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VIBRATIONAL CHARACTERISTICS
OF LINEAR SPACE FRAMES

March 1968

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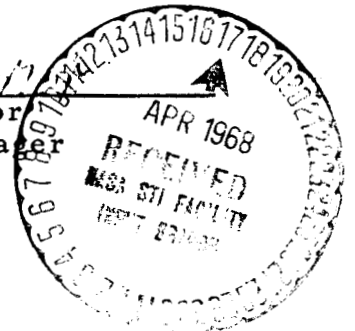
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FOREWORD

The work described in this report was performed by Lockheed Missiles & Space Company, Huntsville Research & Engineering Center, for the George C. Marshall Space Flight Center of the National Aeronautics and Space Administration under Contract NAS8-21095.

The work was administered under the direction of the Aero-Astroynamics Laboratory, NASA/MSFC, with Dr. George F. McDonough as Contracting Officer Representative.

SUMMARY

This report describes a method of analysis and digital program for computing the modes and frequencies of arbitrary linear space frames. Frame members may be either uniform beams or Timoshenko beams with arbitrarily varying mass and stiffness properties. Provision is also included for rigid links, lumped masses, concentrated springs, and other features useful in mathematically modeling complicated linear structures.

The program is applicable to many types of problems, including, for example, spacecraft clusters such as those currently planned for the Apollo Applications Program, and clustered vehicles such as Titan III, Saturn IB, and proposed modified Saturns.

The program is "automatic" in the sense that communications — both input and output — are concise. Input data consists of minimum definitions of particular problems (e.g., joint positions, member properties, restraint conditions, etc.). Output data contains complete solution information, including computer-generated plots of mode shapes. Several examples are presented of solutions computed by the program.

The method upon which the program is based is applicable to more general classes of finite-element structures.

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Section 1
INTRODUCTION

References 1, 2, 4, 5, 6 and 8 discuss studies of the modal characteristics and, in some cases, the dynamic response of complicated linear structures. The same general method was used in each of these studies. Essentially, it is as follows:

- The structure is modeled as an assemblage of several interconnected pieces. Each piece may be a relatively complicated structure modeled by finite element methods, etc.
- The state of the structure is represented by a relatively small number of generalized coordinates, typically including:
 1. The displacement and/or rotation components of the boundary points or surfaces interconnecting the pieces.
 2. The coefficients of displacement functions representing the deformations of individual pieces. Various displacement functions are used, including:
 - a. Free vibrational mode shapes corresponding to particular boundary conditions for individual pieces.
 - b. Displacement fields associated with particular static loadings.

Several such functions are usually employed for each piece. They may be determined in various ways; e.g., closed-form solution, finite element methods, static and dynamic tests, etc.

- The kinetic and potential energies of the structure are evaluated as quadratic forms in the generalized coordinates:

$$T = \frac{1}{2} \dot{Q}^* M \dot{Q},$$

$$V = \frac{1}{2} \dot{Q}^* K Q.$$

Provided the elements of M and K are constant, Lagrange's equation gives:

$$M \ddot{Q} + K Q = F.$$

In the absence of dissipative effects and active externally-applied forces, the generalized force vector F is zero, and solution vectors of the form $Q = \sin \omega t X$ yield the linear,

small-vibration eigenproblem, $\omega^2 MX - KX = 0$.

Several well-known numerical methods may be used to compute precise solutions to equations of this type, provided M and K are not poorly conditioned or of high order.

The procedure outlined above may be viewed as a generalized Rayleigh-Ritz technique. As indicated by the excellent comparisons with experimental results reported in References 4 and 8, very accurate results may be obtained in this way, provided the displacement functions are well-selected.* However, substantial "restraint" errors may result if the displacement functions are not well-chosen, as illustrated by the example given in Section 6.5 of Reference 4.

Relative to finite element or finite difference methods (as usually applied), this procedure affords a substantial reduction in eigenproblem order; however, the order may still be moderately large for complicated structures. For example, the Saturn V analysis described in Reference 8 involved the solution of a 147 degree-of-freedom eigenproblem. Although accurate solutions of well-conditioned eigenproblems this size or larger may be computed by using existing methods,** practical difficulties (principally involving the interrelated factors of numerical accuracy, computer execution time, machine storage capacity, complexity of method, etc.) sharply raise costs of solution as the number of degrees of freedom is increased.

* A good choice of displacement functions is equally, if not more, important in forced vibration analysis; in which case function selection must be based not only upon the system boundary conditions but upon the character of the externally applied forces as well. If coefficients of free vibration modes of the entire system are being used as coordinates in a response analysis, it may under some conditions be highly desirable to use "mixed" modes (that is modes corresponding to several different system boundary conditions, e.g., for a beam-like structure, some free-free modes and some cantilever modes). The fact that such functions are not orthogonal usually adds only slightly to the complexity of the analysis, while significant opportunities are often afforded for improving the accuracy obtainable with a given number of degrees of freedom.

** See e.g., Reference 7 (Wilkinson).

This report presents a modified version of the previously outlined procedure which is applicable to the general linear frame modes-and-frequencies problem. Degree-of-freedom requirements are substantially reduced through the use of whole-structure deformation functions. In addition, a method is described for determining a sequence of improved displacement functions leading to nominally small errors in the solutions. A general purpose digital program was developed to implement this formulation for arbitrary linear space frames. The program includes provision for:

- non-uniform beams,
- rigid links offsetting member end points from joints,
- rigid, pinned or elastic connections of member end points to joints (or rigid links),
- automatic generation of a sequence of improved displacement functions, and
- additional lumped masses at the joints.

As restraint conditions, an arbitrary set of joint motion components may be set identically equal to zero.

Features cited above were incorporated into the program primarily to facilitate mathematical modeling of unusual aerospace structures (such as the orbital workshop-cluster planned for the Apollo Applications Program).

If the lowest N modes and frequencies of the structure are required, the procedure executed by the program involves the solution of eigenproblems of order $1, 2, \dots, N$. Accordingly, if only low-frequency modes are required, only low-order, very well-conditioned eigenproblems are solved.

The procedure implemented by the program to automatically compute successions of improved sets of displacement functions is essentially a generalization of the well-known Stodola-Vianello iterative method. Each modal approximation is used to compute an equivalent static loading for the entire structure from which a new displacement function is calculated for use in the succeeding approximation. The key to the computational efficiency

of the program is the ability of its static analysis routines to economically compute displacement functions. Results obtained by the program indicate that the techniques used are well-suited for extension to more general types of finite-element structures.

Examples are presented of solutions computed by the program.

Elements generally consist of a "member" and two "rigid links", as shown on Figure 1. The rigid links are included in the formulation to facilitate mathematical modeling of certain kinds of real joints.

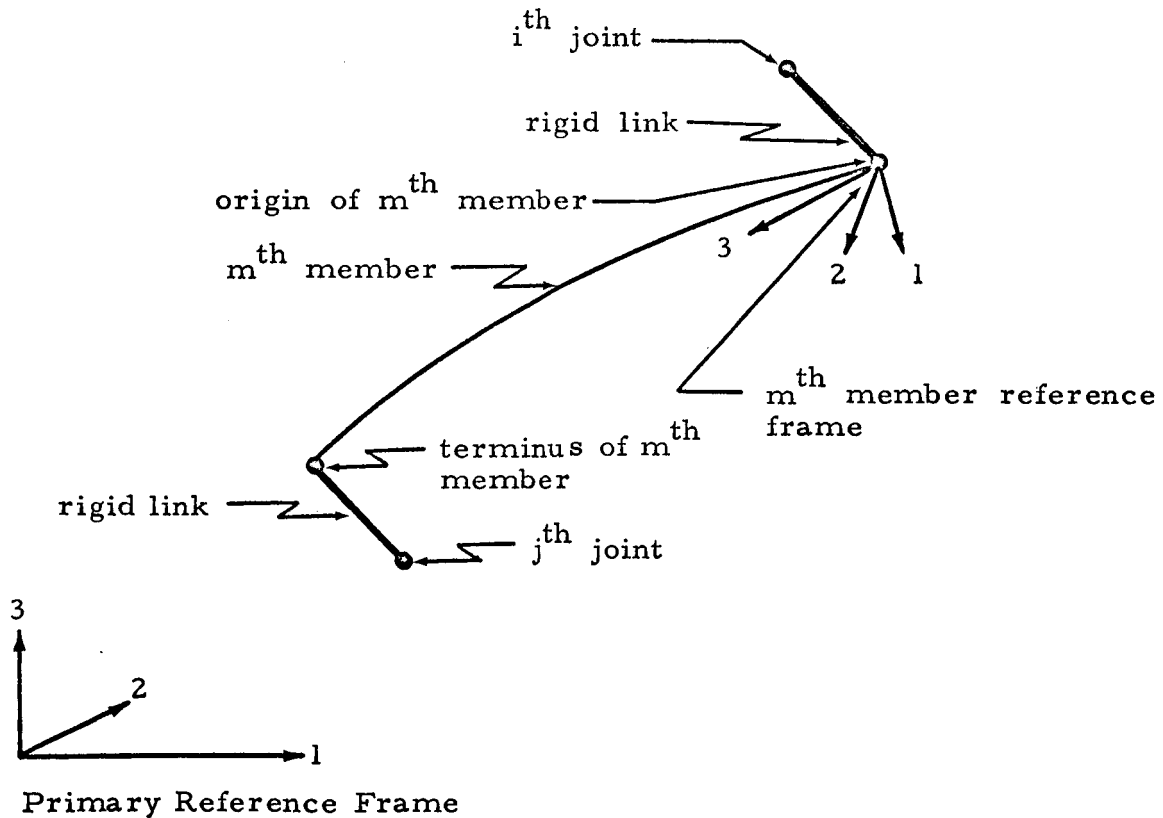


Figure 1 - Typical Frame Element

Section 2
DEFINITIONS

2.1 COORDINATE SYSTEMS

Part of a typical frame, consisting of an array of "joints" interconnected by "elements" is shown on Figure 2. Joints, which are represented geometrically as points, are regarded as very small rigid bodies into which the ends of elements are rigidly embedded.

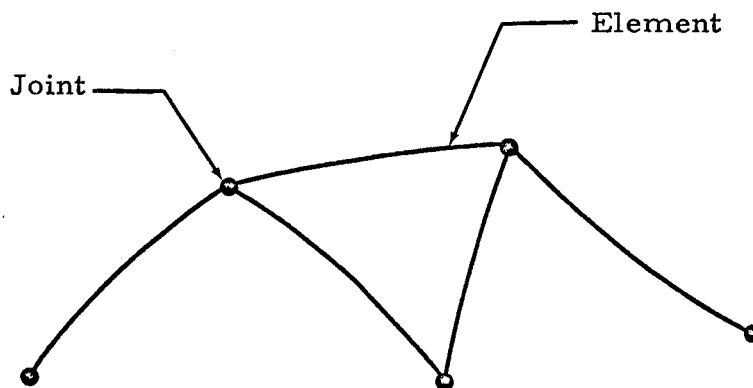


Figure 2 - Typical Array of Joints and Elements

One of the end points of each member is designated the member "origin;" the opposite end is the member "terminus." Position coordinates (relative to the arbitrarily selected "primary reference frame")* of joints and member end points will be represented by the following symbols:

$$X_k^l \equiv k\text{-direction position coordinate of the } l^{\text{th}} \text{ joint.}$$

*This is a right-hand rectangular system.

$Y_k^{mn} \equiv$ k-direction position coordinate of the n^{th} end point of the m^{th} member ($n = 1 \rightarrow$ origin; $n = 2 \rightarrow$ terminus).

As shown on Figure 1, a distinct "member reference frame" is associated with each member.* The orientations of these systems relative to the primary reference frame are represented by the direction cosine matrix defined below. Where R_{ij}^m is the cosine of the angle between the i^{th} axis of the m^{th} member reference frame and the j^{th} axis of the primary reference frame

$$R^m \equiv \begin{bmatrix} R_{11}^m & R_{12}^m & R_{13}^m \\ R_{21}^m & R_{22}^m & R_{23}^m \\ R_{31}^m & R_{32}^m & R_{33}^m \end{bmatrix} .$$

The D^{mn} matrices defined below represent the rigid link offsets of member end points from joints. It is assumed in this definition that the origin and terminus of the m^{th} member are connected by rigid links to the i^{th} and j^{th} joints, respectively.

$$D^{m1} \equiv \begin{bmatrix} 0 & (Y_3^{m1} - X_3^i) & -(Y_2^{m1} - X_2^i) \\ -(Y_3^{m1} - X_3^i) & 0 & (Y_1^{m1} - X_1^i) \\ (Y_2^{m1} - X_2^i) & -(Y_1^{m1} - X_1^i) & 0 \end{bmatrix} .$$

$$D^{m2} \equiv \begin{bmatrix} 0 & (Y_3^{m2} - X_3^j) & -(Y_2^{m2} - X_2^j) \\ -(Y_3^{m2} - X_3^j) & 0 & (Y_1^{m2} - X_1^j) \\ (Y_2^{m2} - X_2^j) & -(Y_1^{m2} - X_1^j) & 0 \end{bmatrix} .$$

*All of these are right-hand rectangular systems.

2.2 MEMBER DISPLACEMENT FUNCTIONS

As illustrated on Figure 3, the total direction- i cross-section displacement and rotation components* of a member are u_i and ϕ_i ($i = 1, 2, 3$), respectively.

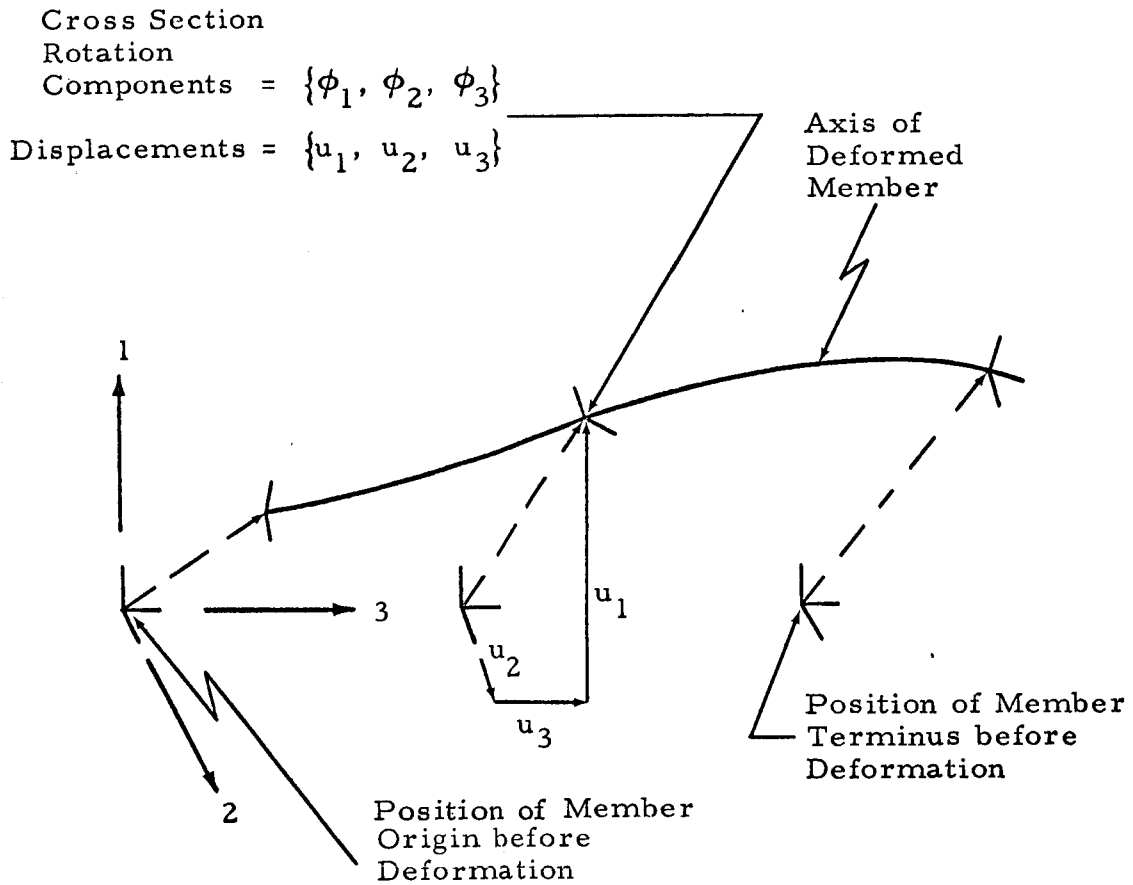


Figure 3 - Member in Deformed State

* Member reference frame; right-hand rule for rotations.

In the solution procedures described in Sections 3, 4 and 5, member deformations are represented by linear combinations of displacement functions. That is, where s is a position coordinate along the m^{th} member's central axis, and the n displacement function coefficients (the c 's) are linearly related to the generalized coordinates,

$$\begin{aligned} u_i^m(s) &= \sum_{j=1}^n c_j^m v_{ij}^m(s), \\ \text{and} \\ \phi_i^m(s) &= \sum_{j=1}^n c_j^m \theta_{ij}^m(s), \quad \text{for } i = 1, 2, 3. \end{aligned} \quad (2-1)$$

Particular displacement functions (the v 's and θ 's) and the associated generalized coordinates are discussed in later sections.

2.3 MEMBER ENERGIES

Where, for the m^{th} member,

$\mu^m(s)$ = mass per unit length,

$\rho_i^m(s)$ = (effective) cross-section mass moment of inertia about the i^{th} member axis,

$E I_i^m$ = bending stiffness about i^{th} axis,

$k_i G A^m$ = transverse shear stiffness in i^{th} direction,

$E A^m$ = axial stiffness,

$G J^m$ = torsional stiffness, and

$$f' = \frac{\partial f}{\partial s},$$

the kinetic and potential energies of the member are:

$$T^m = \frac{1}{2} \int_{\text{length}} \left\{ \mu^m(s) \left[(\dot{u}_1^m)^2 + (\dot{u}_2^m)^2 + (\dot{u}_3^m)^2 \right] + \rho_1^m(s) (\dot{\phi}_1^m)^2 + \rho_2^m(s) (\dot{\phi}_2^m)^2 + \rho_3^m(s) (\dot{\phi}_3^m)^2 \right\} ds, \text{ and}$$

$$V^m = \frac{1}{2} \int_{\text{length}} \left\{ EI_2^m(s) (\phi_2^m)'^2 + k_1 GA^m(s) (u_1^{m'} - \phi_2^m)^2 + EI_1^m(s) (\phi_1^m)'^2 + k_2 GA^m(s) (u_2^{m'} + \phi_1^m)^2 + GJ^m(s) (\phi_3^m)'^2 + EA^m(s) (u_3^{m'})^2 \right\} ds. \quad (2-2)$$

Substitution of Equations (2-1) into (2-2) gives the energies as quadratic forms in the displacement function coefficients:

$$T^m = \frac{1}{2} \int_{\text{length}} \left\{ \mu^m(s) \sum_{i=1}^3 \left[\sum_{j=1}^n \dot{c}_j^m v_{ij}^m(s) \right]^2 + \sum_{i=1}^3 \rho_i^m(s) \left[\sum_{j=1}^n \dot{c}_j^m \theta_{ij}^m(s) \right]^2 \right\} ds,$$

$$V^m = \frac{1}{2} \int_{\text{length}} \left\{ EI_2^m(s) \left[\sum_{j=1}^n c_j^m \theta_{2j}^m(s) \right]^2 + k_1 GA^m(s) \left[\sum_{j=1}^n c_j^m \left[v_{1j}^m(s) - \theta_{2j}^m(s) \right] \right]^2 + \dots + EA^m(s) \left[\sum_{j=1}^n c_j^m v_{3j}^m(s) \right]^2 \right\} ds, \quad (2-3)$$

which may be written as:

$$T^m = \frac{1}{2} \begin{bmatrix} \cdot^m c_1 & \cdot^m c_2 & \cdot & \cdot & \cdot^m c_n \end{bmatrix} \begin{bmatrix} M_{11}^m & M_{12}^m & \cdot & \cdot & \cdot \\ & M_{22}^m & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ \text{Symmetric} & & & & M_{nn}^m \end{bmatrix} \begin{bmatrix} \cdot^m c_1 \\ \cdot^m c_2 \\ \cdot \\ \cdot \\ \cdot^m c_n \end{bmatrix} ,$$

and

$$V^m = \frac{1}{2} \begin{bmatrix} c_1^m & c_2^m & \cdot & \cdot & c_n^m \end{bmatrix} \begin{bmatrix} K_{11}^m & K_{12}^m & \cdot & \cdot & \cdot \\ & K_{22}^m & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ \text{Symmetric} & & & & K_{nn}^m \end{bmatrix} \begin{bmatrix} c_1^m \\ c_2^m \\ \cdot \\ \cdot \\ c_n^m \end{bmatrix} , \quad (2-4)$$

where, for $j=1$ through n and $l=1$ through n

$$M_{jl}^m \equiv \sum_{i=1}^3 \int_{\text{length}} \left[\mu^m(s) v_{ij}^m(s) v_{il}^m(s) + \rho_i^m(s) \theta_{ij}^m(s) \theta_{il}^m(s) \right] ds ,$$

and

$$\begin{aligned}
 K_{jl}^m \equiv & \int_{\text{length}} \left\{ E I_2^m(s) \theta_{2j}^{m'}(s) \theta_{2l}^{m'}(s) + k_1 G A^m(s) \left[v_{1j}^{m'}(s) - \theta_{2j}^m(s) \right] \left[v_{1l}^{m'}(s) - \theta_{2l}^m(s) \right] \right. \\
 & + E I_1^m(s) \theta_{1j}^{m'}(s) \theta_{1l}^{m'}(s) + k_2 G A(s) \left[v_{2j}^{m'}(s) + \theta_{1j}^m(s) \right] \left[v_{2l}^{m'}(s) + \theta_{1l}^m(s) \right] \\
 & \left. + G J^m(s) \theta_{3j}^{m'}(s) \theta_{3l}^{m'}(s) + E A^m(s) v_{3j}^{m'}(s) v_{3l}^{m'}(s) \right\} ds . \quad (2-5)
 \end{aligned}$$

Section 3

COEFFICIENTS OF WHOLE-STRUCTURE DISPLACEMENT
FUNCTIONS AS GENERALIZED COORDINATES

The following discussion deals with the use of whole-structure displacement functions corresponding to particular static loadings.* The use of such functions is practical because of the availability of accurate and economical digital programs for analyzing statically-loaded space frames.

Consider, for example, the specific set of displacement functions shown on Figure 4. (These particular functions have no special significance; they were arbitrarily selected for illustrative purposes.) We use as generalized coordinates the four whole-structure displacement function coefficients q_i corresponding to the illustrated static loadings. The member displacement function coefficients defined in Section 2.2 can be identified simply as follows:

$$\begin{aligned} c_1^m &= q_1, \\ c_2^m &= q_2, \\ c_3^m &= q_3, \text{ and} \\ c_4^m &= q_4. \end{aligned} \tag{3-1}$$

Highly automated methods are available for analyzing arbitrary statically-loaded space frames; accordingly, all of the member displacement and rotation functions, v_{ij}^m and θ_{ij}^m associated with each q_j are easily computed. Reference 3 presents a general formulation of linear static

* Any whole-structure deformation function corresponds to a unique applied loading, of course. This interpretation of the genesis of whole-structure deformation functions will, however, prove convenient in subsequent discussions.

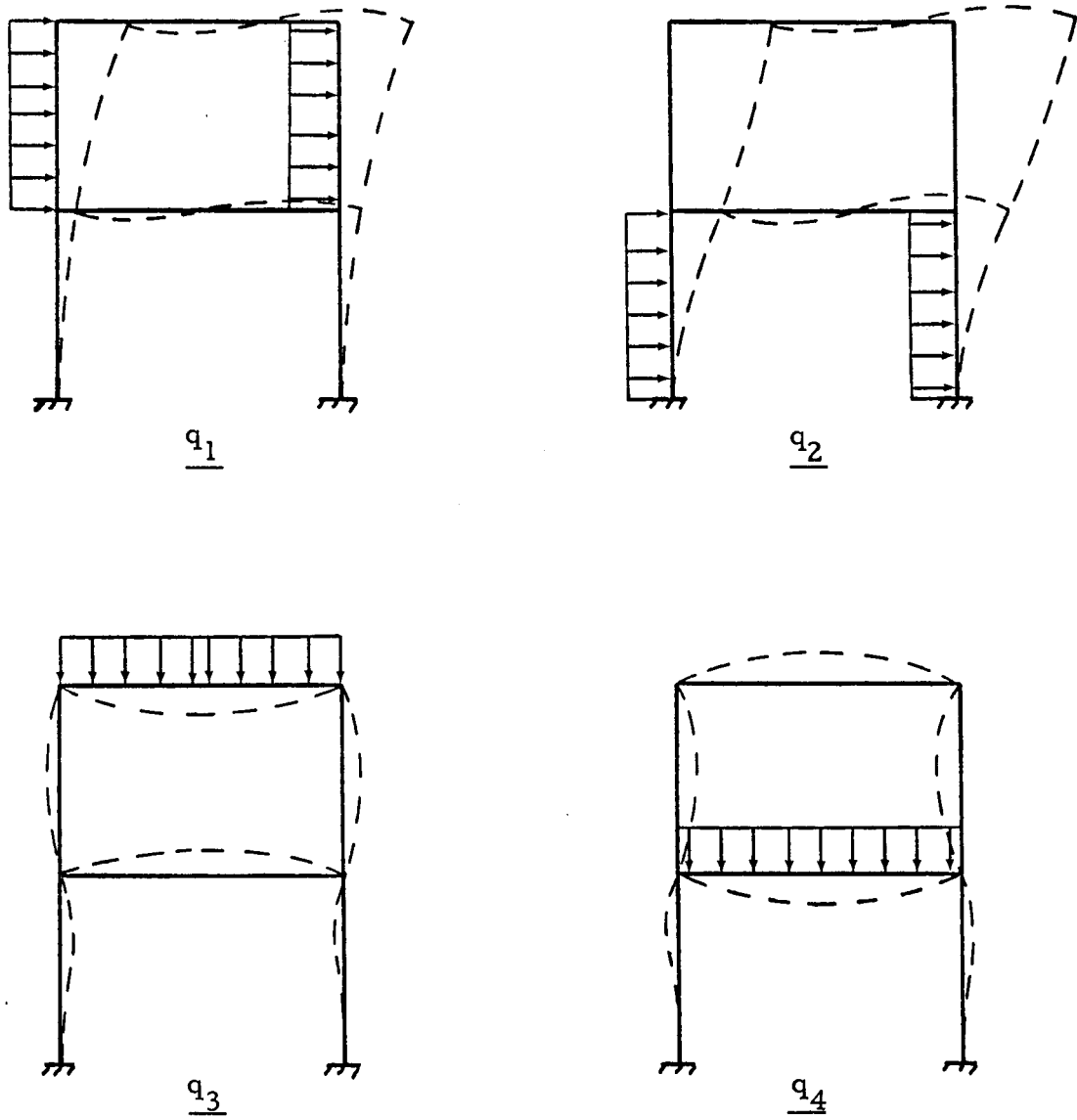


Figure 4 - Example: Static Loadings Corresponding to Whole Structure Deformation Functions Associated with Four Generalized Coordinates, q_1 , q_2 , q_3 and q_4

frame analysis and describes a digital program* which executes solutions to a wide class of problems. After the v 's and θ 's have been evaluated, the M_{jl}^m 's and K_{jl}^m 's of Equations (2-4) may be computed from Equation (2-5).

Where

$$M_{jl} \equiv \sum_{\text{all } m} M_{jl}^m \quad \text{and} \quad K_{jl} \equiv \sum_{\text{all } m} K_{jl}^m, \quad (3-2)$$

the energies of the entire system are:

$$T = \sum_{\text{all } m} T^m = \frac{1}{2} \begin{bmatrix} \dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \dot{q}_4 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ & M_{22} & M_{23} & M_{24} \\ & & M_{33} & M_{34} \\ \text{Symmetric} & & & M_{44} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix},$$

and

$$V = \sum_{\text{all } m} V^m = \frac{1}{2} \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix} \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ & K_{22} & K_{23} & K_{24} \\ & & K_{33} & K_{34} \\ \text{Symmetric} & & & K_{44} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}. \quad (3-3)$$

Instead of using Equations (2-5) and (3-2) to compute the K_{jl} 's, advantage can be taken of knowledge of the static loading distributions corresponding to the whole-structure deformation functions as follows:

The distributed forces and moments acting over the members to comprise the static loading which will produce the whole-structure deformation function associated with the generalized coordinate q_j will be represented by the symbols \mathcal{F}_{ij}^m and \mathcal{M}_{ij}^m , respectively (m^{th} member, direction- i

*This program is incorporated in the dynamic analysis program discussed in Section 6.

in the member reference frame). From external work considerations,

$$K_{jl} = \sum_{\text{all } m} \int_{\text{length}} \left[\sum_{i=1}^3 \mathcal{F}_{ij}^m(s) v_{il}^m(s) + \mathcal{L}_{ij}^m(s) \theta_{il}^m(s) \right] ds . \quad (3-4)$$

It will be convenient to re-write Equations (3-3) in compact form as:

$$2T = \dot{Q}^* M \dot{Q}, \quad \text{and} \quad 2V = Q^* K Q . \quad (3-5)$$

In the absence of dissipative effects and active externally applied forces, Lagrange's equation is

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} + \frac{\partial V}{\partial q_i} = 0 . \quad (3-6)$$

Accordingly, $M\ddot{Q} + KQ = 0$, and solutions of the form $Q = \sin\omega t X$ yield the eigenproblem,

$$\omega^2 M X - K X = 0 . \quad (3-7)$$

Several well-known numerical methods are suitable for solving Equation (3-7); provided it is not of high order nor poorly conditioned.* For the example illustrated on Figure 4, the four solutions to Equation (3-7) would be approximately as follows (depending on the member properties, etc.):

$$X_1 = \text{first lateral mode} \approx \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} ,$$

* Poorly conditioned matrices are frequently a consequence of poor mathematical modeling; e.g., representing a rigid connection by an elastic spring many orders of magnitude stiffer than the connected elements themselves.

$$X_2 = \text{second lateral mode} \approx \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

$$X_3 = \text{first vertical mode} \approx \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and}$$

$$X_4 = \text{second vertical mode} \approx \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

Section 4

REPETITIVE PROCEDURE FOR OBTAINING A SEQUENCE
OF IMPROVED DISPLACEMENT FUNCTIONS

In the following discussion, it is assumed that very accurate approximations of the lowest $N - 1$ frequencies and mode shapes of the system have already been determined, and that an initial approximation* of the N^{th} mode is known. To calculate a precise approximation of the N^{th} mode, a sequence of N -order eigenproblems of the type discussed in the preceding section are solved. Each solution produces an improved approximation of the N^{th} mode. The first $N - 1$ whole-structure deformation functions used in each of these solutions are the previously-determined lowest $N - 1$ system modes, and the N^{th} function is the most recently-obtained approximation of the N^{th} mode. The eigenvector associated with the highest-frequency solution of the ℓ^{th} of these N -order eigenproblems will be represented as:

$$\begin{bmatrix} X_{1N}^{(\ell)} \\ X_{2N}^{(\ell)} \\ \cdot \\ \cdot \\ X_{NN}^{(\ell)} \end{bmatrix} .$$

The superscript (ℓ) indicates that this vector was obtained using the ℓ^{th} approximation of the N^{th} mode (this notation will have the same meaning when used elsewhere in this section). Corresponding to this solution, the ℓ^{th} interim approximation of the N^{th} system mode is comprised of the set

*Initial approximations, which are discussed in Section 5, may be very coarse and still not adversely affect the solution process.

of member cross-section displacement and rotation functions defined below.

$$\begin{aligned}\tilde{v}_{iN}^{m(\ell)} &\equiv \sum_{j=1}^N X_{jN}^{(\ell)} v_{ij}^{m(\ell)} \\ \tilde{\theta}_{iN}^{m(\ell)} &\equiv \sum_{j=1}^N X_{jN}^{(\ell)} \theta_{ij}^{m(\ell)} .\end{aligned}\tag{4-1}$$

From these interim functions, a set of distributed force and moment loadings of the members are computed as follows:

$$\begin{aligned}\text{Distributed force intensity} &= \mathcal{F}_{iN}^{m(\ell)} \equiv \mu_i^m \tilde{v}_{iN}^{m(\ell)} \\ \text{Distributed moment intensity} &= \mathcal{M}_{iN}^{m(\ell)} \equiv \rho_i^m \tilde{\theta}_{iN}^{m(\ell)}\end{aligned}\tag{4-2}$$

The deformation of the structure corresponding to this static loading is the $(\ell + 1)$ -th approximation of the N^{th} system mode.

It is interesting to note that since the first $N-1$ whole-structure displacement functions are the previously-calculated final approximations of the lower system modes, all of the off-diagonal elements of M and K except those in the last column and row are extremely small compared with diagonal elements of the corresponding row and/or column.

Section 5

INITIAL APPROXIMATIONS OF SYSTEM MODES

5.1 GENERAL

Initial approximations of system modes are required for the repetitive procedure outlined in Section 4 to be executed. In Section 5.2, a method is described for obtaining relatively accurate sets of approximate solutions. However, recent experience with the digital program described later in this report indicates that generation and use of a set of accurate initial approximations of the system's modes is not necessary; nor, from the standpoint of computational efficiency, is it desirable. Instead, convergence of the repetitive process previously described has been found to be so rapid that the same whole-structure displacement function — typically one associated with a simple static loading of the structure — may serve quite well as the initial approximation of each of the several system modes to be calculated. Consider the plane-frame example illustrated on Figure 4. The structure was analyzed using the structural deformation corresponding to the static loading illustrated on Figure 5 as the initial approximation of the first mode, then the second, and so-on through the fourth system mode. Frequency approximations obtained at successive steps in the repetitive process are summarized in Table 1 (initial solutions in the top row, first improved solution in the next row, etc.). Figures 6 through 9 are plots of the final approximations of the system modes (corresponding to the last row of frequencies in Table 1).

Table 1

Frequency Approximations

<u>1st Mode</u>	<u>2nd Mode</u>	<u>3rd Mode</u>	<u>4th Mode</u>
7.0233337	13.885242	33.963648	38.535649
3.6164171	12.262470	21.322420	28.435214
3.5823153	12.226653	19.024036	23.568244
3.5820643	12.222883	18.659281	22.949570
3.5820623	12.222252	18.526616	22.892897
3.5820624	12.222137	18.469786	22.887886
3.5820625	12.222115	18.445507	22.887443
3.5820625	12.222111	18.435226	22.887406

All members are tubes,
Inside diameter = 1.0 in.,
Outside diameter = 1.5 in.

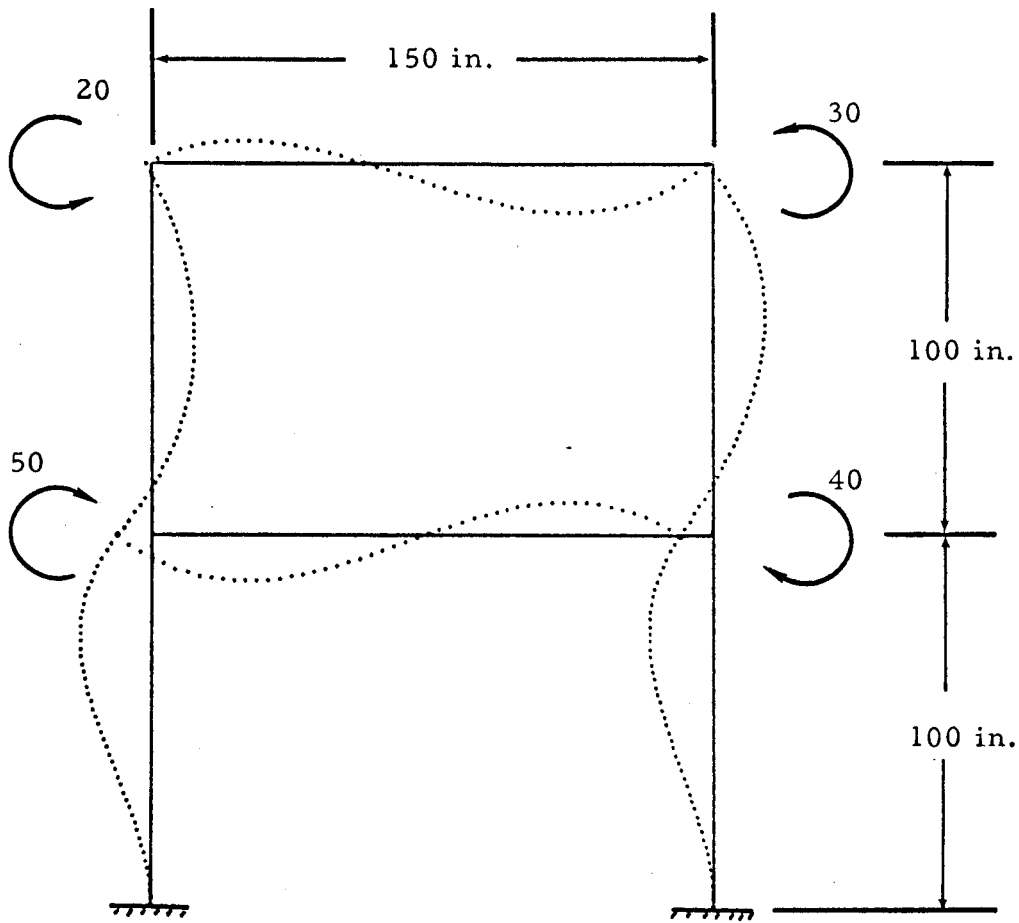


Figure 5 - Plane-Frame Example

MODE NUMBER 1
ITERATION NUMBER 8
FREQUENCY = 3.5820X10⁰⁰ CPS

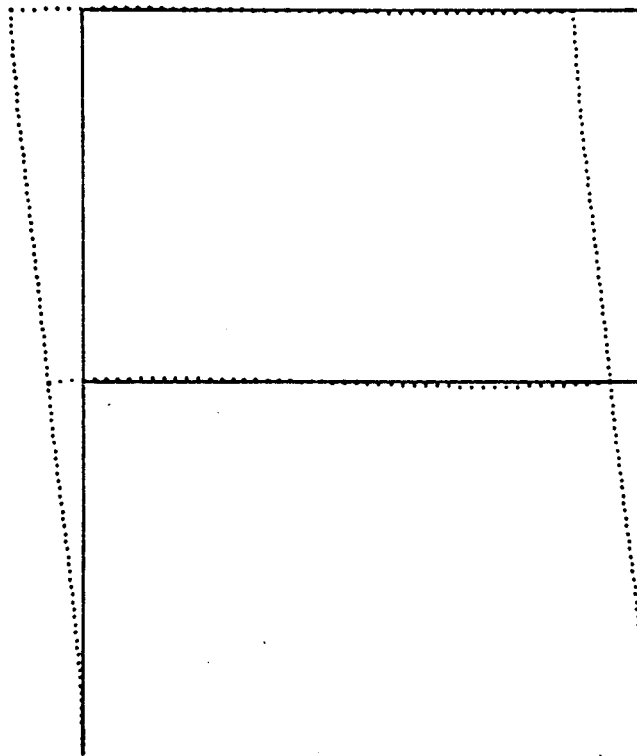


Figure 6

MODE NUMBER 2
ITERATION NUMBER 8
FREQUENCY = $1.2222 \times 10^{+01}$ CPS

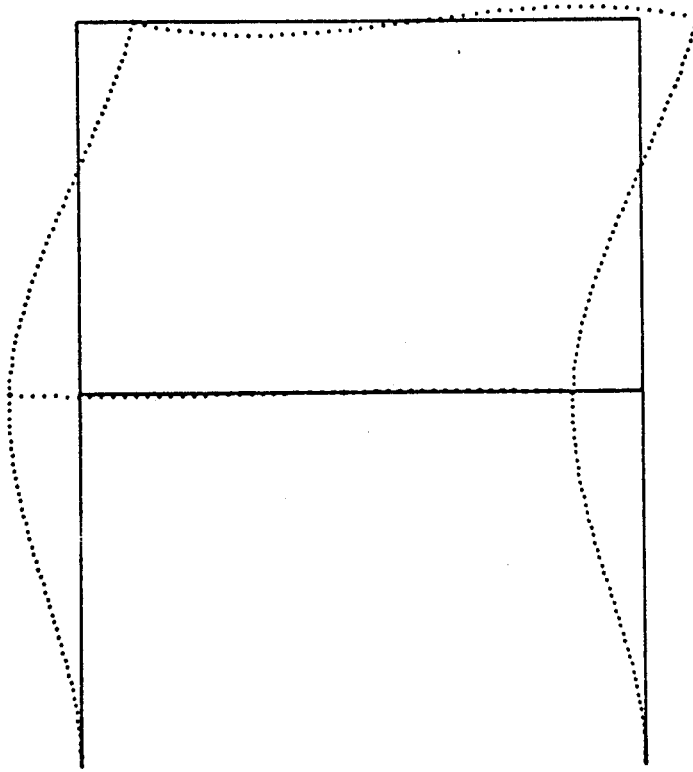


Figure 7

MODE NUMBER 3
ITERATION NUMBER 0
FREQUENCY = $1.8435 \times 10^{+01}$ CPS

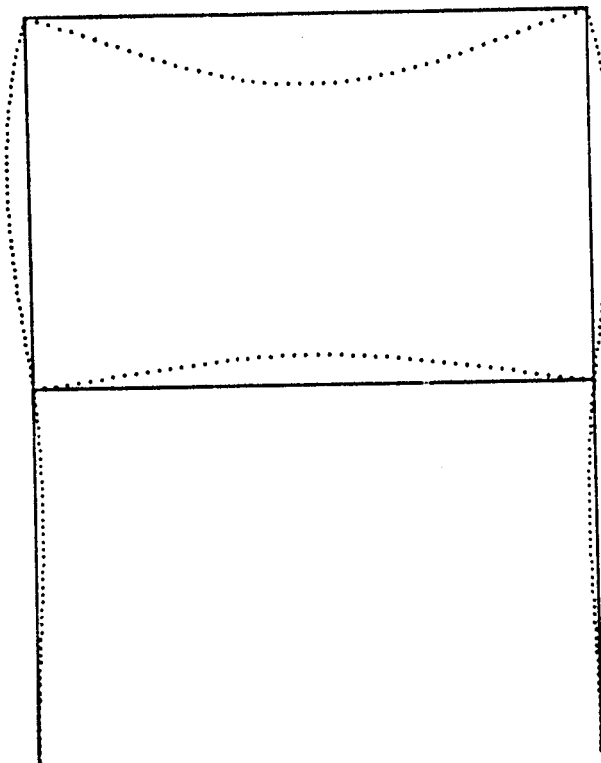


Figure 8

MODE NUMBER 4

ITERATION NUMBER 8

FREQUENCY = $2.2887 \times 10^{+01}$ CPS

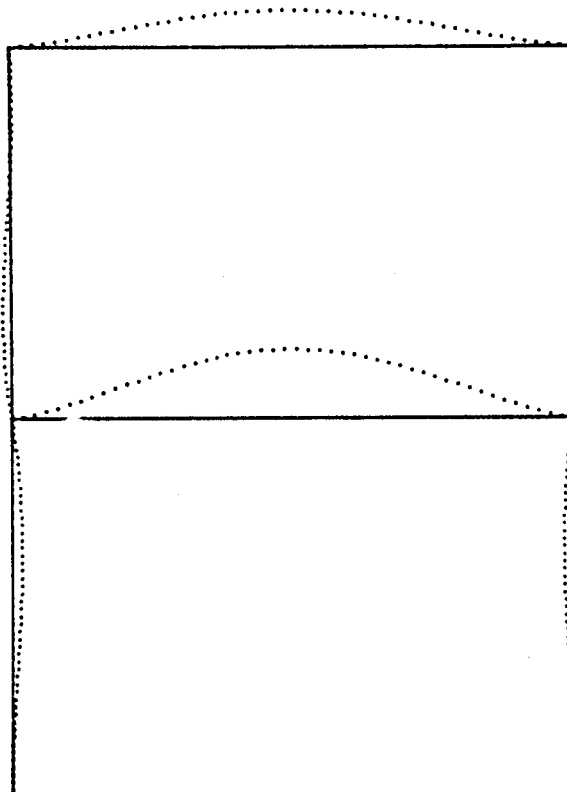


Figure 9

5.2 JOINT MOTIONS AS GENERALIZED COORDINATES

A method of obtaining a relatively accurate set of initial approximations of system modes is presented below. The digital program described in Section 6.1 contains routines for (optionally) executing this procedure; however, as pointed out in the preceding discussion, the use of accurate initial approximations is generally unnecessary in implementing the procedure outlined in Sections 3 and 4. Accordingly, these routines are normally bypassed for economy of computer execution time.

In the method outlined below, the generalized coordinates are the displacement and rotation components of the joints. Member displacement functions are the quasi-static deformations associated with member end motion.

Joint motion components (with respect to the primary reference frame) will be represented by the following symbols:

$U_{1k}^i \equiv$ direction-k displacement of the i^{th} joint,

$U_{2k}^i \equiv$ direction-k rotation of the i^{th} joint.

The motion components (with respect to the m^{th} member reference frame) of the end points (origin: $n = 1$, terminus: $n = 2$) of the m^{th} member are:

$V_{1k}^{mn} \equiv$ direction-k displacement of n^{th} end point,

$V_{2k}^{mn} \equiv$ direction-k rotation components.

The following abbreviated notation will be useful:

For $j = 1$ (displacements) and 2 (rotations),

$$U_j^i \equiv \begin{bmatrix} U_{j1}^i \\ U_{j2}^i \\ U_{j3}^i \end{bmatrix}, \text{ and } V_i^{mn} \equiv \begin{bmatrix} V_{i1}^{mn} \\ V_{i2}^{mn} \\ V_{i3}^{mn} \end{bmatrix} .$$

Also,

$$U^i \equiv \begin{bmatrix} U_1^i \\ U_2^i \end{bmatrix} \text{ and } V^{mn} \equiv \begin{bmatrix} V_1^{mn} \\ V_2^{mn} \end{bmatrix} .$$

Throughout the following discussion, it is assumed that the origin ($n = 1$) and terminus ($n = 2$) of the m^{th} member are connected by rigid links to the i^{th} and j^{th} joints, respectively. Accordingly, the R and D matrices defined in Section 2.1 may be used to express the linearized relation between joint and member end-point motion as follows:

$$\begin{bmatrix} V_1^{m1} \\ V_2^{m1} \end{bmatrix} = \begin{bmatrix} R^m & \phi_3 \\ \phi_3 & R^m \end{bmatrix} \begin{bmatrix} I_3 & D^{m1} \\ \phi_3 & I_3 \end{bmatrix} \begin{bmatrix} U_1^i \\ U_2^i \end{bmatrix} ,$$

$$\begin{bmatrix} V_1^{m2} \\ V_2^{m2} \end{bmatrix} = \begin{bmatrix} R^m & \phi_3 \\ \phi_3 & R^m \end{bmatrix} \begin{bmatrix} I_3 & D^{m2} \\ \phi_3 & I_3 \end{bmatrix} \begin{bmatrix} U_1^j \\ U_2^j \end{bmatrix} . \tag{5-1}$$

In the above equations, and where used subsequently, ϕ_l and I_l are $l \times l$ zero and identity matrices, respectively. Using the definitions,

$$G^{mn} \equiv \begin{bmatrix} R^m & \phi_3 \\ \phi_3 & R^m \end{bmatrix} \begin{bmatrix} I_3 & D^{mn} \\ \phi_3 & I_3 \end{bmatrix},$$

Equations (5-1) may be written compactly as:

$$\begin{aligned} v^{m1} &= G^{m1} U^i \\ v^{m2} &= G^{m2} U^j \end{aligned} \tag{5-2}$$

As discussed later, the kinetic and potential energies of the m^{th} member may be expressed as quadratic forms in its twelve end point motion components. That is,

$$\begin{aligned} T^m &= \frac{1}{2} \begin{bmatrix} (\dot{v}^{m1})^* & (\dot{v}^{m2})^* \end{bmatrix} \begin{bmatrix} A^{m11} & A^{m12} \\ (A^{m12})^* & A^{m22} \end{bmatrix} \begin{bmatrix} \dot{v}^{m1} \\ \dot{v}^{m2} \end{bmatrix}, \text{ and} \\ V^m &= \frac{1}{2} \begin{bmatrix} (v^{m1})^* & (v^{m2})^* \end{bmatrix} \begin{bmatrix} B^{m11} & B^{m12} \\ (B^{m12})^* & B^{m22} \end{bmatrix} \begin{bmatrix} v^{m1} \\ v^{m2} \end{bmatrix}. \end{aligned} \tag{5-3}$$

Using Equations (5-2), T^m and V^m may be expressed in terms of the generalized coordinates as

$$\begin{aligned} T^m &= \frac{1}{2} \begin{bmatrix} (\dot{U}^i)^* & (\dot{U}^j)^* \end{bmatrix} \begin{bmatrix} \bar{A}^{m11} & \bar{A}^{m12} \\ (\bar{A}^{m12})^* & \bar{A}^{m22} \end{bmatrix} \begin{bmatrix} \dot{U}^i \\ \dot{U}^j \end{bmatrix}, \\ V^m &= \frac{1}{2} \begin{bmatrix} (U^i)^* & (U^j)^* \end{bmatrix} \begin{bmatrix} \bar{B}^{m11} & \bar{B}^{m12} \\ (\bar{B}^{m12})^* & \bar{B}^{m22} \end{bmatrix} \begin{bmatrix} U^i \\ U^j \end{bmatrix}, \end{aligned} \tag{5-4}$$

where

$$\begin{bmatrix} \bar{A}^{m11} & \bar{A}^{m12} \\ (\bar{A}^{m12})^* & \bar{A}^{m22} \end{bmatrix} \equiv \begin{bmatrix} (G^{m1})^* & \phi_6 \\ \phi_6 & (G^{m2})^* \end{bmatrix} \begin{bmatrix} A^{m11} & A^{m12} \\ (A^{m12})^* & A^{m22} \end{bmatrix} \begin{bmatrix} G^{m1} & \phi_6 \\ \phi_6 & G^{m2} \end{bmatrix},$$

and

$$\begin{bmatrix} \bar{B}^{m11} & \bar{B}^{m12} \\ (\bar{B}^{m12})^* & \bar{B}^{m22} \end{bmatrix} \equiv \begin{bmatrix} (G^{m1})^* & \phi_6 \\ \phi_6 & (G^{m2})^* \end{bmatrix} \begin{bmatrix} B^{m11} & B^{m12} \\ (B^{m12})^* & B^{m22} \end{bmatrix} \begin{bmatrix} G^{m1} & \phi_6 \\ \phi_6 & G^{m2} \end{bmatrix}. \quad (5-5)$$

Accordingly, the kinetic and potential energies of the entire frame may be written as

$$T = \frac{1}{2} \dot{U}^* M \dot{U}$$

and

$$V = \frac{1}{2} U^* K U,$$

where for an n-jointed frame,

$$U \equiv \begin{bmatrix} U^1 \\ U^2 \\ \cdot \\ \cdot \\ U^n \end{bmatrix}, \quad M \equiv \begin{bmatrix} M^{11} & M^{12} & \cdot & \cdot & \cdot \\ & M^{22} & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ \text{Symmetric} & & & & M^{nn} \end{bmatrix}$$

$$K \equiv \begin{bmatrix} K^{11} & K^{12} & . & . & . \\ & K^{22} & . & . & . \\ & & . & . & . \\ & & & . & . \\ \text{Symmetric} & & & & K^{nn} \end{bmatrix},$$

and the 6 x 6 submatrices M^{ij} and K^{ij} may be constructed by the following algorithm:

- First, set all of the M^{ij} 's and K^{ij} 's identically equal to zero.
- For each member in the frame, add terms to these matrices as follows, based on Equation (5-4):

$$\begin{aligned} &\text{add } \bar{A}^{m11} \text{ to } M^{ii}, \bar{A}^{m12} \text{ to } M^{ij}, \bar{A}^{m22} \text{ to } M^{jj}, \\ &\bar{B}^{m11} \text{ to } K^{ii}, \bar{B}^{m12} \text{ to } K^{ij}, \text{ and } \bar{B}^{m22} \text{ to } K^{jj}. \end{aligned}$$

If some joint motion components are set equal to zero as restraint conditions, the corresponding elements of U and the associated rows and columns of M and K are deleted.

As before, Lagrange's equation leads to the undamped free-vibration eigenproblem, $\omega^2 MX - KX = 0$, where $U = \sin\omega t X$.

The formulation outlined above, which is essentially the one discussed in Section 1, gives excellent approximate solutions to many types of problems. Its primary shortcoming is that for many-jointed frames, it requires solution of high-order eigenproblems of the type $(\omega^2 M - K)X = 0$.

Also, in some applications it is possible that the quasi-static displacement functions may not give suitably accurate results; and, although such "restraint errors" may be reduced by introducing additional joints along members, addition of joints leads to high-order eigenproblems with their associated numerical difficulties and high cost of solution.

For particular types of members, the A^m and B^m matrices appearing in Equations (5-3) are easily evaluated. In general, for static displacement functions associated with member end motion, the c_j^m 's (member displacement function coefficients) defined in Section 2.2 may be identified simply as the end point motions, e.g.,

$$\begin{aligned}
 c_1^m &\equiv v_{11}^{m1} = \text{direction-1 displacement of member origin,} \\
 c_2^m &\equiv v_{12}^{m1} = \text{direction-2 displacement of member origin,} \\
 &\cdot \\
 &\cdot \\
 c_6^m &\equiv v_{23}^{m1} = \text{direction-3 rotation of member origin,} \\
 c_7^m &\equiv v_{11}^{m2} = \text{direction-1 displacement of member terminus,} \\
 c_8^m &\equiv v_{12}^{m2} = \text{direction-2 displacement of member terminus,} \\
 &\cdot \\
 &\cdot \\
 c_{12}^m &\equiv v_{23}^{m2} = \text{direction-3 rotation of member terminus.}
 \end{aligned}$$

Then the coefficient matrices appearing in Equation (2-4) may be interpreted as

$$\begin{bmatrix} A^{m11} & A^{m12} \\ (A^{m12})^* & A^{m22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B^{m11} & B^{m12} \\ (B^{m12})^* & B^{m22} \end{bmatrix},$$

respectively, and Equation (2-5) may be used to compute individual elements of these matrices.

Each of the twelve sets of member displacement functions (v_{ij}^m and θ_{ij}^m for the $j = 1$ through 12) is the deformation of the member associated with a unit value of the corresponding end point motion, with the other

eleven end motion components identically zero. For particular types of members, these displacement functions are easily identified.

Consider for example a straight, uniform simple beam having two planes of cross-section symmetry. If the member origin and terminus are defined to lie on the section centroid, and the member reference frame axes are oriented as indicated on Figure 10, the interior deformation of the

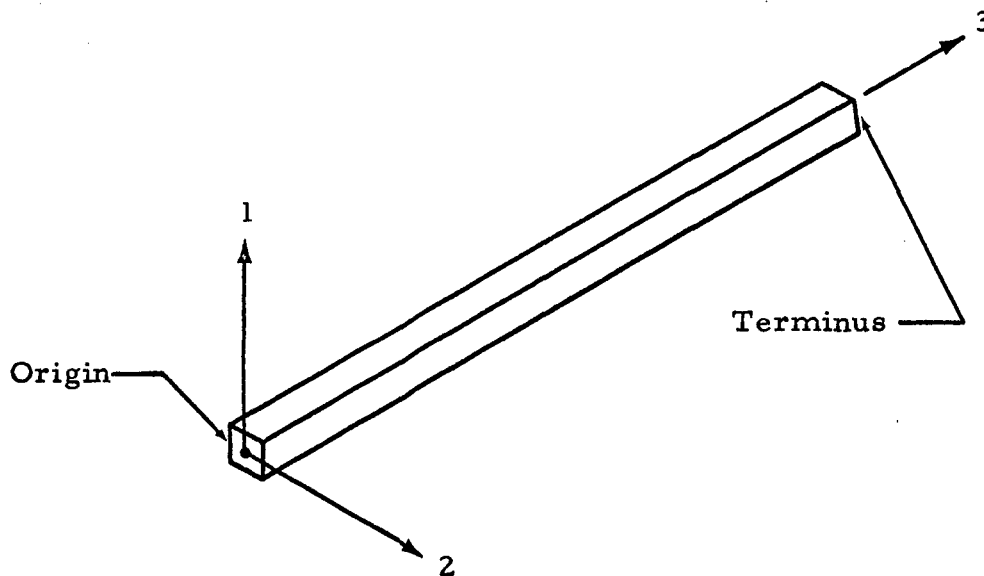


Figure 10 - Prismatic Beam

member (using the notation introduced in Section 2.1) is:

where l = member length,

Bending in plane-1:

$$u_1^m(s) = v_{11}^{m1} + v_{22}^{m1}s + s^2 \left[\frac{3}{l^2} (-v_{11}^{m1} + v_{11}^{m2}) - \frac{1}{l} (2v_{22}^{m1} + v_{22}^{m2}) \right] + s^3 \left[\frac{2}{l^3} (v_{11}^{m1} - v_{11}^{m2}) + \frac{1}{l^2} (v_{22}^{m1} + v_{22}^{m2}) \right]$$

Bending in plane-2:

$$u_2^m(s) = V_{12}^{m1} - V_{21}^{m1}s + s^2 \left[\frac{3}{\ell^2} (-V_{12}^{m1} + V_{12}^{m2}) + \frac{1}{\ell} (2V_{21}^{m1} + V_{21}^{m2}) \right] + s^3 \left[\frac{2}{\ell^3} (V_{12}^{m1} - V_{12}^{m2}) - \frac{1}{\ell^2} (V_{21}^{m1} + V_{21}^{m2}) \right]$$

Axial extension:

$$u_3^m(s) = V_{13}^{m1} + \frac{V_{13}^{m2} - V_{13}^{m1}}{\ell} s .$$

Torsion:

$$\phi_3^m(s) = V_{23}^{m1} + \frac{V_{23}^{m2} - V_{23}^{m1}}{\ell} s .$$

For a simple beam,

$$\phi_1^m(s) = [u_2^m(s)]'$$

$$\phi_2^m(s) = -[u_1^m(s)]'$$

(5-6)

From the preceding equations, all of the member displacement functions are easily identified. For example, $v_{11}^m(s)$, the direction-1 displacement associated with a unit value of V_{11}^{m1} is

$$v_{11}^m(s) = 1 - 3 \frac{s^2}{\ell^2} + 2 \frac{s^3}{\ell^3} . \quad (5-7)$$

Note that as a consequence of the symmetry characteristics of the simple beam, many of the displacement functions are identically zero. The result of substituting the set of relations of the type indicated by Equations (5-7) and (5-6) into Equations (2-5) is indicated below. The mass per unit length is constant, μ . Rotary inertia other than that associated with torsion is neglected.

$$A^{m11} = \frac{\mu l}{420} \begin{bmatrix} 156 & 0 & 0 & 0 & 22l & 0 \\ 0 & 156 & 0 & -22l & 0 & 0 \\ 0 & 0 & 140 & 0 & 0 & 0 \\ 0 & -22l & 0 & 4l^2 & 0 & 0 \\ 22l & 0 & 0 & 0 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 140 \frac{\rho_3}{\mu} \end{bmatrix}$$

$$A^{m22} = \frac{\mu l}{420} \begin{bmatrix} 156 & 0 & 0 & 0 & -22l & 0 \\ 0 & 156 & 0 & 22l & 0 & 0 \\ 0 & 0 & 140 & 0 & 0 & 0 \\ 0 & 22l & 0 & 4l^2 & 0 & 0 \\ -22l & 0 & 0 & 0 & 4l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 140 \frac{\rho_3}{\mu} \end{bmatrix}$$

$$A^{m12} = \frac{\mu l}{420} \begin{bmatrix} 54 & 0 & 0 & 0 & -13l & 0 \\ 0 & 54 & 0 & 13l & 0 & 0 \\ 0 & 0 & 70 & 0 & 0 & 0 \\ 0 & -13l & 0 & -3l^2 & 0 & 0 \\ 13l & 0 & 0 & 0 & -3l^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 70 \frac{\rho_3}{\mu} \end{bmatrix}$$

$$B^{m11} = \frac{1}{l} \begin{bmatrix} \frac{12}{l^2} EI_2^m & 0 & 0 & 0 & \frac{6}{l} EI_2^m & 0 \\ 0 & \frac{12}{l^2} EI_1^m & 0 & -\frac{6}{l} EI_1^m & 0 & 0 \\ 0 & 0 & EA^m & 0 & 0 & 0 \\ 0 & -\frac{6}{l} EI_1^m & 0 & 4EI_1^m & 0 & 0 \\ \frac{6}{l} EI_2^m & 0 & 0 & 0 & 4EI_2^m & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ^m \end{bmatrix}$$

$$B^{m22} = \frac{1}{l} \begin{bmatrix} \frac{12}{l^2} EI_2^m & 0 & 0 & 0 & -\frac{6}{l} EI_2^m & 0 \\ 0 & \frac{12}{l^2} EI_1^m & 0 & \frac{6}{l} EI_1^m & 0 & 0 \\ 0 & 0 & EA^m & 0 & 0 & 0 \\ 0 & \frac{6}{l} EI_1^m & 0 & 4EI_1^m & 0 & 0 \\ -\frac{6}{l^2} EI_2^m & 0 & 0 & 0 & 4EI_2^m & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ^m \end{bmatrix}, \text{ and}$$

$$B^{m12} = \frac{1}{l} \begin{bmatrix} -\frac{12}{l^2} EI_2^m & 0 & 0 & 0 & \frac{6}{l} EI_2^m & 0 \\ 0 & -\frac{12}{l^2} EI_1^m & 0 & -\frac{6}{l} EI_1^m & 0 & 0 \\ 0 & 0 & -EA^m & 0 & 0 & 0 \\ 0 & \frac{6}{l} EI_1^m & 0 & 2EI_1^m & 0 & 0 \\ -\frac{6}{l} EI_2^m & 0 & 0 & 0 & 2EI_2^m & 0 \\ 0 & 0 & 0 & 0 & 0 & -GJ^m \end{bmatrix}$$

Section 6 RESULTS

6.1 PROGRAM DESCRIPTION

A digital program has been developed to implement the formulation described in the preceding sections of this report. It is coded entirely in Fortran IV, and in its present configuration requires approximately 26,000 words of core storage and five scratch files for intermediate data storage.

The program requires as input data a minimum definition of the geometry, boundary conditions, etc., together with control variables governing execution of various options. Figures 11 and 12 illustrate the general arrangement of the program.

Members may be either simple beams or Timoshenko beams with mass and stiffness properties varying piecewise-linearly along their lengths. Rigid masses may be attached to the joints, and member end connections to joints (either directly or via rigid links as described in Section 2.1) may be pinned, rigid, or elastic.

The routine used to solve the $(\omega^2 M - K) X = 0$ eigenproblem is the one used in References 4, 5, and 6. Present DIMENSION statements allow solution of 60-th order systems. These routines are fast and accurate for low-order, well-conditioned eigenproblems of the type required to implement the procedure outlined in Section 4. Since solutions are usually computed only for the lowest few system modes, no more than a few seconds of computer execution time are normally required by this routine.

The routines for analyzing structural deformation due to static loading were extracted from the Lockheed FRAME program described in Reference 3.

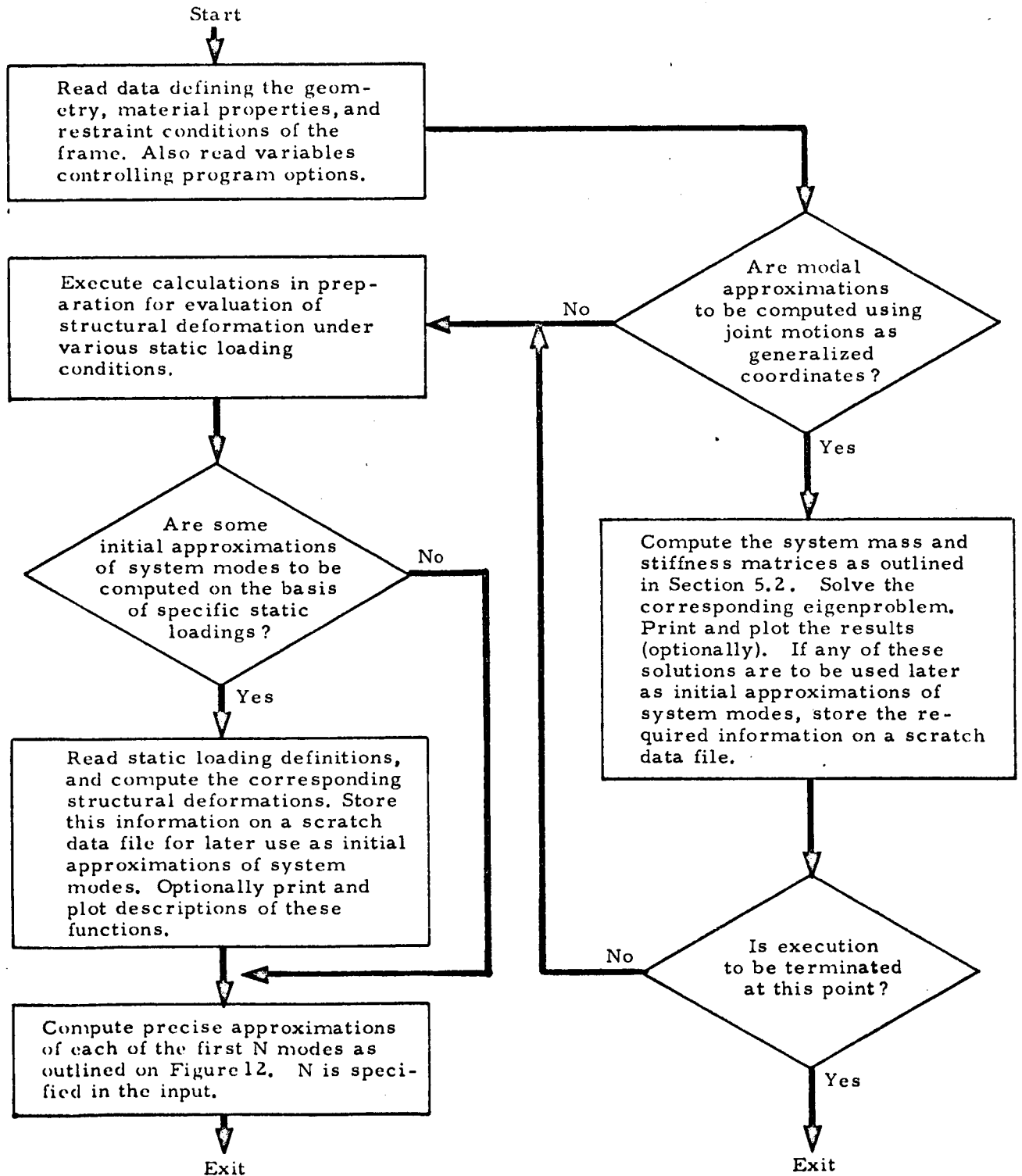


Figure 11 - Flow Chart of Computer Program

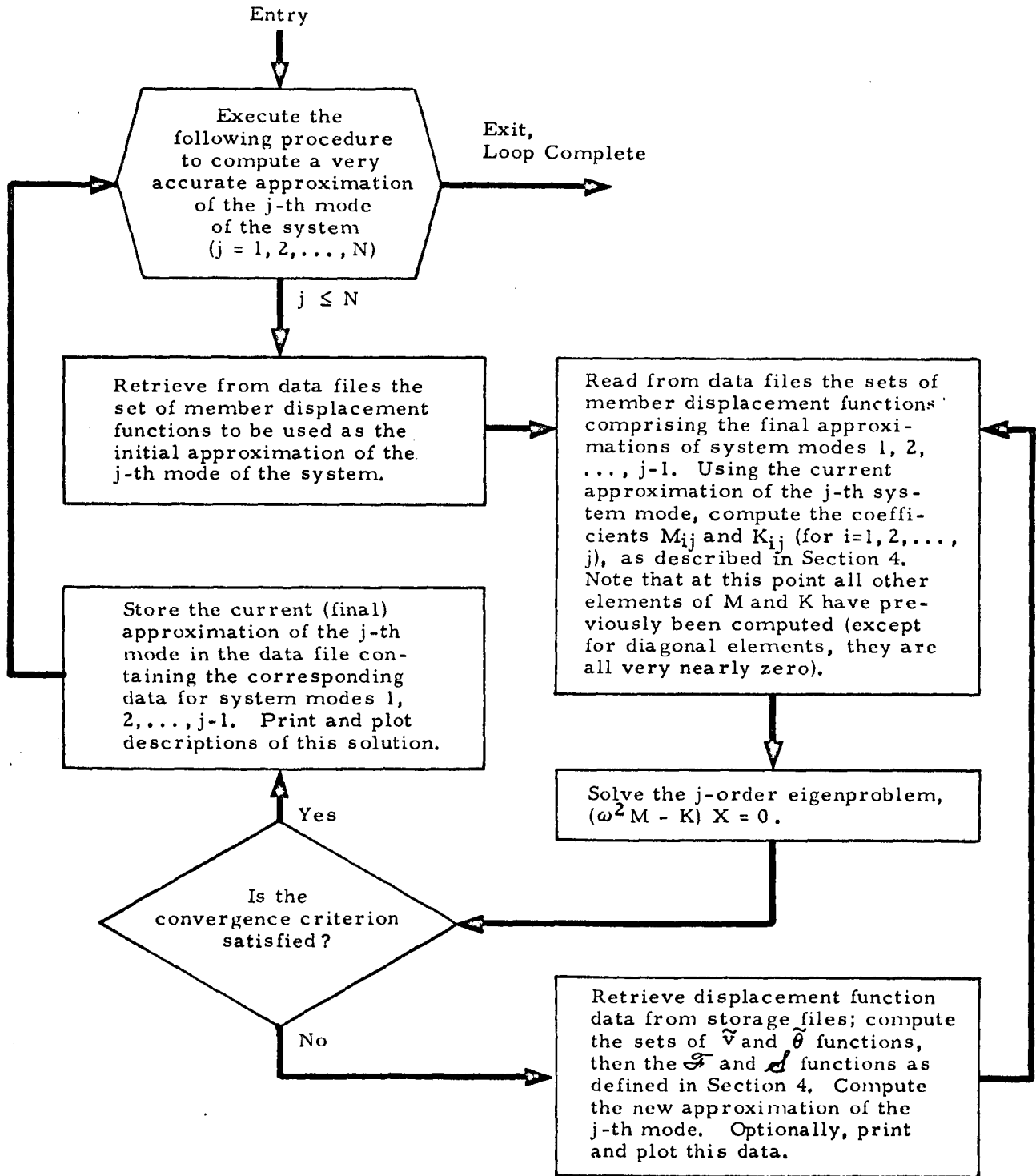


Figure 12 - Flow Chart of Repetitive Technique

The present DIMENSION statements in these routines allow frames having up to 180 non-constrained joint motion components.* Early in execution, the program computes the structure's flexibility matrix for later use in computing deformations under various loading conditions. The execution time required for this step varies approximately as the square of the number of non-constrained joint motion components. About one minute (IBM 7094) is required to compute the flexibility matrix of a space frame having twenty joints.

Deformations of individual members are analyzed, using one of the routines (NUB) developed originally for use in the program described in Reference 6. The solution method used in this routine (which is essentially the one outlined in Reference 3 for analysis of tapered beams) represents non-uniform beams as a sequence of short uniform elements. The interpolating functions used over the interior of the elements are the displacement and cross-section rotation functions associated with edge loading of the elements.

The criteria presently used in the program for terminating the sequence of improved modal approximations are: either (1) a limiting number of approximations have been executed, or (2) the frequency change from one approximation to the next is smaller than a stated fraction of the current frequency approximation.

A certain amount of numerical error will, of course, be present in all solutions. One approach to the identification of this error in physical terms is explained below. For simplicity, abbreviated notation will be used in

* The capacity of the program may be increased substantially by replacing the static analysis routines now used with a large frame analysis routine, such as Lockheed's proprietary FRAME-66 program which is applicable to frames having many thousands of members.

discussing the lateral motion of a simple beam in one of its principal bending planes. (The symbols \tilde{v}_{ij}^{mn} , \mathcal{F}_i^{mn} and $v_{ij}^{m, n+1}$ will be replaced by v^n , \mathcal{F} and v^{n+1} , respectively.)

Suppose that associated with the n^{th} interim approximation of a system mode the lateral motion of a member in one of its principal directions is computed as $v^n(s)$. As the next step in the solution process $\mathcal{F}(s)$ is evaluated:

$$\mathcal{F} = \omega^2 \mu v^n \quad (6-1)$$

where ω is the current frequency approximation. Next, the static analysis problem,

$$EI \frac{\partial^4}{\partial s^4} (v^{n+1}) = \mathcal{F} \quad (6-2)$$

is defined and v^{n+1} is computed. Now, suppose (disregarding momentarily how v^{n+1} was obtained) that the motion of the frame is such that

$$u = \sin \omega t v^{n+1} \quad (6-3)$$

The beam equation is

$$EI \frac{\partial^4}{\partial s^4} (u) = -\mu \ddot{u} + q(t) \quad (6-4)$$

where q is the externally applied distributed force function. Substitution of Equation (6-3) into Equation (6-4) gives:

$$EI \sin \omega t \frac{\partial^4}{\partial s^4} (v^{n+1}) = \omega^2 \mu \sin \omega t v^{n+1} + q(t) \quad (6-5)$$

Substitution of Equations (6-2) and (6-1) into Equation (6-5) gives:

$$q(t) = \omega^2 \mu \sin \omega t (v^n - v^{n+1}) \quad (6-6)$$

If the solution is exactly a free vibration mode, q is identically zero; however, since some numerical error is always present, a reasonable assessment of the physical significance of the error can be made by comparing the magnitude of q as indicated by Equation (6-6) with the corresponding inertia force [Equation (6-1)], since q is the external force which would be necessary to cause the system to actually execute the motion $u = \sin \omega t v^{n+1}$.

The $\omega^2 MX - KX = 0$ eigenproblems solved during the repetitive process outlined in Section 4 have several interesting properties. At each step in the repetitive process, it is necessary to solve for only one mode and frequency (the highest one). Also, only the elements in the last row and column of M and K need be re-evaluated at successive steps in the repetitive process. Since the first $j-1$ generalized coordinates being used during successive approximations of the j^{th} mode are accurate approximations of system modes, all of the off-diagonal elements of M and K are nearly zero, except for those in the last row and column. Furthermore, in solving this eigenproblem the n^{th} time, the $(n-1)^{\text{th}}$ approximation of the j^{th} frequency generally affords a close upper bound to new j^{th} frequency approximation. These special properties afford significant opportunities for economy of computer execution time in calculating eigenproblem solutions.

The repetitive solution procedure occasionally exhibits a certain peculiar effect when solving for higher modes, especially if the structure has one or more symmetry planes. For example, the program may require only a few repetitive steps to solve accurately for each of the first five modes; then obtain in its initial solution for the sixth mode a good approximation of the seventh (or higher) mode. In successive steps, it may then converge to a precise approximation of the seventh mode and subsequently require a number of additional repetitive solutions before the modal approximation begins to shift toward the lower mode.

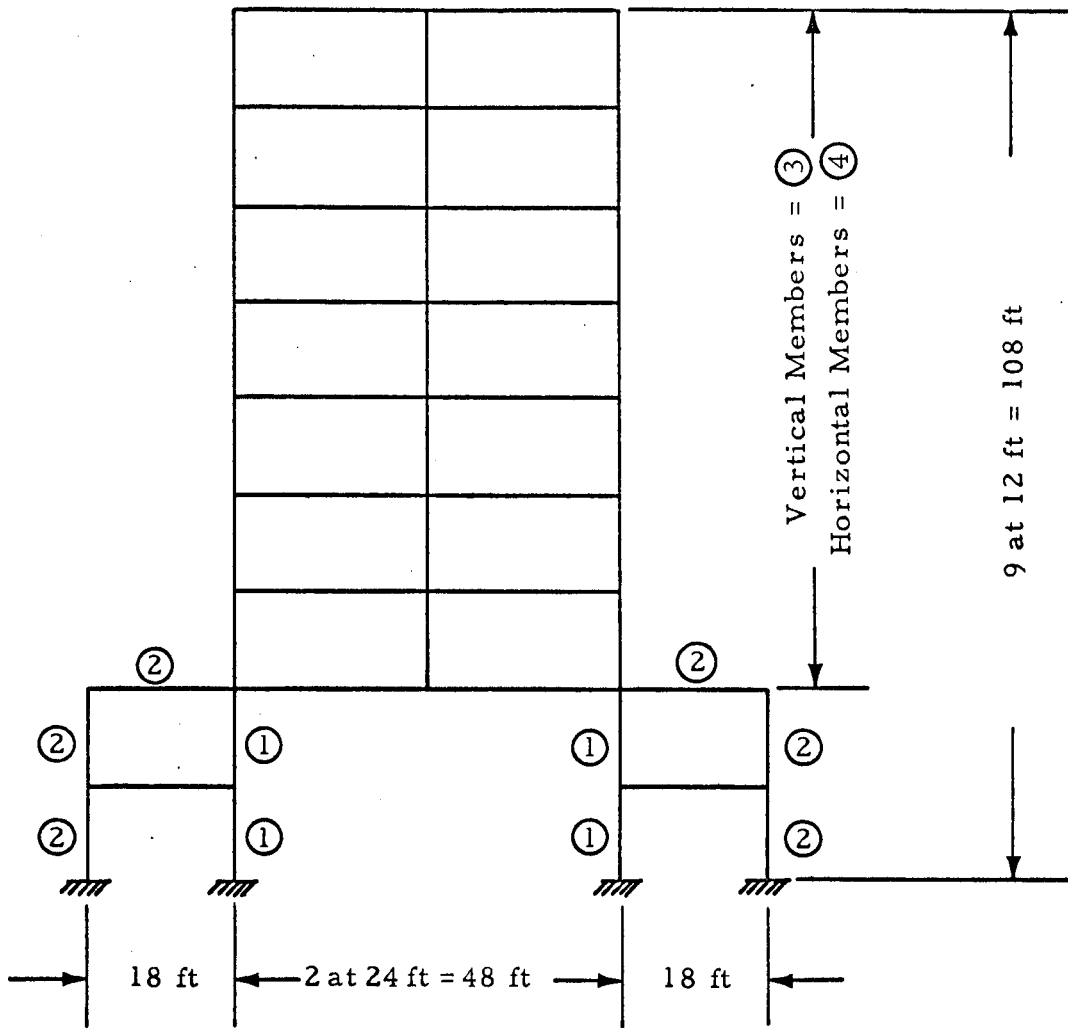
6.2 EXAMPLES

Three examples are presented to illustrate the program's capabilities. All of the mode shape plots shown on Figures 14 through 29 were automatically generated by the program through the use of a Stromberg-Carlson 4020 plotter.

Example A, illustrated on Figure 13, involves a planar structure containing 34 joints and 49 members. The first three planar modes of this structure were computed using the repetitive technique discussed in Section 4. Plots of the final approximations of these modes are shown on Figures 14, 15 and 16.

The geometrical form of Example B, illustrated on Figure 17, resembles approximately a Titan III launch vehicle. For simplicity, uniform beam properties are used in this example; however, since provision is included for arbitrarily varying inertia and stiffness properties, vibration characteristics of vehicles of this type (or even more complicated ones, such as proposed Saturn IB's with additional "strapped-on" solid propellant motors) can be computed by the program. The first six free-free modes (above the six rigid-body modes) of this structure are shown on Figures 18 through 23. Note that the last of these is a torsional mode. These solutions were computed using joint motions as generalized coordinates, as described in Section 5.2. For improved accuracy, intermediate joints were introduced at the mid-spans of each of the four beams.

In Example C, illustrated on Figure 24, joint motions were used as generalized coordinates. The structure's first four modes are shown on Figures 25 through 29. Figure 29 shows the fourteenth mode.



Member Types

- ① 8WF100
- ② 8WF40
- ③ 10WF60
- ④ 8WF67

Figure 13 - Example A

MODE NUMBER 1

ITERATION NUMBER 3

FREQUENCY = 1.2738×10^{00} CPS

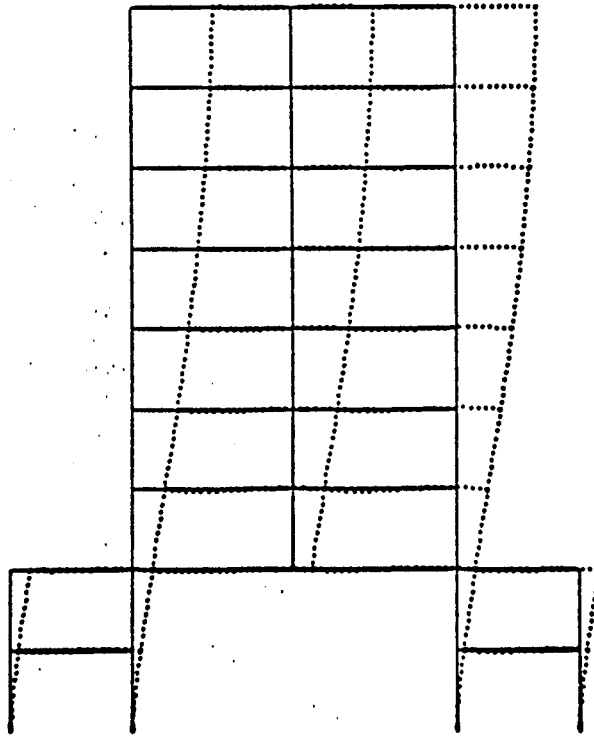


Figure 14

MODE NUMBER 2
ITERATION NUMBER 8
FREQUENCY = 3.8144X10⁰⁰ CPS

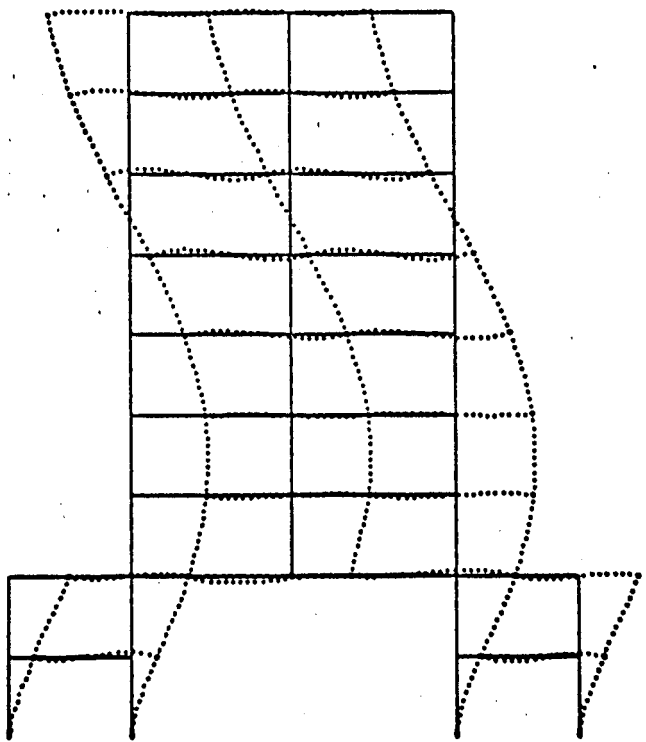


Figure 15

MODE NUMBER 3

ITERATION NUMBER 8

FREQUENCY = 6.0442 X 10⁰⁰ CPS

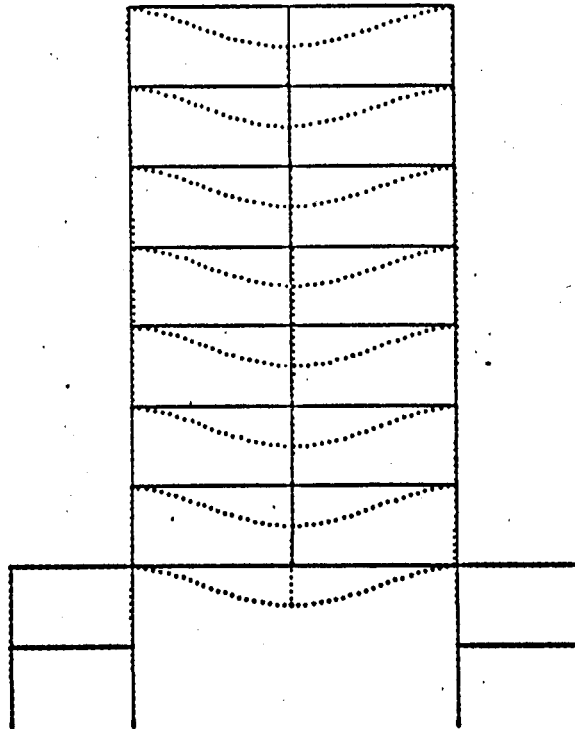


Figure 16

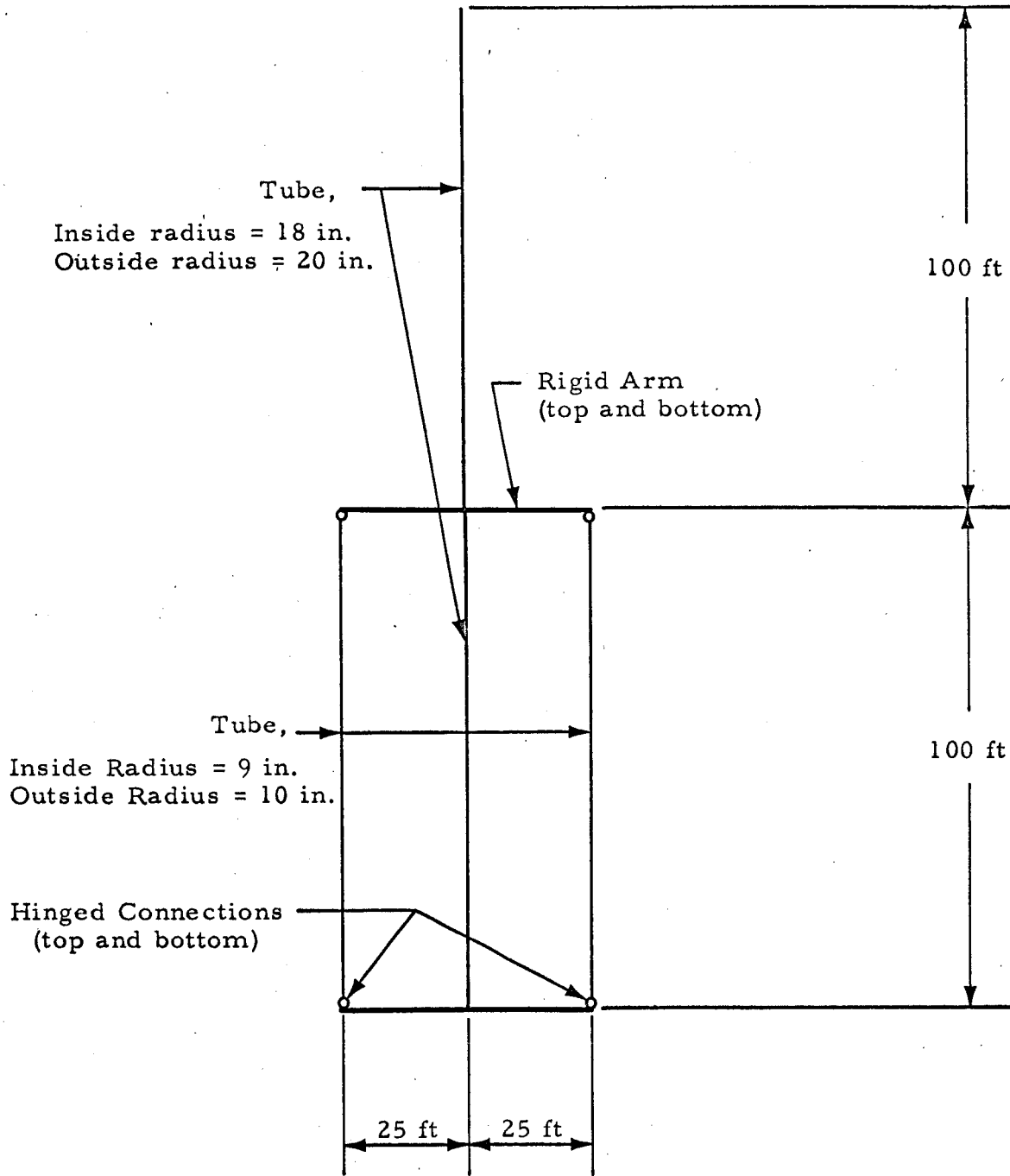


Figure 17 - Example B

MODE NUMBER 7

FREQUENCY = 1.4847×10^{00} CPS

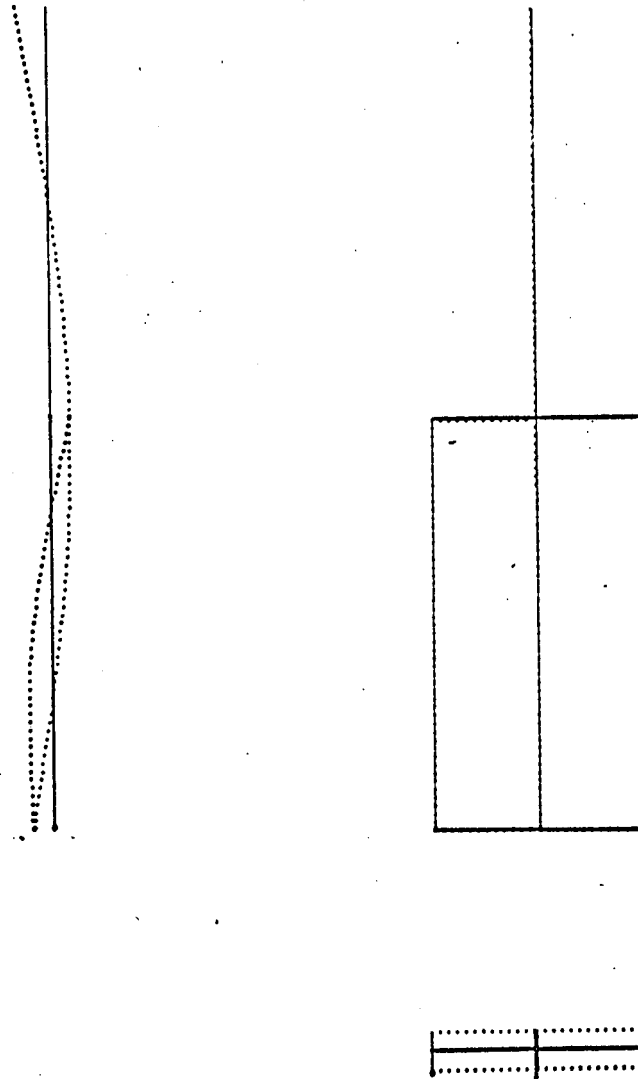


Figure 18

MODE NUMBER 8

FREQUENCY = $1.4886 \times 10^{+00}$ CPS

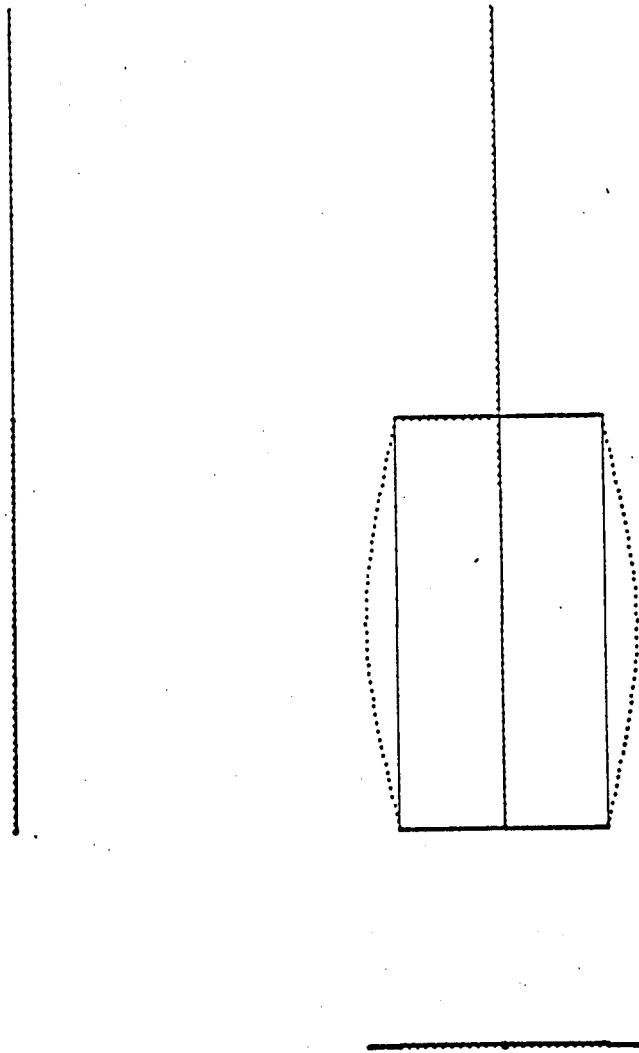


Figure 19

MODE NUMBER 9

FREQUENCY = $1.6971 \times 10^{+00}$ CPS

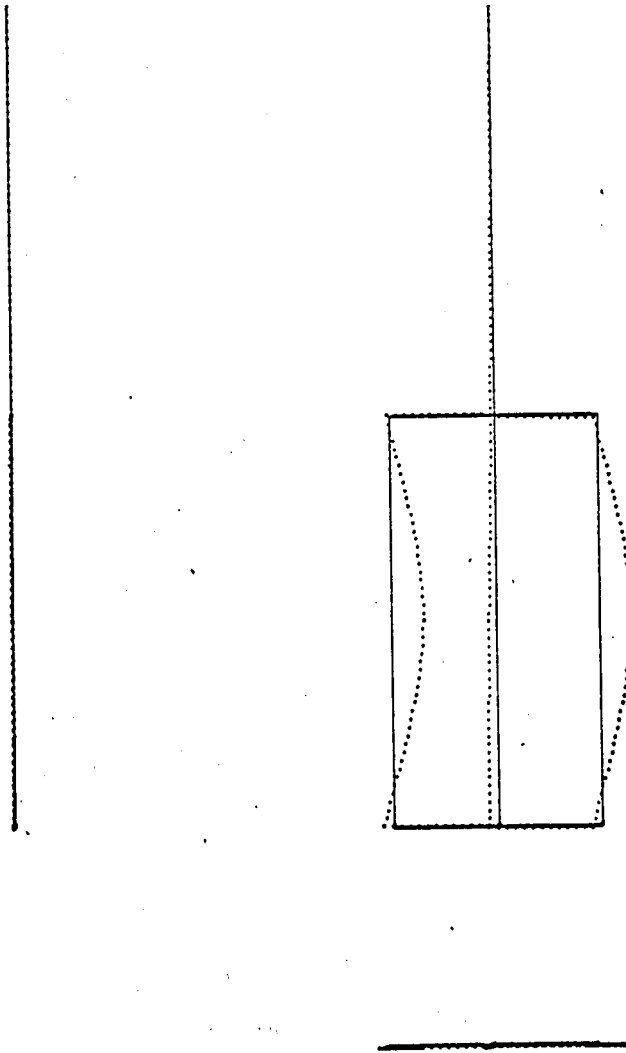


Figure 20

MODE NUMBER 10

FREQUENCY = 1.7209×10^{00} CPS

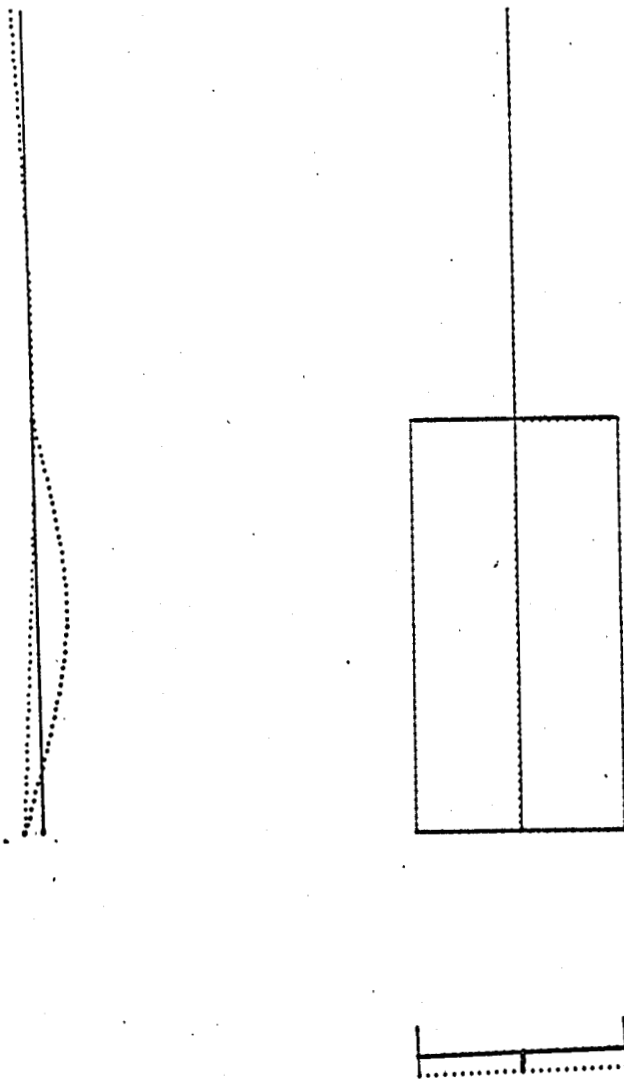


Figure 21

MODE NUMBER 11

FREQUENCY = 1.6703×10^{00} CPS

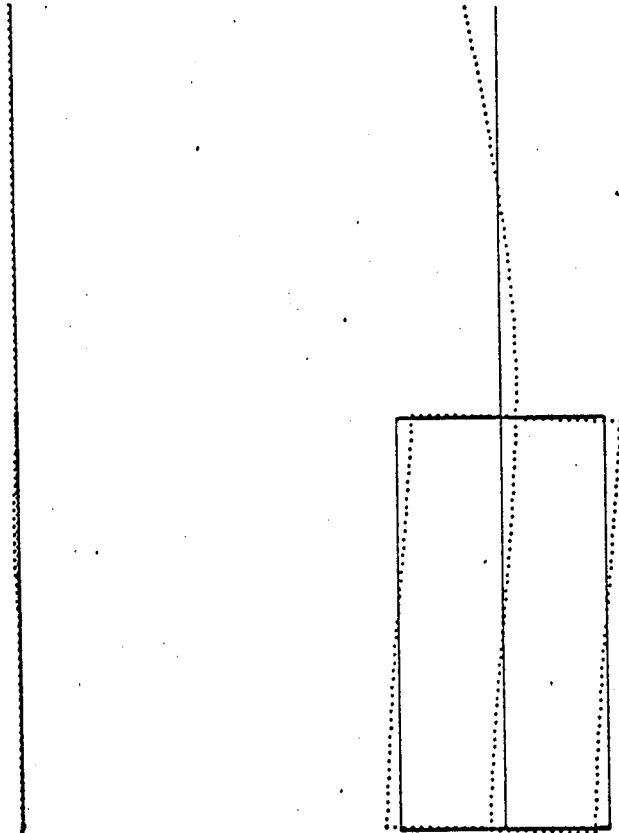


Figure 22

MODE NUMBER 12

FREQUENCY = $2.4608 \times 10^{+00}$ CPS

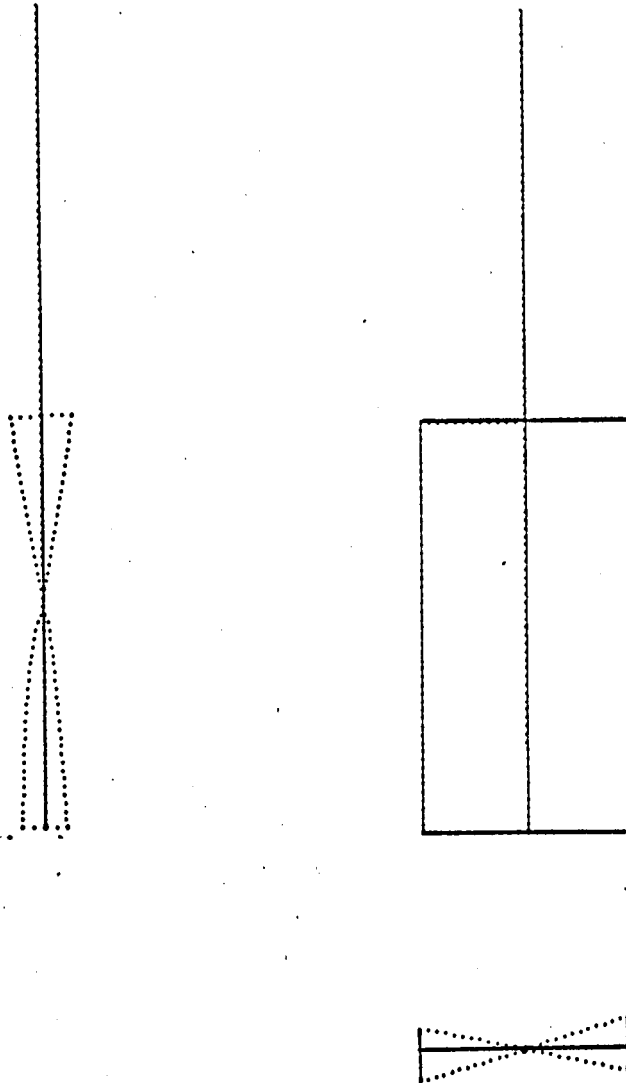
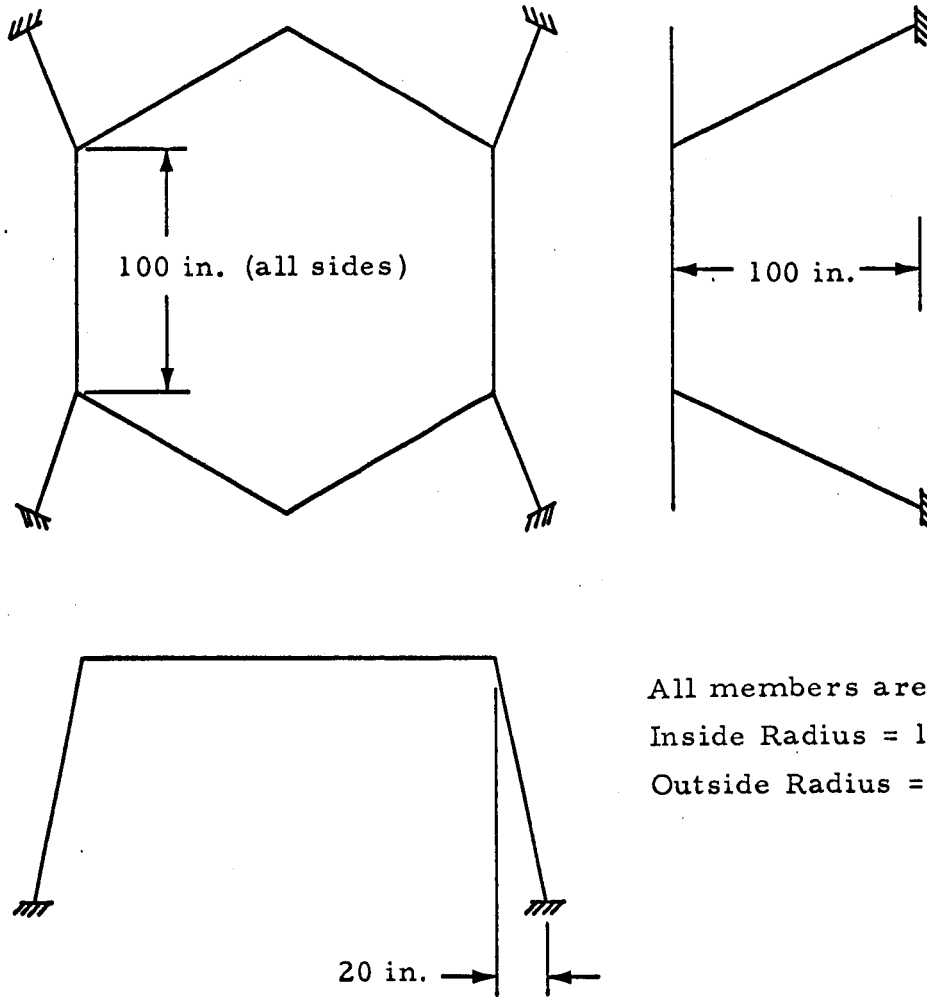


Figure 23



All members are tubes,
Inside Radius = 1.0 in.
Outside Radius = 1.5 in.

Figure 24 - Example C

MODE NUMBER 1

FREQUENCY = $4.0555 \times 10^{+00}$ CPS

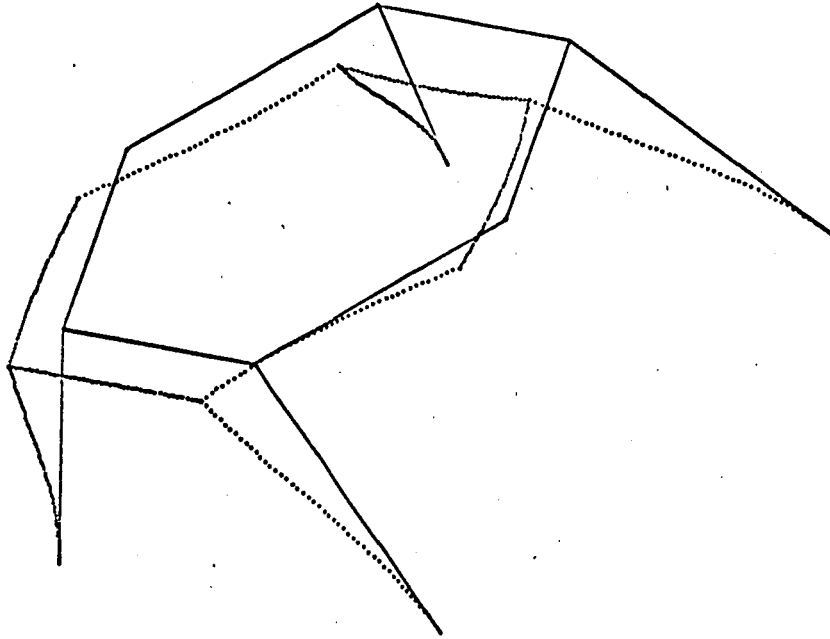


Figure 25

MODE NUMBER 2

FREQUENCY = 6.5539×10^{00} CPS

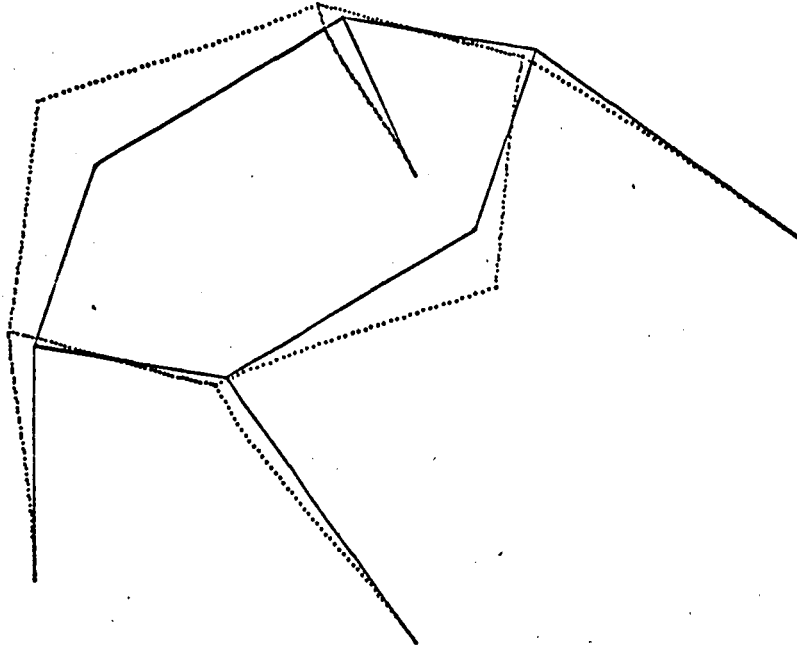


Figure 26

MODE NUMBER 3

FREQUENCY = 7.2452×10^{10} CPS

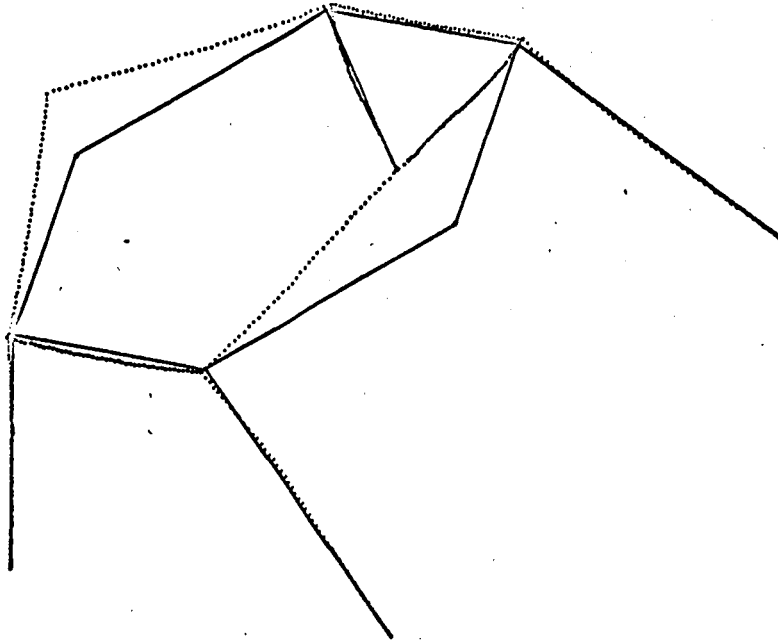


Figure 27

MODE NUMBER 4

FREQUENCY = 7.3930×10^{10} cps

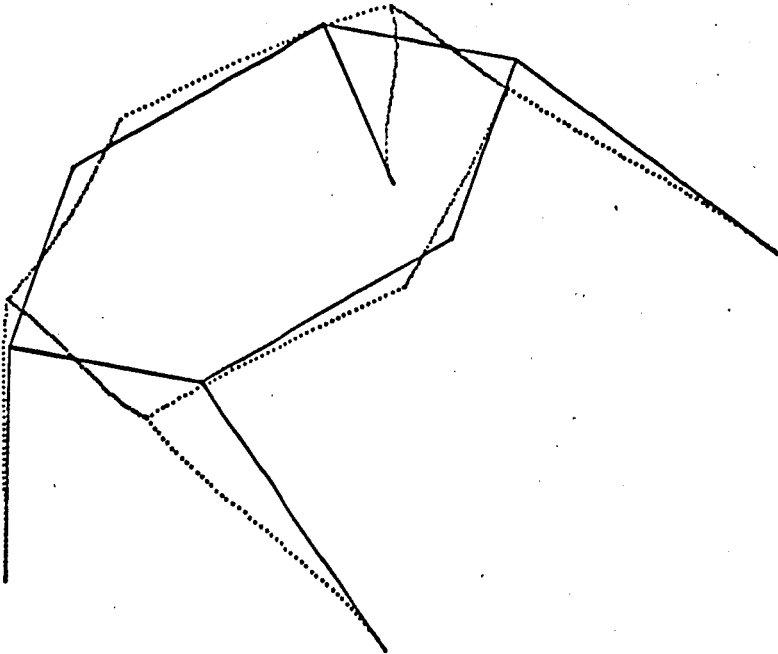


Figure 28

MODE NUMBER 14

FREQUENCY = $4.7362 \times 10^{+03}$ CP8

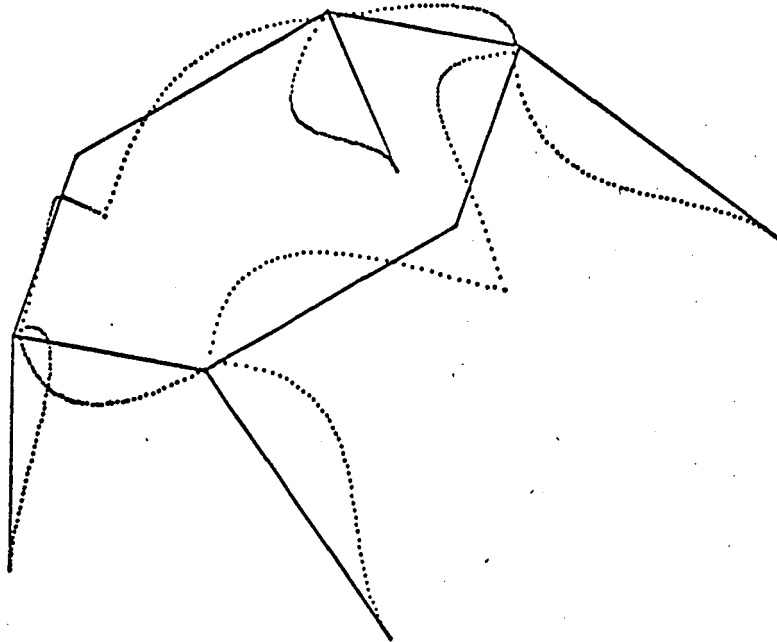


Figure 29

Section 7
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