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On the local structure of hyperbolic points in Banach spaces.

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It has been proved by Hartman [2,3], answering a question raised by Peixoto, that a diffeomorphism or a flow on R^n with a hyperbolic fixed point is locally topologically equivalent to its linear part. This fact, basic for the generic theory of flows and diffeomorphisms was also proved independently by Grobman [1].

In this paper we extend this theorem for Banach spaces.

We shall make use of a key idea presented by Moser in [5], where he gave a new proof of the structural stability of the Anosov diffeomorphisms (see also [4]). As in the finite dimensional case, we do not need to assume the existence of invariant manifolds associated to the fixed point.

I wish to thank M. Hirsch, I. Kupka and M. Shub for helpful conversations.*

2. Preliminaries.

Definition 1. Let E be a Banach space and $L: E \to E$ be a linear map. L is called hyperbolic if its sprectrum is disjoint from the unit circle S^1 in the complex plane.

Definition 2. Let $0 \in E$ and V be a neighborhood of 0 in E. A diffeomorphism $f: V \to E$ has 0 as a hyperbolic fixed point if $(Df)_0$ is hyperbolic.

*After the completion of this paper, I received a manuscript by C. Pugh proving the same result.

Lemma 1. Let L be a hyperbolic linear isomorphism in E, E with norm $\| \|_1$. There exist subspaces E_s and E_u, invariant under L, and a norm $\| \|_2$ equivalent to $\| \|_1$ in E such that $E = E_s \oplus E_u$, $\|L/E_s\|_2 < 1$ and $\|L^{-1}/E_u\|_2 < 1$.

Proof. The existence of E_s and E_u is given by showing that

$$P = -1/2\pi i \int_{S^1} R_\gamma d\gamma$$

is a projection map. R_γ is the resolvent transformation for L (see [6] for more details). Define $E_s = \{z \in E \mid P(z) = z\}$ and $E_u = \{z \in E \mid P(z) = 0\}$. The spectrum of $L_s = L/E_s$ has radius $r_s < 1$. So there exists a positive integer p and a real number a, $r_s < a < 1$, such that $\|L_s^p\|_1 < a^p$. From the convergence of $\sum_{n=0}^{\infty} a^{-pn}\|L^{pn}\|_1$ we get the convergence of $\sum_{n=0}^{\infty} a^{-n}\|L_s^{-n}\|_1$. Set on E_s:

$$\|v\|_2 = \sum_{n=0}^{\infty} a^{-n}\|L_s^n\|_1$$

Clearly $\| \|_2$ is equivalent to $\| \|_1$ and $\|L_s\|_2 < a$. The procedure for $L_u = L/E_u$ is similar. The lemma is proved.

Lemma 2. Let Φ: $V(0) \to E$ be a C^1 map, $\Phi(0) = 0$ and $(D\Phi)_0 = 0$. Given any $\epsilon > 0$, Φ restricted to a smaller neighborhood of 0 can be extended to all of E, the extension being bounded and having a global Lipschitz constant less than ϵ.

Proof. From the continuity of $D\Phi$, we can find $\rho > 0$ so that $\|(D\Phi)_z\| < \epsilon/3$ for $\|z\| < \rho$. Let α be a real C^∞ function satisfying $0 \leqq \alpha(t) \leqq 1$, $\alpha(t) = 1$ for $|t| < 1/2$, $\alpha(t) = 0$ for $|t| \geqq 1$ and $|D\alpha| \leqq 2$. Define

$$\tilde{\Phi}(z) = \alpha(\|z\|/\rho)\ \Phi(z) \quad \text{for} \quad \|z\| \leqq \rho$$

$$\tilde{\Phi}(z) = 0 \qquad\qquad\qquad \text{for} \quad \|z\| > \rho$$

Then $\tilde{\Phi}(z) = \Phi(z)$ for $\|z\| \leqq \rho/2$ and $\tilde{\Phi}$ has Lipschitz constant less than ϵ. The lemma is proved.

3. The main result

Theorem 1. Let $0 \in E$ and V be a neighborhood of 0 in E. Let $f: V \to E$ be a C^1 diffeomorphism having 0 as a hyperbolic fixed point. There exist a neighborhood U of 0 in V and a homeomorphism $h: U \to E$ such that $hL(z) = fh(z)$ for all $z \in U$, where $L = (Df)_0$.

Proof. Let $f = L + \Phi$ be the local expression of f. Extending Φ to $\tilde{\Phi}$ as in lemma 2, we shall look for a homeomorphism $h: E \to E$ satisfying $hL = (L + \tilde{\Phi})h$. Let $C_b^0(E)$ denote the Banach space of continuous bounded functions from E into E with the sup norm. Writting $h = Id + v$, where Id means the identity map in E, the above conjugacy becomes:

$$(Id + v)L = (L + \tilde{\Phi})(Id + v)$$

or equivalently

$$vL - Lv = \tilde{\Phi}(Id + v) \tag{1}$$

We now show that (1) has a unique solution in $C_b^o(E)$, if the Lipschitz constant of $\tilde{\Phi}$ is small enough.

Consider the operator $\mathcal{L}$: $C_b^o(E) \to C_b^o(E)$ defined by $\mathcal{L}(v) = vL - Lv$. $\mathcal{L}$ is an isomorphism. This is a consequence of the fact that the operator $\mathcal{L}^*$: $C_b^o(E) \to C_b^o(E)$, defined by $\mathcal{L}^*(v) = LvL^{-1}$ is hyperbolic. To see this we write $\mathcal{L}_s^* = \mathcal{L}^*/C_b^o(E,E_s)$ and $\mathcal{L}_u^* = \mathcal{L}^*/C_b^o(E,E_u)$ and notice that $\mathcal{L}_s^*$ and $(\mathcal{L}_u^*)^{-1}$ have norm less than one, since from lemma 1 we may assume that this is the case for the norms of L_s and L_u^{-1}. Thus $\mathcal{L}^* = \mathcal{L}_s^* \oplus \mathcal{L}_u^*$ is hyperbolic.

To solve the equation (1) is equivalent to find a fixed point for the map μ: $C_b^o(E) \to C_b^o$ defined by $\mu(v) = \mathcal{L}^{-1}\tilde{\Phi}(Id + v)$. If the Lipschitz constant of $\tilde{\Phi}$ is less than ϵ and $\max(\|L_s\|, \|L_u^{-1}\|) < a < 1$ we get

$$\|\mu(v) - \mu(w)\| \leq \epsilon(a-1)^{-1}\|v-w\|.$$

Thus taking $\epsilon(a-1)^{-1} < 1$ we have, by the Contraction Principle, a unique solution $v \in C_o^b$ for (1).

Reversing the argument we get a unique solution for the equation:

$$(Id + u)(L + \tilde{\Phi}) = L(Id + u) \tag{2}$$

Finally, we notice that from (1) and (2) we have

$$(Id + u)(Id + v)L = L(Id + u)(Id + v)$$

and from the uniqueness above $(Id + u)(Id + v) = Id$. Therefore $Id + v$ is a homeomorphism and the theorem is proved.

4. Local Stability.

Theorem 2. Let f be a local C^1 diffeomorphism as in Theorem 1. There exists a neighborhood N of f, C^1 topology, such that for each $g \in N$ there is a local homeomorphism in E satisfying $gh = hf$.

Proof. It is a consequence of Theorem 1 and the following two lemmas:

Lemma 3. Let f be as in Theorem 1. Then there exist a neighborhood N of f, C^1 topology, a neighborhood U of 0 in E and a continuous map $\gamma: N \to U$ such that $\gamma(T) = 0$ and for each $g \in N$ $g(\gamma(g)) = \gamma(g)$.

Proof. Consider the map $\psi: C_b^1(V,E) \times V \to E$ defined by $\psi(g,x) =$

$g(x)-x$. Clearly $\psi(f,0) = 0$ and $(\partial\psi/\partial x)_{(f,0)} = (Df)_0 - Id$ is an isomorphism for $(Df)_0$ is hyperbolic. The lemma is then proved, using the Implicit Function Theorem.

Lemma 4. In the Banach space of linear transformations of E into E the set of hyperbolic maps is open.

Proof. From the fact that the set of linear isomorphisms is open and S^1 is compact.

5. The flow case. We now extend Theorem 1 to the case of flows in Banach spaces. Let φ_t be a C^1 flow in E having 0 as a critical point. φ_t is hyperbolic at 0 if the induced diffeomorphism at time $t = 1$, φ_1, has 0 as a hyperbolic fixed point. If this is the case, we claim the existence of a local homeomorphism H taking trajectories of φ_t into those of L_t, L_t being the associated linear flow. Using a device appearing in [7], we set

$$H = \int_0^1 L_{-t} h \varphi_t dt$$

where $L_{-1}h\varphi_1 = h$ as in Theorem 1. We verify now that $L_{-s}H\varphi_s = H$ for all $s \in R$. It is enough to consider $s \in [0,1]$. $L_{-s}H\varphi_s =$ $L_{-s}(\int_0^1 L_{-t}h\varphi_t dt)\varphi_s$ or $L_{-s}H\varphi_s = \int_0^1 L_{-s-t}h\varphi_{t+s}dt$. Taking $u = t+s-1$, the left hand side becomes $\int_{-1+s}^{s} L_{-u-1}h\varphi_{u+1}du = \int_{-1+s}^{0} L_{-u}L_{-1}h\varphi\varphi_u du +$ $\int_0^s L_{-u-1}h\varphi_{u+1}du$. And finally, making $v = u+1$ in the second summand we get

$$\int_0^s L_{-u} h\varphi_u du + \int_s^1 L_{-v} h\varphi_v dv = H.$$

In particular, $L_{-1}H\varphi_1 = H$ and from the uniqueness of solution of this equation in $C^0(E)$ at finite distance from the identity map, we have $H = h$. The assertion is proved.

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