of equations (Eqs. (25) with $\rho = \nu = m = 0$) in the xy plane, which becomes uncoupled from the z equation, was shown to give rise to a doubly periodic solution with the eigenvalues $\omega_1 \cong .955$ and $\omega_2 \cong .298$ (these frequencies were nondimensionalized with respect to the mean Earth-Moon angular velocity $n \cong .23$ rad/day). The uncoupled, out of plane, linear equation in the z direction possesses a simple harmonic solution with eigenvalue $\omega_3 = 1$. (The reason for a period of 1 lunar month in the z motion is easy to explain physically if we consider the limiting case of a vanishingly small lunar gravitational force field. In that case the small particle at L_4 follows a near circular planar 2-body orbit around the Earth at the lunar distance, which crosses the Earth-Moon plane twice for each complete particle revolution, thus leading to an orbital period of 1 lunar month, which is also the same as the period of the projected simple harmonic oscillator in the z direction.)

4. A first, and only, integral constant of the motion was found to exist. This so-called Jacobi constant C_J corresponds to our scleronomic (i.e., time independent) Hamiltonian H, and consists of the combination $E - nh_z = \text{constant} = - C_J = H$, where E is the particle's total energy (i.e., kinetic and potential) in a nonrotating baricenter centered coordinate frame, h_z is its angular momentum in the Z direction, and n is the mean angular velocity of the Earth-Moon axis.

B. Some difficulties are encountered if one tries to extend the stability conclusions obtained from linear analysis to predict the behavior of the complete nonlinear system. The main reasons are indicated below.

1. The near commensurability of the eigenvalues $\omega_1 \cong 3\omega_2$ leads to an internal near resonance with a detuning $\epsilon_{12} = \omega_1 - 3\omega_2 \cong .954593-3\cdot.297912 = \cong .06086$. This causes poor convergence of the usual perturbation solutions by means of which one attempts to evaluate the effects of higher order terms, by substituting back the homogeneous solutions into the nonlinear driving terms. Some of them give rise to combination frequencies which are nearly resonant with the natural frequencies of the linear equations, and thus lead to small divisors in the next approximation.

2. The Hamiltonian H is not definite near L_4 or L_5 (positive or negative). This sign indefiniteness has a bearing on the nature of the stability of adjacent motions, as is briefly indicated below.

If we suitably recombine the terms in H of A(4) above we can come up with an equivalent relation for the Hamiltonian $H = \frac{1}{2} v^2 + V_{eff}$, where $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$ and V_{eff} represents an effective potential energy $V_{eff} = - \frac{1}{2} \omega^2 (x^2 + y^2) - \mu_1/r_1 - \mu_2/r_2$. The first term in H thus corresponds to the kinetic energy, as measured in the rotating frame, while the last two terms in V_{eff} represent the usual gravitational potential energy V. In this new form H can be interpreted as being in the nature of an energy integral of the motion. The nature of the stability near $L_{4,5}$ can thus be deduced from the shape of the surfaces V_{eff} = constant in that region. It turns out that near the equilateral points the planar part of V_{eff} has the shape of a "potential hill" rather than the "trough" which is required for stability.

This circumstance raises a question concerning the applicability of the linear-theory stability analysis to the complete nonlinear system, i.e., whether the nonlinear system would exhibit the same kind of stability as predicted by the linear equations for given initial conditions. One may remark at this point, on the basis of work to be presented later, that the answer is yes in a rather small neighborhood of L_4. The nonlinear system will however exhibit instability for certain ranges of initial conditions.

It is also appropriate to remark here that the stability of motion exhibited by the linear system near L_4 and L_5 in the presence of a potential energy "hill" is brought about by the presence of gyroscopic terms in the linear equation (due to the Corioli's force $2(\bar{n} \times \dot{\bar{r}})$ which arise in the rotating frame). When further nonlinear and external effects are included, it is possible for additional energy to be transferred into the system with the result that initially small oscillations may grow in the course of time.

It is interesting to mention that a Taylor series expansion of V_{eff} near L_4 shows the equipotential curves to be extremely elongated ellipses of fineness ratio roughly 1:10 oriented at right angles to the line from barycenter to L_4. The potential field thus falls off quite slowly as we move in a direction perpendicular to the Earth - L_4 line.

3. Another internal resonance occurs because of nonlinear coupling of the z and xy solutions, and the near commensurability of the eigenvalues $\omega_1 \cong \omega_3$, with the resulting detuning $\epsilon_{13} \cong .0454$. This resonance leads again to poor convergence of perturbation type solutions.

4. Although not actually a part of the classical 3-body problem, it might perhaps not be inappropriate to mention at this point also the presence of a third important resonance of an external nature caused by the Sun's perturbative action on a nominally circular lunar orbit, which is an important factor in the subsequent analysis. This indirect solar perturbation leads to a detuning $\epsilon_{\xi 1} = 2[\omega_1 - (1 - m)] = 2[.95459-.92520] \cong .05878$.

5. The additional complications of resonances introduced by the inclusion of lunar eccentricity terms will be taken up later.

2. NUMERICAL APPROACHES (SOLAR EFFECT INCLUDED)

Straightforward integration of the complete set of differential equations, for zero initial conditions, gives rise to particle trajectories, a typical xy projection of which looks roughly like the one shown in Fig. 3 (taken from Ref. 4).

Figure 4 presents schematically another plot due to Feldt and Shulman(5) of total particle displacement d with time t for an integration time period of 5000 days. Initial conditions were the same as those in Fig. 3.

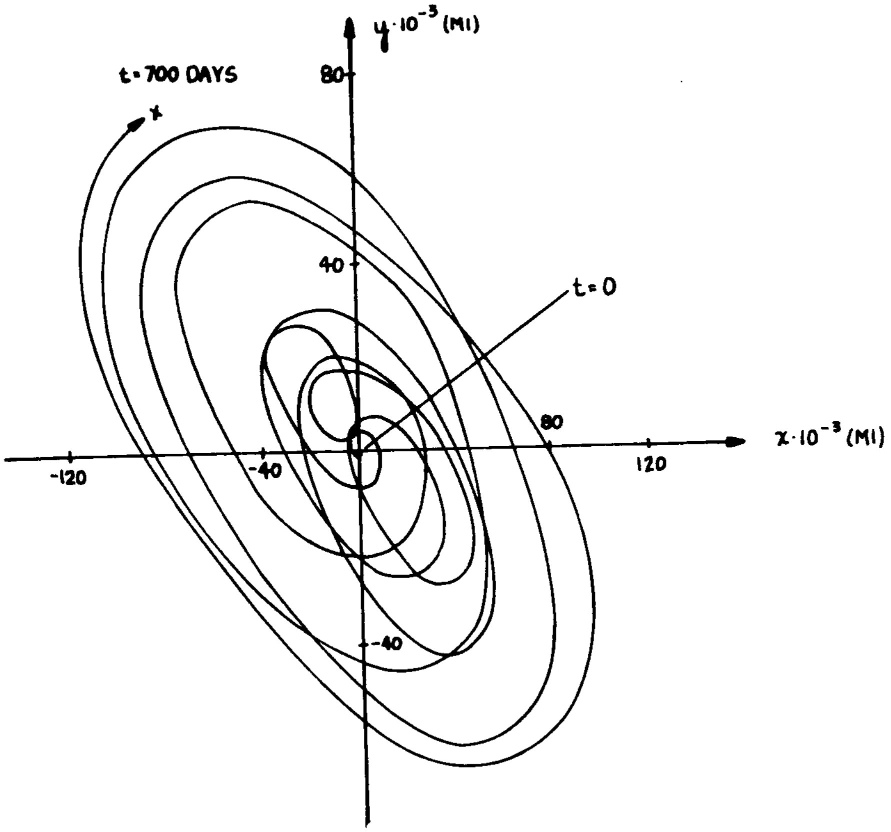


Fig. 3: Typical particle trajectory in xy plane near L_4 (t = 700 days)

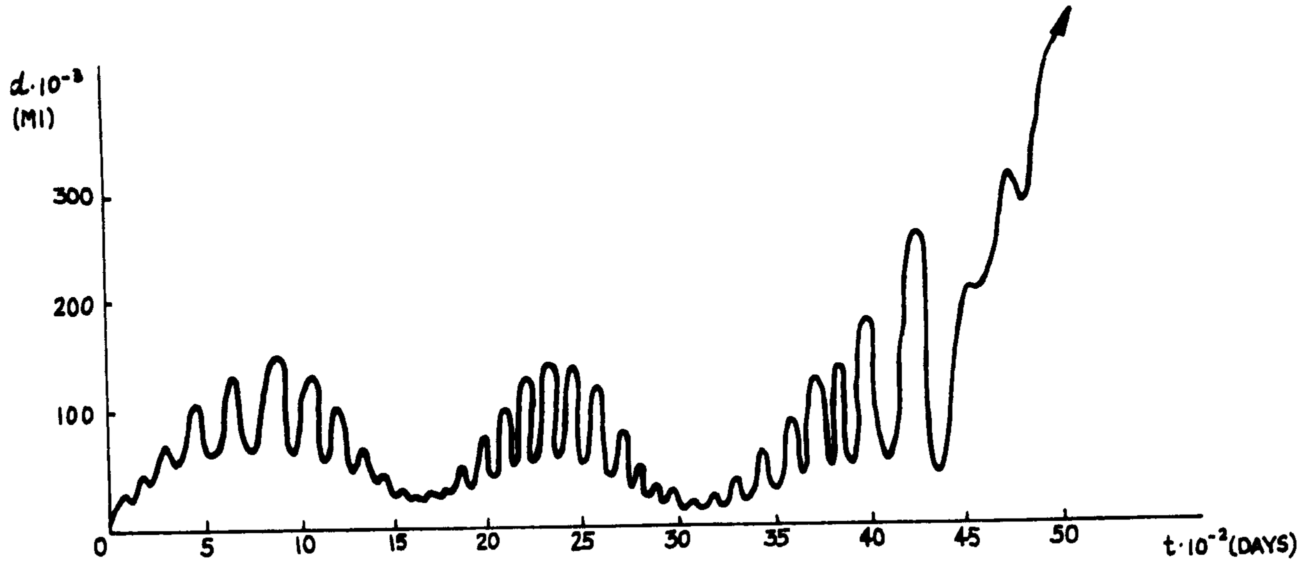


Fig. 4: Displacement-time history

IV. SOME CONCLUSIONS REGARDING PRIOR STATUS OF THE PROBLEM

The following conclusions summarize some of the points which were raised in Sections III(A) and III(B):

1. The past analytical efforts do not resolve in a satisfactory manner the question of boundedness of motion near the equilateral libration points of the Earth-Moon system, with or even without the inclusion of the perturbative effect of the Sun.

2. The numerical results available to date are rather limited in that they were generated only for restricted sets of initial conditions and initial Earth-Moon-Sun configurations. Consequently they do not shed much further light on the question of the possible existence of domains of initial conditions and configurations which allow small amplitude, bounded motions to take place for long time periods.

3. In view of the multiplicity of possible starting conditions and configurations, it is quite clear that a purely numerical search for such initial conditions would be both costly as well as of questionable success, and thus not very attractive.

4. The necessity and usefulness for further analytical groundwork on this problem seems to be clearly indicated.

The above brief rundown will hopefully help to bring into better perspective the difficulties as well as the motivations underlying the present investigation.

V. THE LAGRANGIAN L FOR A PARTICLE NEAR L_4

We shall desire the expression for the Lagrangian of a particle near the L_4 libration point, in the rotating xyz frame centered at L_4, and having its xy plane coincide with the fundamental Earth-Moon orbital plane. To this end it is convenient to start out with an inertial reference frame x_I, y_I, z_I in which the positions of Earth, Moon, Sun and particle P are designated by the numbers 1, 2, 3, and 4, respectively, and by the position vectors $\bar{R}_i$ $(i = 1, \cdots 4)$. The kinetic energy T_I and potential energy V_I of all the masses are then

$$T_I = \frac{1}{2} \sum_{i=1}^{4} m_i \dot{\bar{R}}_i \cdot \dot{\bar{R}}_i$$

$$V_I = -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \frac{Gm_i m_j}{|\bar{R}_i - \bar{R}_j|} = -\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{4} \frac{Gm_i m_j}{r_{ij}} \tag{1}$$

We switch first to an Earth centered rotating coordinate system X_e, Y_e, Z_e with the X_e axis pointing in the direction of the instantaneous position of the Moon (we neglect here the 3000 mi separation of barycenter from the center of the Earth). For a particle of unit mass at point 4 we then have

$$T = \frac{1}{2}\left(\dot{\bar{R}}_1 + \dot{\bar{r}}_{14}\right) \cdot \left(\dot{\bar{R}}_1 + \dot{\bar{r}}_{14}\right)$$

$$V = -\frac{\mu_1}{r_{14}} - \frac{\mu_2}{r_{24}} - \frac{\mu_3}{r_{34}} \tag{3}$$

$$L = \frac{1}{2}\left[\dot{\bar{r}}_{14} \cdot \dot{\bar{r}}_{14} + 2\dot{\bar{R}}_1 \cdot \dot{\bar{r}}_{14}\right] + \frac{\mu_1}{r_{14}} + \frac{\mu_2}{r_{24}} + \frac{\mu_3}{r_{34}} + \frac{1}{2} \dot{\bar{R}}_1 \cdot \dot{\bar{R}}_1$$

where

$\mu_i = Gm_i$

$i = 1,2,3$

The last term in L is independent of particle position and velocity and can be dropped. This follows from our assumption that the particle does not affect the motion of the primary bodies. It is also convenient to remove from L the explicit presence of the Earth's inertial velocity $\dot{\bar{R}}_1$. This can be done via Lagrange's equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\bar{r}}_{14}}\right] - \frac{\partial L}{\partial \bar{r}_{14}} = 0 \tag{4}$$

and the Earth's equation of motion in inertial space

$$\ddot{\bar{R}}_1 = \frac{\mu_2}{r_{12}^3}\,\bar{r}_{12} + \frac{\mu_3}{r_{13}^3}\,\bar{r}_{13} \tag{5}$$

Since $\ddot{\bar{R}}_1 \neq \ddot{\bar{R}}_1\,(\bar{r}_{14})$, one can replace Eq. (5) by the equivalent relation

$$\ddot{\bar{R}}_1 = \frac{\partial}{\partial \bar{r}_{14}}\left[\frac{\mu_2}{r_{12}^3}\,\bar{r}_{12}\cdot\bar{r}_{14} + \frac{\mu_3}{r_{13}^3}\,\bar{r}_{13}\cdot\bar{r}_{14}\right] \tag{6}$$

After substituting Eq. (6) into (4) one can extract from it the expression for L shown in Eq. (7):

$$L = \frac{1}{2}\,\dot{\bar{r}}_{14}\cdot\dot{\bar{r}}_{14} + \frac{\mu_1}{r_{14}} + \mu_2\left(\frac{1}{r_{24}} - \frac{\bar{r}_{12}\cdot\bar{r}_{14}}{r_{12}^3}\right) + \mu_3\left(\frac{1}{r_{34}} - \frac{\bar{r}_{13}\cdot\bar{r}_{14}}{r_{13}^3}\right) \tag{7}$$

The last (solar) term in (7) can be further simplified if we replace it with the solar potential energy gradient evaluated at the position of the Earth, as shown in Appendix A. This neglects terms of magnitude $(r_{14}/r_{13})^3 \cong 1.5 \times 10^{-8}$, which is quite satisfactory in the present case, and leads to the expression

$$L = \frac{1}{2}\dot{\bar{r}}_{14} \cdot \dot{\bar{r}}_{14} + \frac{\mu_1}{r_{14}} + \mu_2\left(\frac{1}{r_{24}} - \frac{\bar{r}_{12} \cdot \bar{r}_{14}}{r_{12}^3}\right) + \frac{\mu_3}{r_{13}^3}\left[\frac{3}{2}\left(\frac{\bar{r}_{13} \cdot \bar{r}_{14}}{r_{13}}\right)^2 - \frac{1}{2}\bar{r}_{14} \cdot \bar{r}_{14}\right] \qquad (8)$$

Expression (8) is still not in the desired final form of a Taylor series expansion around L_4. Before we carry out the expansion it is convenient to nondimensionalize everything, as indicated in the next section.

VI. NONDIMENSIONALIZATION AND EXPANSION AROUND L_4

1. NONDIMENSIONALIZATION

The nondimensionalization is most conveniently carried out by choosing the reference frequency n and length D defined by

$$n = \sqrt{\frac{\mu_1 + \mu_2}{D^3}} = < \omega_M + \dot{\Omega} \cos i > = \text{mean angular velocity of E-M axis } X_e \cong .23 \text{ rad/day} \qquad (9)$$

$$D = < r_{12} > = \text{mean E-M distance} \cong 2.4 \times 10^5 \text{ mi} \qquad (10)$$

It should be pointed out that the only physical quantity which can be measured with any degree of accuracy is n, so that the reference length D is actually a computed, rather than a natural quantity, and is defined by Eq. (9). The averaging of r_{12} in Eq. (10) must therefore be interpreted in the light of the more basic definition (9).

ω_M denotes the mean angular velocity of an isolated Earth-Moon system (no solar perturbations present), and $\dot{\Omega}$ and i are indicated in Fig. 2.

Two basic dimensionless quantities which will appear often in our equations are

$$m = \frac{n_s}{n} = \sqrt{\frac{\mu_3}{r_{13}^3} \cdot \frac{D^3}{\mu_1 + \mu_2}} \cong .074801 \qquad (11)$$

and

$$\mu = \frac{\mu_2}{\mu_1 + \mu_2} \cong \frac{1}{82.45} \qquad (12)$$

where

n_s = angular velocity of the Earth around the Sun

From now on all lengths, velocities and times will be treated as dimensionless quantities, but we shall retain their old symbols.

2. EXPANSION AROUND L_4

Just as n was the basic quantity selected in the nondimensionalization of the equations, we shall select m as the basic quantity, or yardstick, which defines order of magnitude. We shall denote by o(m) a quantity of first order of smallness, $o(m^2)$ of second order, etc...

The Lagrangian L of Eq. (8) can be written in terms of displacements and velocities measured in the L_4 centered xyz frame by writing the dimensionless vector relations

$$\begin{aligned} \bar{r}_{14} &= \bar{r}_{1L} + \bar{r} \\ \dot{\bar{r}}_{14} &= \dot{\bar{r}}_{1L} + \dot{\bar{r}} + \bar{\omega} \times \bar{r} \end{aligned} \qquad (13)$$

where

$$\bar{r} = x\bar{i}_x + y\bar{i}_y + z\bar{i}_z$$

$|\bar{r}_{1L}| = 1 + \rho(t) = |\bar{r}_{12}|$ = instantaneous displacement of libration point L_4 from the Earth

$$\dot{\bar{r}}_{1L} = |\dot{\bar{r}}_{1L}|\left(\frac{1}{2}\,\bar{i}_x + \frac{\sqrt{3}}{2}\,\bar{i}_y\right) + \bar{\omega} \times \bar{r}_{1L}$$

and for the total angular velocity $\bar{\omega}$ of the xyz frame in inertial space

$$\bar{\omega} = \frac{n}{n}\,\bar{i}_z + \bar{\nu}(t) = \bar{i}_z + \bar{\nu}(t) \qquad (14)$$

$\rho(t)$ and $\nu(t)$ are the perturbations of the E-M distance, and angular velocity caused by solar and eccentricity effects, and are provided by classical lunar theory.(7,8)

$$\rho(t) = -\ .0079 \cos 2\xi - .00093 - e \cos \phi + \frac{1}{2} e^2 (1 - \cos 2\phi)$$
$$- \frac{15}{8} em \cos (2\xi - \phi) \qquad (15)$$

$$\bar{\nu}(t) = \left[\dot{\Omega} \sin i \sin \eta_o + \dot{i} \cos \eta_o\right] \bar{i}_x$$
$$+ \left[\dot{\Omega} \sin i \cos \eta_o - \dot{i} \sin \eta_o\right] \bar{i}_y$$
$$+ \left[.0202 \cos 2\xi + 2e \cos \phi + \frac{15}{4} em \cos (2\xi - \phi)\right.$$
$$\left. + \frac{5}{2} e^2 \cos 2\phi\right] \bar{i}_z$$
$$= \nu_x \bar{i}_x + \nu_y \bar{i}_y + \nu_z \bar{i}_z \qquad (16)$$

For additional details regarding the above expressions, and for an explanation of the various angular variables used, the reader is referred to Appendix B. The coordinates of the Sun in the X_e, Y_e, Z_e frame, presented in Eqs. (17) are also developed in this appendix.

The Sun's position coordinates in the rotating frame are

$$x_s \cong r_{13} \cos \xi$$
$$y_s = -\ r_{13} \sin \xi$$
$$z_s = r_{13} \sin i \sin (\Omega - v') \qquad (17)$$

We now stipulate that the following quantities will be treated as being of the first order of smallness:

$$m, e, x, y, z, P_x, P_y, P_z, \sqrt{\rho(t)}, \sqrt{\nu(t)} \qquad (18)$$

The momenta P_x, P_y, P_z conjugate to x,y,z are introduced through the relations

$$p_x = \frac{\partial L}{\partial \dot{x}} = P_x - \frac{\sqrt{3}}{2}$$

$$p_y = \frac{\partial L}{\partial \dot{y}} = P_y + \frac{1}{2} \qquad (19)$$

$$p_z = \frac{\partial L}{\partial \dot{z}} = P_z$$

The terms linear in $e(\cong .055)$ in $\rho(t)$ and $\nu(t)$ are obviously only of $o(m)$, and will have to be treated in a different fashion if we are to retain the definition of Eq. (18). This problem will arise when we include the eccentricity in the canonical transformations to slow variables.

The use of a Taylor series to expand L and H around L_4 in terms of x,y,z, P_x, ... etc ... raises the question of how many terms of the series expansion have to be retained before we truncate it, i.e., what order of nonlinear terms must be retained so as to take into account all the dominant perturbative effects. This question is readily answered by noting that the highest internal resonance is that resulting from the near equality $\omega_1 \cong 3\omega_2$ which indicates that nonlinear terms up to and including the fourth order must be retained in the Taylor expansions of L and H.

When all the steps have been carried out and all the terms collected, as shown in Appendix C, one obtains for the Hamiltonian H, defined as usual by means of

$$H = p_r^T \dot{r} - L \qquad (20)$$

where

$$p_r^T = \left[p_x, p_y, p_z\right] = (1 \times 3) \text{ row matrix of momenta elements}$$

and

$$r = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (3 \times 1) \text{ column matrix of position elements}$$

the expression of Eq. (21)

$$\begin{aligned}
H = H^{(o)} + H' = \Bigg\{ & \frac{1}{2}\left(P_x^2 + P_y^2 + P_z^2\right) + \left(yP_x - xP_y\right) + \frac{1}{8}\left(x^2 - 5y^2 + 4z^2\right) \\
& - \frac{3\sqrt{3}}{4}(1 - 2\mu)xy - \frac{1}{2}\dot{\rho}\left(P_x + \sqrt{3}\,P_y\right) + \frac{1}{2}\left(\rho + \upsilon_z\right)\left(\sqrt{3}\,P_x - P_y\right) \\
& + \frac{1}{2}\left(\upsilon_y - \sqrt{3}\,\upsilon_x\right)P_z - \left(\rho + \frac{1}{2}\upsilon_z\right)\left(x + \sqrt{3}y\right) \\
& - m^2\left[\frac{3}{2r_{13}^2}\left(x_s + \sqrt{3}y_s\right)\left(x_s x + y_s y + z_s z\right) - \frac{1}{2}\left(x + \sqrt{3}y\right)\right] \\
& + \frac{1}{2}\left(\sqrt{3}\,\upsilon_y + \upsilon_x\right)z\Bigg\}^{(o)} \\
& + \Bigg\{\frac{3\sqrt{3}}{16}\left(x^2y + y^3\right) + \frac{1 - 2\mu}{16}\left(33xy^2 - 7x^3 - 12xz^2\right) - \frac{3\sqrt{3}}{4}yz^2\Bigg\}_3 \\
& + \Bigg\{\frac{5\sqrt{3}\,(1 - 2\mu)}{32}\left(5x^3y - 9xy^3 + 12xyz^2\right) + \frac{37}{128}x^4 + \frac{3}{16}x^2z^2 + \frac{33}{16}y^2z^2 \\
& - \frac{123}{64}x^2y^2 - \frac{3}{128}y^4 - \frac{3}{8}z^4\Bigg\}_4 + \Bigg\{-\frac{3}{8}\rho\left(x^2 - 5y^2 + 4z^2 - 6\sqrt{3}\,(1 - 2\mu)xy\right) \\
& + \upsilon_z\left(yP_x - xP_y\right) + \upsilon_y\left(xP_z - zP_x\right) + \upsilon_x\left(zP_y - yP_z\right) \\
& - m^2\left[\frac{3}{2r_{13}^2}\left(x_s x + y_s y\right)^2 - \frac{1}{2}\left(x^2 + y^2 + z^2\right)\right]\Bigg\}_s + 0(m^5,m^6) \text{ etc.}
\end{aligned} \tag{21}$$

In the above expression we have split H' into a cubic part H_3, a quartic part H_4 and a solar part H_s, which in turn is composed of indirect solar effects (via ρ and υ) and a direct solar effect (via the m^2 term).

We shall concern ourselves in Section VII only with the motion resulting from the bracket $\{\ \}^{(o)}$ which represents the linear and quadratic part $H^{(o)}$ of H. These terms give rise to a system of forced linear differential equations which will be discussed below.

The analysis of the effect of the terms in H' on the motion of the particle will be started in Section VIII.

VII. THE LINEAR DIFFERENTIAL EQUATIONS AND THE TRANSFORMATION TO NORMAL CANONICAL COORDINATES

Hamilton's equations can be written down in a very compact form by using the matrix notation. We define the (3 × 1) column matrix for r and P_r in a manner similar to those introduced for r and P_r in connection with Eq. (20), and introduce the additional (1 × 3) row matrix of partial derivatives of $H^{(o)}$

$$
\begin{aligned}
H_r^{(o)} &= \left[\frac{\partial H^{(o)}}{\partial x}, \frac{\partial H^{(o)}}{\partial y}, \frac{\partial H^{(o)}}{\partial z}\right] \\
H_{P_r}^{(o)} &= \left[\frac{\partial H^{(o)}}{\partial P_x}, \frac{\partial H^{(o)}}{\partial P_y}, \frac{\partial H^{(o)}}{\partial P_z}\right]
\end{aligned}
\qquad (22)
$$

The equations of motion then can be written in the form

$$
\begin{bmatrix} \dot{r} \\ \\ \dot{P}_r \end{bmatrix} = \Phi_o \begin{bmatrix} H_{r^T}^{(o)} \\ \\ H_{P_r^T}^{(o)} \end{bmatrix} \qquad \text{(6 x 1) column matrix} \qquad (23)
$$

where $H_{r^T}^{(o)}$ and $H_{P_r^T}^{(o)}$ are the transpose of $H_r^{(o)}$ and $H_{P_r}^{(o)}$ respectively, Φ_o is the (6 × 6) matrix

$$
\Phi_o = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \qquad (24)
$$

I is the (3 × 3) identity matrix, and 0 is the (3 × 3) null matrix.

In component form, Eq. (23) becomes

$$
\dot{x} = H_{P_x}^{(o)} = P_x + y + \left\{- \frac{1}{2} \rho + \frac{\sqrt{3}}{2} \left(\rho + \upsilon_z\right)\right\}
$$

$$
\dot{y} = H_{P_y}^{(o)} = P_y - x + \left\{- \frac{\sqrt{3}}{2} \dot{\rho} - \frac{1}{2} \left(\rho + \upsilon_z\right)\right\}
$$

(cont. on next page)

$$\dot{z} = H^{(o)}_{P_z} = P_z + \left\{\frac{1}{2}\left(\upsilon_y - \sqrt{3}\,\upsilon_x\right)\right\}_s$$

$$\dot{P}_x = -H^{(o)}_x = P_y - \frac{1}{4}x + \frac{3\sqrt{3}}{4}(1 - 2\mu)y + \left\{\left(\rho + \frac{1}{2}\upsilon_z\right) + m^2\left[\frac{3}{2r^2_{13}}\left(x^2_s + \sqrt{3}\,x_s y_s\right) - \frac{1}{2}\right]\right\}_s$$

$$\dot{P}_y = -H^{(o)}_y = -P_x + \frac{5}{4}y + \frac{3\sqrt{3}}{4}(1 - 2\mu)x + \left\{\sqrt{3}\left(\rho + \frac{1}{2}\upsilon_z\right) + m^2\left[\frac{3}{2r^2_{13}}\left(x_s y_s + \sqrt{3}\,y^2_s\right) - \frac{\sqrt{3}}{2}\right]\right\}_s$$

$$\dot{P}_z = -H^{(o)}_z = -z + \left\{-\frac{1}{2}\left(\sqrt{3}\,\upsilon_y + \upsilon_x\right) + m^2\left[\frac{3}{2r^2_{13}}\left(x_s z_s + \sqrt{3}\,y_s z_s\right)\right]\right\}_s \tag{25}$$

The terms in $\{\ \}_s$ contain the direct and indirect solar contributions.

The homogeneous part of Eq. (25) is obtained by setting $\rho = \upsilon = m = o$. The characteristic equation resulting from a trial solution $e^{i\omega t}$ is

$$\left(1 - \omega^2\right)\left[\omega^4 - \omega^2 + \frac{27}{16} - \eta^2\right] = 0 \tag{26}$$

where

$$\eta = \frac{3\sqrt{3}}{4}(1 - 2\mu) = 1.26753$$

The solutions to Eq. (26) are the eigenvalues

$$\begin{aligned} \omega_1 &= \pm .95459 \\ \omega_2 &= \pm .29791 \\ \omega_3 &= \pm 1.0 \qquad \text{(corresponds to an uncoupled z motion)} \end{aligned} \tag{27}$$

The above ω's are the natural frequencies which were used in the discussion of the detunings in Section III.

Let the solutions of the homogeneous set of equations be denoted by $\bar{x},\bar{y},\bar{z}\cdots$and suitable particular integrals by $\tilde{x},\tilde{y},\tilde{z}\cdots$ Thus the complete solutions are

$$\begin{aligned} x &= \bar{x} + \tilde{x} \\ y &= \bar{y} + \tilde{y} \\ \vdots & \quad \vdots \quad \vdots \end{aligned} \tag{28}$$

For later use the 6 constants of integration which appear in the solutions (28) are best introduced by transforming first to a normal canonical set of coordinates Q and momenta P

$$Q^T = \left[Q_1, Q_2, Q_3\right] \qquad P^T = \left[P_1, P_2, P_3\right] \tag{29}$$

which also satisfy Hamilton's equations of motion and represent uncoupled motions in the form of independent simple harmonic oscillations having as frequencies the three eigenvalues ω_i.

The linear equations of transformation can be written in the form

$$\begin{pmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \\ \bar{P}_x \\ \bar{P}_y \\ \bar{P}_z \end{pmatrix} = J \begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \tag{30}$$

where J is a (6 × 6) matrix whose columns consist of the eigenvectors corresponding to the eigenvalues $\pm\ \omega_1$, and which are normalized so as to satisfy Eq. (31) which is the necessary condition for a canonical transformation.

$$J \ \Phi_o \ J^T = \Phi_o \tag{31}$$

The matrix presented in Eq. (32) satisfies this conditions and thus provides the proper coordinate transformation.

$$J = \begin{bmatrix} 0 & 0 & 0 & \frac{K_1}{\omega_1}\left(\omega_1^2 + \frac{9}{4}\right) & -\frac{K_2}{\omega_2}\left(\omega_2^2 + \frac{9}{4}\right) & 0 \\ -2K_1\omega_1 & -2K_2\omega_2 & 0 & -\frac{K_1}{\omega_1}\eta & \frac{K_2}{\omega_2}\eta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -K_1\omega_1\left(\omega_1^2 + \frac{1}{4}\right) & -K_2\omega_2\left(\omega_2^2 + \frac{1}{4}\right) & 0 & \frac{K_1}{\omega_1}\eta & -\frac{K_2}{\omega_2}\eta & 0 \\ K_1\omega_1\eta & K_2\omega_2\eta & 0 & \frac{K_1}{\omega_1}\left(\frac{9}{4} - \omega_1^2\right) & -\frac{K_2}{\omega_2}\left(\frac{9}{4} - \omega_2^2\right) & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

where

$$K_i = \left\{\left|\frac{11}{2}\omega_i^2 + 2\eta^2 - \frac{45}{8}\right|\right\}^{-1/2} \qquad i = 1,2$$

$$K_1 = .62016$$

$$K_2 = .72101$$

The numerical values of the elements in J are

$$J = \begin{bmatrix} 0 & 0 & 0 & 2.05374 & -5.66028 & 0 \\ -1.24032 & -1.44202 & 0 & -.823463 & 3.06768 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -.687459 & -.0727629 & 0 & .823463 & -3.06768 & 0 \\ .750378 & .272262 & 0 & .869732 & -5.23066 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix} \tag{33}$$

In terms of Q and P the Hamiltonian $H^{(o)}$ (for the case $\rho = \nu = m = o$) becomes

$$H^{(o)} = \frac{1}{2}\left(P_1^2 + \omega_1^2 Q_1^2\right) - \frac{1}{2}\left(P_2^2 + \omega_2^2 Q_2^2\right) + \frac{1}{2}\left(P_3^2 + \omega_3^2 Q_3^2\right) \qquad (34)$$

The solutions for the three harmonic oscillators which make up the expression for $H^{(o)}$ in Eq. (34) can be given in the form

$$\begin{aligned}
Q_1 &= \frac{\sqrt{2\alpha_1}}{\omega_1} \sin \omega_1 \beta_1^{\neq} \\
Q_2 &= \frac{\sqrt{2\alpha_2}}{\omega_2} \sin \omega_2 \beta_2^{\neq} \\
Q_3 &= \frac{\sqrt{2\alpha_3}}{\omega_3} \sin \omega_3 \beta_3^{\neq} \\
P_1 &= \sqrt{2\alpha_1} \cos \omega_1 \beta_1^{\neq} \\
P_2 &= - \sqrt{2\alpha_2} \cos \omega_2 \beta_2^{\neq} \\
P_3 &= \sqrt{2\alpha_3} \cos \omega_3 \beta_3^{\neq}
\end{aligned} \qquad (35)$$

where $\beta_1^{\neq} = t + \beta_1$, $\beta_2^{\neq} = t - \beta_2$, $\beta_3^{\neq} = t + \beta_3$, and α_i, β_i are the 6 required constants of integration.

Substitution of Eq. (35) and the J matrix (33) into Eq. (30) gives the homogeneous solutions for the coordinates

$$\begin{aligned}
\bar{x} &= 2.902 \sqrt{\alpha_1} \cos \omega_1 \beta_1^{\neq} + 8.003 \sqrt{\alpha_2} \cos \omega_2 \beta_2^{\neq} \\
\bar{y} &= 2.103 \sqrt{\alpha_1} \cos \left(\omega_1 \beta_1^{\neq} + 123.57^{o}\right) \\
&\quad + 4.793 \sqrt{\alpha_2} \cos \left(\omega_2 \beta_2^{\neq} + 154.82^{o}\right) \\
\bar{z} &= \sqrt{2\alpha_3} \cos \beta_3^{\neq}
\end{aligned} \qquad (36)$$

The particle trajectories in the xy plane corresponding to each of the two coplanar normal modes are ellipses with major axes at right angles to the vector $\bar{r}_{1L}$ and thickness ratios (minor axis/major axis) 1:2 for ω_1 and 1:5 for ω_2 as shown in Fig. 5. Motion proceeds in a clockwise direction.

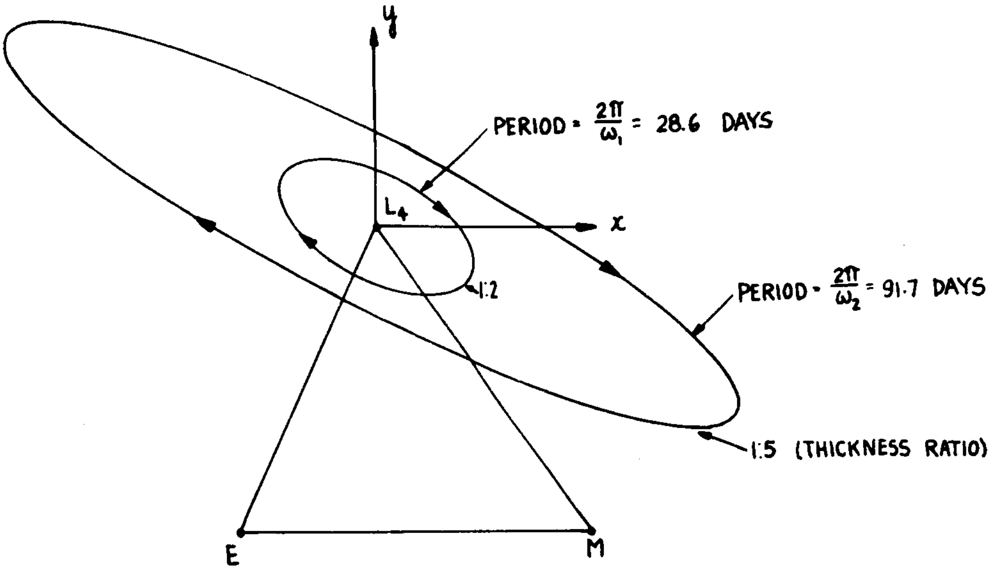


Fig. 5: Trajectories of normal modes.

The complete unperturbed xy motion consists of a weighted superposition of these two normal modes, and is in general not periodic.

The particular solutions $\tilde{x}, \tilde{y}$, corresponding to the forcing functions contained in the { } brackets of Eq. (25) are most readily obtained from the coplanar equations

$$\begin{aligned} \ddot{x} - 2\dot{y} - \frac{3}{4} x - \eta y &= \dot{f}_x - f_y + f_{P_x} \\ \ddot{y} + 2\dot{x} - \eta x - \frac{9}{4} y &= \dot{f}_y + f_x + f_{P_y} \end{aligned} \qquad (37)$$

where f_x, f_y, f_{p_x}, f_{p_y} denote the direct and indirect solar forcing functions of the subscript variables given in equations (25). For our purposes it is sufficient to obtain the particular solutions to $o(m^2)$. After introducing Eqs. (16) into (37) we obtain the solutions

$$\begin{aligned}\tilde{x} &= \tilde{x}_o + \tilde{x}_e + \tilde{x}_{em} + \tilde{x}_{e^2} + \tilde{x}_{m^2} + \tilde{x}_c \\ \tilde{y} &= \tilde{y}_o + \tilde{y}_e + \tilde{y}_{em} + \tilde{y}_{e^2} + \tilde{y}_{m^2} + \tilde{y}_c\end{aligned} \tag{38}$$

where

$$\left.\begin{aligned}\tilde{x}_o &= .01016 \cos (2\xi - 67.2^o) \\ \tilde{y}_o &= .00867 \cos (2\xi + 38.3^o)\end{aligned}\right\} \quad \text{Resulting from the indirect solar terms}$$

$$\begin{aligned}\tilde{x}_e &= .31\, e \cos (\phi - 72.2^o) \\ \tilde{y}_e &= .227\, e \cos (\phi + 50.2^o)\end{aligned}$$

$$\begin{aligned}\tilde{x}_{em} &= 11.1\, em \cos (2\xi - \phi - 75.2^o) \\ \tilde{y}_{em} &= 7.86\, em \cos (2\xi - \phi + 51.76^o)\end{aligned}$$

$$\begin{aligned}\tilde{x}_{e^2} &= 1.274\, e^2 \cos (2\phi + 30.8^o) \\ \tilde{x}_{e^2} &= 1.062\, e^2 \cos (2\phi - 66.0^o)\end{aligned}$$

$$\left.\begin{aligned}\tilde{x}_{m^2} &= 1.697\, m^2 \cos (2\xi - 127.7^o) \\ \tilde{y}_{m^2} &= 1.43\, m^2 \cos (2\xi - 20.83^o)\end{aligned}\right\} \quad \text{Resulting from the direct solar terms}$$

$$\left.\begin{aligned}\tilde{x}_c &= .50\, e^2 \\ \tilde{y}_c &= .2895\, e^2\end{aligned}\right\} \quad \text{Constant displacement}$$

No particular solution for $\tilde{z}$ is retained since it is of $o(m^3)$ or higher, and would lead to terms of $o(m^5)$ when substituted into H'. On this point we shall have something more to say in Section XIII.

The corresponding solutions for $\tilde{P}_r$ are readily obtained from the relations

$$\begin{aligned} P_x &= \dot{x} - y \\ P_y &= \dot{y} + x \\ &\text{etc.} \end{aligned} \tag{39}$$

It is interesting to note that if we substitute Eq. (35) into Eq. (34) we obtain the simple expression

$$H^{(o)} = \alpha_1 - \alpha_2 + \alpha_3 \tag{40}$$

The particular manner of introducing the polar set of integration constants α_i, β_i into Eq. (35) follows from the canonical relationship which they bear the Hamiltonian $H^{(o)}$. The quantities $\beta_1^{\neq}$, $-\beta_2^{\neq}$, $\beta_3^{\neq}$ and α_1, α_2, α_3 form, respectively, a canonical set of coordinates and conjugate momenta with respect to $H^{(o)}$ of Eq. (40).

Thus

$$\begin{array}{llll} \dot{\beta}_1^{\neq} = 1 = H^{(o)}_{\alpha_1} & \dot{\alpha}_1 = - H^{(o)}_{\beta_1^{\neq}} = 0 & \text{or} & \alpha_1 = \text{const} \\ \dot{\beta}_2^{\neq} = -1 = H^{(o)}_{\alpha_2} & \dot{\alpha}_2 = + H^{(o)}_{\beta_2^{\neq}} = 0 & \text{or} & \alpha_2 = \text{const} \\ \dot{\beta}_3^{\neq} = 1 = H^{(o)}_{\alpha_3} & \dot{\alpha}_2 = - H^{(o)}_{\beta_3^{\neq}} = 0 & \text{or} & \alpha_3 = \text{const} \end{array} \tag{41}$$

The above results are in agreement with our stipulation that α_i and β_i be constants.

Furthermore, the quantities α_i and β_i themselves form a canonical set with respect to an unperturbed Hamiltonian $H = 0$.

The above canonical properties will be made use of when we analyze the perturbative effect of H'.

The form of $H^{(o)}$ in Eq. (40) makes it very easy to verify the point made earlier in B(2) of Section III, regarding the sign indeterminacy of H which is seen to depend, for small α_3, on the difference

$\alpha_1 - \alpha_2$. Although in the present case α_1 and α_2 individually are constants, it turns out that for the case $\alpha_3 \equiv 0$ the combination $\alpha_1(t) - \alpha_2(t)$ remains a constant of the motion also when the perturbative effects of higher order internal nonlinearities are included (but external solar perturbations and lunar eccentricity are still neglected). Thus α_1 and α_2 may grow individually as long as their difference remains fixed, which indicates the possibility of an internally generated instability near L_4 also for the classical restricted 3-body problem (for which we use the exact expression for H).

That H is a constant of the motion in the latter case (where $H \neq H(t)$) as stated in A(4) of Section III, is readily verified since

$$\frac{dH}{dt} = \frac{\partial H}{\partial r}\dot{r} + \frac{\partial H}{\partial p}\dot{p} = \text{using Eq. (23)} = -\dot{p}^T\dot{r} + \dot{r}^T\dot{p} = 0 \tag{42}$$

The only existing integral of the motion, the Jacobi constant C_J (see pp. 281 of Ref. 9 where it is denoted by unsubscripted C), is equal to the negative of H

$$H = -C_J \tag{43}$$

VIII. MODIFICATION OF THE LINEAR SOLUTION DUE TO H$'$

The inclusion of the terms in the Hamiltonian H$'$, neglected until now in the previous solution, can be handled by a method equivalent to the customary variations of constants technique by requiring the original constants of integration α and β introduced in Eq. (35) to become functions of time, which then satisfy Hamilton's equations with Hamiltonian H$'$.

Inasmuch as we are not concerned in the present investigation with an exact or detailed determination of the particle's trajectory, but rather in the overall broad features of the motion, we shall desire to obtain only the slowly varying components of α and β which will arise from the secular terms in H$'$, and those terms containing low combination frequencies which arise from the near resonances.

This can be accomplished by means of a suitable canonical transformation of coordinates from the polar canonical set α,β associated with H = 0 to a new slowly varying canonical set α',β' associated with a new slowly varying Hamiltonian K$'$. K$'$ will contain only the lowest frequency terms which arise in H$'$ as a result of the above transformation, all other faster terms having been suitably eliminated. The question as to which frequencies should be retained, and the cutoff point beyond which the periodic terms are dropped cannot be readily answered in general terms, but would depend on the particular problem considered, and also on the density of spacing of the resonance peaks in the lower end of the frequency spectrum. This point will be touched upon again later in connection with the specific form of the expression for K$'$.

Returning once more to the coordinate transformation mentioned earlier, it is reasonable to assume that for relatively small displacements x,y,z of the particle, the effect of H$'$ would be in the nature of a perturbation of the linearized solution found earlier. With this assumption in mind we may now consider a stationary contact transformation

$$\begin{aligned} \alpha_i' &= \alpha_i + \delta\alpha_i \\ \beta_i' &= \beta_i + \delta\beta_i \end{aligned} \tag{44}$$

that may be introduced with the aid of a generating function $G(\beta,\alpha')$

$$G(\beta,\alpha') = \beta\alpha' + S(\beta,\alpha') \tag{45}$$

which satisfies the relations$^{(10)}$

$$\begin{aligned} \beta' &= \frac{\partial G}{\partial \alpha'} = \beta + S_{\alpha'} \\ \alpha &= \frac{\partial G}{\partial \beta} = \alpha' + S_{\beta} \end{aligned} \tag{46}$$

The first term $\beta\alpha'$ in G generates the identity transformation, while the function $S(\beta,\alpha') = S_1 + S_2$ denotes an additional suitably selected generating function which is introduced for the specific purpose of eliminating all the short period terms which occur in H': S_1 is selected to eliminate the terms of $o(m^3)$ and S_2 those of $o(m^4)$.

Since S does not depend explicitly on time t we can write

$$K'(\beta',\alpha',t) = H^{(o)}(\beta',\alpha',t) + H'(\beta',\alpha',t) \tag{47}$$

where H above is evaluated in terms of the new coordinates β' and new momenta α'.

When all the required steps of the transformation are carried out, as indicated in Appendix D, one arrives at the following relation for K'

$$K' = \overset{\approx}{H}_3 + \overset{\approx}{H}_4 + \overline{H}_4 - \frac{1}{2}\overline{\left[H_3, S_1\right]} \tag{48}$$

$\tilde{\bar{H}}_3$ and $\tilde{\bar{H}}_4$ are the long period terms resulting from the Taylor series expansions

$$\tilde{H}_i = \sum_{n=1}^{\infty} \frac{1}{n!} \left[\tilde{x} \frac{\partial}{\partial x} + \tilde{y} \frac{\partial}{\partial y} + \tilde{z} \frac{\partial}{\partial z} \right]^n H_i \tag{49}$$

evaluated at $\bar{x},\bar{y},\bar{z}$.

$\overline{[H_3,S_1]}$ denotes the long period part of the Poisson bracket of H_3 with S_1. $\bar{H}_4$ results from the substitution of the homogeneous solutions $\bar{x},\bar{y},\bar{z}$ into H_4, and consists of an internal part $\bar{H}_{4int}$ and an external part $\bar{H}_{4ext}$ which contains both the direct and indirect solar effects.

The algebraic work needed to express K' in terms of α',β' and t is rather formidable, and is one of the major stumbling blocks in what would otherwise be a relatively straightforward solution. A few representative steps of the required manipulations are briefly demonstrated in Appendix E. If all the manipulations have been successfully carried out, one does eventually come up with an expression for K' which has the general form shown in Eq. (50).

$$\begin{aligned} K' = {} & \alpha_1'\left[b_1 + b_2 C_{2\Delta\xi_1+\lambda_1}\right] + \alpha_2' b_3 + \alpha_3' b_4 + \alpha_1'^{3/2} b_5 C_{\Delta\phi_1+\lambda_2} \\ & + \alpha_2'^{3/2}\left[b_6 C_{\Delta\phi_2+\lambda_3} + b_7 C_{\sigma-\Delta\phi_1+\lambda_4}\right] + \alpha_1'^{1/2}\alpha_2\left[b_8 C_{\Delta\phi_1+\lambda_5}\right] \\ & + \alpha_1'^{1/2}\alpha_3'\left[b_9 C_{\Delta\phi_1+\lambda_6} + b_{10} C_{\Delta_{13}+\Delta\phi_3+\lambda_7}\right] + \alpha_1'\alpha_2' b_{11} + \alpha_1'^2 b_{12} \\ & + \alpha_2'^2 b_{13} + \alpha_2'^{1/2} b_{14} C_{\sigma+\lambda_8} + \alpha_1'\alpha_2'^{1/2}\left[b_{15} C_{\sigma-\Delta\phi_1+\lambda_9} + b_{16} C_{\Delta\phi_2+\lambda_{10}}\right] \\ & + \alpha_3'^2 b_{17} + \alpha_1'\alpha_3'\left[b_{18} + b_{19} C_{2\Delta_{13}+\lambda_{11}}\right] + \alpha_2'\alpha_3' b_{20} \\ & + \alpha_2'^{1/2}\alpha_3'\left[b_{21} C_{\Delta\phi_2+\lambda_{12}} + b_{22} C_{\Delta_{13-\sigma+\Delta\phi_3+\lambda_{13}}}\right] \end{aligned} \tag{50}$$

$b_j (j = 1,\cdots 22)$ are known constants and C_x stands for cos x. The detuning frequencies retained in Eq. (50) have the following magnitudes:

$$\begin{aligned}
&2\Delta\xi_1 \rightarrow 2(1 - m - \omega_1) = -.05878\\
&2\Delta\phi_1 \rightarrow 2(1 - \frac{3}{4} m^2 - \omega_1) = .08242\\
&\Delta\phi_2 = \phi - 3\omega_2\beta_2^{\neq} \rightarrow (1 - .0042 - 3\omega_2) = .10207\\
&\sigma = \omega_1\beta_1^{\neq} - 3\omega_2\beta_2^{\neq} \rightarrow \omega_1 - 3\omega_2 = .06086\\
&\Delta_{13} \rightarrow \omega_1 - 1 = -.04541\\
&\Delta_{13} + \Delta\phi_3 \rightarrow - .04541 - .0042 = -.04961\\
&\sigma - \Delta\phi_1 \rightarrow .06086 - .04121 = .01965
\end{aligned} \tag{50a}$$

and

$$\Delta_{13} - \sigma + \Delta\phi_3 \rightarrow -.1105$$

The terms containing $\Delta\phi$ arise from the lunar eccentricity.

As can be seen from Eqs. (50a) no terms with frequencies larger than .12 have been retained in the expression for K'. Although this choice of cut-off frequency appears at first sight rather arbitrary, it can be argued here that for higher frequencies the resultant detuning would not be narrow enough to introduce the very small divisors which usually lead to divergent solutions, and that consequently their omission should not materially affect the overall features of the resultant particle motion.

The large number of frequencies which still are left in K' pose considerable difficulties in the way of a straightforward analytical treatment. To enable one to carry out nonetheless a reasonably meaningful analysis of the effects of internal resonances and of the solar perturbation, it was found necessary to reduce the number of admissible resonance peaks still further. This was accomplished by disregarding for the present time from further consideration all the terms which arise from the lunar orbital eccentricity. While this step does tend to restrict the present analysis to encompass only circular lunar orbits, it manages to reduce the number of detuning frequencies left down to 3. For this number of resonances an analysis can be carried out.

Eccentricity terms could perhaps be reintroduced at a later time, possibly by means of an additional perturbation of the variational equations which result from the present circular orbit analysis. A possible shortcoming with such a scheme might be that it would probably lead to a set of parametrically excited linear differential equations which would not be readily solvable.

Another somewhat different approach might be attempted, if we recall that the elliptic 3-body problem (no solar perturbation present) admits as a solution an elliptic particle orbit around L_4. This ellipse is identical to the ellipse along which the moon appears to move relative to an observer moving with constant circular velocity along the moon's mean circular reference orbit, but rotated 60^o with respect to it. Stated another way, the particle's motion is synchronized with that of the moon, but takes place 60^o ahead of it. Variational equations for these orbital elements due to the solar perturbation could then be set up and hopefully solved.

The above are just two of the many other different approaches which might have to be explored in greater detail before the more general question of stability of motion could be satisfactorily resolved.

In the present dissertation however, we shall hereafter confine our attention only to the case of zero lunar eccentricity.

IX. THE LONG PERIOD HAMILTONIAN FOR e = o AND THE ELIMINATION OF TIME t

For the case of e = o the expression for K' shown in Eq. (50) is reduced to the simpler form given in Eq. (51) below. The numerical values of the coefficients b, and the phase shifts λ, are determined after one performs all the tedious algebraic manipulations similar to those briefly demonstrated in Appendix E. There results

$$\begin{aligned} K' = \Big\{ & .1266\alpha_1'^2 - 6.000\alpha_1'\alpha_2' + 3.829\alpha_2'^2 \\ & - 29.04\alpha_1'^{1/2}\alpha_2'^{3/2} \cos\left[.06086t + \omega_1\beta_1' + 3\omega_2\beta_2' + 14.2^\circ\right] \\ & + \alpha'\alpha_3'\left[.09316 + .08608 \cos 2\Delta_{13} - .03934 \sin 2\Delta_{13}\right] \\ & + .7554\alpha_2'\alpha_3' - .002231\,\alpha_3'^2\Big\}_{\text{int}} - \Big\{.005394\alpha_1' + .008208\alpha_2' \\ & + .02685\alpha_1' \cos\left[.05878t + 2\omega_1\beta_1' + 29.4^\circ + 2\epsilon' - 2\epsilon\right] + .004193\alpha_3'\Big\}_{\text{ext}} \end{aligned} \tag{51}$$

where

$$\Delta_{13} = \omega_1(t + \beta_1') - (t + \beta_3') = -.04541t + \omega_1\beta_1 - \beta_3'$$

The first bracket contains all the internal terms, while the second bracket includes all the external (solar) terms. The long period contributions to the coplanar (α_1', α_2') terms resulting from the periodic parts of the indirect $\rho(t)$ and $\nu(t)$ terms in H' were found to cancel exactly the indirect periodic terms generated by the linear forced response $\tilde{x}_o$ and $\tilde{y}_o$ of Eq. (38). The external terms displayed in Eq. (51), which are left after the above cancellations, stem from the contribution of the indirect constant component -.00093 in ρ, from the direct (m^2) terms in H, and from the forced responses $\tilde{x}_{m^2}$ and $\tilde{y}_{m^2}$ of the linear system.

Equation (51) shows that the dependence of K' on time t comes about through the presence of three distinct slowly varying trigonometric terms with frequencies .06086, .09082 and .05878, all of which are of o(m). Since the same trigonometric functions also depend on various combinations of the three angular variables $\beta_i'(i = 1,2,3)$, the possibility suggests itself to eliminate the explicit presence of t by means of a suitable redefinition of the β_i' so as to absorb the time dependent terms. Such a transformation would result in a new Hamiltonian K^* which would not depend explicitly on t.

This absorption of the time terms is accomplished by means of a coordinate transformation to a new canonical set of variables α^* and β^* as indicated below.

We define β_1^* via

$$2\beta_1^* = .05878t + 2\omega_1\beta_1' + 29.4^\circ - 2\epsilon + 2\epsilon'$$

or

$$\beta_1^* = .02939t + \omega_1\beta_1' + 14.7^\circ - \epsilon + \epsilon' \tag{52}$$

The conjugate momentum α_1^* is obtained by the introduction of a generating function J_1 defined as

$$J_1 = \alpha_1^*\left[.02939t + \omega_1\beta_1' + 14.7^\circ - \epsilon + \epsilon'\right] \tag{53}$$

so that

$$\alpha_1' = \frac{\partial J_1}{\partial \beta_1'} = \omega_1\alpha_1^* \qquad \text{or} \qquad \alpha_1^* = \frac{\alpha_1'}{\omega_1} \tag{54}$$

For the definition of β_2^* we use the trigonometric argument

$$.06086t + \omega_1\beta_1' + 3\omega_2\beta_2' + 14.2^\circ$$

and substitute for β_1' from Eq. (52). This leads to the expression

$$\left[.03146t + 3\omega_2\beta_2' + \epsilon - \epsilon' - .5^\circ\right] + \beta_1^*$$

which suggests that β_2^* be taken as

$$\beta_2^* = .03146t + 3\omega_2\beta_2' + \epsilon - \epsilon' - .5^\circ \qquad (55)$$

Use of a second generating function

$$J_2 = \alpha_2^*\left[.03146t + 3\omega_2\beta_2' + \epsilon - \epsilon' - .5^\circ\right] \qquad (56)$$

gives for the conjugate momentum α_2^*

$$\alpha_2^* = \frac{\alpha_2'}{3\omega_2} \qquad (57)$$

The expressions for β_3^* and α_3^* can be obtained in a similar fashion with the aid of Δ_{13}. Combining first the cosine and sine terms

$$.08608 \cos 2\Delta_{13} - .03934 \sin 2\Delta_{13} = .09464 \cos\left[2\Delta_{13} + 24.56^\circ\right]$$

we find that

$$\beta_3^* = .074801t + \omega_3\beta_3' - \epsilon + \epsilon' + 2.42^\circ \qquad (58)$$

and after introducing a generating function J_3 we obtain

$$\alpha_3^* = \frac{\alpha_3'}{\omega_3} = \alpha_3' \qquad (59)$$

Letting $J = J_1 + J_2 + J_3$ and noting that $J = J(\beta', \alpha^*, t)$ we determine the transformed time independent Hamiltonia K^* from the relation

$$K^* = K'(\beta^*, \alpha^*) + \frac{\partial J(\beta', \alpha^*, t)}{\partial t} \tag{60}$$

Substitution for α', β' in terms of α^*, β^* in Eq. (51) and use of Eq. (60) results in the desired expression for K^*:

$$\begin{aligned} K^* = &\Big\{.1154\alpha_1^{*2} - 5.1\alpha_1^*\alpha_2^* + 3.059\alpha_2^{*2} - 23.97\alpha_1^{*1/2}\alpha_2^{*3/2}\, C_{\beta_1^*+\beta_2^*} \\ &+ .09035\alpha_1^*\alpha_3^* C_{2(\beta_1^*-\beta_3^*)} + .08893\alpha_1^*\alpha_3^* + .6751\alpha_2^*\alpha_3^* \\ &- .002231\alpha_3^{*2} + .02939\alpha_1^* + .03146\alpha_2^* + .074801\alpha_3^*\Big\}_{\text{int}} \\ &+ \Big\{.004193\alpha_3^* - .007336\alpha_2^* - \alpha_1^*\Big[.005149 + .02563C_{2\beta_1^*}\Big]\Big\}_{\text{ext}} \end{aligned} \tag{61}$$

where the notation $C_x \equiv \cos x$ has again been used for convenience.

X. ANALYSIS OF THE INTERNAL COPLANAR MOTION

1. SIMPLIFICATION OF THE HAMILTONIAN

The analysis of the motion governed by the Hamiltonian K^* of Eq. (61) is made easier, and a greater amount of physical insight is gained, if we treat at first separately the internal terms contained in the first bracket. The modifications required by the presence of the second, external, bracket are then taken up later.

Let us write for convenience

$$K^* = K_i^* + K_e^* \tag{62}$$

where

K_i^* = all the internal terms

K_e^* = all the external terms

and confine our attention in this and the next section to the Hamiltonian K_i^*.

It would help matters appreciably if we could eliminate also for the time being the coupling which exists between the out-of-plane and coplanar terms.

This elimination can be accomplished by a suitable choice of initial conditions which result in $\alpha_3^* \approx 0$, provided we have reason to believe that a physical motion in which α_3^* does not depart much from its initial small value can in fact exist.

The resultant coplanar type of motion can be maintained as long as the nonlinear coupling with the out-of-plane terms does not lead to an appreciable transfer of energy from one mode of motion to the other.

In the next section, where we consider the out-of-plane motion, this situation will be shown to hold true.

On the basis of the foregoing we shall neglect here all the α_3^* terms in K^*, which leaves us with the 2-dimensional Hamiltonian K_{12}^* given by

$$K_{i2}^{*} = .1154\alpha_1^{*2} - 5.1\alpha_1^{*}\alpha_2^{*} + 3.059\alpha_2^{*2} - 23.97\alpha_1^{*1/2}\alpha_2^{*3/2}C_{\beta_1^*+\beta_2^*=}$$

$$+ .02939\alpha_1^{*} + .03146\alpha_2^{*} \qquad (63)$$

Since t is not explicitly present in K_{i2}^{*}, the latter can also be treated as a constant of the motion.

2. INVARIANCE OF THE DIFFERENCE $\alpha_1^{*} - \alpha_2^{*}$ AND BOUNDED MOTIONS

The presence of β_1^{*} and β_2^{*} in K_{i2}^{*} occurs only through the combination $\beta_1^{*} + \beta_2^{*}$. From this one readily sees that

$$\frac{\partial K_{i2}^{*}}{\partial \beta_1^{*}} = \frac{\partial K_{i2}^{*}}{\partial \beta_2^{*}}$$

which implies that

$$\dot{\alpha}_1^{*} = \dot{\alpha}_2^{*} \qquad (64)$$

and after integration results in the additional coplanar integral of the motion

$$\alpha_1^{*} - \alpha_2^{*} = D_1 = \pm |D_1| \qquad (65)$$

Unfortunately, this last integral does not provide any bounds on the magnitude of the coplanar displacements, inasmuch as α_1^{*} and α_2^{*} are not prohibited by Eq. (65) from growing individually as long as their difference remains unchanged.

On the other hand it is clear that the validity of the present fourth order theory would cease to hold long before the α's have grown to very large size, and that additional higher order terms in H would have to be included in the analysis. Equations (64) and (65) are of great use in those cases when α_1^* and α_2^* do not grow without limit.

Let us consider now the question of boundaries of α^*. From Eq. (64) we have

$$\left(\dot{\alpha}_2^*\right)^2 = \left(-\frac{\partial K_{i2}^*}{\partial \beta_2^*}\right)^2 = 23.97^2 \alpha_1 \alpha_1^* \alpha_2^{*3} S^2_{\beta_1^*+\beta_2^*}$$

$$= 23.97^2 \alpha_1^* \alpha_2^{*3} - \left[K_{i2}^* - .1154\alpha_1^{*2} + 5.1\alpha_1^*\alpha_2^* - 3.059\alpha_2^{*2} - .02939\alpha_1^* - .03146\alpha_2^*\right]^2 \qquad (66)$$

We now introduce the new variable

$$\xi = \frac{\alpha_2^*}{|D_1|} \qquad (67)$$

and Eq. (65) into Eq. (66), which can then be written in the form

$$\left(\frac{\xi}{23.97D_1}\right)^2 = f^2(\xi) - \eta^2(\xi) \qquad (68)$$

where

$$f = \begin{cases} \pm\left[\xi^3(\xi + 1)\right]^{1/2} & \text{for} \quad D_1 > 0 \\ \pm\left[\xi^3(\xi - 1)\right]^{1/2} & \text{for} \quad D_1 < 0 \end{cases} \qquad (69)$$

and

$$\eta = \begin{cases} -\left(.2028 - \frac{.00254}{|D_1|}\right)\xi - .0801\ \xi^2 + \text{constant} & D_1 > 0 \\ \left(.2028 + \frac{.00254}{|D_1|}\right)\xi - .0801\ \xi^2 + \text{constant} & D_1 < 0 \end{cases} \qquad (70)$$

The constants in Eq. (70) denote the value of $\eta(o)$ and are related to the value of the Hamiltonian K_{i2}^*.

The points at which the η curve intersects the + or - branch of the f curve

$$\eta = \pm f \tag{71}$$

correspond to points at which $\dot{\alpha}_2^* = 0$ and, by Eq. (65), also $\dot{\alpha}_1^* = 0$. Reality of the particle motions requires that $f^2 \geq \eta^2$.

The gradual changes of the motion of the physical particle in the xy space can be described by observing the motion of a representative mathematical point along a given curve η in a plane in which f and η are plotted as functions of ξ.

If the η curve intersects both branches of the f curve or intersects the same branch at two different points, then $\dot{\alpha}_1^*$ and $\dot{\alpha}_2^*$ will have finite values at intermediate points on η, which tend toward zero as the representative point approaches the f curve. The sense of motion of the point is reversed every time one of the branches of f is reached, so that the point continues to travel back and forth on a given η curve between its points of intersection with f. The turning or extremal values of the momenta α^* are thus fixed by the values which ξ assumes at the points of intersection of η with ± f.

The geometry in the $f(\xi)$ and $\eta(\xi)$ plane is shown schematically in Fig. 6.

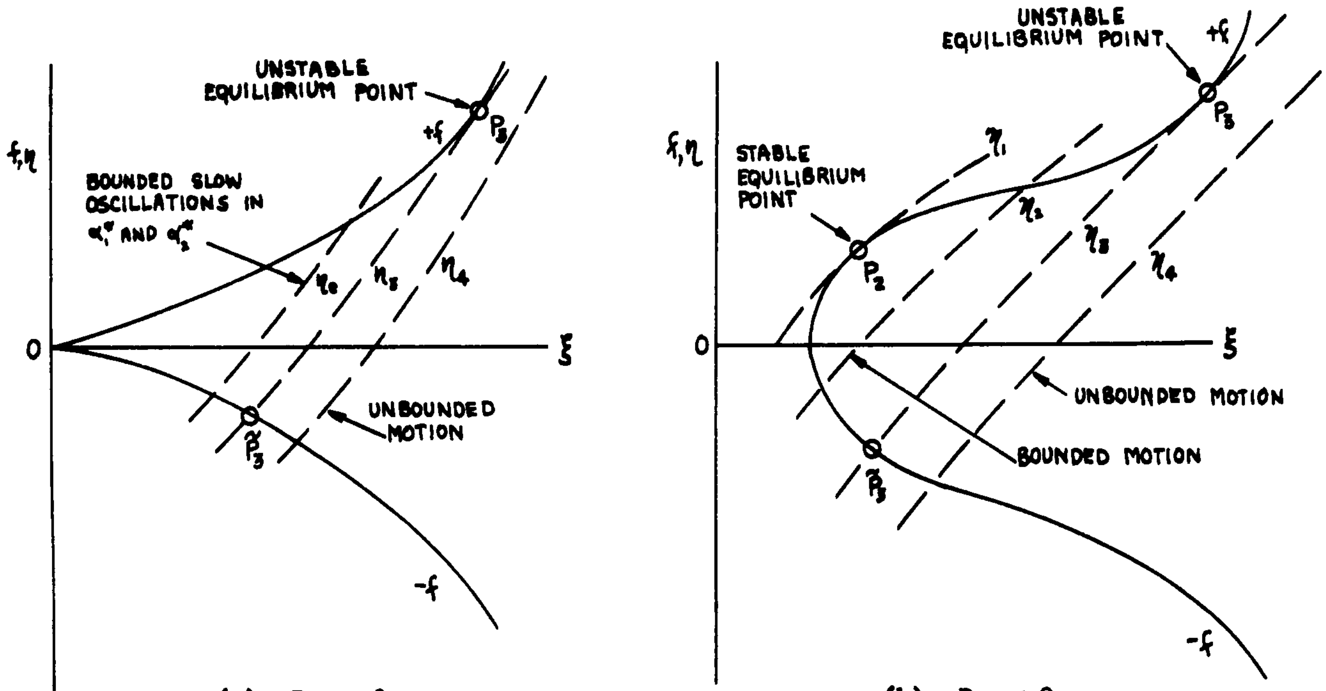
The curves η_2 in Figs. 6(a) and (b) represent bounded particle trajectories in the xy plane. The tangency points P_2,P_3 at which

$$\frac{d}{d\xi}\eta \equiv \eta' = \pm f'$$

and

$$\dot{\alpha}_1^* = \dot{\alpha}_2^* = 0 \tag{72}$$

are equilibrium points in the (α_1^*,α_2^*) plane, and with the aid of Eq. (66) can be shown to correspond to coplanar periodic particle



(a) $D_1 > 0$

(b) $D_1 < 0$

Fig. 6: Geometry of coplanar internal motion in α^* Space.

orbits. Equation (66) requires that $\beta_1^* + \beta_2^* = n\pi$, which can also be written in the form

$$\omega_1\beta_1' + 3\omega_2\beta_2' + .06086t - n\pi + 14.2^\circ = 0$$

Reference to Eq. (51) shows that this condition eliminates the detuning term due to coplanar coupling and indicates commensurability of the internally perturbed coplanar normal frequencies $\omega_1' = \omega_1 + \omega_1\dot{\beta}_1'$ and $\omega_2' = \omega_2 + \omega_2\dot{\beta}_2'$. The periodicity of the coplanar particle orbits follows from here.

The equilibrium is stable at point P_2 and unstable at point P_3, where small disturbances may cause a displacement to a neighboring curve such as η_4 which causes divergence of the physical motion.

Transition from stability to instability occurs at points where

$$\eta'' = \pm f'' \tag{73}$$

When $D_1 > 0$, f'' does not change sign as can be seen in Fig. 6(a), and from this follows that all the periodic particle orbits for which $\alpha_1^* > \alpha_2^*$ would be of the unstable kind. For the case $D_1 < 0$, f'' does change sign at some value $\xi > 1$ and we note accordingly the presence of one stable and one unstable equilibrium point along the +f branch in Fig. 6(b).

3. THE PERIODIC MOTIONS

When one solves the tangency Eq. (72) for the value of ξ which corresponds to every choice of D_1, one can obtain an α_1^* for every α_2^* found. In the α_1^* versus α_2^* plane this solution curve represents the so called "tangency locus" of equilibrium values of α_1^* and α_2^* which designate periodic particle orbits. This curve is presented in Fig. 7, where we have chosen as coordinates the quantities $10\sqrt{\alpha_1^*}$ and $10\sqrt{3\alpha_2^*}$ (α_1^* and $3\alpha_2^*$ are in fact the associated "action variables"). On this curve we have set the angular variables Δ_{12}^*

$$\overset{*}{\Delta}_{12} = \overset{*}{\beta}_1 + \overset{*}{\beta}_2 = \begin{cases} 0 \\ \text{or} \\ \pi \end{cases} \tag{74}$$

This plot is seen to consist of two distinct branches which connect at the point (1.12,0). The left hand branch consists of a segment of stable periodic orbits which is followed by a segment of unstable periodic orbits. On both segments $\overset{*}{\Delta}_{12} = 0$. The unstable branch on the right hand side of (1.12,0) requires a $\overset{*}{\Delta}_{12} = \pi$.

Two more curves passing through (1.12,0) and consisting of left hand and right hand branches are also shown in this figure. The lower (solid) curve denotes the loci of intersection points $\tilde{P}_3$ of η_3 with the second f branch. (For added clarification small inserts of the appropriate geometrical situation described by Fig. 6 are also displayed here in connection with specific segments of the curves.)

The dashed curve lying close to the $\tilde{P}_3$ locus represents the intersection of η_3 with the ξ axis. On this curve $\overset{*}{\Delta}_{12} = \pi/2$. The values of $\overset{*}{\Delta}_{12}$ which allow stable motions to exist in each one of the domains I - IV which are separated by the above curves are indicated in the figure, and also by shaded regions in the small inserts from Fig. (6).

The axis $\overset{*}{\alpha}_2 = 0$ represents the locus of stable periodic particle orbits which are traversed with a mean angular frequency differing but slightly from ω_1. The stable periodic segments along which $10\sqrt{3\overset{*}{\alpha}_2} >> 10\sqrt{\overset{*}{\alpha}_1}$ marks those particle orbits which are traversed with a mean frequency close to ω_2.

Curves of D_1 = constant, intersecting all the above curves are also displayed for a few selected values of D_1.

4. FREQUENCIES OF THE PERIODIC MOTIONS

In the present nonlinear treatment, except for the special periodic motions mentioned above which are described only by one single normal mode, all the other periodic particle orbits are generated by a superposition of both normal modes. Periodicity here is achieved as the result of an adjustment of the natural frequencies via the nonlinear

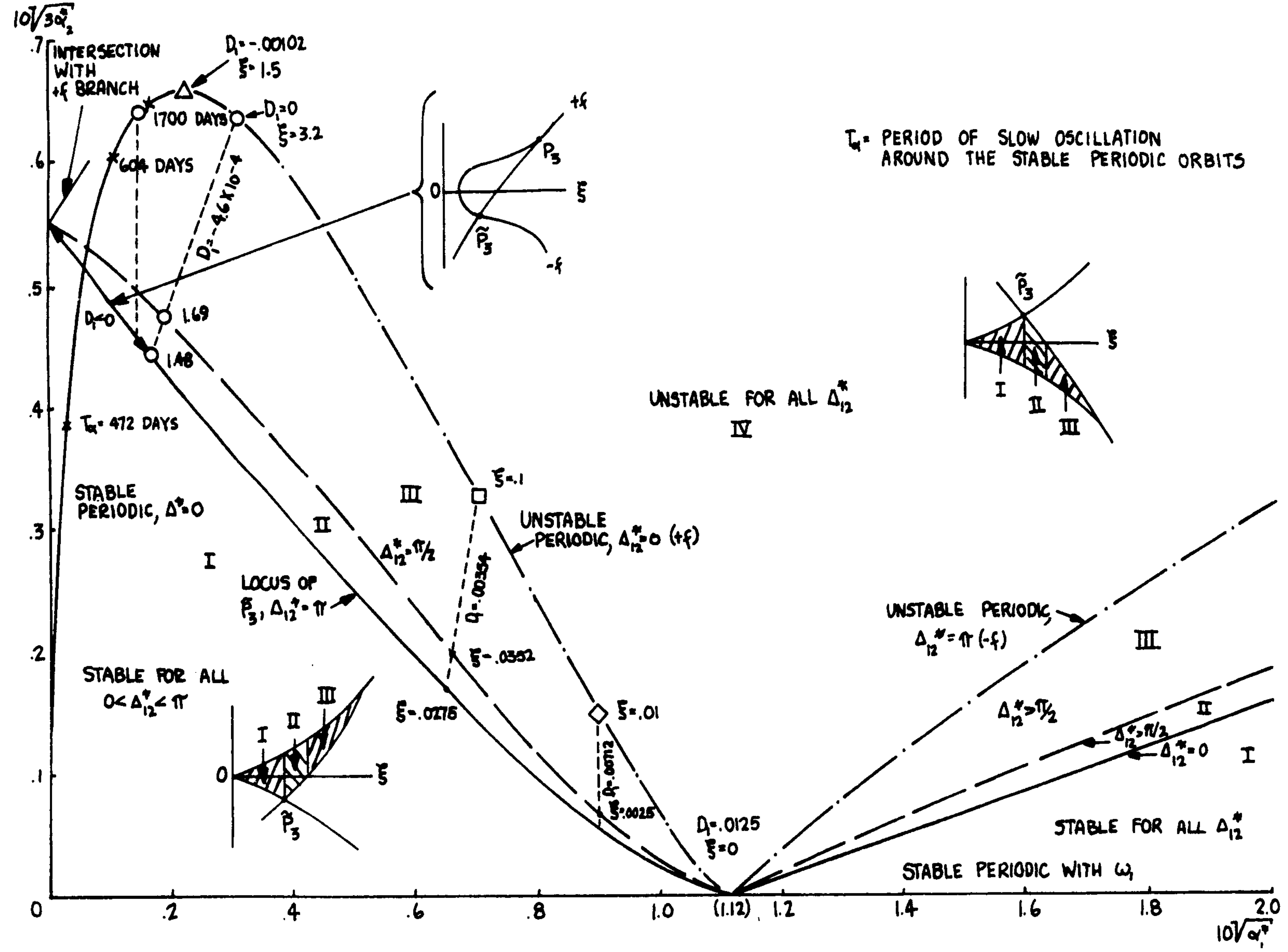


Fig. 7: Tangency Locus of Equilibrium Values for α_1^* and α_2^*

coupling which occurs between the two modes and which makes them exactly commensurable. The resultant frequency shifts $\Delta\omega_1$ and $\Delta\omega_2$ in the original undisturbed frequencies ω_1 and ω_2, lead to normal modes with modified commensurable (3:1) frequencies ω_1' and ω_2'

$$\omega_1' = \omega_1 + \Delta\omega_1 = 3\omega_2' = 3(\omega_2 + \Delta\omega_2) \qquad (75)$$

This point was also raised earlier in the discussion following Eq. (72).

The orbital period T is determined by the slower mode

$$T = \frac{2\pi}{\omega_2'} \qquad (76)$$

During this time T three cycles of the faster mode are completed.

E. Evaluation of the Frequency Shifts for Periodicity

For every point on the "periodic motion" curve of Fig. 7 there exists a unique set of equilibrium values α_{1E}^* and α_{2E}^*.

The shifts $\Delta\omega_1$ and $\Delta\omega_2$ can be estimated by writing

$$\omega_1(1 + \dot{\beta}_1')t = 3\omega_2(1 - \dot{\beta}_2')t \qquad (77)$$

and solving for $\omega_1\dot{\beta}_1'$ and $\omega_2\dot{\beta}_2'$ from the relations

$$\begin{aligned} \dot{\beta}_1^* &= \frac{\partial K_{i2}^*}{\partial \alpha_1'} = .02939 + \omega_1\dot{\beta}_1' \\ \dot{\beta}_2^* &= \frac{\partial K_{i2}^*}{\partial \alpha_2^*} = .03146 + 3\omega_2\dot{\beta}_2' \end{aligned} \qquad (78)$$

evaluated at α_{1E}^* and α_{2E}^*. From here one finds

$$\begin{aligned}\Delta\omega_1 &= \omega_1\dot{\beta}_1' = .2308\alpha^*_{1E} - 5.1\alpha^*_{2E} \mp \alpha^{*-1/2}_{1E}\alpha^{*3/2}_{2E}\\ \Delta\omega_2 &= -\,\omega_2\dot{\beta}_2' = 1.7\alpha^*_{1E} - 2.039\alpha^*_{2E} \pm \alpha^{*1/2}_{1E}\alpha^{*1/2}_{2E}\end{aligned} \tag{79}$$

where the upper sign corresponds to $\Delta^*_{12} = 0$ and the lower to $\Delta^*_{12} = \pi$.

F. Variation of α^*'s Near Equilibrium Points

For small disturbances from the equilibrium points P_2 and P_3 the time dependence of ξ can be approximated by means of a Taylor series expansion of f and η around the equilibrium points.

Letting

$$f = f_E + (\xi - \xi_E)f_E' + \frac{1}{2!}\,(\xi - \xi_E)^2 f_E'' + \ldots$$

and

$$\eta = \eta_E + (\xi - \xi_E)\eta_E' + \frac{1}{2!}\,(\xi - \xi_E)^2 \eta_E'' + \ldots \tag{80}$$

and recalling that $\eta_E = f_E$, $\eta_E' = f_E'$, we can combine Eqs. (68) and (80) to obtain (after approximating 23.97 by 24 for convenience)

$$\frac{\dot{\xi}}{24|D_1|} = \frac{1}{24|D_1|}\frac{d}{dt}\,(\xi - \xi_E) = \sqrt{f^2 - \eta^2} \cong \sqrt{|f_E|(f_E'' - \eta_E'')}\;(\xi - \xi_E) \tag{83}$$

whence

$$\xi - \xi_E = e^{24|D_1|\sqrt{|f_E|(f_E'' - \eta_E'')}\;t} \tag{84}$$

For a stable point such as P_2 in Fig. 6(b), we have $f_E'' < \eta_E''$. This makes the exponent in Eq. (84) imaginary of the form $i\omega_\alpha t$ and indicates a slow oscillatory variation in α. For an unstable periodic point such as P_3, $f_E'' > \eta_E''$ which leads to an exponential growth of α with time.

A few representative values of the period $T_\alpha = 2\pi/\omega_\alpha$ are indicated alongside the stable periodic segment in Fig. 7.

The developments of the present section can now be summarized by means of the following general conclusions:

1. On the assumption that the out of plane terms do not couple strongly with the in-plane terms (which will be proven later) it is possible to reduce the problem to an essentially 2-dimensional one.
2. Initial conditions which lie on an η curve located to the left of the limiting curve of type η_3 will lead to bounded motions of the particle in the xy plane.
3. Depending on whether the η curve is tangent to the f curve at a point such as P_3 or P_2, periodic particle motions of an unstable or a stable type, respectively, may exist.
4. The periodic orbits generally result from a superposition of the two normal modes of vibration in which the nonlinear coupling has brought about commensurability of the basic frequencies by means of appropriate frequency shifts. For special initial conditions, periodic particle motions consisting of only the faster normal mode may exist.
5. In the neighborhood of stable equilibrium points of type P_2, the momenta α_1^* and α_2^* perform low frequency bounded oscillations in time. Near unstable equilibrium points of type P_3, the α^*'s will tend to grow exponentially with time, which results in a large growth of the particle's motion in the physical xy plane.

XI. ANALYSIS OF THE INTERNAL OUT OF PLANE MOTION

The analysis of the out of plane motion is rather simple and straightforward compared to the coplanar analysis of Section X. We shall investigate the coupling of α_3^* and α_1^* in the region where $\alpha_2^* = 0$, by neglecting the α_2^* terms in Eq. (61).

The reason for this particular decision is the result of hindsight, based on a prior preliminary study of the external effects on the coplanar motion which disclosed the presence of a stable equilibrium point $\alpha_1^* \neq 0$, $\alpha_2^* = 0$, for the Sun perturbed problem. This will be discussed in more detail in Section XII.

Let us denote by F^* the internal terms left in the Hamiltonian K^* of Eq. (61) when all α_2^* terms are dropped. We have then

$$\begin{aligned} F^* = {} & .1154\alpha_1^{*2} + .09035\alpha_1^*\alpha_3^* C_{2(\beta_1^* - \beta_3^*)} + .08893\alpha_1^*\alpha_3^* \\ & - .002231\alpha_3^{*2} + .074801\alpha_3^* + .02939\alpha_1^* \end{aligned} \tag{85}$$

Let $\Delta_{13}^* = \beta_1^* - \beta_3^*$.

From Hamilton's equations we then obtain

$$\begin{aligned} \dot{\alpha}_1^* &= 2\cdot .09035\alpha_1^*\alpha_3^* S_{2\Delta_{13}^*} \\ \dot{\alpha}_3^* &= -\ 2\cdot .09035\alpha_1^*\alpha_3^* S_{2\Delta_{13}^*} \end{aligned} \tag{86}$$

This leads to the new integral of motion

$$\alpha_1^* + \alpha_3^* = D_2 \tag{87}$$

with $D_2 > 0$.

As we did before for $\dot{\alpha}_2^*$, we can now write for $\dot{\alpha}_1^{*2}$

$$\left(\dot{\alpha}_1^*\right)^2 = 4\cdot(.09035)^2\alpha_1^{*2}\alpha_3^{*2} - 4\Big[F^* - .1154\alpha_1^{*2} - .08893\alpha_1^*\alpha_3^* + .002231\alpha_3^{*2} - .02939\alpha_1^* - .074801\alpha_3^*\Big]^2 \tag{88}$$

We introduce the auxilliary variable ξ_3

$$\xi_3 = \frac{\alpha_3^*}{D_2} \tag{89}$$

and end up again with the equation

$$\left(\frac{\dot{\xi}_3}{.1807D_2}\right)^2 = f^2 - \eta^2 \tag{90}$$

where this time

$$f = \pm\ \xi_3(1 - \xi_3) \tag{91}$$

and

$$f'(o) = \pm\ 1 \tag{92}$$

$$\eta = \frac{F^*}{.09035D_2^2} - 1.277(1 - \xi_3)^2 - .9843\xi_3(1 - \xi_3) + .02469\xi_3^2 - \frac{.32534}{D_2} - \frac{.50259}{D_2}\xi_3 \tag{93}$$

At the origin, the first and second η derivatives are

$$\eta'(o) = 1.5703 - \frac{.50259}{D_2} \underset{<}{>} 0 \quad \text{depending on } D_2 \tag{94}$$

$$\eta''(o) = \eta''(\xi_3) = -.5366 < 0 \tag{95}$$

$$\eta'(o) < 0 \qquad \text{if} \qquad D_2 < .3201 \tag{96}$$

and

$$\eta'(o) \leq - 1 \qquad \text{if} \qquad D_2 \leq .1955 \tag{97}$$

The magnitude of the slope η' will determine the time history of $\overset{*}{\Delta}_{13}$. In particular, if $\eta' < - 1$ then $\overset{*}{\Delta}_{13}$ will exhibit a circulatory behavior, while a value of $-1 < \eta' < 0$ would lead to a librational behavior.

An upper bound on η' can be established by making a reasonable estimate for an upper value of D_2. Such an estimate can be furnished from some of the mathematical and physical considerations which underlie the present analysis.

From a mathematical standpoint it is clear that in the binominal expansions and truncations used to obtain the expression for the Hamiltonian $H(x,y,z,t)$ of Eq. (21) it was assumed that x,y,z were small compared to unity.

From a physical point of view it is not clear that relatively large displacements away from the Moon would necessarily invalidate the conclusions of the present analysis, but the large accelerations resulting from large displacements towards the Moon or Earth could not be tolerated.

If we assume that the displacements should be limited to values $x,y,z < .5$ (say) then for the excitation mode ω_1 we can obtain from Eqs. (36)

$$\sqrt{\overset{*}{\alpha}_1} < \frac{.5}{2} = .25$$

$$\text{i.e.,} \quad \overset{*}{\alpha}_1 < .0625$$

and also

$$\alpha_3^* < .0625$$

so that for these limits

$$D_2 < .1250 < .1955$$

The slopes of all η curves are thus steeper than $f'(\xi_3)$ from which follows that every η curve will intersect both ± f branches, giving rise to a circulatory motion in Δ_{13}^* as indicated in Fig. 8.

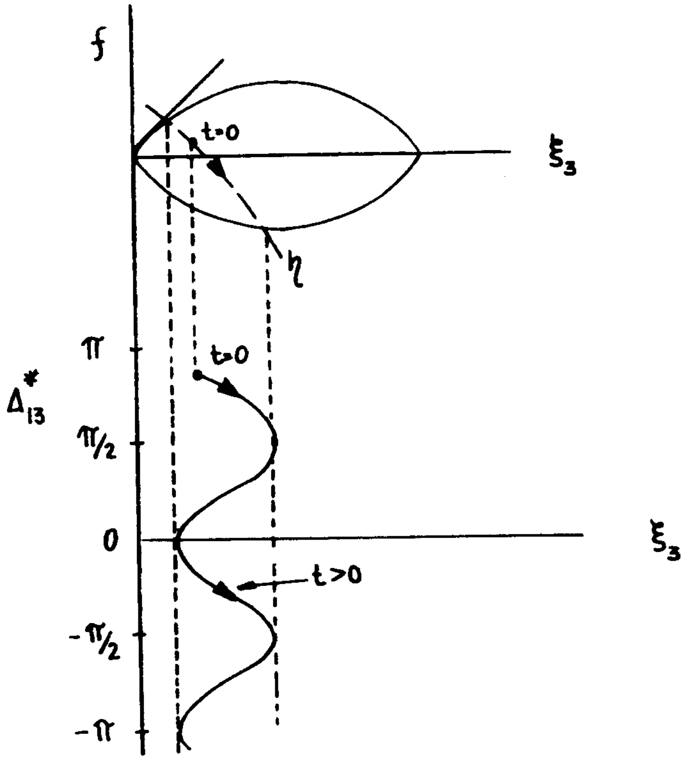


Fig. 8: Geometry in (f,ξ_3) and (Δ_{13}^*,ξ_3) space.

Since $|\eta'|>|f'|$ no equilibrium points with $\alpha_3^* \neq 0$ can exist, and consequently no periodic orbits in xyz space result from the nonlinear coupling of modes 1 and 3.

The actual slope of any η curve would depend of course on the value chosen for D_2, subject to the limits mentioned earlier.

We may choose for example a representative value of $\alpha_1^* = .006$ (say) and assume α_3^* to be of the same magnitude (this α_1^* is very close to the actual coplanar equilibrium value of α_1^* in the externally perturbed case discussed in Section XII). Then we have

$$D_2 \cong 2\alpha_3^* = .012 \tag{98}$$

This results in a slope

$$\eta' \cong 1.57 - \frac{.5026}{.012} = -40.3 = \tan^{-1}\theta$$

$$\text{or} \tag{99}$$

$$\theta \cong 90^{\circ}$$

In other words the η curve intersects the ξ_3 axis nearly vertically, from which one concludes that $\xi_3 \approx$ constant; thus, there is hardly any energy interchange taking place between α_3^* and α_1^*, which shows that the out of plane coupling is not very important in this problem, and that the motion is dominated by the coplanar coupling.

That the out of plane coupling does not introduce any instabilities when $\alpha_3^* <<$ and α_1^* is close to its equilibrium value $\alpha_1^* \cong .006$ could also have been deduced directly from the expression for F^* in Eq. (85). For very small α_3^* it is sufficient to consider only the terms linear in α_3^*, and to evaluate the coefficients at $\alpha_1^* \cong .006$. The resultant Mathieu type Hamiltonian F^* indicates a parametrically excited motion. Such Hamiltonians are discussed more fully in Appendix F, (in connection with the solar effects on the coplanar motion examined in Section XII) but under the assumption that the values of α_1^* and α_2^* are to remain very small (i.e., coplanar particle motions for very small perturbations from rest at L_4).

If one applies the results of Appendix F to the present situation, and notes that the coefficient of $\overset{*}{\alpha}_3 C_{2(\beta_1^*-\beta_3^*)}$ is smaller than that of $\overset{*}{\alpha}_3$ one readily concludes that the parametric resonance present in the out of plane motion does not lead to instability.

XII. ANALYSIS OF EXTERNAL EFFECTS

1. DETERMINATION OF EQUILIBRIUM POINTS

For a complete analysis of the motion in the presence of the external solar effects, one must retain the complete expression for K^* given in Eq. (61).

From the discussion of Section XI it was seen that the α_1^*,α_3^* internal coupling did not lead to any measurable transfer of energy from the out-of-plane mode to the coplanar mode of motion, while from Section X we have established the existence of an appreciable coplanar coupling effect.

The major long term solar effect causes mainly an excitation of the α_1^* mode. The α_3^* mode does not experience any external excitation to the order of magnitude of the terms retained. This latter statement follows from the developments presented in Section XIII.

If a stable motion in the presence of the Sun is possible in which α_1^*,α_2^* and α_3^* remain small, it would suffice to retain only linear terms in K^* in order to determine long term effects. To linear terms we have the simpler Hamiltonian

$$.02425\alpha_1^* + .02412\alpha_2^* + .07899\alpha_3^* - .02563\alpha_1^* C_{2\beta_1^*} \qquad (100)$$

which is of the Mathieu type, as indicated in Appendix F, and leads to parametric resonance in the α_1^* motion.

Since .02563 > .02425, the stability criteria of Appendix F indicate that the motion falls into the unstable region of the Mathieu plane, and that therefore to linear terms no motion can exist for which α_1^* remains very small.

From a physical point of view this means that the libration point L_4 is not stable with respect to small perturbations, when the solar force field is included, and that the higher order terms in K_2^* must be retained in any analysis.

The lack of stability exhibited by the linearized Hamiltonian does not preclude the existence of equilibrium points in the α^* space for the complete Hamiltonian. In view of the negligible effect of α_3^* on the coplanar motion, it is of interest to look for equilibrium points for $\alpha_3^* = 0$. Such points in the (α_1^*, α_2^*) plane are determined by looking for solutions to Hamilton's equations of the form $\dot{\alpha}_1^* = \dot{\alpha}_2^* = 0$.

Once such points are located, it is then necessary to investigate the type of equilibrium which exists there, and to identify the stable ones.

This search is more easily carried out if one switches over to a set of normal canonical coordinates (Q^*, P^*) defined by

$$\begin{pmatrix} Q_1^* \\ Q_2^* \\ P_1^* \\ P_2^* \end{pmatrix} = \begin{pmatrix} \sqrt{2\alpha_1^*} & 0 & 0 & 0 \\ 0 & \sqrt{2\alpha_2^*} & 0 & 0 \\ 0 & 0 & \sqrt{2\alpha_1^*} & 0 \\ 0 & 0 & 0 & \sqrt{2\alpha_2^*} \end{pmatrix} \begin{pmatrix} S_{\beta_1^*} \\ S_{\beta_2^*} \\ C_{\beta_1^*} \\ C_{\beta_2^*} \end{pmatrix} \qquad (101)$$

After setting $\alpha_3^* = 0$, the two dimensional part of K^*, which we denote here by K_2^*, becomes

$$\begin{aligned} K_2^* = {} & \frac{.1154}{4}\left(P_1^{*2} + Q_1^{*2}\right)^2 - \frac{5.1}{4}\left(P_1^{*2} + Q_1^{*2}\right)\left(P_2^{*2} + Q_2^{*2}\right) + \frac{3.059}{4}\left(P_2^{*2} + Q_2^{*2}\right)^2 \\ & - \frac{23.97}{4}\left(P_1^* P_2^* - Q_1^* Q_2^*\right)\left(P_2^{*2} + Q_2^{*2}\right) + \frac{.02425}{2}\left(P_1^{*2} + Q_1^{*2}\right) \\ & + \frac{.02412}{2}\left(P_2^{*2} + Q_2^{*2}\right) - \frac{.02563}{2}\left(P_1^{*2} - Q_1^{*2}\right) \end{aligned} \qquad (102)$$

The equilibrium points (Q_e^*, P_e^*) are obtained from the solution of the equations

$$\begin{pmatrix} \dot{Q}_e^* \\ \dot{P}_e^* \end{pmatrix} = \Phi_o \begin{pmatrix} K_{2P^*T}^* \\ K_{2Q^*T}^* \end{pmatrix} = 0 \qquad (103)$$

From Eq. (103) we have

$$K_{2P_1^*}^* = 0 = .1154P_1^*\left(P_1^{*2} + Q_1^{*2}\right) - \frac{5.1}{2} P_1^*\left(P_2^{*2} + Q_2^{*2}\right)$$

$$- \frac{23.97}{4} P_2^*\left(P_2^{*2} + Q_2^{*2}\right) - .001379P_1^* \qquad (104a)$$

$$K_{2P_2^*}^* = 0 = - \frac{5.1}{2} P_2^*\left(P_1^{*2} + Q_1^{*2}\right) + 3.059P_2^*\left(P_2^{*2} + Q_2^{*2}\right)$$

$$- \frac{23.97}{4} \left(3P_1^*P_2^{*2} + P_1^*Q_2^{*2} - 2P_2^*Q_1^*Q_2^*\right) + .02412P_2^*$$

$$(104b)$$

$$K_{2Q_1^*}^* = 0 = .1154Q_1^*\left(P_1^{*2} + Q_1^{*2}\right) - \frac{5.1}{2} Q_1^*\left(P_2^{*2} + Q_2^{*2}\right)$$

$$+ \frac{23.97}{4} Q_2^*\left(P_2^* + Q_2^*\right) + .04988Q_1^* \qquad (104c)$$

$$K_{2Q_2^*}^* = 0 = - \frac{5.1}{2} Q_2^*\left(P_1^{*2} + Q_1^{*2}\right) + 3.059Q_2^*\left(P_2^{*2} + Q_2^{*2}\right)$$

$$- \frac{23.97}{4} \left(2Q_2^*P_1^*P_2^* - Q_1^*P_2^{*2} - 3Q_1^*Q_2^{*2}\right) + .02412Q_2^*$$

$$(104d)$$

Equations (104c) and (104d) are identically satisfied if we chose $Q_{1e}^* = Q_{2e}^* = 0$. For convenience we shall therefore restrict our search to those equilibrium points for which

$$Q_{1e}^* = Q_{2e}^* = 0 \qquad (105)$$

For the above Q^*'s Eqs. (104a) and (104b) give

$$.1154P_1^{*3} - 2.55P_1^*P_2^{*2} - 5.810P_2^{*3} - .001379P_1^* = 0 \qquad (106a)$$

$$- 2.55P_1^{*2}P_2^* + 3.059P_2^{*3} - 17.43P_1^*P_2^{*2} + .02412P_2^* = 0 \qquad (106b)$$

One equilibrium point can be obtained by setting $P_{2e}^* = 0$ (which automatically satisfies Eq. (106b) and then solving for P_{1e}^* from the relation

$$.1154P_1^{*2} - .001379 = 0 \qquad (107)$$

or

which corresponds to

$$P_{1e}^* = .1093$$
$$\alpha_{1e}^* = .005975 \qquad (108)$$

The above value of α_1^* is the one which was used in earlier sections when representative numerical values were used.

The first equilibrium point, which we denote by E_I, is thus specified by the coordinates

$$E_I\text{:}\quad Q_1^* = Q_2^* = Q_3^* = P_2^* = P_3^* = 0 \qquad \alpha_1^* = .005975$$
$$P_1^* = .1093 \qquad \alpha_2^* = 0 \qquad (109)$$
$$\alpha_3^* = 0$$

Another equilibrium point can be found for which $P_2^* \neq 0$, all other homogeneous coordinates remaining the same as for point E_I. The values of P_1^* and P_2^* result from the solution of the algebraic equations (106a) and (106b), after P_2^* is factored out from the latter. The coordinates of the second equilibrium point E_{II} were found to be

$$E_{II}: \; Q_1^* = Q_2^* = Q_3^* = P_3^* = 0$$

$$P_1^* = .1106 \qquad \alpha_1^* = .006116 \qquad (110)$$

$$P_2^* = -.003675 \qquad \alpha_2^* = 6.753 \times 10^{-6}$$

The two points E_I and E_{II} were the only ones readily found for the present simplified conditions. A machine search of the complete set of Eqs. (103) might reveal the existence of additional roots. The periodic elliptic particle motion of mode close to ω_1 corresponding to conditions at E_I has a semimajor axis of about 60,000 mi and a semiminor axis of half this value. These values were determined by computing $r_{max} = [\bar{x}^2 + \bar{y}^2]^{1/2}_{max}$ where the ω_1 modes of $\bar{x}$ and $\bar{y}$ of Eq. (36) were used, and the maximum determined with respect to $\omega_1 \beta_1^{\neq}$. It can be shown that this requires that $8.422 S_{2\omega_1 \beta_1^{\neq}} + 4.423 S_{2\omega_1 \beta_1^{\neq} + 247.14^o} = 0$ and results in a value $\omega_1 \beta_1^{\neq} \cong 15.62^o$. The dimensionless expression for r_{max} then becomes $r_{max} \cong 3.2\alpha_1^{\prime 1/2}$, and at $\alpha_1^* \cong .006$ amounts to roughly $3.2 \sqrt{.955:006} \times 2.4 \times 10^5 = 58,128 \cong 60,000$ in round numbers.

In a similar manner one finds for the maximum dimensionless displacement in mode ω_2 the semimajor axis $r_{max} \cong 9.1\sqrt{\alpha_2^{\prime}}$ and in miles $r_{max} = 9.1 \sqrt{3\omega_2\alpha_2^*} \times 2.4 \times 10^5 = 9.1\sqrt{.8937}/\sqrt{2}\; P^*_{2_{max}} \cdot 2.4 \times 10^5$ miles.

It is of interest to observe that this result indicates the particles mean motion is synchronized with that of the Sun such that their angular positions coincide closely whenever the particle crosses one of the axes of the ellipse.

We recall that at equilibrium $Q_1^* = 0$ and hence $\beta_1^* = n\pi$ with $n = 0,1\cdots$. For $n = 0$, Eq. (52) gives

$$\beta_1^* = 0 = .02939t + \omega_1\beta_1^{\prime} + 14.7 - \epsilon + \epsilon^{\prime}$$

and from here

$$\omega_1\beta_1^{\neq} = \omega_1 t - .02939t - 14.7 + \epsilon - \epsilon^{\prime}$$

When the particle crosses the major axis we had $\omega_1 \xi_1^* = 15.62$, and from the commensurability of angular velocities at E_I, $(\omega_1 - .02939) = 1 - m$. Substitution above gives

$$15.62 + 14.7 = 30.32^{\circ} = (1 - m)t + \epsilon - \epsilon' \equiv \xi$$

as defined by Eq. (B-9). Equation (17) then shows the Sun to be located 30.32° below the x axis, and therefore closely aligned with the major axis of the particle's orbit.

2. STABILITY OF THE EQUILIBRIUM POINTS

The stability of the slow variations around the above periodic equilibrium motions in the xy plane can be determined by setting up the expression for the variation δK^* which results from taking small displacements δQ^* and δP^* around the equilibrium values $Q^*_{ie} = 0$ and P^*_e. Clearly, since E_I and E_{II} are equilibrium points, the coefficients of the linear terms in δP^* must vanish, and on then obtains in three dimensions

$$\begin{aligned}
\delta K^* = {} & .02885P^{*2}_{1e}\left[6\delta P^{*2}_1 + 2\delta Q^{*2}_1 + \cdots\right] - 1.275\left[P^{*2}_{1e}\left(\delta P^{*2}_2 + \delta Q^{*2}_2\right)\right. \\
& \left. + 4P^*_{1e}P^*_{2e}\delta P^*_1\delta P^*_2 + P^{*2}_{2e}\left(\delta P^{*2}_1 + \delta Q^{*2}_1\right) + \cdots\right] \\
& + .7648P^{*2}_{2e}\left[6\delta P^{*2}_2 + 2\delta Q^{*2}_2 + \cdots\right] - .00039\left[P^{*2}_{1e}\left(\delta P^{*2}_3 + \delta Q^{*2}_3\right)\right. \\
& \left. + 4P^*_{1e}P^*_{3e}\delta P^*_1\delta P^*_3 + P^{*2}_{3e}\left(\delta P^{*2}_1 + \delta Q^{*2}_1\right)\right] \\
& + .04958\left[P^{*2}_{1e}\delta P^{*2}_3 + P^{*2}_{3e}\delta P^{*2}_1 + 2P^*_{1e}P^*_{3e}\left(2\delta P^*_1\delta P^*_3 + \delta Q^*_1\delta Q^*_3\right)\right] \\
& + .2113\left[P^{*2}_{2e}\left(\delta P^{*2}_3 + \delta Q^{*2}_3\right) + 4P^*_{2e}P^*_{3e}\delta P^*_2\delta P^*_3\right. \\
& \left. + P^{*2}_{3e}\left(\delta P^{*2}_2 + \delta Q^{*2}_2\right)\right] - .0005578\left[P^{*2}_{3e}\left(6\delta P^{*2}_3 + 2\delta Q^{*2}_3\right)\right] \\
& + .0395\left[\delta P^{*2}_3 + \delta Q^{*2}_3\right] - 5.810\left[P^*_{1e}P^*_{2e}\left(3\delta P^{*2}_2 + \delta Q^{*2}_2\right)\right.
\end{aligned}$$

(con't on next page)

$$+ P_{2e}^{*2}\left(3\delta P_1^*\delta P_2^* - \delta Q_1^*\delta Q_2^*\right) + .012126\left[\delta P_1^{*2} + \delta Q_1^{*2}\right]$$

$$+ .012062\left[\delta P_2^{*2} + \delta Q_2^{*2}\right] - .012815\left[\delta P_1^{*2} - \delta Q_1^{*2}\right] \qquad (111)$$

Applying expression (111) to point E_I results in

$$\delta K^* = .001380\delta P_1^{*2} + .02563\delta Q_1^{*2} - .003174\delta P_2^{*2} - .003174\delta Q_2^{*2}$$

$$+ .04008\delta P_3^{*2} + .03949\delta Q_3^{*2} \qquad (112)$$

Since for every value of i = 1,2,3 the coefficients of δP_i^* have the same sign as the coefficients of δQ_i^{*2}(i.e., δK^* is either positive or negative definite irrespective of the signs of δP^* or δQ^*) we can conclude that point E_I is stable for small disturbances in all principal directions. The period of the slow variations in δP_1,δQ_1 is approximately 83 months.

It is more convenient to retain only coplanar terms in δK^* for the determination of stability at E_{II}. We then obtain the expression

$$\delta K^* = .001411\delta P_1^{*2} + .02563\delta Q_1^{*2} + .001838\delta P_1^*\delta P_2^* + .003652\delta P_2^{*2}$$

$$+ 7.847 \times 10^{-5}\ \delta Q_1^*\delta Q_2^* - .001150\delta Q_2^{*2} \qquad (113)$$

If we now assume P_1^* and Q_1^* to remain unchanged while we introduce variations δP_1^* and δQ_1^* we have

$$\delta K^* = .001411\delta P_1^{*2} + .02563\delta Q_1^{*2} \qquad (114)$$

where $\delta Q_2^* = \delta P_2^* = 0$.

Thus δK^* is positive definite for variations in the first set of coordinates and hence δQ_1^* and δP_1^* remain bounded.

Repeating the same steps for δP_2^* and δQ_2^* while keeping P_1^* and Q_1^* fixed gives

$$\delta K^* = .003652\delta P_2^{*2} - .001150\delta Q_2^{*2} \tag{115}$$

where $\delta P_1^* = \delta Q_1^* = 0$.

Since δK^* is not definite for arbitrary choices of δP_2^* and δQ_2^* we conclude that point E_{II} is not stable in δP_2^* and δQ_2^*, and hence is an unstable equilibrium point. The equilibrium for variations δP_3^* and δQ_3^* was found to be stable, which is in agreement with the findings of the last section.

The above conclusion could have been reached also more rigorously in a somewhat lengthier fashion by writing down the complete system of first order linear differential equations for $\delta\dot{Q}^*$ and $\delta\dot{P}^*$ obtained from δK^* of Eq. (113), and examining the roots of the appropriate characteristic equation. We would find that

$$\begin{pmatrix} \delta\dot{Q}_1^* \\ \delta\dot{Q}_2^* \\ \delta\dot{P}_1^* \\ \delta\dot{P}_2^* \end{pmatrix} = \begin{pmatrix} 0 & 0 & .002822 & .001838 \\ 0 & 0 & .001838 & .007304 \\ -.05126 & -7.85\cdot10^{-5} & 0 & 0 \\ -7.85\cdot10^{-5} & -.0023 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta Q_1^* \\ \delta Q_2^* \\ \delta P_1^* \\ \delta P_2^* \end{pmatrix} \tag{116}$$

A trial solution of the form e^{st} would lead to the characteristic equation

$$s^4 + 1.282\cdot10^{-4}s^2 - 2.031\cdot10^{-8} = 0 \tag{117}$$

which has one positive root because of the negative constant term. Equation (117) thus bears out the conclusions reached from Eq. (115).

A simple geometrical description of the stable and unstable regions in the 6 dimensional P^*, Q^* space is of course not feasible. On the other hand it is possible to take advantage of the fact that the stable point E_I is noticed to lie very close to the unstable point E_{II}.

It is thus of particular interest to determine the extent of the stable region around E_I, by expanding K^* up to cubic powers in δP^* and δQ^* around E_I.

The intersection of surfaces of constant K^* with the (P_2^*,Q_2^*) plane, for a value of $P_1^* = .11$, is shown in Fig. (9). The dashed curve shows the separatrix which passes through E_{II} and separates the stable from the unstable regions.

In the physical xy plane, a point in the stable region gives rise to slow variations of the elements of the periodic particle orbit corresponding to E_I. A point in the unstable region of the (P_2^*,Q_2^*) plane would lead to large particle departures from the equilibrium orbit, and thus indicate a possible divergence.

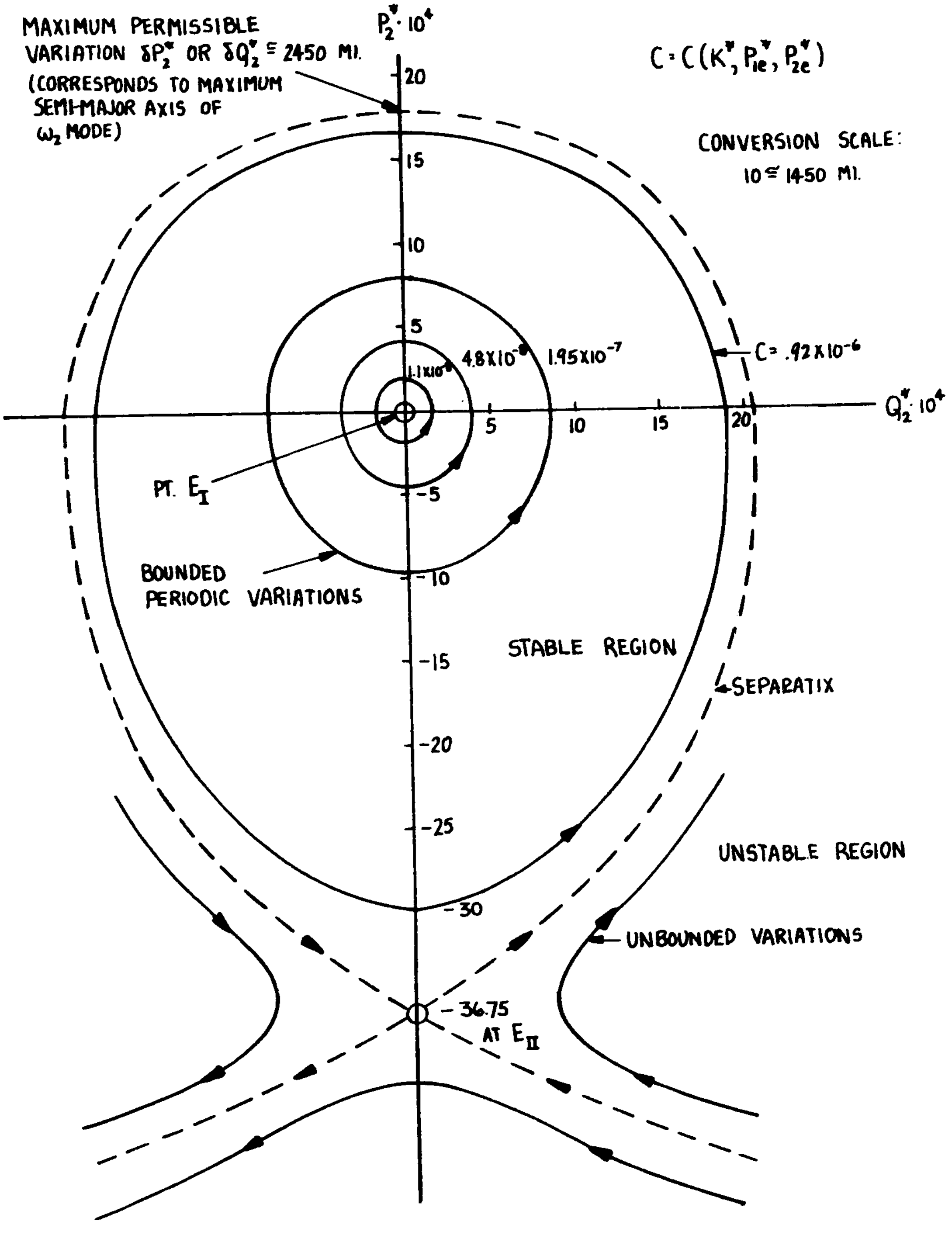


Fig. 9: Stability Regions in the (P_2^*, Q_2^*) Plane Near The Coplanar Equilibrium Points.

XIII. EVALUATION OF THE EFFECT OF THE RESONANCE CAUSED BY THE FORCED SOLUTION $\tilde{z}$

We had alluded on page 23 to the fact that no forced solutions in z, i.e., $\tilde{z}$, had been retained since they are of $o(m^3)$ and would thus give rise to terms of $o(m^5)$ or higher in H when one went on to derive the long period contributions.

A closer second look at the external z terms in $H^{(o)}$ disclosed the existence of a very closely tuned forcing term in the linearized out of plane z motion which could introduce perhaps small divisors in the solution for $\tilde{z}$ and thus depress the order of magnitude of that solution. This would introduce another important long period term into the Hamiltonian K. The resonance in question arises for example from a term such as

$$x_s z_s = -r_{13}^2 \cdot \frac{1}{2} \sin i_o \sin [1.0040212t + \epsilon]$$

which would lead to a detuning of magnitude

$$1.0040212 - 1 = .0040212 \qquad (118)$$

This value would introduce a much slower term in K than any of the terms previously retained, and might conceivably require a redefinition of the angular variable β_3^* introduced earlier.

The developments indicated briefly below disclosed that the z resonance terms cancel each other exactly, and consequently do not contribute a term slower than the one already considered. No further modifications to the analysis of the out-of-plane motion of Section XI were thus required. The steps leading to the above mentioned cancellation were nevertheless found interesting enough to justify their inclusion here.

The z portion of the external part of $H^{(o)}$ of Eq. (21) was

$$\overset{(o)}{H(z)} = -m^2\left[\frac{3}{2r_{13}^2}\left(x_s + \sqrt{3}y_s\right)z_s z\right] + \frac{1}{2}\left(\upsilon_x + \sqrt{3}\upsilon_y\right)z + \frac{1}{2}\left(\upsilon_y - \sqrt{3}\upsilon_x\right)P_z \qquad (119)$$

For a coordinate system with its x axis pointing at the instantaneous position of the Moon, the angular velocity components υ_x and υ_y are given by

$$\begin{aligned} \upsilon_x &= \dot{i} \cos \eta + \dot{\Omega} \sin i \sin \eta \\ \upsilon_y &= \dot{\Omega} \sin i \cos \eta - \dot{i} \sin \eta \end{aligned} \qquad (120)$$

These are the same as Eqs. (B-3) except that η_o has now been replaced by $\eta = gnt + \epsilon - \Omega$ and $g = 1.0040212$.

The angular velocities $\dot{\Omega}$ and $\dot{i}$ can be expressed in terms of η,i and the solar acceleration component W normal to the Earth-Moon plane at the Moon's position, by means of the variational equations on page 404 of Ref. 9, in which a corresponds to $\langle r_{12} \rangle$ here

$$\begin{aligned} \dot{\Omega} &= \frac{r_{12} \sin \eta}{na^2 \sin i} W \\ \dot{i} &= \frac{r_{12} \cos \eta}{na^2} W \end{aligned} \qquad (121)$$

By our nondimensionalization convention $na^2 = 1$, so that

$$\upsilon_y = \frac{r_{12}}{D} W\left[\sin \eta \cos \eta - \cos \eta \sin \eta\right] = 0 \qquad (122)$$

and the angular velocity $\bar{\omega}$ has thus no component in the y direction.

One can also write for υ_x

$$\upsilon_x = \frac{r_{12}}{Dna^2} W\left[\sin^2\eta + \cos^2\eta\right] = (1 + \rho)W \qquad (123)$$

A suitable expression for W can be obtained by differentiation from the potential energy V_s near the Moon. If we let x,y,z denote small displacements from the instantaneous position of the Moon, we have

$$W = -\left.\frac{\partial V_s}{\partial z}\right|_{x,y,z=0} = \frac{\partial}{\partial z}\left\{ m^2 \left[\frac{3}{2r_{13}^2} \left(\bar{r}_{13} \cdot \bar{r}_{14}\right)^2 - \frac{1}{2} r_{14}^2 \right] \right\}_{x,y,z=0}$$

$$= \frac{\partial}{\partial z}\left\{ m^2 \left[\frac{3}{2r_{13}^2} \left(X_s(1 + \rho + x) + y_s y + z_s z\right)^2 - \frac{1}{2}\left((1 + \rho + x)^2 + y^2 + z^2\right) \right] \right\}_{x,y,z=0}$$

$$= 3m^2 \frac{x_s z_s}{r_{13}^2} + o(m^5) = 3m^2 \sin i \cos \xi \sin (\Omega - v') \qquad (124)$$

To sufficient accuracy then

$$\upsilon_x \cong W = 3m^2 \sin i_o \cos \xi \sin (\Omega - v') \qquad (125)$$

To check if $\overset{(o)}{H}(z)$ would in fact lead to the presence of small divisors in the solution for $\tilde{z}$ it suffices to check if $\overset{(o)}{H}(z)$ contains slowly varying terms of frequency .0040212 when we replace in it z and P_z by the homogeneous solutions $\bar{z}$ and $\bar{P}_z$.

$$\overset{(o)}{H}(z) = \frac{3}{4} m^2 \sin i_o \sqrt{2\alpha_3} \cos \beta_3^{\neq} \Big\{\sin (1.0040212t + \epsilon) + \sqrt{3} \cos (1.0040212t + \epsilon)\Big\} + \frac{1}{2} \upsilon_x \sqrt{2\alpha_3} \left[\cos \beta_3^{\neq} + \sqrt{3} \sin \beta_3^{\neq}\right] \qquad (126)$$

Since

$$\Omega - v' - \xi = -1.0040212t - \epsilon \tag{127}$$

we may retain in υ_x only the dominant resonance term

$$\upsilon_x \cong - \frac{3}{2} m^2 \sin i_o \sin (1.0040212t + \epsilon) \tag{128}$$

(o)

When Eq. (128) is substituted into H(z) of Eq. (126) and all the terms combined it is found that all the long period terms cancel each other exactly and only fast terms remain. From this one can conclude that the forcing function of the linearized z equation does not contain a resonance term which is close enough to introduce small divisors into the forced response $\tilde{z}$ and thereby lower its order of magnitude from $o(m^3)$ to $o(m^2)$ or less.

Based on the foregoing we can conclude that the neglect of the contribution of $\tilde{z}$ to the long period terms $\overline{(\partial H/\partial z)\tilde{z}}$ was consistent with our convention of neglecting terms of order higher than $o(m^4)$.

This analysis shows that although the Sun has an appreciable long term effect on the changes in inclination of the lunar orbital plane, it has the same effect also on the orbital plane of the librating particle, with the net result that any relative long term out-of-plane responses vanish. Short period, fast, relative terms do not cancel out though.

XIV. SUMMARY AND CONCLUSIONS

In the present dissertation, the 3-dimensional stability of the motion of a particle near the equilateral libration points of the Earth-Moon system, in the presence of the Sun, has been investigated.

Because the inclusion of lunar eccentricity would have introduced into the problem a larger number of internal and external resonances than could have been handled by the present method of approach, it was found necessary to restrict the stability analysis to a lunar orbit perturbed by the Sun but without eccentricity.

Four major conclusions emerge from the present study. First, small coplanar motions near L_4 or L_5 will grow large because of parametric excitation by the Sun, as a result of nonlinear resonance. In fact, the growth of the energy in the faster normal mode of the linearized theory is found to be governed by a Mathieu equation.

Second, the out-of-plane motion is not seriously excited by the Sun, and has a negligible effect on the coplanar motion, which is the dominant factor as far as stability is concerned.

Third, a stable periodic coplanar orbit can exist in the presence of the Sun. It consists of a clockwise motion along the 1:2 ellipse corresponding to the first (or faster) normal mode, and has a semimajor axis of approximately 60,000 mi. The external nonlinear excitation causes the mean angular motion of the particle to become synchronized with that of the Sun. Thus to an observer located at L_4 and looking continuously in the direction of the Sun, the particle would appear to move back and forth across his line of sight in the manner of a simple harmonic oscillator. The times of crossing of the line of sight coincide closely with the times at which the line of sight is aligned with the major or minor axis of the ellipse.

Fourth, the presence of the internal resonant excitation, resulting from the near commensurability (3:1) of the two coplanar normal models makes the stability somewhat delicate. As a consequence, the semimajor axis of the second mode is limited to magnitudes less than approximately 2450 mi. For larger values the motion becomes unstable and may result in very large displacements which would exceed the range of applicability of the present theory.

Appendix A

SOLAR GRAVITATIONAL GRADIENT CONTRIBUTION

Consider the term

$$\frac{1}{r_{34}} - \frac{\bar{r}_{13} \cdot \bar{r}_{14}}{r_{13}^3} \tag{A-1}$$

of Eq. (7), and decompose $\bar{r}_{34}$ into

$$\bar{r}_{34} = \bar{r}_{31} + \bar{r}_{14} = \bar{A} \text{ (for simplicity)} \tag{A-2}$$

For $(r_{14}/r_{31}) \ll 1$ we can expand $1/r_{34}$ into a Taylor series around r_{31} as shown:

$$\begin{aligned} \frac{1}{r_{34}} = \frac{1}{\left[\bar{r}_{34} \cdot \bar{r}_{34}\right]^{1/2}} &= \frac{1}{r_{31}} + \bar{r}_{14} \cdot \nabla \left. \frac{1}{\left[\bar{A} \cdot \bar{A}\right]^{1/2}} \right|_{\bar{r}_{14}=o} \\ &+ \frac{1}{2}\left(\bar{r}_{14} \cdot \bar{r}_{14}\right) \nabla^2 \left. \frac{1}{\left[\bar{A} \cdot \bar{A}\right]^{1/2}} \right|_{\bar{r}_{14}=o} + O\left(\frac{r_{14}}{r_{13}}\right)^3 + \cdots \end{aligned} \tag{A-3}$$

where

$$\nabla = \frac{\partial}{\partial \bar{r}_{31}} \quad \text{and} \quad \bar{r}_{31} = - \bar{r}_{13}$$

$$\nabla \frac{1}{\left[\bar{A} \cdot \bar{A}\right]^{1/2}} = - \frac{\bar{A} \cdot \nabla \bar{A}}{\left[\bar{A} \cdot \bar{A}\right]^{3/2}} = - \frac{\dot{A} \cdot \underline{\underline{I}}}{\left[\bar{A} \cdot \bar{A}\right]^{3/2}} \tag{A-4}$$

and

$$\bar{r}_{14} \cdot \nabla \frac{1}{\left[\bar{A} \cdot \bar{A}\right]^{1/2}}\Bigg|_{\bar{r}_{14}=o} = \frac{\bar{r}_{13} \cdot \bar{r}_{14}}{r_{13}^3} \qquad \text{(A-5)}$$

where

$\underline{\underline{I}}$ = unit diadic

Similarly

$$\frac{1}{2}\left(\bar{r}_{14} \cdot \bar{r}_{14}\right) \nabla^2 \frac{1}{\left[\bar{A} \cdot \bar{A}\right]^{1/2}} \rightarrow \frac{1}{2}\,\bar{r}_{14} \cdot \nabla \left(-\frac{\bar{r}_{31}}{r_{31}^3}\right) \cdot \bar{r}_{14}$$

$$= \frac{1}{2}\,\bar{r}_{14} \cdot \left[\frac{3}{r_{31}^4}\,\bar{i}_{31}\bar{r}_{31} - \frac{\underline{\underline{I}}}{r_{31}^3} \cdot \bar{r}_{14}\right] = \frac{1}{r_{31}^3}\left[\frac{3}{2}\left(\frac{\bar{r}_{14} \cdot \bar{r}_{31}}{r_{31}}\right)^2 - \frac{1}{2}\,\bar{r}_{14} \cdot \bar{r}_{14}\right]$$

$$= \frac{1}{r_{13}^3}\left[\frac{3}{2}\left(\frac{\bar{r}_{13} \cdot \bar{r}_{14}}{r_{13}}\right)^2 - \frac{1}{2}\,\bar{r}_{14} \cdot \bar{r}_{14}\right] \qquad \text{(A-6)}$$

Combining (A-1), (A-3), (A-5) and (A-6) and neglecting the first term of the series, $1/r_{13}$, which makes no contribution to the equations of motion, we end up with the last term of Eq. (8) which is the expression of (A-6), and represents the solar gradient force near the Earth.

Appendix B

THE EXPRESSIONS FOR $\rho(t)$ AND $\nu(t)$ FROM LUNAR THEORY

The expression for $\rho(t)$ is readily obtained from Eq. (1), p. 281 of Ref. 7, after computing $(a/r)^{-1} = 1 + \rho(t) = r_{12}$ and retaining only terms of $o(m^2)$ or lower. The term -.00093 of our Eq. (15) corresponds to $-\frac{1}{6}m^2$ in the series for $(a/r)^{-1}$. The semimajor axis a is set equal to the reference length D in our notation.

Derivation of the expression for $\nu(t)$ requires a few more algebraic manipulations. We shall make use for this of Fig. 2 (p. 13) and Fig. 4 (p. 38) of Ref. 8, which are combined for convenience in Fig. B-1, and also Fig. B-2 which shows the lunar orbital plane as viewed from above (i.e., looking in the direction of the negative Z axis). In order to facilitate the derivation we shall retain (in this Appendix only) the notation and symbols of Ref. 8 irrespective of the use to which some of the letters have been put in the main body of the present report. Where necessary, the corresponding letters in our notation will be pointed out.

In dimensional symbols we now have

$$\begin{aligned}
\bar{\omega} &= \left(n + \nu_z\right)\bar{i}_{z_e} + \dot{i}\bar{i}_N + \dot{\Omega}\sin i\left(\bar{i}_z \times \bar{i}_N\right)\\
\bar{i}_N &= \cos\eta_o\bar{i}_{x_e} - \sin\eta_o\bar{i}_{y_e}\\
\bar{i}_z \times \bar{i}_N &= \sin\eta_o\bar{i}_{x_e} + \cos\eta_o\bar{i}_{y_e}\\
\eta_o &= nt + \epsilon - \Omega
\end{aligned} \qquad \text{(B-1)}$$

so that

$$\begin{aligned}
\bar{\omega} = {} & \left(n + \nu_z\right)\bar{i}_{z_e} + \left[\dot{i}\cos\eta_o + \dot{\Omega}\sin i\sin\eta_o\right]\bar{i}_{x_e}\\
& + \left[\dot{\Omega}\sin i\cos\eta_o - \dot{i}\sin\eta_o\right]\bar{i}_{y_e}
\end{aligned} \qquad \text{(B-2)}$$

The dimensionless form of $\bar{\omega}$ results if we set n = 1 in (B-2). Noting that $\bar{i}_{x_e}$, $\bar{i}_{y_e}$, $\bar{i}_{z_e}$ are parallel respective to the unit vectors $\bar{i}_x$, $\bar{i}_y$, $\bar{i}_z$ of our L_4 centered coorindate frame, we have

$$\begin{aligned} \upsilon_x &= \dot{\Omega} \sin i \sin \eta_o + \dot{i} \cos \eta_o \\ \upsilon_y &= \dot{\Omega} \sin \cos \eta_o - \dot{i} \sin \eta_o \end{aligned} \qquad \text{(B-3)}$$

both of which are of $o(m^3)$ or higher. The expression for υ_z can be obtained by taking the time derivative of the true anomaly v in either one of the expressions on p. 110 of Ref. 8 or Eq. (2), p. 281 of Ref. 7. This results in the coefficient of $\bar{i}_z$ of our expression (16).

With the aid of Fig. (B-1) it is also relatively straightforward to determine the components of $\bar{r}_{13}$ in the $\bar{i}_x$, $\bar{i}_y$ and $\bar{i}_z$ directions.

We refer the reader to pp. 38, 41, and 79 of Ref. 8 for a more detailed presentation of the relations summarized here. For convenience the following explanatory relations for the various angular arcs are summarized below.

Ex or Eγ = fixed reference line in ecliptic

$\Omega m'(t) = v' - \Omega$ where $x\,\Omega \equiv \gamma\,\Omega$ = arc of nodal regression

$\varpi = x\,\Omega + \Omega\,A$ (measured in two planes) = γEA

$\epsilon = \gamma\, EM_o(o)$ i.e., at t = o

$s = \tan M'M$

$v = xM'$ = ecliptic projection of xM

$i = M'\,\Omega\,M$

$\gamma = \tan i \cong \sin i$

$\eta_o = \Omega\, M_o = nt + \epsilon - \Omega$

$\epsilon' = \gamma\, Em'(o)$ at t = o if $e_s \neq o$

One can then show that

$$\cos Mm' = \cos (v - v') \cos M'M = \left(1 - \frac{1}{2} s^2 + \frac{3}{8} s^4 - \ldots\right) \cos (v - v') \qquad \text{(B-4)}$$

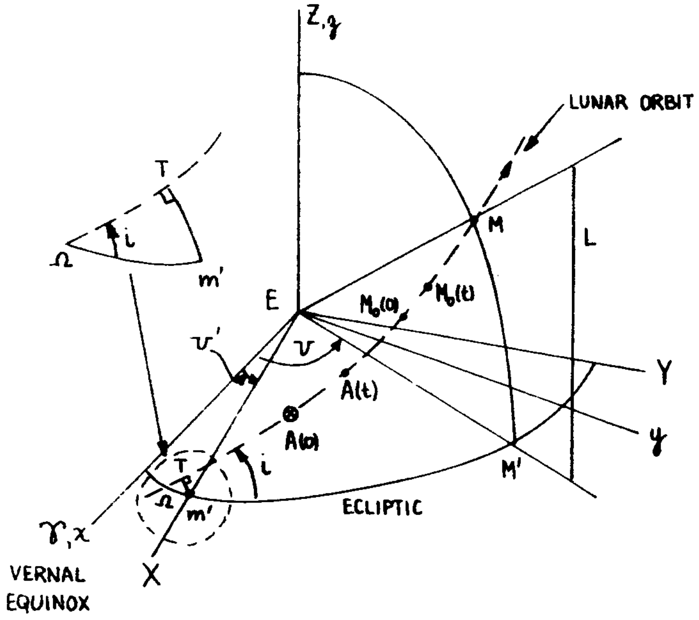


Fig. B-1: Lunar Motion Geometry in 3-Dimension.

m' = SUN

M = MOON

M_0 = MEAN LUNAR POSITION

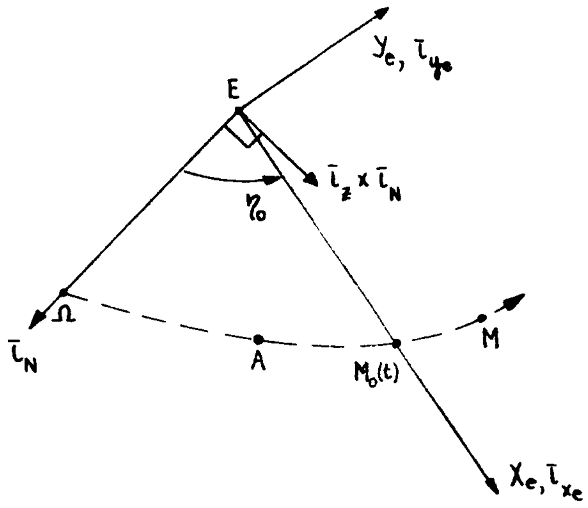
$v = XM'$ - LUNAR LONGITUDE

v' = SOLAR LONGITUDE

A = POSITION OF PERILUNE

Ω = DESIGNATES NODAL CROSSING (USED BY US TO DESIGNATE ANGLE $\sphericalangle \Upsilon E \Omega$)

$\bar{\iota}_N$ = UNIT VECTOR FROM E TO Ω



$$\bar{\iota}_{z_e} = \bar{\iota}_{x_e} \times \bar{\iota}_{y_e}$$

Fig. B-2: Planar view of lunar orbital plane and since

$$s = \left(1 - e^2 - \frac{1}{8}\gamma^2\right) \gamma \sin \eta_o + e\gamma \sin \ldots + \frac{1}{8} e^2\gamma \sin \ldots + \ldots \qquad \text{(B-5)}$$

(from p. 41 of Ref. 8).
while

$$s^2 \leq \gamma^2 \cong \sin^2 i \cong \sin^2 i_o = \sin^2 (5^o8'43'') \cong .008 \ll 1 \qquad \text{(B-6)}$$

we can approximate to sufficient accuracy

$$\begin{aligned} \cos Mm' \cong \cos (v - v') &= \cos [nt + \epsilon - n't - \epsilon' \\ &\quad + e, e' \text{ times periodic terms}] \\ &\cong \cos [(1 - m)t + \epsilon - \epsilon'] + \text{higher order terms} \\ &\quad \text{(we have divided by } n = 1) \qquad \text{(B-7)} \end{aligned}$$

$$\sin Mm' \cong - \sin [(1 - m)t + \epsilon - \epsilon'] + \text{H.O.T.} \qquad \text{(B-8)}$$

$$\text{Define:} \quad \xi = \sphericalangle Mm' = (1 - m)t + \epsilon - \epsilon' \qquad \text{(B-9)}$$

and note that

$$\sin m'T = \sin i \sin \Omega m' = \sin i \sin (v' - \Omega) \qquad \text{(B-10)}$$

With the above relations we can now obtain Eqs. (17) of the text

$$\begin{aligned} x_s &= r_{13} \cos \xi \\ y_s &= - r_{13} \sin \xi \qquad \text{(B-11)} \\ z_s &= - r_{13} \sin i \sin (v' - \Omega) = r_{13} \sin i \sin (\Omega - v') \end{aligned}$$

Appendix C

TAYLOR SERIES EXPANSION AROUND L_4

The steps needed in the expansion of the various terms in the Lagrangian L of Eq. (8) up to fourth order terms [i.e., $o(m^4)$] are indicated below. In dimensionless notation we have

$$\bar{r}_{14} = \bar{r}_{1L} + \bar{r} \qquad \text{where} \qquad \bar{r}_{1L} = \frac{1}{2}(1+\rho)\,\bar{i}_x + \frac{\sqrt{3}}{2}(1+\rho)\,\bar{i}_y$$

and thus

$$r_{14}^2 = \left[\frac{1}{2}(1+\rho) + x\right]^2 + \left[\frac{\sqrt{3}}{2}(1+\rho) + y\right]^2 + z^2 = \ldots \text{ algebra}$$

$$= 1 + \rho(2 + x + \sqrt{3}\,y) + \left(x + \sqrt{3}\,y + x^2 + y^2 + z^2\right)$$

$$= 1 + (a + b) = 1 + I \tag{C-1}$$

where a and b refer to the two terms following 1.

This enables us to write r_{14}^{-1} in the form

$$r_{14}^{-1} = [1 + I]^{-1/2} = 1 - \frac{1}{2}I + \frac{3}{8}I^2 - \frac{15}{48}I^3 + \frac{5}{16}\cdot\frac{7}{8}I^4 + \ldots \tag{C-2}$$

A similar expression applies also to r_{24}^{-1} after replacing x by -x in Eq. (C-1).

Evaluate now the various terms in (C-2).

$$\left[I^2\right]: \qquad = \overset{o(m^5)}{\cancel{a^2}} + 2ab + b^2$$

$$2ab = 2\rho(2 + x + \sqrt{3}\,y)\left(x + \sqrt{3}y + x^2 + y^2 + z^2\right)$$

$$= 4\rho\left(x + \sqrt{3}y + x^2 + y^2 + z^2\right) + 2\rho\left(x + \sqrt{3}y\right)^2 + o(m^5)$$

$$= 2\rho\left[2(x + \sqrt{3}y) + 3x^2 + 5y^2 + 2z^2 + 2\sqrt{3}\,xy\right]$$

Terms independent of x, y, or z have been dropped since they don't contribute to the final D.E.

$$b^2 = x^2 + 2\sqrt{3}\,xy + 3y^2 + 2(x + \sqrt{3}y)\left(x^2 + y^2 + z^2\right)$$

$$+ \left(x^2 + y^2\right)^2 + 2\left(x^2 + y^2\right) z^2 + z^4$$

$\left[I^3\right]$: $$= \cancel{a^3} + \cancel{3a^2b} + 3ab^2 + b^3$$

neglect as H.O.T.

$$3ab^2 \to 6\rho(x + \sqrt{3}y)^2 + o(m^5)$$

$$b^3 \to (x + \sqrt{3}y)^3 + 3(x + \sqrt{3}y)^2\left(x^2 + y^2 + z^2\right) + o(m^5)$$

$\left[I^4\right]$: only b^4 contributes

$$b^4 = (x + \sqrt{3}y)^4 + o(m^5)$$

Combining the above terms and neglecting noncontributing factors gives

$$r_{14}^{-1} = -\frac{1}{2}\left[\rho(x + \sqrt{3}y) + \left(x + \sqrt{3}y + x^2 + y^2 + z^2\right)\right]$$

$$+ \frac{3}{8}\left\{2\rho\left[2(x + \sqrt{3}y) + 3x^2 + 5y^2 + 2z^2 + 2\sqrt{3}\,xy\right]\right.$$

$$+ \left[(x + \sqrt{3}y)^2 + 2(x + \sqrt{3}y)\left(x^2 + y^2 + z^2\right) + \left(x^2 + y^2\right)^2\right.$$

$$+ 2\left(x^2 + y^2\right) z^2 + z^4\Big\} - \frac{15}{48}\Big\{6\rho(x + \sqrt{3}y)^2 + (x + \sqrt{3}y)^3$$

$$+ 3(x + \sqrt{3}y)^2 \left(x^2 + y^2 + z^2\right)\Big\} + \frac{35}{128}(x + \sqrt{3}y)^4 \qquad \text{(C-3)}$$

Furthermore

$$\bar{r}_{12} \cdot \bar{r}_{14} = (1 + \rho)\left[\frac{1}{2}(1 + \rho) + x\right] = \frac{1}{2}(1 + \rho)^2 + (1 + \rho)\, x$$

$$r_{12}^3 = (1 + \rho)^3 = 1 + 3\rho + \ldots$$

Thus

$$\frac{\bar{r}_{12} \cdot \bar{r}_{14}}{r_{12}^3} = \frac{1}{2}(1 + \rho)^{-1} + (1 + \rho)^{-2} \rightarrow x - 2\rho x + \text{noncontributing terms}$$

$$\text{(C-4)}$$

The Lagragian L in Eq. (8) is made dimensionless by multiplying it by $D/(\mu_1 + \mu_2)$. Let us multiply Eq. (8) by this factor and then set

$$\mu_1 + \mu_2 = 1$$
$$D = 1$$

and introduce the dimensionless quantity

$$\bar{\mu}_1 = \frac{\mu_1}{\mu_1 + \mu_2} \qquad \text{(C-5)}$$

while from before we had defined already the quantity μ

$$\mu = \frac{\mu_2}{\mu_1 + \mu_2}$$

by Eq. (12).

It then follows that

$$\bar{\mu}_1 + \mu = 1 \tag{C-6}$$

Consider the contribution V_{EM} of Earth and Moon to the potential energy term in L of Eq. (8)

$$V_{EM} = -\left[\frac{\bar{\mu}_1}{r_{14}} + \frac{\mu}{r_{24}}\right] + \mu\left[\frac{\bar{r}_{12} \cdot \bar{r}_{14}}{r_{12}^3}\right] \tag{C-7}$$

We recall that only the x coordinate changes sign when we use the expression for r_{14}^{-1} to obtain r_{24}^{-1}. A convenient expression for V_{EM} can be obtained by making in r_{24}^{-1} the following substitution for all odd powers of x

$$-x^{2n+1} = x^{2n+1} - 2x^{2n+1} \qquad (n = 0,1)$$

When use is made of Eq. (C-6) and the lengthy algebraic manipulations are carried out, one ends up with a V_{EM} given by

$$\begin{aligned} V_{EM} = &\left\{\frac{1}{8} x^2 - \frac{5}{8} y^2 + \frac{1}{2} z^2 - \frac{3\sqrt{3}}{4} (1 - 2\mu) xy \right.\\ &\left. - \rho(x + \sqrt{3}y)\right\}^{(o)} + \left\{- 7 \frac{1 - 2\mu}{16} x^3 + \frac{3\sqrt{3}}{16} y^3 \right.\\ &\left. + \frac{1 - 2\mu}{16} \cdot 33 xy^2 - \frac{1 - 2\mu}{16} \cdot 12 xz^2 + \frac{3\sqrt{3}}{16} x^2y - \frac{12\sqrt{3}}{16} yz^2\right\}_3 \\ &+ \left\{\frac{37}{128} x^4 - \frac{123}{64} x^2y^2 + \frac{3}{16} x^2z^2 + \frac{33}{16} y^2z^2 - \frac{3}{128} y^4 - \frac{3}{8} z^4 \right.\\ &\left. + \frac{25\sqrt{3}}{32} (1 - 2\mu) x^3y - \frac{45(1 - 2\mu)\sqrt{3}}{32} xy^3 + \frac{15(1 - 2\mu)}{8} xyz^2\right\}_4 \\ &+ \rho \left\{- \frac{3}{8} x^2 + \frac{15}{8} y^2 - \frac{3}{2} z^2 + \frac{9}{4} (1 - 2\mu) xy\right\}_s \end{aligned} \tag{C-8}$$

The solar contribution V_s to the potential energy term in L is found below.

$$\bar{r}_{13} = x_s \bar{i}_x + y_s \bar{i}_y + z_s \bar{i}_z$$

$$\bar{r}_{13} \cdot \bar{r}_{14} = \left[\frac{1}{2}(1+\rho)\, x_s + x_s x\right] + \left[\frac{\sqrt{3}}{2}(1+\rho)\, y_s + y_s y\right] + z_s z$$

Now

$$V_s = -\, m^2 \left[\frac{3}{2r_{13}^2}\left(\bar{r}_{13} \cdot \bar{r}_{14}\right)^2 - \frac{1}{2}\, r_{14}^2\right] \qquad \text{(C-9)}$$

so that only terms of $o(m^2)$ or lower must be retained inside the bracket. After dropping all terms which do not contain the particle's coordinates we get

$$\left(\bar{r}_{13} \cdot \bar{r}_{14}\right)^2 \to \left(x_s x + y_s y + \cancel{z_s z}\right)^2 + \left(x_s + \sqrt{3}\, y_s\right)\left(x_s x + y_s y + z_s z\right)$$

will lead to $o(m^5)$ terms (C-10)

and

$$r_{14}^2 \to x + \sqrt{3}y + x^2 + y^2 + z^2 \qquad \text{(C-11)}$$

Substitution of (C-10) and (C-11) into (C-9) results in

$$V_s = -m^2 \left\{ \frac{3}{2r_{13}^2}\left[\left(x_s x + y_s y\right)^2 + \left(x_s + \sqrt{3}\, y_s\right)\left(x_s x + y_s y + z_s z\right)\right] - \frac{1}{2}\left[(x + \sqrt{3}y) + \left(x^2 + y^2 + z^2\right)\right] \right\} \qquad \text{(C-12)}$$

The Hamiltonian H is defined by the relation

$$H = \bar{p}_r \cdot \dot{\bar{r}} - L = p_r^T \dot{r} - L \qquad \text{(C-13)}$$

where

$$\bar{p} = \left[\frac{\partial L}{\partial \dot{\bar{r}}_{14}}\right] \cdot \left[\frac{\partial \dot{\bar{r}}_{14}}{\partial \dot{\bar{r}}}\right] = \dot{\bar{r}}_{14} \cdot \underline{\underline{I}} = \dot{\bar{r}}_{14} \qquad \text{(C-14)}$$

$\underline{\underline{I}}$ denotes the identity tensor. The equality $\bar{p} = \dot{\bar{r}}_{14}$ is a consequence of the linear dependence of $\dot{\bar{r}}_{14}$ on the velocity components $\dot{x},\dot{y},\dot{z}$, in the rotating coordinate frame. Writing $\dot{\bar{r}}_{14}$ as

$$\dot{\bar{r}}_{14} = \dot{\bar{r}}_{1L} + \dot{r}\bar{i}_r + \bar{\omega} \times \bar{r}$$

$$\dot{\bar{r}}_{1L} = \dot{r}_{1L}\bar{i}_{1L} + \bar{\omega} \times \bar{r}_{1L}$$

$$\dot{r}\bar{i}_r = \dot{x}\bar{i}_x + \dot{y}\bar{i}_y + \dot{z}\bar{i}_z$$

we get

$$\therefore \dot{\bar{r}}_{14} = \text{... algebra ...}$$

$$= \left[\frac{1}{2}\dot{\rho} - \frac{\sqrt{3}}{2}\left(1 + \nu_z\right)(1 + \rho) + \dot{x} + z\nu_y - y(1 + \nu_z)\ \bar{i}_x\right.$$

$$+ \left[\frac{\sqrt{3}}{2}\dot{\rho} + \frac{1}{2}(1 + \rho)\left(1 + \nu_z\right) + \dot{y} + x\left(1 + \nu_z\right) - z\nu_x\right]\bar{i}_y$$

$$+ \left[\frac{\sqrt{3}}{2}(1 + \rho)\ \nu_x - \frac{1}{2}(1 + \rho)\ \nu_y + \dot{z} + y\nu_x - x\nu_y\right]\bar{i}_z$$

(C-15)

We now introduce the momenta P via Eq. (19), solve for $\dot{r}$ from (C-14) and obtain the expressions

$$\dot{x} = P_x - \frac{\sqrt{3}}{2} + \left[\frac{\sqrt{3}}{2}(1 + \rho) + y\right]\left(1 + \upsilon_z\right) - \frac{1}{2}\dot{\rho} - z\upsilon_y$$

$$\dot{y} = P_y + \frac{1}{2} - \frac{\sqrt{3}}{2}\dot{\rho} - \left[\frac{1}{2}(1 + \rho) + x\right]\left(1 + \upsilon_z\right) + z\upsilon_x$$

$$\dot{z} = P_z + \left[\frac{1}{2}(1 + \rho) + x\right]\upsilon_y - \left[\frac{\sqrt{3}}{2}(1 + \rho) + y\right]\upsilon_x \qquad \text{(C-16)}$$

Also from Eq. (C-14) and Eq. (19) we can write L in the form

$$L = \frac{1}{2}\left(P_x - \frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2}\left(P_y + \frac{1}{2}\right)^2 + \frac{1}{2}P_z^2 - V_{EM} - V_s \qquad \text{(C-17)}$$

If we now substitute (C-17) into (C-13), make use of (C-14) and (C-16), and neglect all the terms which do not depend on the momenta P or the particle's position $\bar{r}$ we end up after a lot of algebra with the expression for the Hamiltonian H presented in Eq. (21) of the text.

Appendix D

CANONICAL TRANSFORMATION TO SLOW VARIABLES

We shall outline here the steps which underlie the canonical transformation from the variables α,β, to the slow set α',β'. These variables are analogous to polar coordinates where $\sqrt{\alpha}$ corresponds to an amplitude and β to a phase shift. We shall find it convenient to use also a cartesian set of generalized coordinates q,p in terms of which the transformation relations will be developed.

Let

$$S = S(q,p') = S_1 + S_2 \qquad \text{(D-1)}$$

be a generating function from the set q,p to a second slowly varying set q',p' where S_1 will be selected to remove from H the 3rd order terms (all of which are short period) and S_2 to remove all 4th order short period, and define

$$S_q = \left[\frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}, \frac{\partial S}{\partial q_3}\right] = (1 \times 3) \text{ row matrix of partial derivatives of } S$$

S_{q^T} = (3 X 1) column matrix of partial derivatives

Then

$$\begin{aligned} p &= p' + S_{q^T}(q,p') \\ q &= q' - S_{p'^T}(q,p') \end{aligned} \qquad \text{(D-2)}$$

Let

$$\begin{aligned} \Delta q &= q - q' \\ \Delta p &= p - p' \end{aligned} \qquad \text{(D-3)}$$

and expand S_{q^T} and $S_{p'^T}$ in a Taylor series around the values of q′,p′. To second order in Δq and Δp

$$S_{q^T}(q,p') = \frac{\partial}{\partial q^T}\left[S(q',p') + S_{q'}\Delta q + \frac{1}{2}\,\Delta q^T S_{q'^T q'}\,\Delta q + \cdots\right] \tag{D-4}$$

The expressions for q = q(q′,p′) and p = p(q′,p′) can then be developed via (D-2) and (D-4)

$$q = q' - S_{p'^T}(q,p') = q' - \frac{\partial}{\partial_{p'^T}}\left[S(q',p') + S_{q'}\Delta q + \frac{1}{2}\,\Delta q^T S_{q'^T q'}\,\Delta q + \cdots\right]$$

$$\cong q' - S_{p'^T}(q',p') - S(q',p')_{p'^T q'}\Delta q + \cdots \tag{D-5}$$

In order to prevent carrying unnecessary terms along let us estimate the order of magnitude of the above terms. Since S will be used to perform a transformation of variables in H′ which contains terms of $o(m^3)$ and $o(m^4)$ then the lowest terms in S will be of $o(m^3)$. We might also use the notation $o(x^3)$ since the x,y,z coordinates are the ones to be transformed.

Let us view q as equivalent to β and p as equivalent to α. Then the derivatives result in the following orders of magnitude

$$\begin{aligned} S &= o(H_3) \cong o(x^3) = o(\alpha^{3/2}) \\ S_p &\rightarrow S_\alpha = o(\alpha^{1/2}) = o(x) \\ S_q &\rightarrow S_\beta = o(\alpha^{3/2}) = o(x^3) \\ S_{p^T q} &\rightarrow o(S_p) = o(x) \end{aligned} \tag{D-6}$$

We assume also that

$$\Delta q \cong o(x) + \text{higher order terms} \tag{D-7}$$

We thus note that the term $\Delta q^T S_{q'T_q'}\,\Delta q = o(x^5)$ and after operating on it with ∂/∂_p, it becomes $o(x^3)$.

Equation (D-5) can then be written as

$$\Delta q = - S_{p'T} - S_{p'T_q'}\,\Delta q + o(x^3) \cdots \tag{D-8}$$

and terms of $o(x^3)$ are not carried along. Thus to $o(x^2)$ we can write the following relation

$$\left[I + S_{p'T_q'}\right] \Delta q \cong - S_{p'T}$$

which can be inverted to solve for Δq

$$\Delta q \cong - \left[I + S_{p'T_q'}\right]^{-1} S_{p'T} \cong - S_{p'T} + S_{p'T_q'}\, S_{p'T} + o(x^3) \tag{D-9}$$

where I is the identity matrix.

From Eq. (D-2) we also note that

$$\Delta p \approx o(S_{qT}) = o(x^3) + \text{H.O.T.}$$

Expanding for Δp as was done in (D-9) for Δq we find

$$\Delta p \cong S_{q'T} - S_{q'T_q'}\, S_{p'T} + o(x^5) \tag{D-10}$$

The partial derivatives of H can also be treated similarly to the partials of S. Thus

$$\begin{aligned} H_q &= o(H) \\ H_p &= o\left(\frac{H}{x^2}\right) \end{aligned} \qquad \text{(D-11)}$$

For a scleronomic generating function S we have the transformation relation for Hamiltonians

$$K(q',p') = H(q,p) \qquad \text{(D-12)}$$

and expanding H in a Taylor series around q',p' gives

$$\begin{aligned} K(q',p') &= H(q',p') + H_{q'} \Delta q + H_{p'} \Delta p + \\ &\quad + \frac{1}{2!} \left[\Delta q^T \frac{\partial}{\partial q'^T} + \Delta p^T \frac{\partial}{\partial p'^T}\right]^2 H(q',p') + \cdots \\ &= H(q',p') + H_{q'}\Delta q + H_{p'}\Delta p + \frac{1}{2}\left[\Delta q^T H_{q'^T q'} \Delta q\right. \\ &\quad \left. + \Delta q^T H_{q'^T p'} \Delta p + \Delta p^T H_{p'^T q'} \Delta q + \Delta p^T H_{p'^T p'} \Delta p\right] + \cdots \\ &= H(q',p') + H_{q'}\left[- S_{p'^T} + S_{p'^T q'} S_{p'^T}\right]_A + H_{p'}\left[S_{q'^T} - S_{q'^T q'} S_{p'^T}\right]_B \\ &\quad + \frac{1}{2}\left[- S_{p'} + S_{p'} S_{p' q'^T}\right]_C H_{q'^T q'}\left[- S_{p'^T} + S_{p'^T q'} S_{p'^T}\right] \\ &\quad + \frac{1}{2}\left[- S_{p'} + S_{p'} S_{p' q'^T}\right]_D H_{q'^T p'}\left[S_{q'^T} - S_{q'^T q'} S_{p'^T}\right] \\ &\quad + \frac{1}{2}\left[S_{q'} - S_{p'} S_{q' q'^T}\right] H_{p'^T q'}\left[- S_{p'^T} + S_{p'^T q'} S_{p'^T}\right] \\ &\quad + \frac{1}{2}\left[S_{q'} - S_{p'} S_{q' q'^T}\right] H_{p'^T p'}\left[S_{q'^T} - S_{q'^T q'} S_{p'^T}\right] + \cdots \end{aligned}$$

(D-13)

The subscript letters A,B,C,D have been introduced merely for ease of subsequent identification of the respective brackets [ ].

We recall that

$$H = \underset{o(x^2)}{H^{(o)}} + \underset{o(x^3)}{H_3} + \underset{o(x^4)}{H_4} \qquad \text{(D-14)}$$

where H_3 contains x^3 terms and H_4 denotes the 4th order terms like x^4, m^2x^2, ρx^2 etc. This breakdown into H_3 and H_4 will be made use of in choosing the relations defining $S_1(q',p')$ and $S_2(q',p')$.

We shall assume S_1 to contain only $o(x^3)$ terms and S_2 only terms of $o(x^4)$, because we shall select S_2 so as to remove all 4th order short period terms from the Hamiltonian K.

$$[\ ]_A \rightarrow - \left[S_{1_{p'T}} + S_{2_{p'T}} - \left(S_{1_{p'Tq'}} + S_{2_{p'Tq'}}\right)\left(S_{1_{p'T}} + S_{2_{p'T}}\right)\right]$$

$$= \underset{o(x)}{- S_{1_{p'T}}} \underset{o(x^2)}{- S_{2_{p'T}}} + \underset{o(x^2)}{S_{1_{p'Tq'}} S_{1_{p'T}}} + \underset{o(x^3)}{S_{1_{p'Tq'}} S_{2_{p'T}}} + \underset{o(x^3)}{S_{2_{p'Tq'}} S_{1_{p'T}}} + o(x^4)$$

(D-15)

Thus to $o(x^4)$, which is the highest order retained in all terms,

$$H_{q'}[\ ]_A \rightarrow H^{(o)}_{q'} \left[- S_{1_{p'T}} - S_{2_{p'T}} + S_{1_{p'Tq'}} S_{1_{p'T}}\right] + H_{3_{q'}}\left[- S_{1_{p'T}}\right]$$

(D-16)

Similarly, the following expressions can be derived

$$[\ ]_B \rightarrow S_{1_{q'T}} + S_{2_{q'T}} - S_{1_{q'Tq'}} S_{1_{p'T}} + \text{H.O.T.} \qquad \text{(D-17)}$$

$$H_{p'}[\;]_B \to H^{(o)}_{p'}\left[S_{1_{q'T}} + S_{2_{q'T}} - S_{1_{q'Tq'}}S_{1_{p'T}}\right] + H_{3_{p'}}S_{1_{q'T}} + \text{H.O.T.}$$

(D-18)

$$\frac{1}{2}[\;]_C H_{q'Tq'}[\;]_A \to \frac{1}{2}S_{1_{p'}}H^{(o)}_{q'Tq'}S_{1_{p'T}} \qquad \text{(D-19)}$$

$$\frac{1}{2}[\;]_C H_{q'Tp'}[\;]_B \to -\frac{1}{2}S_{1_{p'}}H^{(o)}_{q'Tp'}S_{1_{q'T}} \qquad \text{(D-20)}$$

$$\frac{1}{2}[\;]_D H_{p'Tq'}[\;]_A \to -\frac{1}{2}S_{1_{q'}}H^{(o)}_{p'Tq'}S_{1_{p'T}} \qquad \text{(D-21)}$$

$$\frac{1}{2}[\;]_D H_{p'Tp'}[\;]_B \to \frac{1}{2}S_{1_{q'}}H^{(o)}_{p'Tp'}S_{1_{q'T}} \qquad \text{(D-22)}$$

Substituting (D-16) and (D-18) through (D-22) into (D-13) results in the expression

$$K = H^{(o)} + H_3 + H_4 + H^{(o)}_{q'}\left[-S_{1_{p'T}} - S_{2_{p'T}} + S_{1_{p'Tq'}}S_{1_{p'T}}\right]$$

$$- H_{3_{q'}}S_{1_{p'T}} + H^{(o)}_{p'}\left[S_{1_{q'T}} + S_{2_{q'T}} - S_{1_{q'Tq'}}S_{1_{p'T}}\right] + H_{3_{p'}}S_{1_{q'T}}$$

$$+ \frac{1}{2}S_{1_{p'}}H^{(o)}_{q'Tq'}S_{1_{p'T}} - \frac{1}{2}S_{1_{p'}}H^{(o)}_{q'Tp'}S_{1_{q'T}} - \frac{1}{2}S_{1_{q'}}H^{(o)}_{p'Tq'}S_{1_{p'T}}$$

$$+ \frac{1}{2}S_{1_{q'}}H^{(o)}_{p'Tp'}S_{1_{q'T}} \qquad \text{(D-23)}$$

Recalling the definition of the Poisson bracket

$$[H,S] = H_q S_p^T - H_p S_q^T \qquad \text{(D-24)}$$

and applying it to the terms of Eq. (D-23) one can obtain after a lengthy series of manipulations and combinations of terms the expression

$$\begin{aligned} K = {} & H^{(o)} + H_3 + H_4 - \left[H^{(o)},S_1\right] - \left[H^{(o)},S_2\right] - \left[H_3,S_1\right] \\ & + \frac{1}{2}\left[H^{(o)},S_{1_{q'}} S_{1_{p'}}^T\right] + \frac{1}{2}\left[\left[H^{(o)},S_1\right], S_1\right] \end{aligned} \qquad \text{(D-25)}$$

in terms of q' and p' only.

Let us choose for the definition of S_1 the relation

$$\left[H^{(o)},S_1\right] - H_3 = 0 \qquad \text{(D-26)}$$

and thereby remove H_3 which contains only short period terms.

Then

$$\begin{aligned} K = {} & H^{(o)} + H_4 + \frac{1}{2}\left[H^{(o)},S_{1_{q'}} S_{1_{p'}}^T\right] + \frac{1}{2}\left[H_3,S_1\right] - \left[H_3,S_1\right] \\ & - \left[H^{(o)},S_2\right] = H^{(o)} + H_4 + \frac{1}{2}\left[H^{(o)},S_{1_{q'}} S_{1_{p'}}^T\right] \\ & - \frac{1}{2}\left[H_3,S_1\right] - \left[H^{(o)},S_2\right] \end{aligned} \qquad \text{(D-27)}$$

The third and fifth final terms in Eq. (D-27) can be combined into the one bracket

$$- \left[H^{(o)}, S_2 - \frac{1}{2} S_{1_q}, S_{1_p}, T \right] \qquad \text{(D-28)}$$

We now define S_2 such as to remove all remaining 4th order short period terms from K. Now both H_4 and $[H_3, S_1]$ will contain both long period $\overline{(\ )}$ and short period (s·p) terms, which can all be eliminated by letting S_2 be defined via

$$H_{4_{s \cdot p}} - \frac{1}{2} \left[H_3, S_1 \right]_{s \cdot p} - \left[H^{(o)}, S_2 - \frac{1}{2} S_{1_q}, S_{1_p}, T \right] = 0 \qquad \text{(D-29)}$$

This leaves the long period form of the Hamiltonian K as

$$K = H^{(o)} + \overline{H}_4 - \frac{1}{2} \overline{\left[H_3, S_1 \right]} \qquad \text{(D-30)}$$

and if α' and β' are selected as the canonical variables, rather than α' and $(t + \beta')$, the long period perturbation Hamiltonian K' is obtained as

$$K' = \overline{H}_4 - \frac{1}{2} \overline{\left[H_3, S_1 \right]} \qquad \text{(D-31)}$$

To this expression one must still add the contribution from the linear forced solutions $\tilde{x}, \tilde{y}$ due to $H^{(o)}$ as indicated in Eq. (49) which then finally leads to the relation presented in Eq. (48).

Comparison of the K from Eq. (D-31) with the K presented on p. 63 of Ref. 6 shows that the two Hamiltonians are not alike. This difference can be traced to the particular way in which the time dependent generating function $S_1(q, p', t)$ of Ref. 6 was defined there by means of an equation in the mixed variables q, p', instead of first carrying out the transformation to the new set of coordinates q', p' shown in this appendix in Eqs. (D-9) to (D-13).

As a consequence of the use of mixed variables, some of the terms which would have appeared from the additional Taylor series expansion over q were thus missing and only one half of the terms of the Poisson bracket $[H_3,S_1]$ of Eq. (D-31) showed up in the function Φ introduced in Ref. 6. The absence of these additional terms prevented the cancellation of nonpolynomial terms (i.e., terms which do not arise from binomial expansions such as $(x + y)^n$, where n is some finite integer) and led to the presence of an extraneous term such as the $\alpha_1'^{3/2}\alpha_2'^{1/2}$ term in Eq. (10) of Ref. 6.

The source of the incorrect results, which can arise when one operates with mixed variables anytime terms higher than of first order are retained in the Hamiltonian, were recognized by Prof. Breakwell, who then suggested that the correct procedure in choosing the function S_1 would be to transform first to the new set of coordinates q',p'. The implementation of this suggestion led to the developments presented in this appendix, and avoided here the presence of the inadmissible nonpolynomial terms.

The derivation of the Mathieu type Hamiltonian in Appendix F does make use of mixed variables. However, the results obtained there are correct since only linear terms were retained in H.

A last comment should be made regarding the slow variables q',p', or α',β'. It turns out that it is impossible to prevent the presence of some higher order long period terms in S_2 which arise because the term $S_{1_{q'}}S_{1_{p'T}}$ may contain also long period parts. From this it follows that in the expression for, say, q

$$q = q' - S_{1_{p'T}} + S_{1_{p'Tq'}} S_{1_{p'T}} - S_{2_{p_1'T}} \qquad \text{(D-32)}$$

the last two terms may also make some long period contributions to q, which would tend to contradict the assertion that q' (and also p') are the only long period variables. This situation is unfortunately unavoidable and cannot be circumvented by redefining S_1 or S_2, since the elimination of the extra long period terms in q' or p' via S would automatically result in the introduction of unwanted higher order short period terms into K that S would be incapable of suppressing simultaneously.

Fortunately this impasse is not too serious since the bothersome long period terms in Eq. (D-32) are of $o(m^4)$ or higher and may be safely disregarded within the extent of the present theory inasmuch as q' does not appear in a linear manner in H. They would pose a problem however if the present approach were to be extended to encompass some of the higher order terms currently neglected.

Appendix E

SOME ILLUSTRATIVE STEPS IN THE DERIVATION OF LONG PERIOD TERMS IN K′

The steps leading from Eq. (48) to Eq. (50) required by far the most time consuming, tedious and exacting manipulations and computations of the whole investigation. We shall indicate here only briefly as an example a few representative intermediate steps so as to provide the reader with a feeling for what is involved here.

First a general remark concerning the Poisson bracket $\overline{[H_3,S_1]}$. In the expanded form, and using the polar canonical variables α and β, this becomes

$$\left[H_3,S_1\right] = H_{3\beta_i'}S_{1\alpha_i'} - H_{3\alpha_i'}S_{1\beta_i'} \qquad \text{(E-1)}$$

where the tensor notation for summation over i = 1,2,3 has been used. The same bracket, when H_3 and S_1 are expressed in cartesian coordinates x,y,z,P_x,P_y,P_z, can also be written as

$$\left[H_3,S_1\right] = H_{3x}S_{1P_x} + H_{3y}S_{1P_y} + \cdots - H_{3P_x}S_{1x} - \cdots - H_{3P_z}S_{1z} \qquad \text{(E-2)}$$

which indicates that the bracket will give rise only to polynomial terms of the form $x^2P_z^2$, x^3P_y, y^4, etc.

From this it follows that when one evaluates the long period terms in the polar coordinates used in Eq. (E-1) one must be careful to observe that only polynomial type terms should be retained. Thus, one can obtain secular terms like $5\alpha_1'^2$, $7\alpha_3'^2$ $\cdots$ etc. or slowly varying terms like $(\cdots)\ \alpha_1'\alpha_3' \cos[(\omega_1 - \omega_3)t + \cdots]$ or $(\cdots)\ \alpha_1'^{1/2}\alpha_2'^{3/2} \times \cos[(\omega_1 - 3\omega_2)t + \cdots]$, but not terms such as $(\cdots)\ \alpha_1'^{3/2}\alpha_2'^{1/2} \times \cos[(\omega_1 - 3\omega_2)t + \cdots]$ because such a term could not arise from products of the form $x_1y_2^3$ or $y_1x_2^3$ which are the only kind that could give rise to long period trigonometric terms with a frequency $\omega_1 - 3\omega_2$. The quantities x_1, y_2, etc. represent the ω_1 term in $\overline{x}$ and the ω_2 term in $\overline{y}$ of Eq. (36), respectively.

This polynomial requirement is not satisfied in Eq. (10) of Ref. (6) which contains the term $16.2\alpha_1'^{3/2}\alpha_2'^{1/2}\cos[.0609t + \omega_1\beta_1' + 3\omega_2\beta_2' - 4.48^\circ]$.

We shall indicate now a few steps in the evaluation of one of the long period terms in the Poisson bracket. For convenience we let

$$H_3 = H_{32} + H_{33} \tag{E-3}$$

where

H_{32} = coplanar (x,y) terms in H_3
H_{33} = out-of-plane (z) terms in H_3

Similarly

$$S_1 = S_{12} + S_{13} \tag{E-4}$$

where

$$S_1 = -\int^t H_3 dt \tag{E-5}$$

from Appendix D. (We recall that $[H,S] = -\partial S/\partial t$ when H is treated as the momentum conjugate to the coordinate t.)

Then

$$\begin{aligned}\overline{[H_3,S_1]} &= \overline{[H_{32} + H_{33},\ S_{12} + S_{13}]} = \\ &= \overline{[H_{32},S_{12}]} + \overline{[H_{32},S_{13}]} + \overline{[H_{33},S_{12}]} + \overline{[H_{33},S_{13}]}\end{aligned} \tag{E-6}$$

Let us take the first bracket in Eq. (E-6), and consider for example only the component $H_{\alpha_1}S_{1\beta_1}$ in it. It can be shown that it arises from the product of the two parts

$$
\begin{aligned}
H_{32\alpha_1} &= \alpha_1^{1/2}\left\{\frac{3}{2}\,M(a_1 + a_2) + \frac{3}{2}\,N(33a_3 - 7a_4)\right\}_A \\
&+ \alpha_2^{1/2}\left\{M(b_1 + b_2) + N(33b_3 - 7b_4)\right\}_B \\
&+ \alpha_1^{-1/2}\alpha^2\left\{\frac{1}{2}\,M(c_1 + c_2) + \frac{1}{2}\,N(33c_3 - 7c_4)\right\}_C \qquad \text{(E-6)}
\end{aligned}
$$

and

$$
\begin{aligned}
S_{12\beta_1} &= -\,\alpha_1^{3/2}\left\{M(a_1' + a_2') + N(33\alpha_3' - 7\alpha_4')\right\}_D \\
&- \alpha_1\alpha_2^{1/2}\left\{M(b_1' + b_2') + N(33b_3' - 7b_4')\right\}_E \\
&- \alpha_1^{1/2}\alpha_2\left\{M(c_1' + c_2') + N(33c_3' - 7c_4')\right\}_F
\end{aligned}
$$

where we have dropped for convenience the primes on the α's and β's.

The quantities $a_1 \cdots a_4$, $b_1 \cdots b_4$, $\cdots c_1' \cdots c_4'$ are defined in terms of $\bar{x} = A_1C_{(1)} + A_2C_{(2)}$ and $\bar{y} = A_1'C_{(1)+\delta_1} + A_2'C_{(2)+\delta_2}$, where $(1) \equiv \omega_1\beta_1^{\neq}$, $(2) \equiv \omega_2\beta_2^{\neq}$ and $A_1, A_1' \cdots \delta_1, \delta_2$ are obtained from Eq. (36). M and N are two constants defined as $M = 3\sqrt{3}/16$ and $N = (1 - 2\mu)/16$.

In terms of the above constants one can obtain the following expressions

$$
\begin{aligned}
a_1 = a_1' &= \frac{1}{2}\,A_1^2A_1^1\left(C_{(1)+\delta_1} + \frac{1}{2}\,C_{3(1)+\delta_1} + \frac{1}{2}\,C_{(1)-\delta_1}\right) \\
b_1' &= A_1A_1'A_2\left[\frac{\omega_1}{2\omega_1 + \omega_2}\,C_{2(1)+(2)+\delta_1} + \frac{\omega_1}{2\omega_1 - \omega_2}\,C_{2(1)-(2)+\delta_1}\right] \\
&+ \frac{1}{2}\,A_1^2A_2^1\left[\frac{\omega_1}{2\omega_1 + \omega_2}\,C_{2(1)+(2)+\delta_2} + \frac{\omega_1}{2\omega_1 - \omega_2}\,C_{2(1)-(2)-\delta_2}\right] \qquad \text{(E-7)}
\end{aligned}
$$

$\vdots$

and similar expressions for all the other quantities inside the ( ) brackets.

We observe from Eq. (E-6) that in the Poisson bracket, the coefficient of α_1^2 would arise from the product of brackets $\{\ \}_A$ with $\{\ \}_D$, and in the same manner we note that the

$$\begin{array}{lll} \text{coefficient of} & \alpha_1\alpha_2 & \rightarrow \text{results from } \{\ \}_A\cdot\{\ \}_F;\ \{\ \}_B\cdot\{\ \}_E;\ \{\ \}_C\cdot\{\ \}_D \\ \quad '' & \alpha_1^{1/2}\alpha_2^{3/2} & \rightarrow \text{results from } \{\ \}_B\cdot\{\ \}_F;\ \{\ \}_C\cdot\{\ \}_E \\ \quad '' & \alpha_2^2 & \rightarrow \text{results from } \{\ \}_C\cdot\{\ \}_F \end{array} \qquad \text{(E-8)}$$

Products of brackets $\{\ \}$ as indicated in Eq. (E-8) arise in all the partial derivative products of H_3 with S_1, and must be summed up for every combination $\alpha_i^n\alpha_j^m$ to obtain the final value of the coefficient for that particular combination of α's.

For example, to obtain the coefficient of α_1^2 in $H_{32\alpha_1}S_{12\beta_1}$ we have

$$\alpha_1^2:\ -\{\ \}_A\ \{\ \}_D = -\frac{3}{2}\left\{M(a_1 + a_2) + N(33a_3 - 7a_4)\right\}$$

$$\cdot\left\{M(a_1' + a_2') + N(33a_3' - 7a_4')\right\} \qquad \text{(E-9)}$$

where the following relations among the a's apply here

$$\begin{array}{ll} a_1 = a_1' & a_3 = a_3' \\ a_2 = a_2' & a_4 = a_4' \end{array} \qquad \text{(E-10)}$$

Expanding and using the appropriate relations for the a's (not shown here) gives

$$\alpha_1^2: \; - \left\{\frac{3}{2} M^2(a_1 + a_2)^2 + 3MN(a_1 + a_2)(33a_3 - 7a_4) + \frac{3}{2} N^2(33a_3 - 7a_4)^2\right\} \tag{E-11}$$

$$\begin{aligned} \frac{3}{2} M^2(a_1 + a_2)^2 &= \frac{3}{2} M^2(a_1^2 + 2a_1a_2 + a_2^2) \\ &= \frac{3}{2} M^2\left[\frac{1}{4} A_1^4A_1'^2\left(\frac{3}{4} + \frac{1}{2} C_{2\delta_1}\right) + \frac{1}{4} A_1^2A_1'^4\left(\frac{3}{2} + C_{2\delta_1}\right) + \frac{1}{16} A_1'^6 \cdot 5\right] \end{aligned} \tag{E-12}$$

$$\begin{aligned} 3MN(a_1 + a_2)(33a_3 - 7a_4) = 3MN\Bigg[&33\left\{\frac{1}{4} A_1^3A_1'^3\left(\frac{9}{8} C_{\delta_1} + \frac{1}{8} C_{3\delta_1}\right) + \frac{1}{8} A_1A_1'^5 \cdot \frac{10}{4} C_{\delta_1}\right\} \\ &- 7\left\{\frac{1}{8} A_1^5A_1' \frac{10}{4} C_{\delta_1} + \frac{1}{16} A_1^3A_1'^3\left(\frac{9}{2} C_{\delta_1} + \frac{1}{2} C_{3\delta_1}\right)\right\}\Bigg] \end{aligned} \tag{E-13}$$

$$\begin{aligned} \frac{3}{2} N^2(33a_3 - 7a_4)^2 = \frac{3}{2} N^2\Bigg[&1089 \frac{1}{4}\left\{A_1^2A_1'^4\left(\frac{3}{4} + \frac{1}{2} C_{2\delta_1}\right)\right\} - 462\left\{\frac{1}{8} A_1^4A_1'^2\left(\frac{3}{2} + C_{2\delta_1}\right)\right\} \\ &+ 49\left\{\frac{1}{16} A_1^6 \cdot 5\right\}\Bigg] \end{aligned} \tag{E-14}$$

When the numerical values for $A_1, A_1', A_2, A_2', C\delta_1, C2\delta_1$, etc. are substituted into Eqs. (E-12) through (E-14) and all the terms added, one obtains the result

$$-92.871\alpha_1^2 \tag{E-15}$$

This same, or a similar, procedure must be repeated for every combination of α's which arises from all the terms of the Poisson bracket.

Appendix F

MATHIEU TYPE HAMILTONIANS

Consider the near-resonant Mathieu equation

$$\ddot{X} + \omega_o^2\left[1 + \eta \cos 2\omega_o t \; (1 + \epsilon)\right]X = 0 \qquad \text{(F-1)}$$

where η, ϵ, $<< 1$. The effect of the trigonometric coefficient is to introduce a parametric excitation term into the simple harmonic oscillator model.

It is easily seen that as $\epsilon \rightarrow 0$ a resonant forcing term will arise in case a perturbation solution is attempted, after the $X^{(o)}$ solution to the equation

$$\ddot{X} + \omega_o^2 X = 0 \qquad \text{(F-2)}$$

is substituted back into Eq. (F-1) to provide the next higher term.

The Hamiltonian of system (F-1) is

$$\begin{aligned} H &= \frac{1}{2}\left[p^2 + \omega^2 X^2\right] = H^{(o)} + H' \\ \omega &= \omega_o\left[1 + \eta \cos 2\omega_o t(1 + \epsilon)\right]^{1/2} \end{aligned} \qquad \text{(F-3)}$$

X,p = generalized coordinate and momentum, respectively

$H^{(o)}$ = Hamiltonian of simple harmonic oscillator of frequency ω_o

H' = perturbation Hamiltonian (for $\eta << 1$)

The solution corresponding only to $H^{(o)}$ is

$$X^{(o)} = \frac{\sqrt{2\alpha}}{\omega_o} \sin \omega_o \; (t + \beta) \qquad \text{(F-4)}$$

where α, β are constants of integration.

When H' is included, α and β become functions of time t. It is useful to consider mainly the long period variations in α and β, since these basically tell us the most about the "averaged" long term behavior of the system.

The canonical transformations of variables shown next serve the purpose of suppressing all short period terms in H. We assume that α and β can be decomposed into short period and long period (α', β') components, i.e.,

$$\alpha \rightarrow \alpha' + \alpha_{s.p.}$$

$$\beta \rightarrow \beta' + \beta_{s.p.}$$

Introducing $X^{(o)}$ into the Hamiltonian H'

$$H' = \frac{1}{2} \omega_o^2 \eta X^2 \cos 2\omega_o t \ (1 + \epsilon) \qquad \text{(F-5)}$$

and rearranging terms gives

$$H' = \frac{\alpha\eta}{2} \left\{ \cos 2\omega_o t(1 + \epsilon) - \frac{1}{2} \cos \left[2\omega_o t(1 + \epsilon) + 2\omega_o(t + \beta) \right] \right.$$
$$\left. - \frac{1}{2} \cos 2\omega_o(\epsilon t - \beta) \right\} \qquad \text{(F-6)}$$

The last term with angular velocity $2\omega_o\epsilon << 2\omega_o$ is of low frequency and thus gives a contribution to the long period part of H'.

Let this term be designated by $\overline{H}'$

$$\overline{H}' = - \frac{\alpha\eta}{4} \cos 2\omega_o(\epsilon t - \beta) \qquad \text{(F-7)}$$

Note that α still contains s.p. terms so that $\overline{H}'$ still is not the final form of the desired long period Hamiltonian.

To obtain the D.E. for α' we introduce a generating function $S(\alpha',\beta,t)$ of the Hamilton-Jacobi equation which, to first order terms can be written in the form

$$S = \alpha'\beta + \eta S_1(\alpha',\beta,t) \tag{F-8}$$

in terms of the new momenta α' and the old coordinates β. Thus

$$\begin{aligned} \alpha &= \frac{\partial S}{\partial \beta} = \alpha' + \eta \frac{\partial S_1}{\partial \beta} \\ \beta' &= \frac{\partial S}{\partial \alpha'} = \beta + \eta \frac{\partial S_1}{\partial \alpha'} \end{aligned} \tag{F-9}$$

relate the old and new coordinates and momenta. α',β' form a canonical set with respect to a long period Hamiltonian K', such that

$$\begin{aligned} \dot{\alpha}' &= -\frac{\partial K'}{\partial \beta'} \\ \dot{\beta}' &= \frac{\partial K'}{\partial \alpha'} \end{aligned} \tag{F-10}$$

where

$$K' = H' + \frac{\partial S}{\partial t} \tag{F-11}$$

The Hamiltonian K' of Eq. (F-11) is treated as a function of the coordinates α',β' and t, after the transformation relations (F-9) are substituted into the right hand side of (F-11).

To linear terms only

$$H'(\alpha,\beta,t) = H'(\alpha' + \eta S_{1\beta},\beta,t) \cong H'(\alpha',\beta,t) + \text{H.O.T.} \tag{F-12}$$

and thus

$$K' \cong H'_{sp}(\alpha',\beta,t) + \overline{H}'(\alpha',\beta,t) + \eta S_{1_t} + \text{H.O.T.} \qquad \text{(F-13)}$$

The function S_1 is chosen in such a way as to eliminate all s.p. terms from (F-13). We thus require

$$\eta S_{1_t} + H'_{sp}(\alpha',\beta,t) = 0 \qquad \text{(F-14)}$$

from which results

$$S_1 = \frac{\alpha'}{8\omega_o(2+\epsilon)} \sin 2\omega_o(2t + \epsilon t + \beta) - \frac{\alpha'}{4\omega_o(1+\epsilon)} \sin 2\omega_o t(1+\epsilon) \qquad \text{(F-15)}$$

Expanding β around β' in $\overline{H}'$ of Eq. (F-13) gives, again to first order

$$K' = \overline{H}'(\alpha',\beta',t) = -\frac{\alpha'\eta}{4} \cos 2\omega_o(\epsilon t - \beta') \qquad \text{(F-16)}$$

with the aid of Eq. (F-7).

Equation (F-16) defines a long period, time dependent, Mathieu type Hamiltonian.

Stability Analysis

The differential equations for α' and β' are summarized by the matrix equation

$$\begin{pmatrix} \dot{\beta}' \\ \dot{\alpha}' \end{pmatrix} = \Phi_o \begin{pmatrix} \partial/\partial\beta' \\ \partial/\partial\alpha' \end{pmatrix} K' \qquad \text{(F-17)}$$

Rather than solve Eq. (F-17) directly for α' and β' it is more convenient to introduce a further generating functions $S^*(\alpha^*,\beta',t)$

so as to eliminate the explicit dependence on time and thereby transform K' to a new Hamiltonian K^* which is a constant of the motion, i.e., $K^* = K^*_o$ = constant.

Let us define a variable β^* by the relation

$$\beta^* = \beta' - \epsilon t \tag{F-18}$$

and then take S^* to be given by

$$S^* = \alpha^*(\beta' - \epsilon t) \tag{F-19}$$

Since

$$S^*_{\alpha^*} = \beta' - \epsilon t = \beta^*$$

and

$$S_{\beta'} = \alpha' = \alpha^*$$

the two variables α^* and β^* are canonically related to the new Hamiltonian K^* which becomes

$$K^* = K' + S^*_t = K' - \epsilon\alpha^* = -\frac{\alpha^*\eta}{4}\cos 2\omega_o\beta^* - \epsilon\alpha^* \tag{F-20}$$

That K^* is an integral constant of the motion is evident from the fact that

$$\frac{dK^*}{dt} = \dot{K}^* = \frac{\partial K^*}{\partial \alpha^*}\dot{\alpha}^* + \frac{\partial K^*}{\partial \beta^*}\dot{\beta}^* + \cancelto{0}{\frac{\partial K^*}{\partial t}}$$

$$= \text{by Hamilton's equation} = \frac{\partial K^*}{\partial \alpha^*}\left(-\frac{\partial K^*}{\partial \beta^*}\right) + \frac{\partial K^*}{\partial \beta^*}\left(\frac{\partial K^*}{\partial \alpha^*}\right) = 0$$

$$\therefore K^* = K^*_o = \text{constant of the motion}$$

The stability or instability of systems governed by Hamiltonians of the form (F-20) can readily be established based on a comparison of the magnitude of the coefficients of α^* and α^* cos $2\omega_o\beta^*$ in (F-20).

The relative magnitude required of these coefficients for instability or stability to exist can be determined as shown below, and the conclusions then checked by referring to the known stability regions of the Mathieu plane.

From Eq. (F-20) we obtain the differential equations for α^*

$$\dot{\alpha}^* = - \frac{\partial K^*}{\partial \beta^*} = - \frac{\omega_o \alpha^* \eta}{2} \sin 2\omega_o \beta^* \qquad \text{(F-21)}$$

and squaring,

$$\dot{\alpha}^{*2} = \frac{\omega_o^2 \alpha^{*2} \eta^2}{4} \left\{ 1 - \left[\frac{4}{\alpha^* \eta} K_o^* + \frac{4\epsilon}{\alpha^* \eta} \alpha^* \right]^2 \right\} \qquad \text{(F-22)}$$

after $\sin^2 2\omega_o\beta^*$ is replaced from Eq. (F-20) and the constancy of K^* is made use of.

The condition necessary for $\dot{\alpha}^*$ to vanish is obtained by setting the right hand side of (F-22) equal to zero, i.e., at the intersection of the two lines.

$$\left.\begin{aligned} & y = \pm \alpha^* \\ \text{and} \quad & \\ & y = \frac{4}{\eta} K_o^* + \frac{4\epsilon}{\eta} \alpha^* \end{aligned}\right\} \qquad \text{(F-23)}$$

This is shown in the next sketch, Fig. (F-1).

From this sketch we see that for $\alpha_o^* > \alpha_{cr}^*$ and $\dot{\alpha}_o^* > 0$ the variation of α^* is bounded by the lines $y = \pm \alpha^*$ if $4\epsilon/\eta > 1$, thus implying a stable motion, while if $4\epsilon/\eta < 1$, α^* grows without limit.

Hence, if

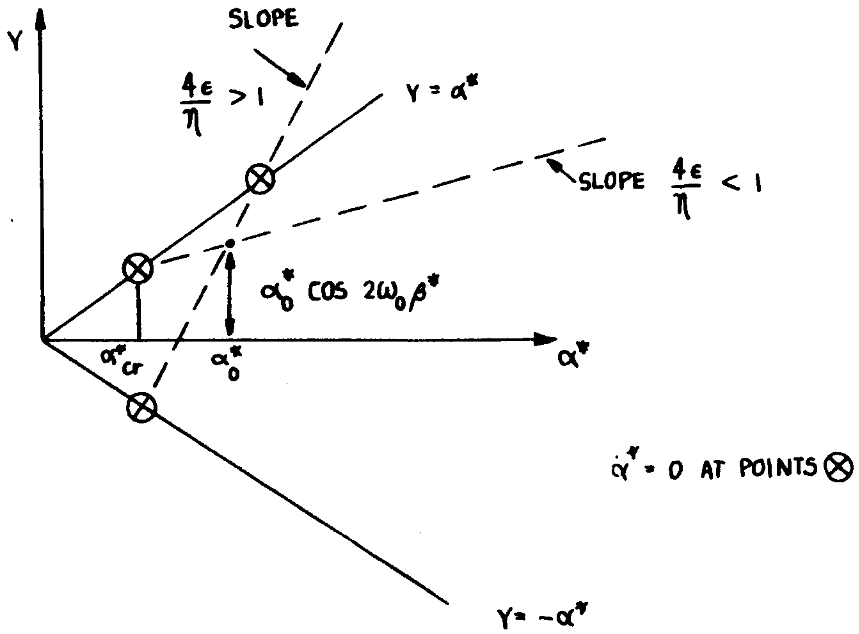


Fig. (F-1): Stability Conditions for Mathieu Type Hamiltonians

$\frac{4\epsilon}{\eta} > 1 \rightarrow$ stability exists (i.e., x = bounded)

$\frac{4\epsilon}{\eta} < 1 \rightarrow$ instability exists (i.e., $x \rightarrow \infty$ as $t \rightarrow \infty$)

This leads to the conclusion that the motion is unstable if in the Hamiltonian K^* the coefficient of α^* is smaller than the coefficient of $\alpha^* \cos 2\omega_0\beta^*$.

The above conclusion is also borne out by considering the Mathieu Equation in the standard form,

$$\frac{d^2v}{dz^2} + (a - 2q \cos 2z)v = 0 \qquad \text{(F-24)}$$

Referring to Eq. (F-1) and introducing a new time variable τ

$$\tau = \omega_o t(1 + \epsilon) + \frac{\pi}{2}$$

and (F-25)

$$\frac{d}{dt} = \omega_o (1 + \epsilon) \frac{d}{d\tau}$$

Equation (F-1) reduces to (F-24) if

$$a = \frac{1}{(1 + \epsilon)^2}$$

and (F-26)

$$q = \frac{\eta}{2(1 + \epsilon)^2} = \frac{\eta}{2} a$$

The stability boundaries of the Mathieu plane (q,a) in the vicinity of the region a ≅ 1 are shown below (see for instance p. 114 of Ref. 11).

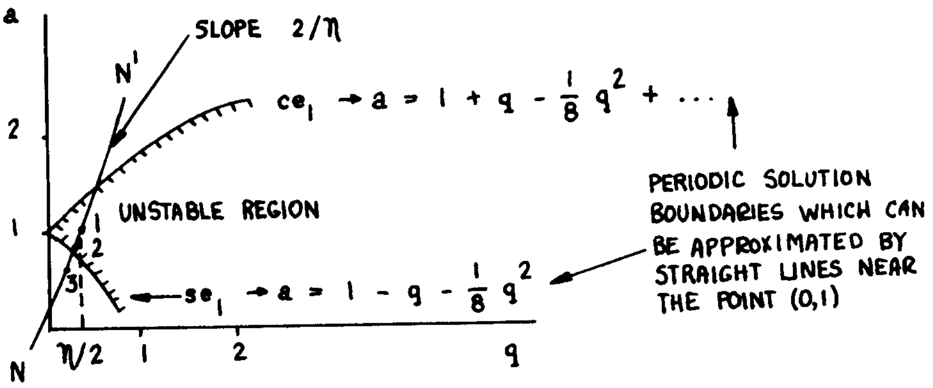


Fig. (F-2): Stability Boundaries in Mathieu Plane (q,a)

On se_1 the slope $(da/dq)_{q \to 0} \to -1$.

The slope of line N - N' is, from Eq. (F-26), $(da/dq)_{N-N'} = 2/\eta$, and $q = \eta/2$ for $a = 1$ (pt. 1).

As the value of (a) changes from 1 to $1/(1+\epsilon)^2 \simeq 1 - 2\epsilon + 3\epsilon^2$, point 1 translates along line N - N$'$ to either point 2 or 3, depending on whether the solution remains unstable (pt. 2) or enters the stable region (pt. 3).

Comparing the values of q on se_1 and N-N$'$ for the same change $\Delta a \cong -2\epsilon$ + H.O.T. we have

<u>On se_1</u> $$q_{se} \cong \left(\frac{dq}{da}\right)_{q=0} \Delta a = (-1)(-2\epsilon) = 2\epsilon$$

<u>On N-N$'$</u> $$q_N \cong \frac{\pi}{2} + \left(\frac{dq}{da}\right)_{N-N} \Delta a = \frac{\pi}{2} - \pi\epsilon$$

Hence, if $q_N > q_{se}$, i.e., if $\pi/2 - \overset{\text{H.O.T.}}{\cancel{\pi\epsilon}} > 2\epsilon$, or $4\epsilon/\pi < 1$, point 1 moves to point 2 and indicates an unstable solution. This is in agreement with the conclusion reached earlier via the Hamiltonian approach.

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