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REPRESENTATIONS FOR SOLUTIONS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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FUNCTIONAL DIFFERENTIAL EQUATIONS

by

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Introduction. An interest in problems involving functional differential equations has been evidenced recently and several papers have been written using results about these systems which were previously published. In particular, the book [4] by Halanay has become an often used reference. The purpose of this note is to point out a mistake in [4] and to present the correct results.

The systems under consideration here are linear differential systems with time delays (Chapter 4, §3 of [4]); namely, systems of the form (see p. 362-368 in [4])

$$\dot{x}(t) = \int_{-\infty}^{0} x(t+s)\, d_s \eta(t,s) + f(t) \tag{1}$$

where the function η, which is of bounded variation (B.V.) in s, may contain a singular part. The case where η has no singular part is considered correctly in [4]. We shall derive the correct results for the general case under slightly less restrictive hypotheses than in [4]. The systems (1), under the hypotheses stated below, appear as the linear variational equations in the study of general nonlinear variational (optimal control) problems (see [2]).

The error in [4] is due to an incorrect interchange of the order of integration involving a Stieltjes integral. If the Stieltjes integral can be written as a sum plus a Lebesgue integral (i.e., if η has no singular part and can be written as a saltus plus an absolutely continuous function) then the ordinary Fubini theorem can be used as was done in [4]. However, in the general case where the measure depends on both variables of integration and contains a singular part, an unsymmetric Fubini type theorem (see [3]) must be used. To use this correctly, one must always interchange over regions which are rectangles (the cross-product of intervals).

For those readers who are not familiar with Lebesgue-Stieltjes integration, the following reasoning will show that some of Halanay's results on the general systems (1) are incorrect. The system (1) includes as a special case a system with a simple time lag

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\theta) + f(t) \tag{2}$$

and η is very easy to find in this case. The adjoint system for (2) is well-known and is given correctly in [4]. If the adjoint system given by Halanay for the general case of (1) is correct, it should reduce to the adjoint system for (2) whenever (2) is considered as a special case of (1). It is not hard to show that it does not and hence must be incorrect.

In what follows we shall use vector matrix notation and a vector and its transpose will not be distinguished when it is clear what is meant. For the convenience of the reader, we have used the notation of Halanay in [4] whenever possible.

<u>§1. Representation of solutions.</u> In this section we derive a representation theorem (a variation-of-constants formula) for solutions x of (1) in terms of solutions of an adjoint system under the following assumptions on η. As a function of t, $\eta(t,s)$ is measurable for each fixed s, and for each t, it is of bounded variation in s. We suppose that there exists a finite constant $-\tau$ such that $\eta(t,s) = \eta(t,-\tau)$ for $s \leqq -\tau$ and every t, and also that $\eta(t,s) = 0$ for $s \geqq 0$. We further assume that there is an L_1 function $m(t)$ such that

$$|\eta(t,s)| \leqq m(t) \quad \text{for all} \quad s$$

and

$$\bigvee_{s=-\infty}^{\infty} \eta(t,s) = \bigvee_{s=-\tau}^{0} \eta(t,s) \leqq m(t)$$

for every t.

Thus we are considering essentially the same η as in [4] except the condition that $\eta(t,s)$ be continuous in t, uniformly in s, has been dropped. Our assumptions on η are quite reasonable and systems satisfying these hypotheses actually appear in applications (see [2]) where η is a measure (in s) obtained from the Riesz theorem.

Under the above assumptions the system (1) can be written

$$\dot{x}(t) = \int_{-\tau}^{0} x(t+s)d_s\eta(t,s) + f(t)$$

or

(3) $$\dot{x}(t) = \int_{t-\tau}^{t} x(\beta)d_\beta\eta(t,\beta-t) + f(t)$$

where we assume that f is a given L_1 function. In the discussions below, it will often be convenient to use a function p which we now define as $p(t,s) \equiv \eta(t,s-t)$. Then $p(t,s) = 0$ for $s \geqq t$ and $p(t,s) = p(t,t-\tau)$ for $s \leqq t-\tau$.

Let $x(\alpha)$ be a solution to (3) for $\alpha > \sigma$ which satisfies $x(\alpha) = \varphi(\alpha)$ on $[\sigma-\tau,\sigma]$, where φ is in $BV[\sigma-\tau,\sigma]$. That is, $x(\alpha)$ is an absolutely continuous function for $\alpha > \sigma$, satisfying (3) a.e. on some interval $[\sigma,T]$ and agreeing with the function φ of bounded variation on $[\sigma-\tau,\sigma]$. For each t in $(\sigma,T]$, let $Y(\alpha,t)$ be B.V. in α and satisfy

(4) $$Y(s,t) + \int_{s}^{t}\eta(\alpha,s-\alpha)Y(\alpha,t)d\alpha = E \qquad s \in [\sigma,t]$$

where E is the $n \times n$ identity matrix. We also assume that $Y(s,t) \equiv 0$ for $s > t$. This assumption is not necessary in order to find the solution of (4), but is convenient for the representation to be found below. Theorems guaranteeing the existence and uniqueness of a solution to (4) may be found in [2]. Similar theorems for the system (3) can be obtained by an easy generalization of Theorems 1 and 2 in [1]. (The requirement that the initial function φ be continuous in [1] is not necessary; φ of bounded variation is sufficient.)

Using the integration by parts formula for Lebesgue-Stieltjes integrals, we obtain

$$\int_\sigma^t x(\alpha)\, d_\alpha Y(\alpha,t) + \int_\sigma^t \dot{x}(\alpha) Y(\alpha,t)\, d\alpha = x(\alpha)Y(\alpha,t)\Big|_\sigma^t .$$

This gives

$$x(t) = x(\sigma)Y(\sigma,t) + \int_\sigma^t x(\alpha)\, d_\alpha Y(\alpha,t) + \int_\sigma^t \dot{x}(\alpha) Y(\alpha,t)\, d\alpha$$

or

$$x(t) = x(\sigma)Y(\sigma,t) + \int_\sigma^t f(\alpha) Y(\alpha,t)\, d\alpha + \int_\sigma^t x(\alpha)\, d_\alpha Y(\alpha,t) + \int_\sigma^t \{ \int_{\alpha-\tau}^{\alpha} x(s)\, d_s \eta(\alpha, s-\alpha)\} Y(\alpha,t)\, d\alpha. \tag{5}$$

Let us consider the last term on the right side of equation (5). This term may be written

$$\int_{\sigma}^{t}\{\int_{\alpha-\tau}^{\alpha} x(s)d_s p(\alpha,s)\}Y(\alpha,t)d\alpha \quad ,$$

which is equal to

$$\int_{\sigma}^{t}\{\int_{\sigma-\tau}^{t} x(s)d_s p(\alpha,s)\}Y(\alpha,t)d\alpha$$

since $p(\alpha,s) = \text{constant} = 0$ for $s \geqq \alpha$ and $p(\alpha,s) = \text{constant} = p(\alpha,\alpha-\tau)$ for $s \leqq \alpha-\tau$. We next use an unsymmetric Fubini type theorem due to Cameron and Martin [3] to interchange the order of integration in the above integral. This theorem says that under certain hypotheses on the functions involved (these hypotheses are easily shown to be satisfied in our case) one has

$$\int_a^b s(u)d_u\int_c^d p(x,u)dk(x)$$
$$= \int_c^d dk(x)\int_a^b s(u)d_u p(x,u).$$

Applying this we have

$$\int_{\sigma}^{t}\{\int_{\sigma-\tau}^{t} x(s)d_s p(\alpha,s)\}Y(\alpha,t)d\alpha$$
$$= \int_{\sigma-\tau}^{t} x(s)d_s\int_{\sigma}^{t} p(\alpha,s)Y(\alpha,t)d\alpha.$$

Since $p(\alpha,s) = 0$ for $\alpha \leqq s$, this integral may be written

$$\int_{\sigma-\tau}^{\sigma} x(s)d_s\int_{\sigma}^{t} p(\alpha,s)Y(\alpha,t)d\alpha$$

$$+ \int_{\sigma}^{t} x(s)d_s\int_{s}^{t} p(\alpha,s)Y(\alpha,t)d\alpha.$$

Thus equation (5) becomes

$$\begin{aligned} x(t) &= x(\sigma)Y(\sigma,t) + \int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha \\ &+ \int_{\sigma}^{t} x(s)d_sY(s,t) \\ &+ \int_{\sigma-\tau}^{\sigma} \varphi(s)d_s\int_{\sigma}^{t} p(\alpha,s)Y(\alpha,t)d\alpha \\ &+ \int_{\sigma}^{t} x(s)d_s\int_{s}^{t} p(\alpha,s)Y(\alpha,t)d\alpha \end{aligned}$$

or

$$\begin{aligned} x(t) &= \varphi(\sigma)Y(\sigma,t) + \int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha \\ &+ \int_{\sigma-\tau}^{\sigma} \varphi(s)d_s\int_{\sigma}^{t} p(\alpha,s)Y(\alpha,t)d\alpha \\ &+ \int_{\sigma}^{t} x(s)d_s\{Y(s,t) + \int_{s}^{t} p(\alpha,s)Y(\alpha,t)d\alpha\}. \end{aligned} \tag{6}$$

Since $Y(s,t)$ satisfies (4) we obtain

$$\begin{aligned} x(t) &= \varphi(\sigma)Y(\sigma,t) + \int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha \\ &+ \int_{\sigma-\tau}^{\sigma} \varphi(s)d_s\int_{\sigma}^{t} p(\alpha,s)Y(\alpha,t)d\alpha. \end{aligned}$$

If σ and t are such that $\sigma+\tau < t$, then the properties of η imply that $\int_{\sigma+\tau}^{t} p(\alpha,s)Y(\alpha,t)d\alpha$ is a constant function of s for s in $[\sigma-\tau,\sigma]$ so that the last term above may be written as

$$\int_{\sigma-\tau}^{\sigma} \varphi(s)d_s \int_{\sigma}^{\sigma+\tau} p(\alpha,s)Y(\alpha,t)d\alpha.$$

If $\sigma+\tau > t$, then we may still write the last term this way since $Y(\alpha,t) = 0$ for $\alpha > t$. Hence we have

$$x(t) = \varphi(\sigma)Y(\sigma,t) + \int_{\sigma}^{t} f(\alpha)Y(\alpha,t)d\alpha + \int_{\sigma-\tau}^{\sigma} \varphi(s)d_s \int_{\sigma}^{\sigma+\tau} \eta(\alpha,s-\alpha)Y(\alpha,t)d\alpha. \tag{7}$$

This is a variation-of-constants formula for solutions x of (3) in terms of a particular adjoint solution Y of (4). It should be noted that both the adjoint (4) and the representation in (7) differ from those of Halanay in [4]. This author has been unable to find any meaningful representations using the type of systems given as the adjoint system in [4].

We next let $X(t,\sigma)$ be a solution as a function of t for $t > \sigma$ to (3) with $f \equiv 0$ and

$$(8) \qquad \begin{aligned} X(t,\sigma) &= 0 \quad \text{for} \quad t < \sigma \\ X(\sigma,\sigma) &= E. \end{aligned}$$

From the representation given in (7) we have that the rows $x_i(t)$ of $X(t,\sigma)$ are given by

$$x_i(t) = x_i(\sigma)Y(\sigma,t)$$

for $t > \sigma$. Hence we obtain

$$X(t,\sigma) = X(\sigma,\sigma)Y(\sigma,t)$$

or

$$X(t,\sigma) = Y(\sigma,t) \quad \text{for} \quad t > \sigma.$$

From the definitions of Y and X for $\sigma \geq t$ we find that $X(t,\sigma) = Y(\sigma,t)$ for all σ,t. Using this result in (7) we find that the solution x of (3) with $x = \varphi$ on $[\sigma-\tau,\sigma]$ is given by

$$\text{(9)} \qquad x(t) = \varphi(\sigma)X(t,\sigma) + \int_\sigma^t f(\alpha)X(t,\alpha)d\alpha$$

$$+ \int_{\sigma-\tau}^{\sigma} \varphi(s)d_s\int_\sigma^{\sigma+\tau} \eta(\alpha, s-\alpha)X(t,\alpha)d\alpha.$$

§2. The general adjoint system and the associated bilinear form.

In this section we shall discuss the general adjoint system to (3) (of which (4) is a special case). A bilinear form associated with the system (3) and its adjoint will be given.

Let $-\infty < a < b < \infty$ with $a << b$. Let x be a solution to (3) on $[a,b]$ with $f \equiv 0$; that is

$$\dot{x}(t) = \int_{t-\tau}^{t} x(s)\,d_s p(t,s) \quad t \in [a,b] \tag{10}$$

with $x(t) = \varphi(t)$ for $a-\tau \leqq t \leqq a$, where φ is of bounded variation. Let y be a solution of

$$\frac{d}{ds}\{y(s) + \int_{s}^{\infty} p(\alpha,s)y(\alpha)d\alpha\} = 0 \tag{11}$$

for $s \in [a,b]$ and $y(\alpha) = \psi(\alpha)$ for $\alpha \in [b,\infty)$, where ψ is of bounded variation. Note that for $s < b$, equation (11) may be written

$$\frac{d}{ds}\{y(s) + \int_{s}^{b+\tau} p(\alpha,s)y(\alpha)d\alpha\} = 0$$

since $\int_{b+\tau}^{\infty} p(\alpha,s)y(\alpha)d\alpha$ is constant in s for $s < b$. Thus to solve (11) for $s < b$, one need give the initial function ψ only on $[b,b+\tau]$; that is, one may essentially take $\psi = 0$ on $(b+\tau,\infty)$.

Let $a < \sigma < t < b$. At this point we assume that $\sigma+\tau < t$. We shall later remove this restriction. Using integration by parts as in §1 we obtain

$$(12) \qquad x(t)y(t)-x(\sigma)y(\sigma) = \int_{\sigma}^{t} x(\alpha)dy(\alpha)$$

$$+ \int_{\sigma}^{t}\{ \int_{\alpha-\tau}^{\alpha} x(s)d_s p(\alpha,s)\} y(\alpha)d\alpha.$$

We now consider the last term on the right in (12). As in §1, this may be written

$$\int_{\sigma}^{t}\{ \int_{\sigma-\tau}^{t} x(s)d_s p(\alpha,s)\} y(\alpha)d\alpha$$

which becomes, when one uses the unsymmetric Fubini type theorem,

$$\int_{\sigma-\tau}^{t} x(s)d_s \int_{\sigma}^{t} p(\alpha,s)y(\alpha)d\alpha.$$

Using the fact that $\sigma+\tau < t$ and $p(\alpha,s) = 0$ for $\alpha \leqq s$, this term can be written

$$\int_{\sigma-\tau}^{\sigma} x(s)d_s \int_{\sigma}^{t} p(\alpha,s)y(\alpha)d\alpha$$

$$+ \int_{\sigma}^{t-\tau} x(s)d_s \int_{s}^{t} p(\alpha,s)y(\alpha)d\alpha$$

$$+ \int_{t-\tau}^{t} x(s)d_s \int_{s}^{t} p(\alpha,s)y(\alpha)d\alpha.$$

Since for s in $[\sigma-\tau,\sigma]$, the term

$$\int_{\sigma+\tau}^{t} p(\alpha, s)y(\alpha)d\alpha$$

is a constant function of s, the three integrals above are equal to

$$\begin{aligned}(13)\qquad &\int_{\sigma-\tau}^{\sigma} x(s)d_s\int_{\sigma}^{\sigma+\tau} p(\alpha, s)y(\alpha)d\alpha \\ &+ \int_{\sigma}^{t-\tau} x(s)d_s\int_{s}^{t} p(\alpha, s)y(\alpha)d\alpha \\ &+ \int_{t-\tau}^{t} x(s)d_s\int_{s}^{t} p(\alpha, s)y(\alpha)d\alpha.\end{aligned}$$

Adding and subtracting the term

$$\int_{t-\tau}^{t} x(s)d_s\int_{t}^{t+\tau} p(\alpha, s)y(\alpha)d\alpha$$

to (13) we obtain

$$\begin{aligned}&\int_{\sigma-\tau}^{\sigma} x(s)d_s\int_{\sigma}^{\sigma+\tau} p(\alpha, s)y(\alpha)d\alpha \\ &+ \int_{\sigma}^{t-\tau} x(s)d_s\int_{s}^{t} p(\alpha, s)y(\alpha)d\alpha \\ &- \int_{t-\tau}^{t} x(s)d_s\int_{t}^{t+\tau} p(\alpha, s)y(\alpha)d\alpha \\ &+ \int_{t-\tau}^{t} x(s)d_s\int_{s}^{t+\tau} p(\alpha, s)y(\alpha)d\alpha\end{aligned}$$

which may be written

$$(14)\quad \begin{aligned} &\int_{\sigma-\tau}^{\sigma} x(s)\,d_s\int_{\sigma}^{\sigma+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &+ \int_{\sigma}^{t-\tau} x(s)\,d_s\int_{s}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &- \int_{t-\tau}^{t} x(s)\,d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &+ \int_{t-\tau}^{t} x(s)\,d_s\int_{s}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha \quad , \end{aligned}$$

since for s in $[\sigma,t-\tau]$, the term

$$\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha$$

is constant in s.

Combining two of the terms in (14), one obtains

$$(15)\quad \begin{aligned} &\int_{\sigma-\tau}^{\sigma} x(s)\,d_s\int_{\sigma}^{\sigma+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &+ \int_{\sigma}^{t} x(s)\,d_s\int_{s}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &- \int_{t-\tau}^{t} x(s)\,d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha \quad . \end{aligned}$$

Substituting (15) for the last term in (12) gives

$$(16)\quad \begin{aligned} x(t)y(t) - x(\sigma)y(\sigma) &= \int_{\sigma}^{t} x(s)\,dy(s) \\ &+ \int_{\sigma}^{t} x(s)\,d_s\int_{s}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &+ \int_{\sigma-\tau}^{\sigma} x(s)\,d_s\int_{\sigma}^{\sigma+\tau} p(\alpha,s)y(\alpha)\,d\alpha \\ &- \int_{t-\tau}^{t} x(s)\,d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha . \end{aligned}$$

Now for s in $[\sigma,t]$ we have that

$$\int_{t+\tau}^{\infty} p(\alpha,s)y(\alpha)d\alpha$$

is constant in s. Hence the second term on the right side of (16) may be written

$$\int_{\sigma}^{t} x(s)d_s\int_{s}^{\infty} p(\alpha,s)y(\alpha)d\alpha.$$

Thus equation (16) may be written

$$x(t)y(t) + \int_{t-\tau}^{t} x(s)d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)d\alpha$$

$$= x(\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} x(s)d_s\int_{\sigma}^{\sigma+\tau} p(\alpha,s)y(\alpha)d\alpha$$

$$+ \int_{\sigma}^{t} x(s)d_s\{y(s) + \int_{s}^{\infty} p(\alpha,s)y(\alpha)d\alpha\}.$$

If y is a solution of (11), we obtain

$$x(t)y(t) + \int_{t-\tau}^{t} x(s)d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)d\alpha$$

$$= x(\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} x(s)d_s\int_{\sigma}^{\sigma+\tau} p(\alpha,s)y(\alpha)d\alpha. \tag{17}$$

We have thus shown: If we define $\mathscr{B}_t(x,y)$ as

$$\mathscr{B}_t(x,y) = x(t)y(t) + \int_{t-\tau}^{t} x(s)d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)d\alpha, \tag{18}$$

then along solutions x and y of (10) and (11) we have

$$\mathscr{B}_\sigma(x,y) = \mathscr{B}_t(x,y)$$

whenever $\sigma < t$ and $\sigma+\tau < t$.

It is easy to see that if $F(\sigma) = F(t)$ for σ and t arbitrary such that $\sigma < t$ and $\sigma+\tau < t$, then $F(\sigma) = F(t)$ for arbitrary σ and t, $\sigma < t$. This implies that F is a constant function.

Hence, along solutions x and y of (10) and (11), the function $\mathscr{B}_t(x,y)$ defined in (18) is a constant in t, $a < t < b$.

We now have (17) holding for arbitrary $\sigma < t$, σ,t in $[a,b]$. We next fix t. We are interested in a representation for $y(\sigma)$ for arbitrary $\sigma < t$.

For any $\sigma < t$, let $X(\alpha,\sigma)$ be a solution of (10) as a function of α for $\alpha > \sigma$, satisfying $X(\alpha,\sigma) = 0$ for $\alpha < \sigma$, $X(\sigma,\sigma) = E$. Then (17) gives

$$X(t,\sigma)y(t) + \int_{t-\tau}^{t} X(s,\sigma)\,d_s\int_t^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha$$

$$= X(\sigma,\sigma)y(\sigma) + \int_{\sigma-\tau}^{\sigma} X(s,\sigma)\,d_s\int_\sigma^{\sigma+\tau} p(\alpha,s)y(\alpha)\,d\alpha.$$

This is the same as

$$y(\sigma) = X(t,\sigma)y(t) + \int_{t-\tau}^{t} X(s,\sigma)\,d_s\int_t^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha. \tag{19}$$

From this one sees that knowing y on $[t,t+\tau]$ and $X(s,\sigma)$ for each $\sigma < t$ is sufficient to find $y(\sigma)$ for $\sigma < t$.

As we have already seen, if $Y(s,t)$ is a solution of (4), (this is actually the system (11) with $Y(s,t)$ specified as $Y(s,t) = 0$ for $s > t$, $Y(t,t) = E$ and $Y(s,t)$ satisfying (11) as a function of s for $s < t$), then Y and the matrix function X in (19) are related by $X(t,\alpha) = Y(\alpha,t)$. Hence, (19) may be written

$$(20) \qquad y(\sigma) = Y(\sigma,t)y(t) + \int_{t-\tau}^{t} Y(\sigma,s)\,d_s\int_{t}^{t+\tau} p(\alpha,s)y(\alpha)\,d\alpha.$$

The formula (20) expresses $y(\sigma)$ for $\sigma < t$ as a function of $y(\alpha)$, $\alpha \in [t,t+\tau]$ and the solution matrices $Y(s,\alpha)$, $\alpha \in [t-\tau,t]$, of (11) with "initial functions" $\Phi(s) = 0$ for $s > \alpha$, $\Phi(\alpha) = E$.

Note that these results agree with the previous remarks about system (11) being written as

$$\frac{d}{ds}\{y(s) + \int_{s}^{b+\tau} p(\alpha,s)y(\alpha)\,d\alpha\} = 0$$

for $s < b$, where one specifies ψ on $[b,b+\tau]$ to obtain a solution $y(s)$, $s < b$.

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