A Note on Volterra Integral Equations and Topological Dynamics*

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*The first author was supported in part by the Air Force Office of Scientific Research, Office of Aerospace Research, United States Air Force under AFOSR Grant No. AF-AFOSR-693-67 and by the National Aeronautics and Space Administration under Contract No. NAS8-11264 and Grant No. NGR 40-002-015. The second author was supported in part by the National Science Foundation under Grant No. GP-7041X and the United States Army under Contract No. DA-31-124-ARO-D265.

1. Introduction. In a recent paper, G. R. Sell [5], [6], has developed methods which allow one to apply the theory of topological dynamics to a very general class of nonautonomous ordinary differential equations. The purpose of this note is to illustrate how the methods of Sell can be extended to nonlinear Volterra integral equations of the form

$$x(t) = f(t) + \int_0^t a(t, s)g(x(s), s)ds \quad (1)$$

A complete discussion of our results along with the proofs of the theorems noted here will appear in [3] and [4]. In this note we shall restrict ourself to a description of the semi-flow generated by (1), and we do this in the case where x, f, a and g are real-valued.

Because of the generality of our methods, they can be applied to many problems. Some of these applications are treated in [4]. We shall illustrate our techniques by analyzing a problem of J. Levin [1] in Section 5.

2. Construction of the Semi-Flow. A flow is defined to be a mapping $\pi : X \times R \longrightarrow X$, where X is a topological space and R the reals, that satisfies (i) $\pi(x, 0) = x$, (ii) $\pi(\pi(x, t), s) = \pi(x, t + s)$ and (iii) π is continuous. A (local) flow was defined in [5], and for this note we need the concept of a (local) semi-flow, in which we restrict t to be

nonnegative. A local flow differs from a flow in the sense that motions $\pi(x, t)$ may fail to exist for all time t. We refer the reader to [5] and [7] for details.

For Equation (1), the semi-flow is constructed as follows: Let $\varphi(t) = \varphi(f, g, a; t)$ denote the solution of (1). Under hypotheses on f, g and a which are stated below, it is shown in [3], that φ is uniquely determined and depends continuously on f, g, a and t. Now define the function $T_\tau f$ by

$$(2) \qquad T_\tau f(\theta) = f(\tau + \theta) = \int_0^\tau a(\tau + \theta, s) g(\varphi(s), s) ds$$

for $\theta \geq 0$ and τ in the interval of definition of φ. Define g_τ and a_τ by

$$g_\tau(x, s) = g(x, \tau + t)$$
$$a_\tau(t, s) = a(\tau + t, \tau + s)$$

for $\tau \geq 0$, $0 \leq s \leq t < \infty$ and all x. A topological space X, which is defined below, consists of ordered triples (f, g, a), and the mapping π is defined by

$$(3) \qquad \pi(f, g, a; \tau) = (T_\tau f, g_\tau, a_\tau), \qquad (\tau \geq 0).$$

Our object is to show that the mapping π defines a semi-flow. Most of the defining conditions are easily checked. The continuity of π is the only difficult item, and this, of course, depends on the topology on X.

3. Admissible and Compatible Topologies. Let $\mathcal{C} = \mathcal{C}(R^+, R)$ denote the space of real-valued continuous functions defined on R^+. Assume that $\mathcal{C}$ has the topology of uniform

convergence on compact sets. We shall assume that the term $f(t)$, from (1), lies in $\mathcal{C}$. The terms $g(x,t)$ will be assumed to be in a linear topological space $\mathcal{G}$, and the kernels $a(t,s)$ belong to a linear topological space $\mathcal{A}$.

Definitions. The space $\mathcal{G}$ is said to be admissible if the mapping

$$(g, \tau) \longrightarrow g_\tau$$

of $\mathcal{G} \times R^+$ into $\mathcal{G}$ is continuous. The space $\mathcal{A}$ is said to be admissible if the two mappings

$$(a, \tau) \longrightarrow a_\tau \qquad \text{and} \qquad \tau \longrightarrow a(\tau + \cdot, \cdot)$$

of $\mathcal{A} \times R^+$ into $\mathcal{A}$ and R^+ into $\mathcal{A}$ are continuous. We say that the pair $(\mathcal{G}, \mathcal{A})$ is compatible if

(1) Both $\mathcal{G}$ and $\mathcal{A}$ are admissible and

(2) For every $f \in \mathcal{C}$, $g \in \mathcal{G}$, $a \in \mathcal{A}$, Equation (1) admits a unique solution $\varphi(f, g, a; t)$ and furthermore φ depends continuously on f, g, a and t.

Theorem 1. Let $(\mathcal{G}, \mathcal{A})$ be compatible linear topological spaces. Then the mapping π given by (3) defines a semi-flow on $X = \mathcal{C} \times \mathcal{G} \times \mathcal{A}$.

The question of admissible and compatible spaces is discussed at length in [3] and [4]. We present here just two examples of compatible spaces. These are chosen because many applications fit into this format.

The space $\mathcal{G}_p$, $1 < p < \infty$. The collection of all measurable functions $g(x,t) : R \times R^+ \longrightarrow R$ such that for each compact set $K \subset R$ there exist locally L_p functions $m(t)$ and $k(t)$ such that

$$|g(x,t)| \leq m(t)$$

$$|g(x,t) - g(y,t)| \leq k(t)|x - y|$$

for all x, y in K and t in R^+. We define a topology on $\mathcal{G}_p$ by saying that a generalized sequence $\{g_n\}$ converges to g if for every compact interval $I \subset R^+$ and every compact set $\mathcal{X} \subset \mathcal{C}(I, R)$ (where $\mathcal{C}(I, R)$ denotes the Banach space of real-valued continuous functions defined on I) one has

$$\int_I |g_n(x(s), s) - g(x(s), s)|^p \, ds \longrightarrow 0$$

uniformly for $x(\cdot)$ in $\mathcal{X}$.

<u>The space</u> $\mathcal{G}_\infty$. The collection of all continuous functions $g(x,t) : R \times R^+ \longrightarrow R$ such that for each compact set $K \subset R$ there exists a continuous function $k(t)$ such that

$$|g(x,t) - g(y,t)| \leq k(t)|x - y|$$

for all x, y in K and t in R^+. The topology on $\mathcal{G}_\infty$ is the topology of uniform convergence on compact sets.

<u>The space</u> $\mathcal{A}_p$, $1 < p < \infty$. The collection of all real-valued measurable functions $a(t,s)$ defined for $0 \leq s \leq t \leq \infty$ such that (i) for each t, $a(t,s)$ is locally L_q in s where $p^{-1} + q^{-1} = 1$ and (ii) for each compact interval $I \subset R^+$ and every $t \geq 0$ one has

$$\int_I |a(t+h, s) - a(t,s)|^q \, ds \longrightarrow 0$$

as $h \longrightarrow 0$. The topology on $\mathcal{A}_p$ is defined by saying that a generalized sequence $\{a_n\}$ converges to a if for every compact interval $I \subset R^+$ one has

$$\int_I |a_n(t, s) - a(t, s)|^q \, ds \longrightarrow 0$$

uniformly for t in compact sets.

<u>The space</u> $\mathcal{A}_\infty$. The collection of all real-valued measurable functions $a(t, s)$ defined for $0 \leq s \leq t < \infty$ and such that (i) for each t, $a(t, s)$ is locally L_1 in s and (ii) for each compact interval $I \subset R^+$ the mapping $t \longrightarrow a(t, \cdot)$ is continuous in the weak*-topology on $\mathcal{C}(I, R)^*$. The topology for $\mathcal{A}_\infty$, which is a weak topology, is defined by saying that a generalized sequence $\{a_n\}$ converges to a if for every compact interval $I \subset R^+$ and every $x(\cdot) \in \mathcal{C}(I, R)$ one has

$$\int_I [a_n(t, s) - a(t, s)]x(s)ds \longrightarrow 0$$

uniformly for t in compact sets.

Theorem 2. <u>The spaces</u> $(\mathcal{G}_p, \mathcal{A}_p)$ <u>and</u> $(\mathcal{G}_\infty, \mathcal{A}_\infty)$ <u>are compatible.</u>

Other topologies are given in [3] and [4]. An interesting feature that arises in our study is that as one weakens the topology on the kernels $a(t, s)$, it is necessary to strengthen the topology on the terms $g(x, t)$ in order to preserve the compatibility of the pair $(\mathcal{G}, \mathcal{A})$. We refer the reader to the main papers for details.

4. <u>Compact Motions and Limiting Equations</u>. Two of the basic problems in applying topological dynamics is to determine conditions under which a motion $\pi(x, t)$ is compact, that is, $\pi(x, t)$ remains in a compact set for all $t \geq 0$, and then to analyze the asymptotic behavior of $\pi(x, t)$ in terms of the ω-limit set. In [4] sufficient (and sometimes necessary) conditions are given in order that the motions g_τ and a_τ be compact in $\mathcal{G}$ and $\mathcal{A}$. Because of lack of space, we will not formulate these results here. As for the motion $\pi(f, g, a; \tau)$, one can prove the following result:

Theorem 3. <u>Let</u> $(\mathcal{G}, \mathcal{A})$ <u>be a compatible pair and assume that the motions</u> g_t <u>in</u> $\mathcal{G}$ <u>and</u> a_t <u>in</u> $\mathcal{A}$ <u>are compact. If the solution</u> $\varphi(t) = \varphi(f, g, a; t)$ <u>of</u> (1) <u>lies in a compact set for all</u> $t \geq 0$, <u>then the motion</u> $\pi(f, g, a; t)$ <u>remains in a compact set for all</u> $t \geq 0$.

The problem here is to show that the family $\{T_\tau f : \tau \geq 0\}$ lies in a compact set in $\mathcal{C}$. This follows from the notion of compatibility and the fact that $T_\tau f(\theta)$ can be formulated as

$$T_\tau f(\theta) = \varphi(\tau + \theta) - \int_0^\theta a_\tau(\theta, s) g_\tau(\varphi_\tau(s), s) ds. \tag{4}$$

Once one knows that a motion $\pi(f, g, a; \tau)$ is compact, then its ω-limit set Ω is compact and nonempty. A typical point (F^*, g^*, a^*) in Ω is characterized as the limit of $\pi(f, g, a; \tau_n)$ for some sequence $\tau_n \longrightarrow \infty$. One can show that in this case, the translates $\varphi(t + \tau_n)$ converge uniformly for t in compact sets to a function $\varphi^*(t)$. Under appropriate integrability conditions on $a(t, s)$ and $g(x(s), s)$ one can show that $\varphi^*(t)$ satisfies

(5) $$\varphi^*(t) = f^*(t) + \int_{-\infty}^{t} a^*(t,s)g^*(\varphi^*(s),s)ds$$

where $f_{\tau_n} \longrightarrow f^*$ in $\mathcal{C}$. In other words, the function F^* is of the form

$$F^*(\theta) = f^*(\theta) + \int_{-\infty}^{0} a^*(\theta,s)g^*(\varphi^*(s),s)ds.$$

This generalizes a result of R. K. Miller [2].

5. An Application. The following problem, which is a generalization of a problem of J. Levin [1] follows easily with our techniques.

Theorem 4. *Consider the equation*

(6) $$x(t) = f(t) - \int_0^t \{a(t-s) + b(t,s)\} g(x(s))ds$$

where (1) $f(t)$ *is continuous for* $t \geq 0$ *and* $f(t) \rightarrow f_0$ *as* $t \longrightarrow \infty$,

(2) $a(r) \in \mathcal{C}'[0,\infty) \cap L_1(0,\infty)$ *and* $a(r) \geq 0$,

(3) $a'(r) \leq 0$ *and* $a'(r) \not\equiv 0$ *on any interval* $[0,T]$, $T > 0$,

(4) $b \in \mathcal{A}_\infty$ *and* $\int_\tau^{\tau+1} |b(s+r,s)|ds \longrightarrow 0$, *as* $\tau \longrightarrow \infty$, *uniformly for* r *in compact sets,*

(5) $g(x)$ *is locally Lipschitzian and strictly increasing.*

Assume that the solution $\varphi(t) = \varphi(f,g,a;t)$ *of* (6) *is bounded for all* $t \geq 0$ *and let* x_0 *be the solution of*

$$x_0 = f_0 - Ag(x_0)$$

where $A = \int_0^\infty a(r)dr$. *Then* $\varphi(t) \longrightarrow x_0$ *as* $t \longrightarrow \infty$.

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