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# A REVIEW OF THE PERTURBATION THEORY AS APPLIED TO THE DETERMINATION OF GEOPOTENTIAL

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## ABSTRACT

*A review of close satellite theory as applied to determine geopotential from the orbital motion of artificial earth satellites is given. The most recent dynamical solutions (as of June, 1966) for geopotential coefficients obtained from the optical and Doppler tracking of satellites are used to compute the earth's anomalous gravity field in terms of geoidal undulations and gravity anomalies with respect to the various reference ellipsoids. A 'mean solution,' obtained from a linear combination of three of the available solutions, is also used for the same purpose. The maximum departure of the geoid from the surface of the international reference ellipsoid is much greater than that usually obtained from surface gravity data. As expected, the shorter wavelength component seems to be most poorly represented in the satellite data.*

## Introduction

This report is composed of three sections. In the first section we present a summary of the theory of the method used to determine the geopotential coefficients from orbital perturbations.

In the second section we describe, very briefly, the process of determining geoidal undulations and gravity anomalies from a known set of geopotential coefficients. We explain the various assumptions that are usually made in the actual computational procedures and explore their influence on the results.

In the third section we describe the results we obtained and tabulate the data used in the computation of the results.

One purpose of this report is to provide a readily available reference to the students of satellite geodesy at this Institute.

## Perturbed Equations of Motion

Let the rectangular coordinates of a system of particles be expressed by $x_i$, $y_i$, $z_i$ where i = 1, 2,...i. Let these coordinates be functions of independent quantities $q_s$, (s = 1, 2,...) and possibly of time t. The $q_s$ are called the generalized coordinates and as noted above, are the minimum number of independent coordinates required to define the system. We consider only holonomic systems. A system in which arbitrary infinitesimal changes in the coordinates describing it are possible is said to be holonomic. The time derivatives of the generalized coordinates are called generalized velocities and are customarily denoted by $\dot{q}_s$ . Denote the Lagrangian function of the system by L. Then with the limitation that the potential energy V is not a function of velocities, we obtain the Lagrangian equations of motion in the form (Becker, 1954, p. 331)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_s}\right) - \frac{\partial L}{\partial q_s} = 0, \quad s = 1,2,3 \tag{1}$$

where

T = kinetic energy of the system

V = potential energy

L = T - V

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Define a new set of variable p, given by

$$p_s = \frac{\partial L}{\partial \dot{q}_s} \tag{2}$$

$p_s$ is called generalized momentum corresponding to $q_s$ or the momentum conjugate of $p_s$.

If in the system under discussion, the explicit dependence upon the generalized velocities enters through T only, then the above definition of $p_s$ reduces to (Becker, 1954, p. 335)

$$p_s = \frac{\partial T}{\partial \dot{q}_s} \tag{3}$$

In the expression for Lagrangian L, the potential energy V depends on position and hence is a function of $q_s$ only, while the kinetic energy T is a function of both $q_s$ and $\dot{q}_s$. Hence we can write L as

$$L = L(q_s, \dot{q}_s, t), \; s = 1, \ldots r. \tag{4}$$

Then

$$dL = \sum_{s=1}^{r} \left( \frac{\partial L}{\partial q_s} dq_s + \frac{\partial L}{\partial \dot{q}_s} d\dot{q}_s \right) + \frac{\partial L}{\partial t} dt \tag{5}$$

Eqs. (1) and (2) give

$$\dot{p}_s = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_s} \right) = \frac{\delta L}{\partial q_s} \tag{6}$$

Substituting Eqs. (3) and (6) in (5) we get

$$dL = \sum_{s=1}^{r} (\dot{p}_s \, dq_s + p_s \, d\dot{q}_s) + \frac{\partial L}{\partial t} dt \tag{7}$$

which can be readily rearranged as

$$d\left(\sum_{s=1}^{r} p_s \dot{q}_s - L\right) = \sum_{s=1}^{r} (\dot{q}_s \, dp_s - \dot{p}_s \, dq_s) - \frac{\partial L}{\partial t} dt \qquad (8)$$

Now define a quantity H such that

$$H = \sum_{s=1}^{r} p_s \dot{q}_s - L \qquad (9)$$

Then Eq. (8) becomes

$$dH = \sum_{s=1}^{r} (\dot{q}_s \, dp_s - \dot{p}_s \, dq_s) - \frac{\partial L}{\partial t} dt \qquad (10)$$

The quantity H, defined by Eq. (9) is called the Hamiltonian function of the system and is $H = T + V$ for the system under discussion.

Although L is an explicit function of $q_s$, $\dot{q}_s$ and t and hence is expressible as

$$L = L(q_s, \dot{q}_s, t), \; s = 1, \ldots r$$

the Hamiltonian function H, for the same system can be expressed as

$$H = H(q_s, p_s, t), \; s = 1, \ldots r \qquad (11)$$

This conversion is done by the elimination of the generalized velocities with the help of Eq. (2).

Differentiation of Eq. (11) yields

$$dH = \sum_{s=1}^{r} \frac{\partial H}{\partial q_s} dq_s + \frac{\partial H}{\partial p_s} dp_s + \frac{\partial H}{\partial t} dt \qquad (12)$$

Comparison of Eqs. (10) and (12) gives

$$\frac{\partial H}{\partial p_s} = \dot{q}_s \qquad \text{and} \qquad \frac{\partial H}{\partial q_s} = -\dot{p}_s \tag{13}$$

These equations are called Hamilton's canonical equations of motion. Our problem now is to choose those values of the canonical variables $p_s$ and $q_s$, with the help of which the Hamiltonian equations of motion can be transformed in terms of the Keplerian elements. Many sets of values exist for these variables and each has its own suitability under different conditions. A commonly used set is (Smart, 1961, p. 169) that of Delaunay variables:

$$\begin{aligned} L &= \sqrt{\mu a} = p_1 & \ell &= M = q_1 \\ G &= \sqrt{\mu a\,(1 - e^2)} = p_2 & g &= \omega = q_2 \\ H &= \sqrt{\mu a\,(1 - e^2)}\cos i = p_3 & h &= \Omega = q_3 \end{aligned} \tag{14}$$

In case of a central force field the force F can be expressed as gradient of a scalar U, called the potential, i.e.,

$$F = \nabla U$$

where

$$U = \frac{\mu}{r} \tag{15}$$

The earth's gravity field is, however, non-central so that the actual potential departs from the form given in Eq. (15) by, say, R and assumes the form

$$U = \frac{\mu}{r} + R \tag{15a}$$

The small departure R from the spherical form is known as the disturbing potential.

The kinetic energy T of a particle of unit mass moving with a velocity v is given as

$$T = \frac{1}{2} v^2 \tag{16}$$

The energy equation (Sterne, 1960, p. 9) is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{16a}$$

The Hamiltonian function for the type of system being considered here, can be written as

$$H = T + V$$

or in the sign convention of geodesy, as

$$H = T - U \tag{16b}$$

The mean motion $\bar{n}$ is given by

$$\bar{n} = \mu^{\frac{1}{2}} a^{\frac{-3}{2}} \tag{16c}$$

Substitution of Eqs. (16) and (16a) in Eq. (16b) along with the first member of Eq. (14), gives the Hamiltonian function H in the form

$$H = \frac{\mu^2}{2L^2} - R \tag{16d}$$

With the help of (14), (16c), and (16d), the equations of motion in terms of the Keplerian elements are (Smart, 1961, p. 69) as follows:

$$\dot{a} = \frac{2}{\bar{n}a} \frac{\partial R}{\partial M}$$

$$\dot{e} = \frac{1}{\bar{n}a^2 e} \left[ (1 - e^2) \frac{\partial R}{\partial M} - \sqrt{1 - e^2} \frac{\partial R}{\partial \omega} \right]$$

$$\dot{M} = \bar{n} - \frac{1 - e^2}{\bar{n}a^2 e} \frac{\partial R}{\partial e} - \frac{2}{\bar{n}a} \frac{\partial R}{\partial a}$$

$$\dot{\Omega} = \frac{1}{\bar{n}a^2 (1 - e^2)^{\frac{1}{2}} \sin i} \frac{\partial R}{\partial a} \qquad (17)$$

$$\dot{\omega} = \frac{\sqrt{1 - e^2}}{\bar{n}a^2 e} \frac{\partial R}{\partial e} - \frac{\cot i}{\bar{n}a^2 \sqrt{1 - e^2}} \frac{\partial R}{\partial i}$$

$$\frac{di}{dt} = \frac{1}{\bar{n}a^2 \sqrt{1 - e^2}} \left[ \cot i \frac{\partial R}{\partial \omega} - \operatorname{cosec} i \frac{\partial R}{\partial \Omega} \right]$$

Where a, e, $\Omega$, $\omega$, i, M are the usual orbital elements and $\bar{n}$ the mean motion is also defined as

$$\bar{n} = \frac{M}{t - \tau}$$

where $\tau$ is the epoch of perigee passage.

In Eqs. (17), R has been regarded as the disturbing gravity potential but the above equations are also valid when R is the general disturbing function.

## Conversion of Spherical Harmonic Disturbing Potential to Keplerian Elements

In terms of spherical harmonics the gravity potential V of the earth at an external point can be expressed as

$$V = \frac{GM}{r}\left[1 + \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin\phi)\right] \quad (18)$$

The disturbing potential R is then

$$R = \frac{GM}{r}\left[\sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\ P_{nm}\ (\sin\phi)\right] \quad (19)$$

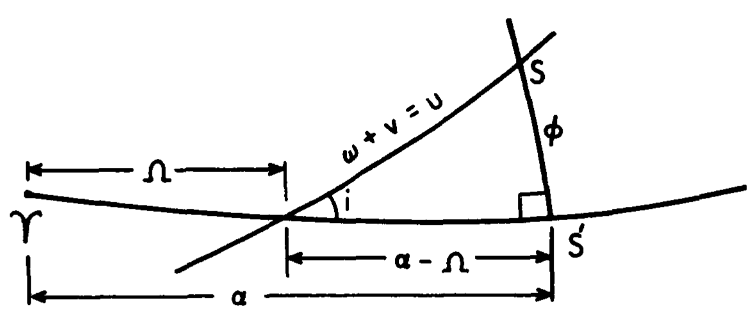
and a general term of R is

$$R_{nm} = \frac{GM}{r}\left(\frac{a_e}{r}\right)^n (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda)\ P_{nm}\ (\sin\phi) \quad (20)$$

In the above expressions $a_e$ is the equatorial radius of the earth and the other notations have the usual meaning.

We wish to express the general term $R_{nm}$ of the disturbing potential in terms of the Keplerian elements.

To do this, first consider the spherical triangle formed by the orbit, the equator, and the satellite meridian.




From well-known formulas of spherical trigonometry, we obtain

$$\frac{\sin i}{\sin \phi} = \frac{\sin u}{\sin \acute{s}}$$ (Smart, 1965, p. 10, Formula B) (21a)

Which gives $\sin \phi = \sin i \sin u$

Also

$$\cos u = \cos(\alpha - \Omega)\cos\phi + \sin(\alpha - \Omega)\sin\phi\cos \acute{s}$$

(Smart, 1965, p. 7, Formula A)

Which gives $\cos(\alpha - \Omega) = \dfrac{\cos u}{\cos \phi}$ (21b)

Again from the same spherical triangle, we can get

$$\sin u \cos i = \cos\phi \sin(\alpha - \Omega) - \sin\phi\cos(\alpha - \Omega)\cos \acute{s}$$

(Smart, 1965, p. 10, Formula C)

which gives $\sin(\alpha - \Omega) = \dfrac{\sin u \cos i}{\cos \phi}$ (21c)

Now let

$\theta$ = the Greenwich sidereal time

then

$\theta + \lambda = \alpha$ or $\lambda = \alpha - \theta$

Write

$$m\lambda = m(\alpha - \theta) = m(\alpha - \Omega) + m(\Omega - \theta)$$

then

$$\cos m\lambda = \cos m(\alpha - \Omega) \cos m(\Omega - \theta) - \sin m(\alpha - \Omega) \sin m(\Omega - \theta)$$

and (22)

$$\sin m\lambda = \sin m(\alpha - \Omega) \cos m(\Omega - \theta) + \cos m(\alpha - \Omega) \sin m(\Omega - \theta).$$

Further, if in the complex variable notation, we mean that

Re = the real part of a complex expression

then

$$\cos m x = \text{Re}(\cos m x + i \sin m x) = \text{Re}(\cos x + i \sin x)^m$$
$$= \text{Re} \sum_{s=0}^{m} \binom{m}{s} i^s \cos^{m-s} x \sin^s x \qquad (23a)$$

and similarly

$$\sin mx = \text{Re} \left[-i(\cos x + i \sin x)^m\right]$$
$$= \text{Re} \sum_{s=0}^{m} \binom{m}{s} i^{s-1} \cos^{m-s} x \sin^s x \qquad (23b)$$

Where $\binom{m}{s}$ is the binomial coefficient and

$$\binom{m}{s} = \frac{m!}{s!\ (m - s)!} \qquad \text{and} \qquad i = \sqrt{-1} \qquad (24)$$

Applying Eqs. (21b), (21c), (23a), and (23b) to Eq. (22) we get (after a little algebraic simplification)

$$\cos m\lambda = \mathrm{Re} \sum_{s=0}^{m} \binom{m}{s} i^{s} \frac{\cos^{m-s} u \sin^{s} u \cos^{s} i}{\cos^{m} \phi} [\cos m(\Omega - \theta) + i \sin m(\Omega - \theta)] \tag{25}$$

$$\sin m\lambda = \mathrm{Re} \sum_{s=0}^{m} \binom{m}{s} i^{s} \frac{\cos^{m-s} u \sin^{s} u \cos^{s} i}{\cos^{m} \phi} [\sin m(\Omega - \theta) - i \cos m(\Omega - \theta)] \tag{26}$$

$P_{nm}$ $(\sin \phi)$ is defined to be (Hobson, 1931, p. 91)

$$P_{nm} (\sin \phi) = \frac{\cos^{m} \phi}{2^{n} n!} \frac{d^{n+m}}{d(\sin \phi)^{n+m}} (\sin^{2} \phi - 1)^{n}$$

Expanding $(\sin^{2} \phi - 1)^{n}$ by binomial theorem, we get

$$P_{nm} (\sin \phi) = \frac{\cos^{m} \phi}{2^{n} n!} \frac{d^{n+m}}{d(\sin \phi)^{n+m}} \sum_{t=0}^{n} \binom{n}{t} (-1)^{t} \sin^{2n-2t} \phi$$

and by successive differentiation

$$P_{nm} (\sin \phi) = \frac{\cos^{m} \phi}{2^{n} n!} \sum_{t=0}^{k} \frac{(2n-2t)!}{(n-m-2t)!} \binom{n}{t} (-1)^{t} \sin^{n-m-2t} \phi$$

where $k = \frac{(n-m)}{2}$ , for (n-m) even

$= \frac{(n-m-1)}{2}$ , for (n-m) odd

Using Eq. (21a) we obtain

$$P_{nm}(\sin\phi) = \frac{\cos^m\phi}{2^n}\sum_{t=0}^{k}\frac{(2n-2t)!\,(-1)^t}{(n-m-2t)!\,n!}\binom{n}{t}\sin^{n-m-2t}i\,\sin^{n-m-2t}u \tag{27}$$

Substituting Eqs. (25), (26), and (27) in Eq. (20) we obtain

$$R_{nm} = \frac{GM}{r}\left(\frac{a_e}{r}\right)^n\frac{1}{2^n}\sum_{t=0}^{k}\frac{(2n-2t)!\,(-1)^t}{(n-m-2t)!\,n!}\binom{n}{t}\sin^{n-m-2t}i$$

$$\cdot\,\mathrm{Re}[(C_{nm} - i\,S_{nm})\cos m(\Omega-\theta) + (iC_{nm} + S_{nm}) \tag{28}$$

$$\cdot\,\sin m(\Omega-\theta)]\sum_{s=0}^{m}\binom{m}{s}i^s\cos^s i\,\underline{\cos^{m-s}u\,\sin^{n-m-2t+s}u}$$

Digress for a moment to consider the identities

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = \frac{-i}{2}(e^{ix} - e^{-ix})$$

Then

$$\sin^a x\,\cos^b x = \left[\frac{-i}{2}(e^{ix} - e^{-ix})\right]^a\left[\frac{e^{ix} + e^{-ix}}{2}\right]^b$$

Expanding the exponential expressions in the brackets by binomial theorem and applying a little algebraic simplification, we obtain

$$\sin^a x \cos^b x = \frac{(-i)^a}{2^{a+b}} \sum_{c=0}^{a} \sum_{d=0}^{b} \binom{a}{c}\binom{b}{d} (-1)^c \,[\cos(a+b-2c-2d)\, x$$

$$+ \, i \sin(a+b-2c-2d)\, x] \tag{29}$$

where use has been made of the identity

$$e^{imx} = \cos mx + i \sin mx$$

Now, applying Eq. (29) to the underlined part of Eq. (28), we get

$$R_{nm} = \frac{GM}{r}\left(\frac{a_e}{r}\right)^n \sum_{t=0}^{k} \frac{(2n-2t)!\,(-1)^t}{2^{2n-2t}(n-m-2t)!\,n!} \binom{n}{t} \sin^{n-m-2t} i$$

$$\cdot \, \mathrm{Re}[(C_{nm} - i\, S_{nm}) \cos m(\Omega - \theta) + (i\, C_{nm} + S_{nm}) \sin m(\Omega - \theta)]$$

$$\sum_{s=0}^{m} \binom{m}{s} \cos^s i \; (-1)^{n-m-2t+s} (i)^{n-m-2t+2s} \sum_{c=0}^{n-m-2t+s} \tag{30}$$

$$\sum_{d=0}^{m-s} \binom{n-m-2t+S}{c} \binom{m-s}{d} (-1)^c \,[\cos(n-2t-2c-2d)u + i$$

$$\sin(n-2t-2c-2d)u]$$

Now apply the usual identities for the products and summations of the trigonometrical functions to that part of Eq. (30) which contains the products of the trigonometrical functions of $(\Omega - \theta)$ and u. Further, since $R_{nm}$ is real, all the terms containing odd powers of $i$ will cancel out. To elaborate:

for n-m even,

$$i^{n-m-2t+2s} = (-1)^{\frac{n-m}{2} - t+s}$$

and as stated above, all other terms containing $i$ to the first power will drop out. Thus $(-1)^{\frac{n-m}{2} - t+s}$ will be multiplied by $(-1)^{n-m-2t+s}$ and $(-1)^{t}$, and the final result will be $(-1)^{3/2\,(n-m)-2(t-s)} = (-1)^{k}$.

For n-m odd, the factor $i^{n-m-2t+2s}$ will reduce to $\pm i$ and hence the terms containing an additional $i$ will have $i$ to an even power after multiplication with this factor, i.e., $i^{n-m-2t+2s}\, i = (-1)^{\frac{n-m+1}{2} - t+s}$ and will thus be retained, while the terms that do not contain an additional $i$ will now have odd powers of $i$ (after multiplication with this factor) and hence will drop out. The factor $(-1)^{\frac{n-m+1}{2} - t+s}$ will be multiplied, as before, with $(-1)^{t}$ and $(-1)^{n-m-2t+s}$. The final result in this case will be $(-1)^{k}$.

All the above considerations applied to Eq. (30) give, after some albegraic simplification

$$R_{nm} = \frac{GM}{r}\left(\frac{a_e}{r}\right)^{n} \sum_{t=0}^{k} \frac{(2n-2t)!}{2^{2n-2t}(n-m-2t)!\; n!} \binom{n}{t} \sin^{n-m-2t} i \sum_{s=0}^{m} \binom{m}{s}$$

$$\cdot (-1)^{k} \cos^{s} i \sum_{c=0}^{n-m-2t+s} \sum_{d=0}^{m-s} \binom{n-m-2t+s}{c} \binom{m-s}{d} (-1)^{c} \tag{31}$$

$$\cdot \left[ \begin{pmatrix} C_{nm} \\ -S_{nm} \end{pmatrix}_{\substack{n-m \text{ even} \\ n-m \text{ odd}}} \cos[(n-2t-2c-2d)u + m(\Omega - \theta)] \right.$$

$$\left. + \begin{pmatrix} S_{nm} \\ C_{nm} \end{pmatrix}_{\substack{n-m \text{ even} \\ n-m \text{ odd}}} \sin[(n-2t-2c-2d)u + m(\Omega - \theta)] \right]$$

Now let t + c + d = p and replace the d-summation with a new p-summation, which is to be placed before t-summation so that like arguments can be combined. The limits of various summations are then changed as follows:

p-summation:

$$o \leq p \leq n$$

c-summation:

$$\left.\begin{matrix} p - t \leq m - s, & o \\ p - t \geq m - s, & p-t-m+s \end{matrix}\right\} \leq c \leq \left\{\begin{matrix} n-m-2t+s, & p-t \geq n-m-2t+s \\ p-t, & p-t \leq n-m-2t+s \end{matrix}\right.$$

t-summation:

$$o \leq t \leq \begin{matrix} p, & p \leq k \\ k, & p \geq k \end{matrix} \quad .$$

With these notations, the general term of the disturbing function can be written as

$$R_{nm} = \frac{GM}{r}\left(\frac{a_e}{r}\right)^n \sum_{p=0}^{n} F_{nmp}(i) \left\{ \begin{pmatrix} C_{nm} \\ -S_{nm} \end{pmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \quad \cos[(n-2p)u + m(\Omega - \theta)] + \begin{pmatrix} S_{nm} \\ C_{nm} \end{pmatrix} \begin{matrix} n-m \text{ even} \\ n-m \text{ odd} \end{matrix} \quad \sin[(n-2p)u + m(\Omega - \theta)] \right\} \tag{32}$$

where

$$F_{nmp}(i) = \sum_t \frac{(2n-2t)!}{2^{2n-2t}(n-m-2t)!\ n!} \binom{n}{t} \sin^{n-m-2t} i \sum_{s=0}^{m} \binom{m}{s}$$

$$\cdot (-1)^k \cos^s i \sum_c \binom{n-m-2t+s}{c} \binom{m-s}{p-t-c} (-1)^c$$

In order to make Eq. (32) consistent with Eq. (17), we now have to replace r and v (v = u-w) by a, M, and e, i.e., to transform the following part of Eq. (32) as follows:

$$\frac{1}{r^{n+1}} \begin{bmatrix} \sin \\ \cos \end{bmatrix} [(n-2p)(\omega+v) + m(\Omega - \theta)] = \frac{1}{a^{n+1}} \sum_{q=-\infty}^{+\infty} G_{npq}(e) \begin{bmatrix} \sin \\ \cos \end{bmatrix} \tag{33}$$

$$[(n-2p)w + (n-2p+q)M + m(\Omega - \theta)]$$

where q is a new summation parameter to be defined later.

The development of the coefficients $G_{npq}$ for long-period terms is as follows: consider the equation of the ellipse, with origin at the center

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Change it to polar coordinates by letting x = r cos v, y = r sin v, and $b^2 = a^2(1-e^2)$. Then solution of the resulting equation gives

$$r = \frac{a(1-e^2)}{1+e \cos v}$$

or

$$\frac{1}{r} = \frac{1+e \cos v}{a(1-e^2)}$$

or

$$\frac{1}{r^{n-1}} = \left( \frac{1+e \cos v}{a(1-e^2)} \right)^{n-1} \tag{34}$$

From the law of conservation of angular momentum and from Keppler's equation, it can be shown that

$$dM = \frac{r^2 dv}{a^2(1-e^2)^{\frac{1}{2}}} \tag{35}$$

Now take Eq. (32), substitute Eq. (34) into it, apply the binomial theorem to the factor $(1+e \cos v)^{n-1}$, apply the results of Eq. (29), change the products of the trigonometrical functions to summations wherever possible, integrate from 0 to $2\pi$ with respect to the mean anomaly M, making use of the relationship between the mean anomaly and true anomaly v, given by Eq. (35), and then divide by $2\pi$. This gives (we write below only that part of Eq. (32) which is affected by these changes):

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{r^{n+1}} \begin{bmatrix} \sin \\ \cos \end{bmatrix} [(n-2p)(\omega+v) + m(\Omega - \theta)] \, dM$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{a^{n+1}(1-e^2)^{n-\frac{1}{2}}} \sum_{b=0}^{n-1} \binom{n-1}{b} e^b \frac{1}{2^b} \sum_{d=0}^{b} \binom{b}{d}$$

$$\cdot \frac{1}{2} \left\{ \begin{pmatrix} \sin \\ \cos \end{pmatrix} [(n-2p)\omega + (n-2p-2d+b)v + m(\Omega - \theta)] \right.$$

$$\left. + \begin{pmatrix} \sin \\ \cos \end{pmatrix} [(n-2p)\omega + (n-2p+2d-b)v + m(\Omega - \theta)] \right\} dv$$

and since for long-period terms the coefficient of v should vanish, the above expression reduces to

$$\frac{1}{a^{n+1}} G_{np(2p-n)}(e) \begin{pmatrix} \sin \\ \cos \end{pmatrix} [(n-\omega p)\omega + m(\Omega - \theta)]$$

where

$$G_{np(2p-n)}(e) = \frac{1}{(1-e^2)^{n-\frac{1}{2}}} \sum_{d=0}^{\acute{p}-1} \binom{n-1}{n+2d-2\acute{p}} \binom{n+2d-2\acute{p}}{d} \left(\frac{e}{2}\right)^{n+2d-2\acute{p}}$$

$$\acute{p} = p \text{ for } p \leq \frac{n}{2}$$

and

$$\acute{p} = n-p \text{ for } p \geq \frac{n}{2}$$

For short-period terms, the terms in the true anomaly v, do not drop out, i.e., n-2p+q = 0 and the development of $G_{npq}$ becomes rather complicated. The resulting expressions are, however, of the form

$$G_{npq}(e) = (-1)^{\bar{q}} \left[1 + \frac{e^2}{(1+\sqrt{1-e^2})^2}\right]^n \left(\frac{e}{1+\sqrt{1-e^2}}\right)^{\bar{q}} \sum_{k=0}^{\infty} P_{npqk}$$

$$\cdot\, Q_{npqk} \left(\frac{e}{1+\sqrt{1-e^2}}\right)^{2k}$$

where

$$P_{npqk} = \sum_{r=0}^{h} \binom{2\acute{p}-2n}{h-r} \left[\frac{(n-2\acute{p}+\acute{q})(1+\sqrt{1-e^2})}{2}\right]^r \frac{(-1)^r}{r!}$$

and

$$Q_{npqk} = \sum_{r=0}^{\ell} \binom{-2\acute{p}}{\ell-r} \left[\frac{(n-2\acute{p}+\acute{q})(1+\sqrt{1-e^2})}{2}\right]^r \frac{1}{r!}$$

where

$$\bar{q} = |q|$$

$$h \begin{cases} = k+\acute{q} \text{ for } \acute{q} > 0 \\ = k \text{ for } \acute{q} < 0 \end{cases}$$

$$\ell \begin{cases} = k \text{ for } \acute{q} > 0 \\ = k-\acute{q} \text{ for } \acute{q} < 0 \end{cases}$$

$$\left.\begin{aligned} \acute{p} &= p \\ \acute{q} &= q \end{aligned}\right\} \text{for } p \leq \frac{n}{2} \quad \text{and} \quad \left.\begin{aligned} \acute{p} &= \ell-p \\ \acute{q} &= -q \end{aligned}\right\} \text{for } p > \frac{n}{2}$$

Now substitute the transformation (33) in Eq. (32), denote $Z_{nmpq}$ as

$$Z_{nmpq}(\omega,m,\Omega,\theta) = \begin{bmatrix} C_{nm} \\ -S_{nm} \end{bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \cos[(n-qp)\omega + (n-2p+q)M + m(\Omega - \theta)]$$
$$+ \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \sin[(n-2p)\omega + (n-2p+q)M + m(\Omega - \theta)] \tag{35a}$$

and obtain Eq. (32) in the final form

$$R_{nm} = \frac{GMa_e^{\,n}}{a^{n+1}} \sum_{p=0}^{n} F_{nmp}(i) \sum_{q=-\infty}^{+\infty} F_{npq}(e)\, Z_{nmpq}(\omega,M,\Omega,\theta) \tag{36}$$

Thus we have succeeded in transforming the general term of the disturbing function from spherical coordinates to orbital coordinates.

Denote one term of $R_{nm}$ in Eq. (36) by $R_{nmpq}$. Then

$$R_{nmpq} = \frac{GMa_e^{\,n}}{a^{n+1}} F_{nmp}(i)\, G_{npq}(e)\, Z_{nmpq}(\omega,M,\Omega,\theta) \tag{36a}$$

Differentiate $R_{nmpq}$ with respect to various orbital elements and substitute the result in Eq. (17). This would give the time rate of change of orbital elements. The reduced equations of motion would then be

$$\frac{d\Omega_{nmpq}}{dt} = \frac{GMa_e^{\,n} F'_{nmp} G_{npq} Z_{nmpq}}{na^{n+3}(1-e^2)^{\frac{1}{2}} \sin i}$$

$$\frac{d\omega_{nmpq}}{dt} = GMa_e^{\,n}\left[\frac{(1-e^2)^{\frac{1}{2}}}{e} F_{nmp} G'_{npq} - \frac{\cot i}{(1-e^2)^{\frac{1}{2}}} F'_{nmp} G_{npq}\right] \frac{Z_{nmpq}}{\bar{n}a^{n+3}}$$

$$\frac{di_{nmpq}}{dt} = \frac{GMa_e^{\,n} F_{nmp} G_{npq} Z'_{nmpq}}{\bar{n}a^{n+3}(1-e^2)^{\frac{1}{2}} \sin i}[(n-2p)\cos i-m] \tag{37}$$

$$\frac{da_{nmpq}}{dt} = \frac{2GMa_e^{\,n} F_{nmp} G_{npq} Z'_{nmpq}}{\bar{n}a^{n+2}}(n-2p+q)$$

$$\frac{de_{nmpq}}{dt} = \frac{GMa_e^{\,n} F_{nmp} G_{npq} Z'_{nmpq}}{\bar{n}a^{n+3} e}[(1-e^2)(n-2p+q) - (1-e^2)^{\frac{1}{2}}(n-2p)]$$

$$\frac{dM^*_{nmpq}}{dt} = \frac{GMa_e^{\,n} F_{nmp} Z_{nmpq}}{\bar{n}a^{n+3}}\left[2(n+1)G_{npq} - \frac{1-e^2}{e} G'_{npq}\right]$$

In the foregoing expressions:

$\Omega$ = right ascension of the ascending node

$a$ = semi-major axis of the satellite orbit

$\bar{n}$ = mean motion

$e$ = eccentricity

$\omega$ = argument of perigee

$i$ = inclination

$M^*$ = perturbation of the mean anomaly = $\int_\tau^t \bar{n}\, dt - \bar{n}(t-\tau)$

and

$\tau$ = time of perigee passage.

Functions $F_{nmp}$, $G_{npq}$, and $Z_{nmpq}$ are as defined earlier in the text, and

$$F'_{nmp} = \frac{dF_{nmp}}{di}$$

and

$$G'_{npq} = \frac{dG_{npq}}{de}$$

and $Z'_{nmpq}$ is the derivative of $Z_{nmpq}$ with respect to its argument.

Eq. (37) gives the time rate of change of various orbital elements due to one term $R_{nmpq}$ of the disturbing potential.

If it is now assumed that the dominant perturbations of the orbital elements are secular (this assumption will be valid for most of the geodetically useful satellites), it is possible to integrate the equations of motion to obtain the integrated changes in the orbital elements caused by a certain perturbing function. To get the expressions for the integrated changes, substitute Eq. (36a) in the equations of motion, i.e., Eq. (17), integrate with respect to time and get

$$\Delta\Omega_{nmpq} = GMa_e^{\,n} \frac{F'_{nmp} G_{npq} Z^{i}_{nmpq}}{\bar{n}a^{n+3}(1-e^2)^{\frac{1}{2}} \sin i[(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})]}$$

$$\Delta\omega_{nmpq} = GMa_e^{\,n} \frac{[e^{-1}(1-e^2)^{\frac{1}{2}} F_{nmp} G'_{npq} - \cot i(1-e^2)^{-\frac{1}{2}} F'_{nmp} G_{npq}] Z^{i}_{nmpq}}{\bar{n}a^{n+3}[(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})]}$$

$$\Delta i_{nmpq} = GMa_e^{\,n} \frac{F_{nmp} G_{npq}[(n-2p)\cos i - m] Z_{nmpq}}{\bar{n}a^{n+3}(1-e^2)^{\frac{1}{2}} \sin i[(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})]}$$

$$\Delta a_{nmpq} = GMa_e^{\,n} \frac{2F_{nmp} G_{npq} Z_{nmpq}(n-2p+q)}{\bar{n}a^{n+2}[(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})]} \qquad (38)$$

$$\Delta e_{nmpq} = GMa_e^{\,n} \frac{F_{nmp} G_{npq} Z_{nmpq}[(1-e^2)(n-2p+q) - (1-e^2)^{\frac{1}{2}}(n-2p)]}{na^{n+3}\, e[(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})]}$$

$$\Delta M_{nmpq} = GMa_e^{\,n} \frac{F_{nmp} Z^{i}_{nmpq}[2(n+1)G_{npq} - (1-e^2)e^{-1} G'_{npq}]}{\bar{n}a^{n+3}[(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})}$$

where $Z^{i}$ = the integral of Z with respect to its argument.

The above development closely follows that of Kaula (1961). We include below a few remarks about the functions $F_{nmp}$, $G_{npq}$, and $Z_{nmpq}$.

For long-period variations, the coefficient of M in Eq. (35a) must vanish, i.e.,

$$n-2p+q = 0$$

$$q = 2p-n$$

With this condition $Z_{nmpq}$ becomes $Z_{nmp(2p-n)}$ and is now independent of the terms in M. Further, if we are interested only in the effect of zonal harmonics, we have m = 0 and for long-period effects of zonal harmonics, the function $Z_{nmpq}$ becomes:

$$Z_{n0p(2p-n)} \begin{cases} = C_{n0} \cos(n-2p)\omega, \text{ n even} \\ \\ = C_{n0} \sin(n-2p)\omega, \text{ n odd} \end{cases}$$

For long-period effects, $G_{npq}$ becomes $G_{np(2p-n)}$. While computing G, considerable labor can be saved by remembering that

$$G_{np(2p-n)} = G_{n(n-p)(n-2p)}$$

In the case of zonal harmonics, m=0 and $F_{nmp}$ becomes

$$F_{n0p} = \sum_t \frac{(2n-2t)!}{2^{2n-2t}(n-2t)!n!} \binom{n}{t} \sin^{n-2t} i \; (-1)^k \sum_c \binom{n-2t}{c}\binom{0}{p-t-c} (-1)^c$$

The last binomial coefficient will be non-zero only if p-c-t = 0 or c = p-t. Thus for a particular p, there is only one value of c corresponding to every t and the c-summation can then be substituted by that value. Thus $F_{n0p}$ finally becomes

$$F_{n0p} = \sum_t \frac{(2n-2t)!}{2^{2n-2t}(n-2t)!n!} \binom{n}{t} \sin^{n-2t} i \; (-1)^k \binom{n-2t}{p-t} (-1)^{p-t}$$

Now examine Eq. (37). Let m=0, for convenience of discussion. Then the expressions for $\dot{\Omega}$ and $\dot{\omega}$ can be written as

$$\frac{d\Omega}{dt} (=\dot{\Omega}) = \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} C_{n0} X_{1n} X_{1nj} \begin{bmatrix} \cos j \omega \\ \sin j \omega \end{bmatrix} \begin{matrix} \text{j and n even} \\ \text{j and n odd} \end{matrix} \qquad (39)$$

and

$$\frac{d\omega}{dt} (=\dot{\omega}) = \sum_{n=2}^{\infty} \sum_{j=0}^{n-2} C_{n0} X_{2n} X_{2nj} \begin{matrix} \cos j\,\omega & j \text{ and } n \text{ even} \\ \sin j\,\omega & j \text{ and } n \text{ odd} \end{matrix}$$

where the X's can be obtained by comparing these expressions with the original Eq. (37).

Similar expressions can be written for the time-rate of change of other orbital elements and their integrated changes.

It can be verified easily that $X_{1n1}$ and $X_{2n1}$, the coefficients of the term sin ω, contain a factor 1/sin i. Also $X_{2n1}$, the coefficient of sin ω in the expression for $\dot{\omega}$, contains 1/e. Hence the equations in the form given in (37) break down for sin i = 0 or *i* = 0 and e = 0. Different sets of equations are available for use under such special circumstances.

The physical explanation is simple. For zero inclination, the position of the ascending node and for circular orbits, the position of the perigee cannot be defined.

All expressions in Eq. (38), for the case of secular or long-period effects of zonal harmonics, contain $\dot{\omega}$ in the denominator. The expression for $\dot{\omega}$ for the particular case of $C_{20}$ is

$$\dot{\omega} = K\, C_{20}(1 + \cos^2 i - \frac{3}{2} \sin^2 i)$$

where K is independent of *i*. For $\dot{\omega}$ = 0, *i* is roughly 63° 26'. This value of *i* is called the critical inclination. Thus the equations for integrated changes (in the form given above) are not valid for orbits with critical inclination.

Note that in Eq. (39), the X coefficients represent the amplitudes of different perturbations of 2π/j wavelength. For j = 0, cos j ω = 1 and hence the perturbations are secular. Note also that it is only for even values of n that the secular terms appear in the expressions.

For odd values of n, the expressions contain only long-period terms.

Similar remarks can be made for other orbital elements.

To sum up, the even zonal harmonics give rise to:

1. secular and long-period changes in Ω, ω, and M.

2. long-period changes in e and $i$.

3. no change in a.

The odd zonal harmonics produce long-period perturbations in $\Omega$, $i$, e, $\omega$, and M.

In practice, the even zonal harmonics are determined from the secular motion of the right ascension of the ascending node and argument of perigee, and the odd zonal harmonics, from the long-period changes in inclination and eccentricity.

The actual procedure of computation involves considerable detail.

## Determination of the Anomalous Gravity Field of the Earth

After having determined the geopotential coefficients from the variation of the orbital elements, we now discuss formulas for finding the gravity anomalies and geoidal undulations.

Gravity anomalies and geoidal undulations are referred to some standard surface. One such standard surface is the international reference ellipsoid. Since gravity anomalies computed from surface gravity measurements are always referred to the international reference ellipsoid, it will be necessary for us to also refer the satellite-computed gravity anomalies to the same standard surface for purposes of comparison.

In the case of geoidal undulations, however, there has been no consistency in reference to a standard surface. Most of the satellite-computed geoidal undulations are referred to an ellipsoid whose $C_{20}$ and $C_{40}$ are equal to the observed ones. Since there is some variation in the observed values of $C_{20}$ and $C_{40}$, depending upon the nature, quality, and quantity of the observations used and the methods of computation adopted, each set of geoidal undulations refers, in theory at least, to a different reference ellipsoid. For purposes of comparison, therefore, it becomes necessary to refer each of these sets to the same standard ellipsoid. Further, since the geoid obtained from surface gravity measurements is usually referred to the international reference ellipsoid, it is desirable to refer our satellite-computed geoid to the same surface. This refinement, however, may not be significant in view of the present accuracy of the satellite results.

In order to compute gravity anomalies and geoidal undulations with respect to any reference ellipsoid, we need to compute the $C_{20}$ and $C_{40}$ coefficients for the reference ellipsoid in question.

### Geopotential Coefficients of a Reference Ellipsoid

The polar equation of an oblate ellipsoid of revolution is

$$r - b(1-e^2 \cos^2\phi)^{-\frac{1}{2}} \qquad (40)$$

where

$b$ = semi-minor axis

$\phi$ = goecentric latitude

$e$ = eccentricity

Now let

$f$ = flattening of this ellipsoid

$a_e$ = its semi-major axis

then

$$e^2 = f - 2f^2$$

and

$$b = a_e\ (1-f)$$

Substituting these relations in Eq. (40) above, we obtain

$$r = a_e(1-f)\ \underline{[1-(2f-f^2)\ \cos^2\phi]}^{-\frac{1}{2}}$$

Using the binomial theorem to expand the underlined expression, we get

$$r = a_e[1-f\ \sin^2\phi - \frac{3}{2} f^2 \sin^2\phi + \frac{3}{2} f^2 \sin^4\phi + 0(f^3)] \tag{40a}$$

and hence

$$\frac{a_e}{r} = 1+ (f+ \frac{3}{2} f^2)\sin^2\phi - \frac{1}{2} f^2 \sin^4\phi + 0(f^3) \tag{41}$$

If we square both sides of (40), we get

$$r^2 = a_e^2[1-2f\ \sin^2\phi - 3f^2 \sin^2\phi + 4f^2 \sin^4\phi + 0(f^3)] \tag{42}$$

The symbol $0(f^3)$ in (40), (41), and (42) means that we have ignored terms containing cubed or higher powers of f.

The reference gravitational potential V, of an ellipsoid of revolution, symmetrical with respect to the equatorial and polar diameters, can be written to the order of accuracy required here, as

$$V = \frac{GM}{r} [1 + C_{20} \left(\frac{a_e}{r}\right)^2 P_{20}(\sin\phi) + C_{40} \left(\frac{a_e}{r}\right)^4 P_{40}\ \sin\phi + 0(f^3)] \tag{43}$$

In the above equation

$$P_{20}\ (\sin\phi) = \frac{1}{2} (3\ \sin^2\phi - 1)$$

$$P_{40}\ (\sin\phi) = \frac{1}{8} (35\ \sin^4\phi - 30\ \sin^2\phi + 3)\ .$$

Now, the gravity force on the surface of the earth arises from

(1) potential V, due to attraction of mass M, of the earth

(2) potential of the centrifugal force due to the rotation of the earth.

Let this combined potential be denoted by W, then

$$W = V + \frac{1}{2}\,\omega^2\, r^2\, (1-\sin^2\phi) \tag{44}$$

where

$\omega$ = rotational velocity of the earth.

The condition that a surface be equipotential is

$$W = \text{constant}. \tag{45}$$

Now the condition that the surface described by r of Eq. (40) or (42), i.e., the surface of the oblate ellipsoid of revolution, be an equipotential, is obtained by substituting the value of r from (40a) in (44) and then applying the condition (45). Also since we are developing only a second-order theory, we will neglect all the terms with magnitude $O(f^3)$. To be more precise, we will neglect terms containing such factors as $C_{40}f$, $C_{40}f^2$, $C_{20}f^2$, etc. With this in mind then, we have from Eq. (42)

$$\left\{\begin{aligned} \left(\frac{a_e}{r}\right)^2 &= 1 + 2f\,\sin^2\phi \\ \left(\frac{a_e}{r}\right)^4 &= 1 \end{aligned}\right\} \tag{46}$$

Substituting (41) and (46) in (43), retaining the terms to $O(f^3)$ only, and combining the coefficients of $\sin^2\phi$ and $\sin^4\phi$, we obtain

$$V = \frac{GM}{a_e}\left\{ (1 - \frac{1}{2} C_{20} + \frac{3}{8} C_{40}) + [f + \frac{3}{2} f^2 + \frac{3}{2} (1-f)C_{20} - \frac{15}{4} C_{40}] \sin^2\phi \right.$$

$$\left. + [-\frac{1}{2} f^2 + \frac{9}{2} f\, C_{20} + \frac{35}{8} C_{40}] \sin^4\phi + 0(f^3)\right\} \qquad (47)$$

Substituting the value of $r^2$ from (42) in the second term of Eq. (44), we get

$$\frac{1}{2} \omega^2 r^2 (1-\sin^2\phi) = \frac{1}{2} \omega^2 a_e^2 [1-\sin^2\phi\, (1+2f+3f^2) + \sin^4\phi\, (2f+7f^2)$$

$$+ 0(f^3)]$$

If we now introduce a quantity

$$m_1 = \frac{\omega^2 a_e^3}{GM}$$

which is of the order of f such that $m_1 f^2 \sim f^3$, the above expression becomes

$$\frac{1}{2} \omega^2 r^2 (1 - \sin^2\phi) = \frac{GM}{a_e} \cdot \frac{m_1}{2} [1 - (1+2f) \sin^2\phi + 2f \sin^4\phi + 0(f^3)] \qquad (48)$$

Substituting (47) and (48) in (44), and combining the coefficients of $\sin^2\phi$ and $\sin^4\phi$, we now get

$$W = \frac{GM}{a_e}\left\{ (1 - \frac{1}{2} C_{20} + \frac{3}{8} C_{40} + \frac{m_1}{2}) + [f + \frac{3}{2} f^2 - \frac{m_1}{2} (1+2f) \right.$$

$$+ \frac{3}{2} (1-f)C_{20} - \frac{15}{4} C_{40}] \sin^2\phi + [-\frac{1}{2} f^2 + m_1 f + \frac{9}{2} f\, C_{20} \qquad (49)$$

$$\left.\frac{35}{8} C_{40}\right] \sin^4\phi + 0(f^3) \Bigg\}$$

The imposition of condition (45) on Eq. (49) means that the coefficients of $\sin^2\phi$ and $\sin^4\phi$ should vanish, that is

$$f + \frac{3}{2} f^2 - \frac{m_1}{2} (1 + 2f) + \frac{3}{2} (1 - f)C_{20} - \frac{15}{4} C_{40} = 0$$

and

$$- \frac{1}{2} f^2 + m_1 f + \frac{9}{2} f \; C_{20} + \frac{35}{8} C_{40} = 0$$

which gives

$$C_{20} = - \frac{2}{3} f + \frac{1}{3} f^2 + \frac{m_1}{3} - \frac{3}{7} m_1 f + 0(f^3)$$

$$C_{40} = \frac{4}{5} f^2 - \frac{4}{7} m_1 f + 0(f^3) \qquad (50)$$

In case we introduce a parameter m such that

$$m = \frac{\omega^2 r_m^3}{GM}$$

where $r_m$ is the mean radius of the earth, $m_1$ becomes

$$m_1 = m(1 + f + \frac{19}{15} f^2)$$

and the Eqs. (50) reduce to the form

$$C_{20} = -\frac{2}{3} f + \frac{1}{3} f^2 + \frac{m}{3} - \frac{2}{21} mf + 0(f^3)$$

(50a)

$$C_{40} = \frac{4}{5} f^2 - \frac{4}{7} mf + 0(f^3)$$

Eq. (50) or (50a) gives the formulas for determing $C_{20}$ and $C_{40}$ for an ellipsoid whose f and m are known.

In the following discussion, we will refer to the coefficients $C_{20}$ and $C_{40}$ of the reference ellipsoid as reference $C_{20}$ or reference $C_{40}$.

Gravity Anomalies

The gravity anomaly $\Delta g$, is the departure of the observed gravity value reduced to the geoid from the corresponding theoretical gravity value on the reference ellipsoid. It is given by

$$\Delta g = -\frac{\partial R}{\partial r} + \frac{2R}{r} \qquad (51)$$

where

R = the disturbing potential.

Substituting the value of R in terms of spherical harmonic expansion in (51) and assembling similar terms, we get

$$\Delta g = \frac{GM}{a_e^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[ (n-1) \left( \frac{a_e}{r} \right)^{n+2} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} (\sin\phi) \right]$$

(52)

Where r should be obtained from Eq. (40), and $C_{20}$ and $C_{40}$ should be replaced by $\Delta C_{20}$ and $\Delta C_{40}$ where

$\Delta C_{20}$ = observed $C_{20}$ - reference $C_{20}$

$\Delta C_{40}$ = observed $C_{40}$ - reference $C_{40}$

Theoretically, $\Delta C_{nm}$ should be substituted for every $C_{nm}$ in Eq. (52), where $\Delta C_{nm}$ is the difference between the observed $C_{nm}$ and the corresponding reference $C_{nm}$. But since for our reference ellipsoid, the only non-zero $C_{nm}$ are $C_{20}$ and $C_{40}$, we have to consider this substitution only for these two terms.

Note that if the value of a particular reference $C_{nm}$ is taken equal to the observed value, the corresponding $\Delta C_{nm}$ vanishes. We have used this fact to define the best-fitting satellite spheroid as given in section III.

Geoidal Undulations

The geoidal undulation N, is given in terms of the disturbing potential R, by

$$R = \int_0^N g \, d \, N$$

Assume g constant over N and equal to the theoretical gravity $g_0$, on the reference ellipsoid, then

$$N = \frac{R}{g_0}$$

Substituting the spherical harmonic expansion of R in the foregoing relation, we get

$$N = \frac{GM}{a_e g_0} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left[ \left( \frac{a_e}{r} \right)^{n+1} (C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} (\sin\phi) \right] \quad (53)$$

where $\Delta C_{20}$ and $\Delta C_{40}$ should be substituted for $C_{20}$ and $C_{40}$, as already discussed for the case of gravity anomalies.

Computational Procedures and Discussion of Results

The zonal harmonic coefficients of the geopotential are given in Table 1 and the tesseral harmonic coefficients, in Table 2. Since some of the higher degree zonal harmonic coefficients and most of the tesseral coefficients given in these tables show considerable discrepancies from

TABLE 1. Normalized Zonal Harmonic Coefficients $C_{n0}$ of the Geopotentials

| | Smith (1963, 1965) | Kozai (1964) | King Hele et al. (1965) |
|---|---|---|---|
| n | $C_{n0}10^{-6}$ | $C_{n0}10^{-6}$ | $C_{n0}10^{-6}$ |
| 2 | -484.172 | -484.174 | -484.172 |
| 3 | 0.923 | 0.963 | 0.967 |
| 4 | 0.567 | 0.550 | 0.507 |
| 5 | 0.054 | 0.063 | 0.045 |
| 6 | -0.202 | -0.179 | -0.158 |
| 7 | 0.077 | 0.086 | 0.114 |
| 8 | 0.112 | 0.065 | -0.107 |

TABLE 2. Normalized Tesseral Harmonic Coefficients of the $C_{nm}$, $S_{nm}$ of the Geopotential

| | | Anderle (1966) | | Guier & Newton (1965) | | Gaposhkin (1966) | |
|---|---|---|---|---|---|---|---|
| n | m | $C_{nm}$ | $S_{nm}$ | $C_{nm}$ | $S_{nm}$ | $C_{nm}$ | $S_{nm}$ |
| | | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ | $10^{-6}$ |
| 2 | 2 | 2.45 | -1.52 | 2.38 | -1.20 | 2.38 | -1.35 |
| 3 | 1 | 2.15 | 0.27 | 1.84 | 0.21 | 1.94 | 0.27 |
| | 2 | 0.98 | -0.91 | 1.22 | -0.68 | 0.73 | -0.54 |
| | 3 | 0.58 | 1.62 | 0.66 | 0.98 | 0.56 | 1.62 |
| 4 | 1 | -0.49 | -0.57 | -0.56 | -0.44 | -0.57 | -0.47 |
| | 2 | 0.27 | 0.67 | 0.42 | 0.44 | 0.33 | 0.66 |
| | 3 | 1.03 | -0.25 | 0.84 | 0.00 | 0.85 | -0.19 |
| | 4 | -0.41 | 0.34 | -0.21 | 0.19 | -0.05 | 0.23 |
| 5 | 1 | 0.03 | -0.12 | 0.14 | -0.17 | -0.08 | -0.10 |
| | 2 | 0.64 | -0.33 | 0.27 | -0.34 | 0.63 | -0.23 |
| | 3 | -0.39 | -0.12 | 0.09 | 0.10 | -0.52 | 0.01 |
| | 4 | -0.55 | 0.15 | -0.49 | -0.26 | -0.26 | 0.06 |
| | 5 | 0.21 | -0.59 | -0.03 | -0.67 | 0.16 | -0.59 |
| 6 | 1 | -0.08 | 0.19 | 0.00 | 0.10 | -0.05 | -0.03 |
| | 2 | 0.13 | -0.46 | -0.16 | -0.16 | 0.07 | -0.37 |
| | 3 | -0.02 | -0.13 | 0.53 | 0.05 | -0.05 | 0.03 |
| | 4 | -0.19 | -0.32 | -0.31 | -0.51 | -0.04 | -0.52 |
| | 5 | -0.09 | -0.79 | -0.18 | -0.50 | -0.31 | -0.46 |
| | 6 | -0.32 | -0.36 | 0.01 | -0.23 | -0.04 | -0.16 |
| 7 | 1 | 0.33 | 0.08 | 0.13 | 0.09 | 0.20 | 0.16 |
| | 2 | 0.35 | -0.19 | 0.46 | 0.06 | 0.36 | 0.16 |
| | 3 | 0.32 | 0.04 | 0.39 | -0.21 | 0.25 | 0.02 |
| | 4 | -0.47 | -0.24 | -0.14 | 0.00 | -0.15 | -0.10 |
| | 5 | 0.05 | 0.02 | -0.06 | -0.19 | 0.08 | 0.05 |
| | 6 | -0.48 | -0.24 | -0.45 | -0.75 | -0.21 | 0.06 |
| | 7 | | | 0.09 | -0.14 | 0.06 | 0.10 |
| 8 | 1 | | | -0.15 | -0.05 | -0.08 | 0.07 |
| | 2 | | | 0.09 | -0.04 | 0.03 | 0.04 |
| | 3 | | | -0.05 | 0.22 | -0.04 | 0.00 |
| | 4 | | | -0.07 | -0.04 | -0.21 | -0.01 |
| | 5 | | | 0.08 | 0.00 | -0.05 | 0.12 |
| | 6 | | | -0.02 | 0.67 | -0.02 | 0.32 |
| | 7 | | | 0.17 | -0.07 | -0.01 | 0.03 |
| | 8 | | | -0.15 | 0.09 | -0.25 | 0.10 |

solution to solution, a "mean solution" has been obtained from the three sets of zonal harmonic coefficients given in Table 1 and the three sets of tesseral coefficients in Table 2. These "mean coefficients" are given in Table 3.

$C_{20}$ and $C_{40}$ parameters of the international reference have been computed from Eq. (50). The results, fully normalized, are:

$$C_{20} = -488 \cdot 375 \times 10^{-6}$$

$$C_{40} = 0.804 \times 10^{-6}$$

Computation of Geoidal Undulations

Eq. (53) is the basic equation used in our computations. We have simplified it further by assuming that:

$$\frac{GM}{a_e^2} = g_0$$

and

$$r = a_e$$

on the surface of the earth.

The equation thus reduced to

$$N = a_e \sum_{n=2}^{\infty} \sum_{m=0}^{n} [C_{nm} \cos m\lambda + S_{nm} \sin m\lambda) P_{nm} (\sin\phi)] \qquad (54)$$

It is this form of the basic equation which has been used to compute the different sets of geoids reported here.

Let us examine the errors, if any, arising from the above assumptions. The first of the two assumptions introduces a maximum error $e_1$, of

TABLE 3. Normalized Spherical Harmonic Coefficients of the Geopotential $C_{nm}$, $S_{nm}$

(Mean Solution)

| n | m | $C_{nm}$ $10^{-6}$ | $S_{nm}$ $10^{-6}$ | n | m | $C_{nm}$ $10^{-6}$ | $S_{nm}$ $10^{-6}$ |
|---|---|---|---|---|---|---|---|
| 2 | 0 | -484.173 | -- | 6 | 3 | 0.15 | -0.02 |
| | 2 | 2.40 | -1.36 | | 4 | -0.18 | -0.45 |
| | | | | | 5 | -0.19 | -0.58 |
| 3 | 0 | 0.951 | -- | | 6 | -0.12 | -0.25 |
| | 1 | 1.98 | 0.25 | | | | |
| | 2 | 0.98 | -0.71 | 7 | 0 | 0.092 | -- |
| | 3 | 0.60 | 1.41 | | 1 | 0.22 | 0.11 |
| | | | | | 2 | 0.39 | 0.01 |
| 4 | 0 | 0.541 | -- | | 3 | 0.32 | -0.05 |
| | 1 | -0.54 | -0.49 | | 4 | -0.25 | -0.11 |
| | 2 | 0.34 | 0.59 | | 5 | 0.02 | -0.04 |
| | 3 | 0.91 | -0.22 | | 6 | -0.38 | -0.31 |
| | 4 | -0.22 | 0.25 | | 7 | 0.07 | -0.02 |
| 5 | 0 | 0.054 | -- | 8 | 0 | 0.024 | -- |
| | 1 | 0.03 | -0.13 | | 1 | -0.11 | 0.01 |
| | 2 | 0.51 | -0.30 | | 2 | 0.06 | 0.0 |
| | 3 | -0.33 | 0.0 | | 3 | -0.04 | 0.11 |
| | 4 | -0.43 | -0.02 | | 4 | -0.14 | -0.02 |
| | 5 | 0.11 | -0.62 | | 5 | 0.01 | 0.06 |
| | | | | | 6 | -0.02 | 0.49 |
| 6 | 0 | -0.180 | -- | | 7 | 0.08 | -0.02 |
| | 1 | -0.04 | 0.09 | | 8 | -0.20 | 0.09 |
| | 2 | 0.01 | -0.33 | | | | |

$\pm$ 0.5%. The error is defined as positive if, to obtain the corrected value, the correction to be applied to the computed value is negative. This error turns out to be dominantly positive.

To examine the error introduced by the second assumption, then, let

$$r = a_e - \Delta r$$

where $\Delta r$ is the departure which r has from the value of $a_e$ in its total range of variation. Then

$$\left(\frac{a_e}{r}\right)^{n+1} = \left(\frac{r}{a_e}\right)^{-(n+1)} = \left(1 - \frac{\Delta r}{a_e}\right)^{-(n+1)} = 1 + (n+1)\frac{\Delta r}{a_e}$$

and the error introduced in the <u>nth</u> term by assuming $a_e/r = 1$, is

$$(n+1)\frac{\Delta r}{a_e} \times 10^2\% \quad .$$

Since $a_e \geq r$, the correction arising from this factor should be added to the computed value of geoidal undulation to obtain the correct value. The correction, however, is not significant in view of the present accuracy of satellite results.

Neither of the above errors is significant in view of the present accuracy of the satellite results.

<u>Computation of the Gravity Anomalies</u>

Eq. (52) was simplified by assuming that $a_e = r$. Eq. (52) then reduces to the following form

$$\Delta g = \frac{GM}{a_e^2} \sum_{n=2}^{\infty} \sum_{m=0}^{n} [(n-1)(C_{nm} \cos M\lambda + S_{nm} \sin m\lambda) P_{nm}(\sin\phi)] \qquad (55)$$

This simplified equation was used for computing the different sets of gravity anomalies given in this report.

As pointed out earlier in the case of geoidal undulations, the error arising from the above assumption is negligible in view of the present accuracy of the satellite results.

## Discussion of Results

### Geoidal Undulations

Table 4 summarizes the magnitude and location of the maximum elevations and depressions of the different geoids obtained using the different sets of geopotential coefficients given in Tables 1 to 3. It should be noted that Table 4 is made up of three sections. In the first section each geoid is referred to a reference ellipsoid whose $C_{20}$ and $C_{40}$ parameters are those defined by the set of zonal coefficients that are used to compute the individual geoid under consideration. Although each set of geoidal undulations was thus derived from a different reference surface, this difference is not significant because the $C_{20}$ and $C_{40}$ values for the different sets of zonal coefficients are in fairly good agreement. Figures 1 to 4 show the geoidal undulations obtained in this manner using the various sets of geopotential coefficients given in Tables 1 to 3.

In the second section each derived geoid is referred to the international reference ellipsoid. Figures 5 to 7 are plots of the geoids determined on this basis. For comparative purposes data are also given for a geoid derived using the mean coefficients and best-fitting satellite-derived reference spheroid with a polar flattening value of 1/298.25. This solution is shown in Figure 8. A similar geoid obtained from Gaposhkin's(1966) tesserals is shown in Figure 9.

In the third section comparative geoidal data are given as obtained by Kaula (1966), Uotila (1962), and Zongolovich (1952).

Kaula (1966) used a combination of satellite and gravimetric data to obtain his results (see Fig. 10). Uotila (1962) and Zongolovich (1952) both used surface gravimetric data. For comparative purposes, Uotila's (1962) geoid is shown in Figure 11 and Zongolovich's (1952) geoidal map is shown in Figure 12.

From an inspection of Table 4 it is seen that broadly speaking, the area of maximum geoidal depression defined by each geoid has much the same location, although there is some variation in magnitude values. The significantly lower magnitude found with the gravity solutions, Uotila (1962) and Zongolovich (1952), can be attributed to the paucity and poor distribution of the available gravity data. In the case of the area of maximum geoidal elevation there is a significant dependence on the reference ellipsoid used. However, the scatter is restricted to one or two areas--the Solomon Islands-New Guinea region and the North Atlantic area immediately south of Iceland. This would suggest the

TABLE 4. Comparison of Geoidal Undulations in Different Representations

| Ref. of the Geo-potential Coeff. used to compute the Geoidal Undulation | Maximum Height Above the Reference Ellipsoid | | | Maximum Depression Below the Ref. Ellipsoid | | | Total Range in Meters | Parameters of the Reference Ellipsoid |
|---|---|---|---|---|---|---|---|---|
| | Magnitude in Meters | Location Long. | Location Lat. | Magnitude in Meters | Location Long. | Location Lat. | | |
| Kozai (1964); Gaposhkin (1966) | +81 | 140° to 150° | 0° to 5°N | -98 | 75° to 80° | 0° to 10°N | 179 | $C_{20}$ & $C_{40}$ of Table 1 (col. 2) |
| Smith (1963, 65); Guier & Newton (1965) | +65 | 350° to 355° | 55°N to 60°N | -91 | 75° to 80° | 10°N to 15°N | 156 | $C_{20}$ & $C_{40}$ of Table 1 (col. 1) |
| King-Hele (1965); Anderle (1966) | +83 | 140° to 150° | 0° to 5°N | -98 | 70° to 80° | 5° to 10°N | 181 | $C_{20}$ & $C_{40}$ of Table 1 (col. 3) |
| Table 3: Mean Coefficients | +70 | 145° to 155° | 5°S to 5°N | -95 | 75° to 80° | 5°N to 10°N | 165 | $C_{20}$ & $C_{40}$ of Table 3 |
| Kozai (1964); Gaposhkin (1966) | +94 | 360° to 355° | 60°N to 70°N | -128 | 75° to 80° | 0° to 10°N | 222 | International reference ellipsoid |
| Smith (1963, 65); Guier & Newton (1965) | +101 | 345° to 355° | 60°N to 65°N | -118 | 75° to 80° | 5°N to 10°N | 219 | -do- |
| Mean Coefficients | +99 | 340° to 355° | 60°N to 70°N | -125 | 75° to 80° | 50°N to 10°N | 224 | -do- |
| Mean Coefficients | +69 | 135° to 165° | 50°S to 15°N | -93 | 65° to 75° | 0° to 20°N | 162 | Best fit Satellite Spheroids Polar Flattening = 1/298.25 |
| **Other Results** | | | | | | | | |
| Uotila's geoid (1962) obtained from free air gravity anomalies | +60 | 130° to 150° | 10°S to 10°N | -60 | 60° to 80° | 10°N to 40°N | 120 | Ellipsoid with flattening = 1/298.24 |
| Kaula's map (1966) obtained from a combination of satellite and gravimetric data. (Geopotential coefficients for this case not given) | +76 | 135° to 165° | 15°S to 5°N | -90 | 65° to 85° | 10°S to 10°N | 166 | Ellipsoid with flattening = 1/298.25 |
| Zongolovich Geoid based on surface gravity data | +80 | 120° to 140° | 3°S to 12°N | -60 | 3 locations of equal value: 55° to 80° | 0 to 22°N | 140 | Russian ellipsoid |
| | | | | | 300° to 340° | 18°S to 20°N | | |
| | | | | | 235° to 265° | 12°S to 35°S | | |

gravity anomaly in the two areas is of similar magnitude. Actually the available data indicate the free-air anomaly values in the Solomons area is considerably higher than in the North Atlantic area. On a 22° x 22° size area (corresponding to an 8th degree fit) the mean gravity values, however, are not significantly different. A 15th degree fit of the data presumably would give a consistent pattern, with the Solomons region being the area of maximum geoidal rise.

It is to be noted that the geoidal undulations referred to the 'best-fitting satellite ellipsoid' (whose reference $C_{20}$ and $C_{40}$ are equal to the observed ones) show a consistently different pattern from those referred to the international reference ellipsoid. As the equatorial radius and flattening of the 'satellite ellipsoid' are smaller than the corresponding parameters of the international reference ellipsoid, the geoidal undulations referred to the 'satellite ellipsoid', appear to show some accentuation of equatorial "highs" and damping of polar "highs." However, this argument may hold only for the general pattern of these differences and not give a systematic change in magnitude.

The data of Table 4 bring out one important point. Until recently it had been believed on the basis of gravimetric data that the maximum deviation of the geoid from the reference surface of the international ellipsoid was not more than 30 to 40 meters. As seen, the differences obtained for an 8th degree fit are of the order of 100 meters or more.

In connection with Figures 8 and 9 in which the geoidal undulations are referred to a best-fitting satellite-derived spheroid the reference geopotential ($V_1$) was defined by

$$V_1 = \frac{GM}{a_e}\left[1 + \sum_{n=2}^{\infty}\left(\frac{a_e}{r}\right)^{n+1} C_n P_n\ (\sin\phi)\right]$$

where $C_n$ are the zonal harmonic coefficients and $P_n$ $(\sin\phi)$ the Legendre polynomials. As is obvious, it is an axially symmetrical surface but not an equatorially symmetrical one. The maximum geoidal deviations are +69 meters and -93 meters in the case of Figure 8 and are of the same magnitude as those computed by setting both $C_{20}$ and $C_{40}$ equal to zero.

Gravity Anomalies

Figures 13 to 15 are the free-air anomaly maps obtained from the mean coefficients and referred to the international gravity formula. Figure 16 is a similar map obtained by Kaula (1966) from a combination of surface gravity data and satellite gravity information and referred to the international gravity formula. Although these maps show broad-scale agreement on some features, there are significant differences on others. At long. 140° and lat. 5°, Figure 13 shows positive values ranging from 10 to 15 milligals; Figure 14, negative values of -2 to -6 milligals; and Figure 15, again positive values of 4 to 8 milligals. At long. 165° and lat. 15°S, the gravity anomaly is 0 milligal, according to Figure 13; -17 milligals, according to Figure 14; and about -1 to -2 milligals, according to Figure 15. To the SW of Australia, at long. 115° and lat. 35°S, Figure 13 shows a gravity anomaly of -27 milligals; Figure 14, -17 milligals; and Figure 15, -23 milligals.

The magnitude and position of the negative anomaly over and around Ceylon in the Indian Ocean agrees within reasonable limits on the three maps.

At the west coast of the North American continent around lat. 50°N, Figure 13 shows an anomaly of 3-5 milligals; Figure 14, 8 to 10 milligals; and Figure 15, 7 to 9 milligals.

Now compare these maps with the gravity anomaly map given in Figure 16. Considerable discrepancies in the locations and in the sizes of the gravity anomalies can be seen at once. These differences can be related to the difference in input data and the fact that some of Kaula's coefficients were for a 14th degree fit.

The short wavelength component of the gravity field which is of interest to the geophysicist, is the one most poorly represented in these results, for as indicated earlier, with an 8th degree fit the results represent average values for areas 22° x 22° in size. They therefore can only outline regional areas of anomalous gravity. That they do accomplish this purpose is shown by a comparison of these results with the available surface gravity information expressed as free-air anomalies. Figure 17 shows a free-air anomaly map for the North Atlantic Ocean which takes in a portion of the gravity "high" defined south of Greenland and the gravity "low" defined in the eastern North Atlantic Ocean on both Figure 15 and Figure 16. Figure 18 shows regional variations in free-air anomaly values in the Pacific Ocean in terms of areas having anomalies > +20 mgals or > -20 mgals, and with no dominant sign. The agreement of the satellite-derived maps with the surface-gravity anomaly maps is on the whole good, and as would be expected, Kaula's map (Fig. 16) appears to be somewhat better, especially in the Atlantic Ocean, since his map was derived using both the available surface gravity data and coefficients for a higher degree of fit.

There appears to be little question about the application of satellite data for determining areas of anomalous mass associated with the earth, or in the use of satellite data in using Stokes' theorem for defining the gravity field for areas remote from a point.

The significance of the anomalous areas of gravity, however, is not too clear. Because of the long wavelengths portrayed, the anomalous mass could be deep seated or represent the integrated effect of a number of shallow mass anomalies located in the upper mantle or crust. In either case there would also be a contribution from surface topography. The fact that the topographic effect only appears to be of secondary importance stresses the need for geophysical investigations in these areas. Some of them such as the positive anomaly area over the Mid-Atlantic Ridge are known to be characterized by anomalous geophysical relations: a sub-normal mantle velocity, pronounced magnetic anomalies, high heat flow along the crest of the ridge but sub-normal heat flow along the flanks. However, it is difficult to reconcile these observations with the anomalous gravity field which conforms closely with the regional topographic relief and which the Bouguer anomalies indicate is compensated without postulating that the sub-normal mantle velocity values are indicative of higher than normal density values or that there is deeper, as yet undiscovered, layering in the upper mantle. Worzel (1965) has shown three possible theoretical mass distributions, all in the upper 30 km of the crust to explain the observed gravity relations over the Mid-Atlantic Ridge. Cook (1962) has postulated that the apparent sub-normal mantle velocities are due to a mixture of crustal and mantle materials as a result of convection with attendant high heat flow. While eminently reasonable for the Mid-Atlantic Ridge, these explanations do not explain the relations in the Indian Ocean area where the satellite data define a broad negative anomaly area that appears to be related to a stable ocean basin region lying between a narrow volcanic ridge and a rise (of the Mid-Atlantic Ridge type) which has many of the geophysical associations noted for the Mid-Atlantic Ridge.

It is this lack of consistency between gravity and other geophysical relations on a regional scale that raises doubts as to interpretations that have been placed on the data and point up the need for more extensive geophysical studies in areas of anomalous gravity. It is only after such extensive geophysical data have become available, that a sophisticated statistical analysis will give meaningful results.

## Acknowledgment

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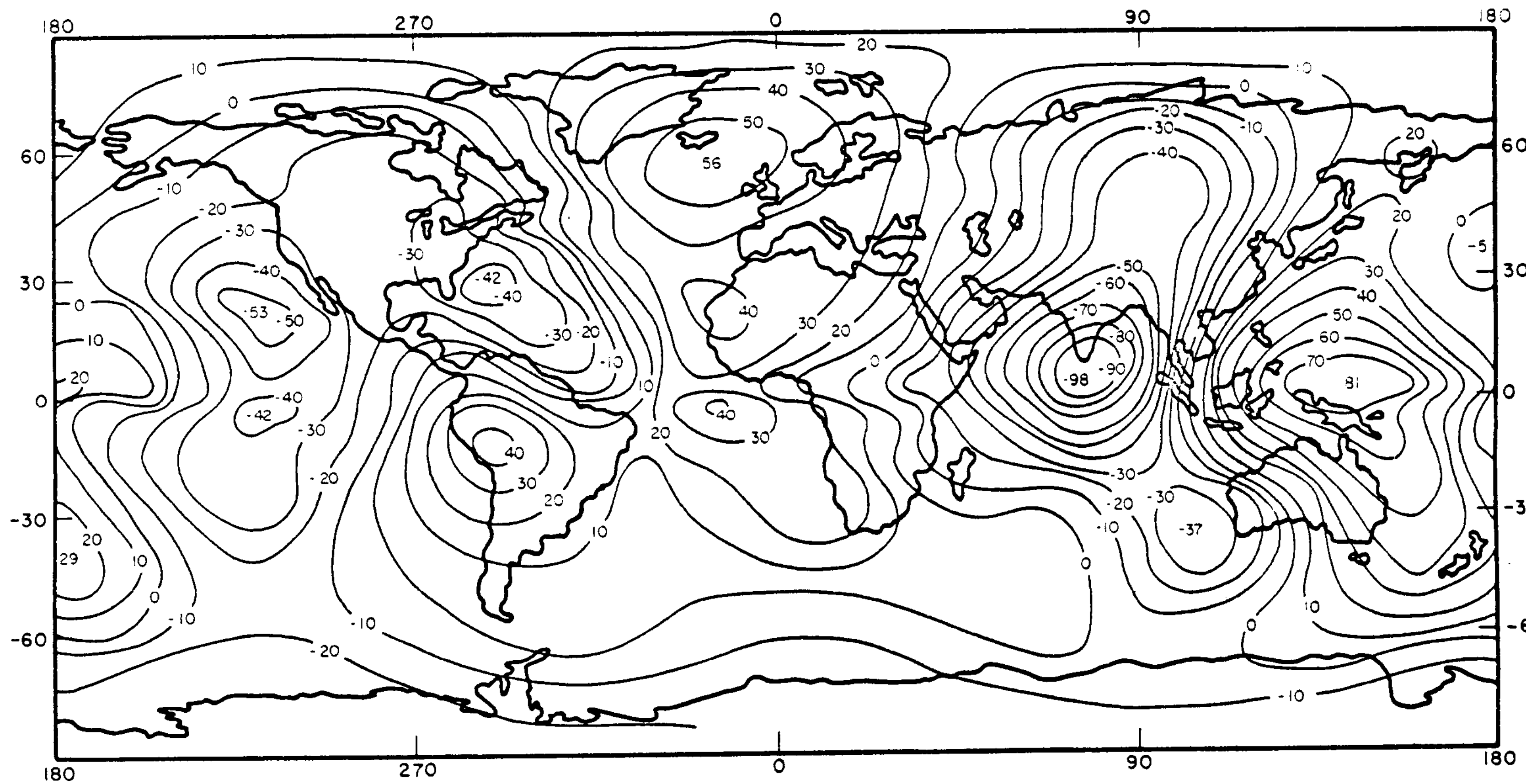


Fig. 1. Geoidal undulations in meters referred to an ellipsoid whose $C_{20}$ and $C_{40}$ coefficients coincide with the satellite-observed ones, computed using Kozai's (1964) zonal harmonic coefficients and Gaposhkin's (1966) tesseral harmonic coefficients up to 8th degree of order. Contour interval, 10 meters. (June 1966.)

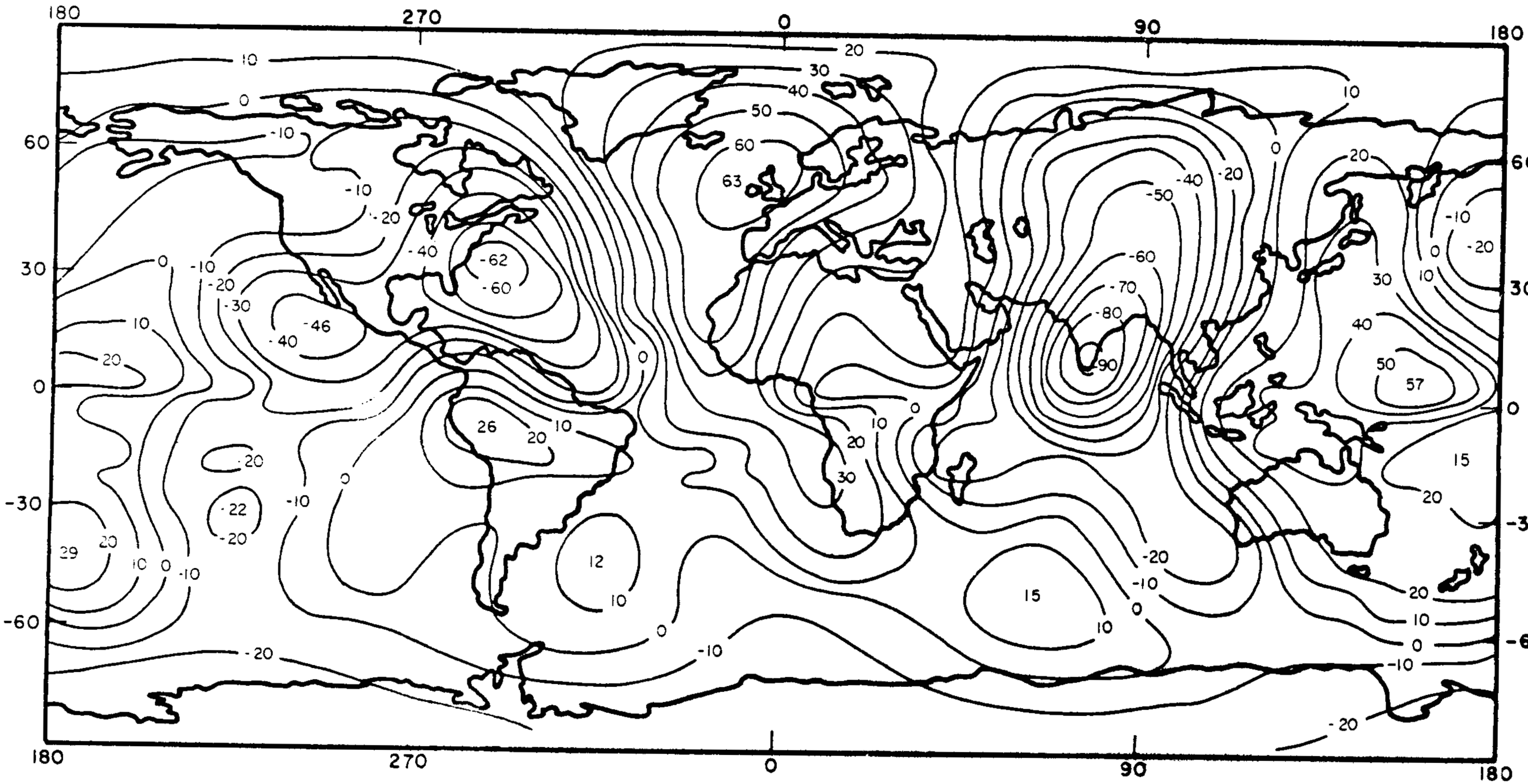


Fig. 2. Geoidal undulations in meters referred to an ellipsoid whose $C_{20}$ and $C_{40}$ coefficients coincide with the satellite-observed ones, computed using Smith's (1963, 1965) zonal harmonic coefficients and Guier and Newton's (1965) tesseral harmonic coefficients up to 8th degree of order. Contour interval, 10 meters. (June 1966.)

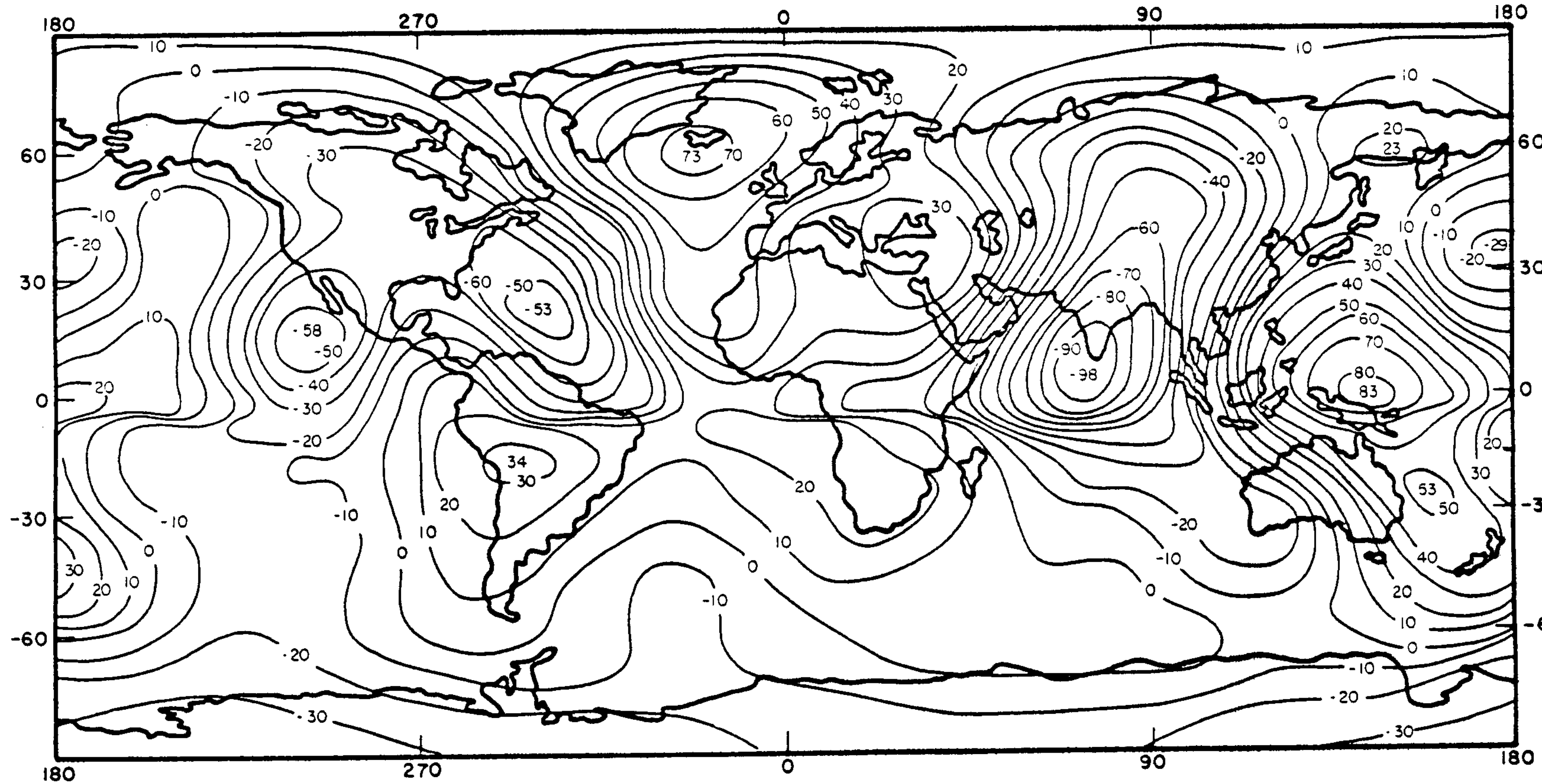


Fig. 3. Geoidal undulations in meters referred to an ellipsoid whose $C_{20}$ and $C_{40}$ coefficients coincide with the satellite-observed ones, computed using King-Hele's (1965) zonals and Anderle's (1966) tesserals up to $P_{76}$. Contour interval, 10 meters. (June 1966.)

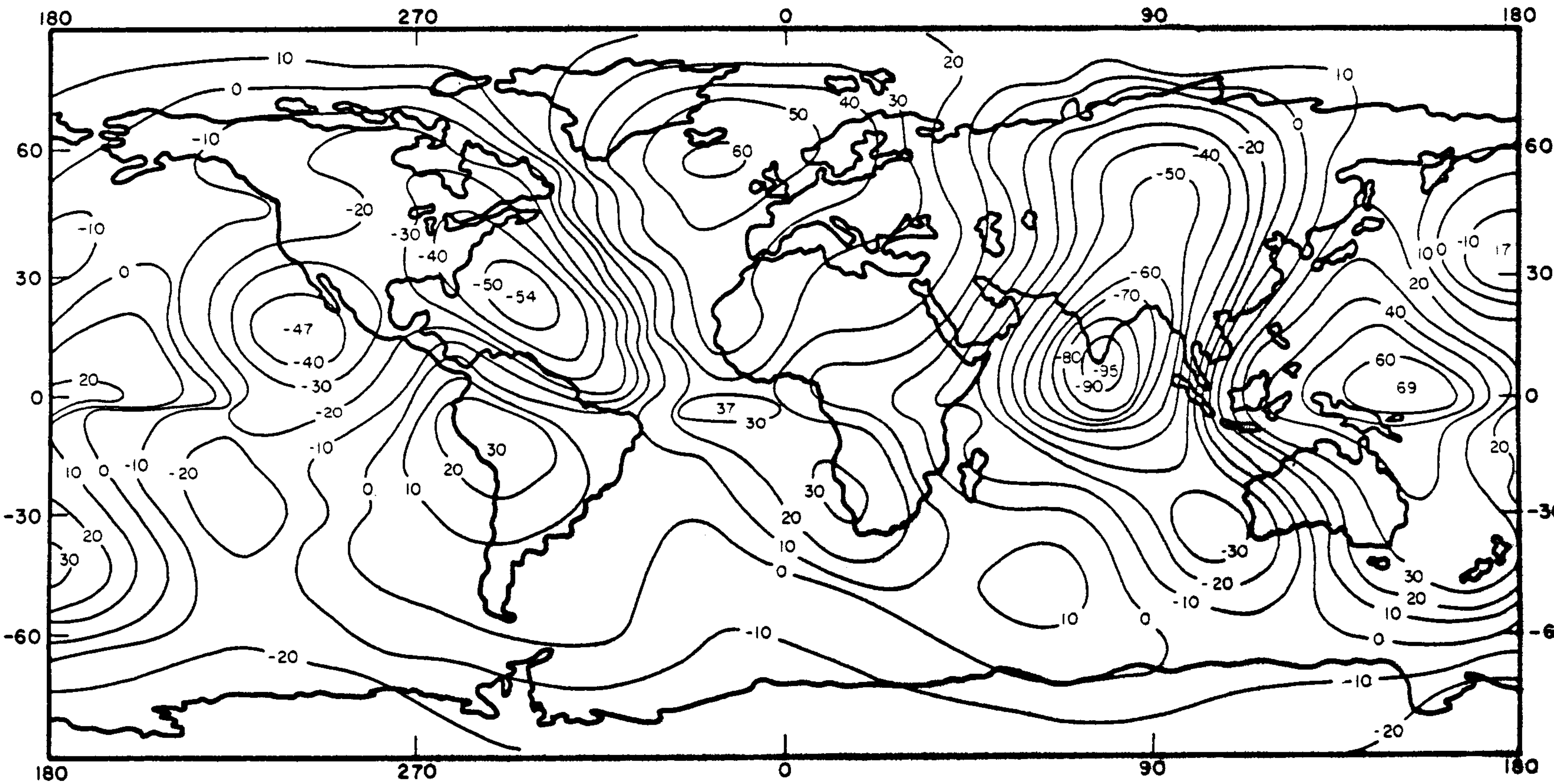


Fig. 4. Geoidal undulations in meters referred to an ellipsoid whose $C_{20}$ and $C_{40}$ coefficients coincide with the satellite-observed ones, computed using mean zonals and tesserals up to $P_{88}$, as defined in the text. Contour interval, 10 meters. (June 1966.)

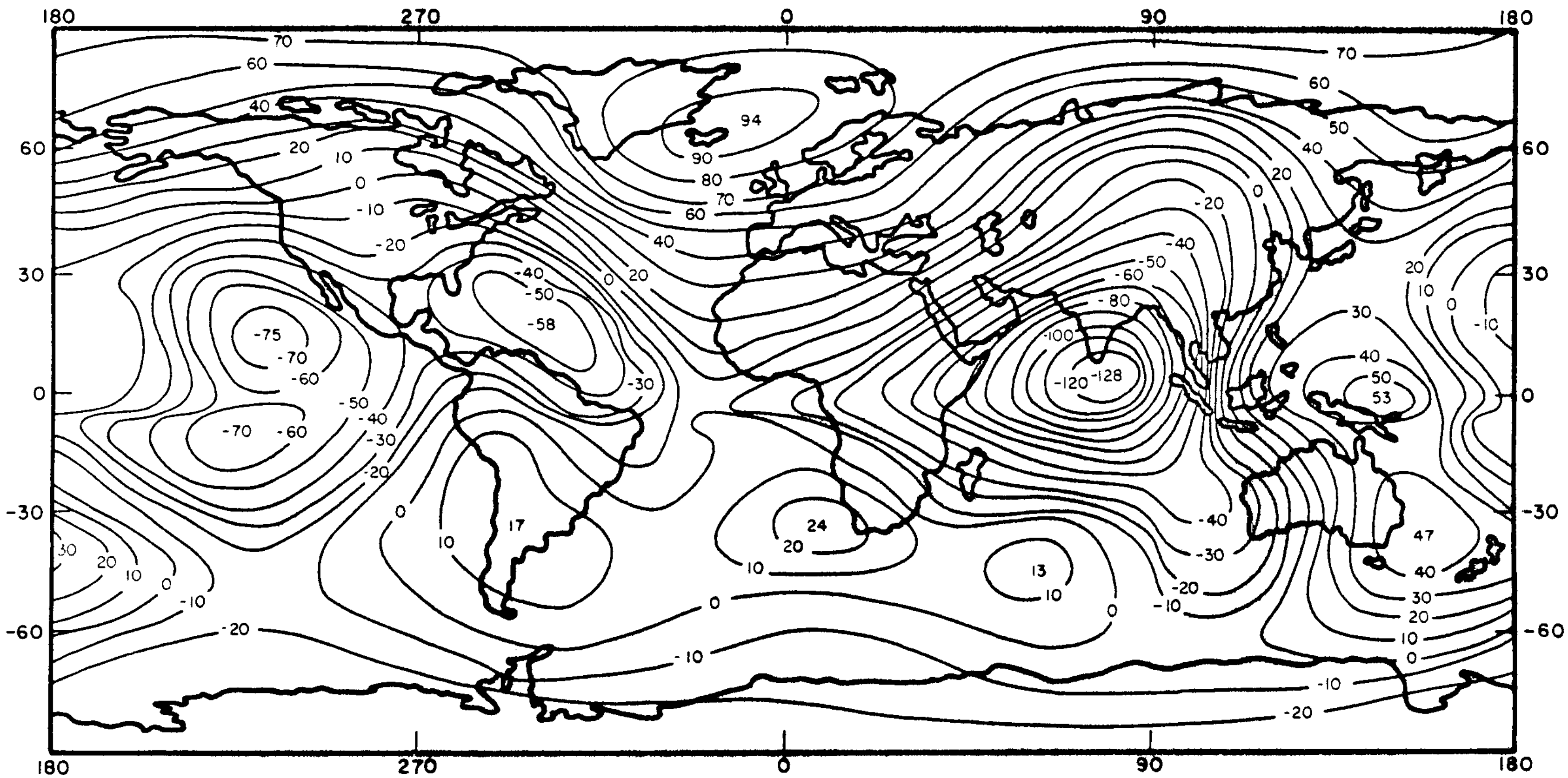

Fig. 5. Geoidal heights in meters referred to the international reference ellipsoid, computed using Kozai's (1964) zonals and Gaposhkin's (1966) tesseral harmonic coefficients up to $P_{88}$. Contour interval, 10 meters. (June 1966.)

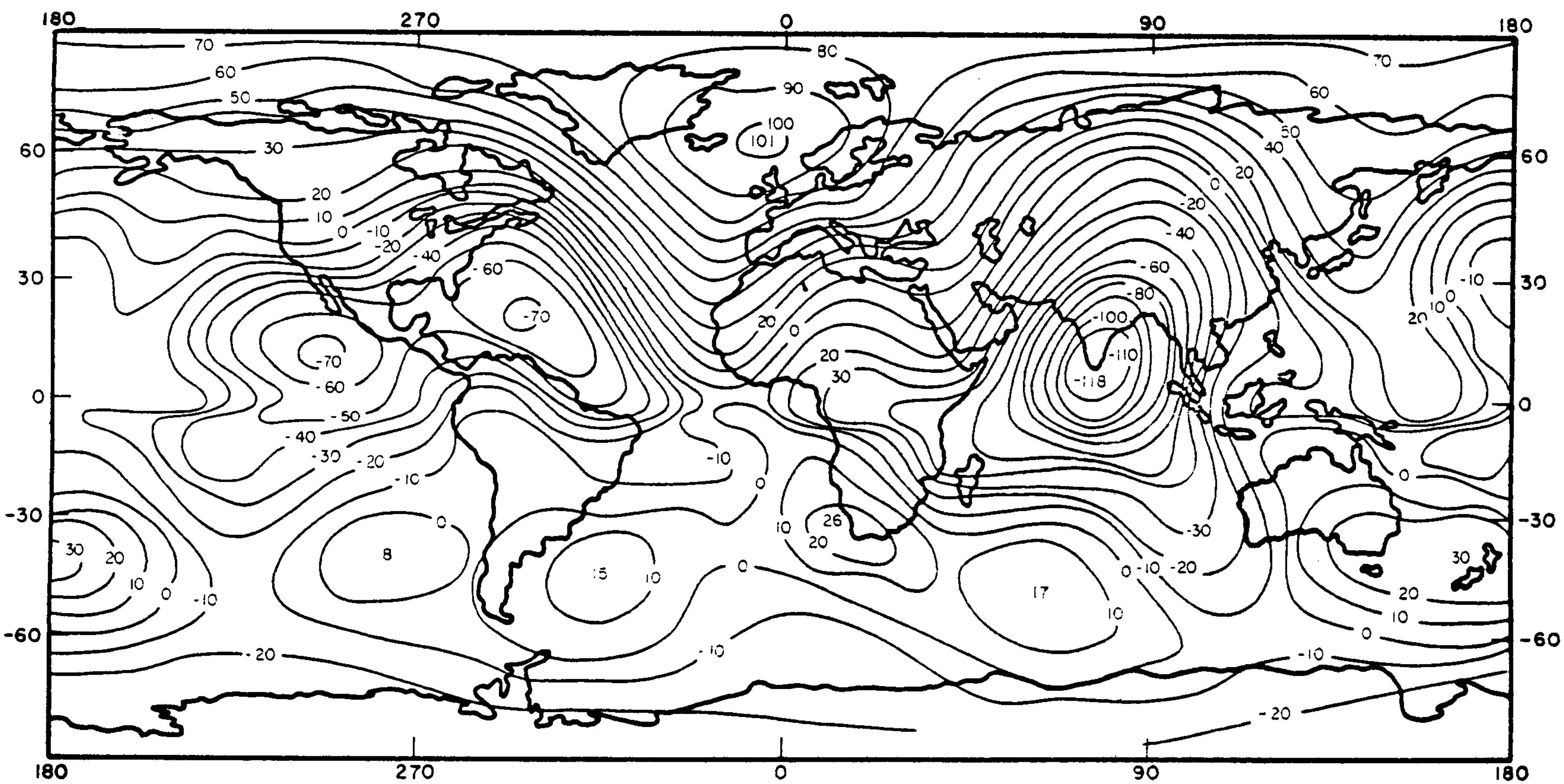


Fig. 6. Geoidal undulations in meters referred to the international reference ellipsoid, computed from Smith's (1963, 1965) zonals and Guier and Newton's (1965) tesserals to $P_{88}$. Contour interval, 10 meters. (June 1966.)

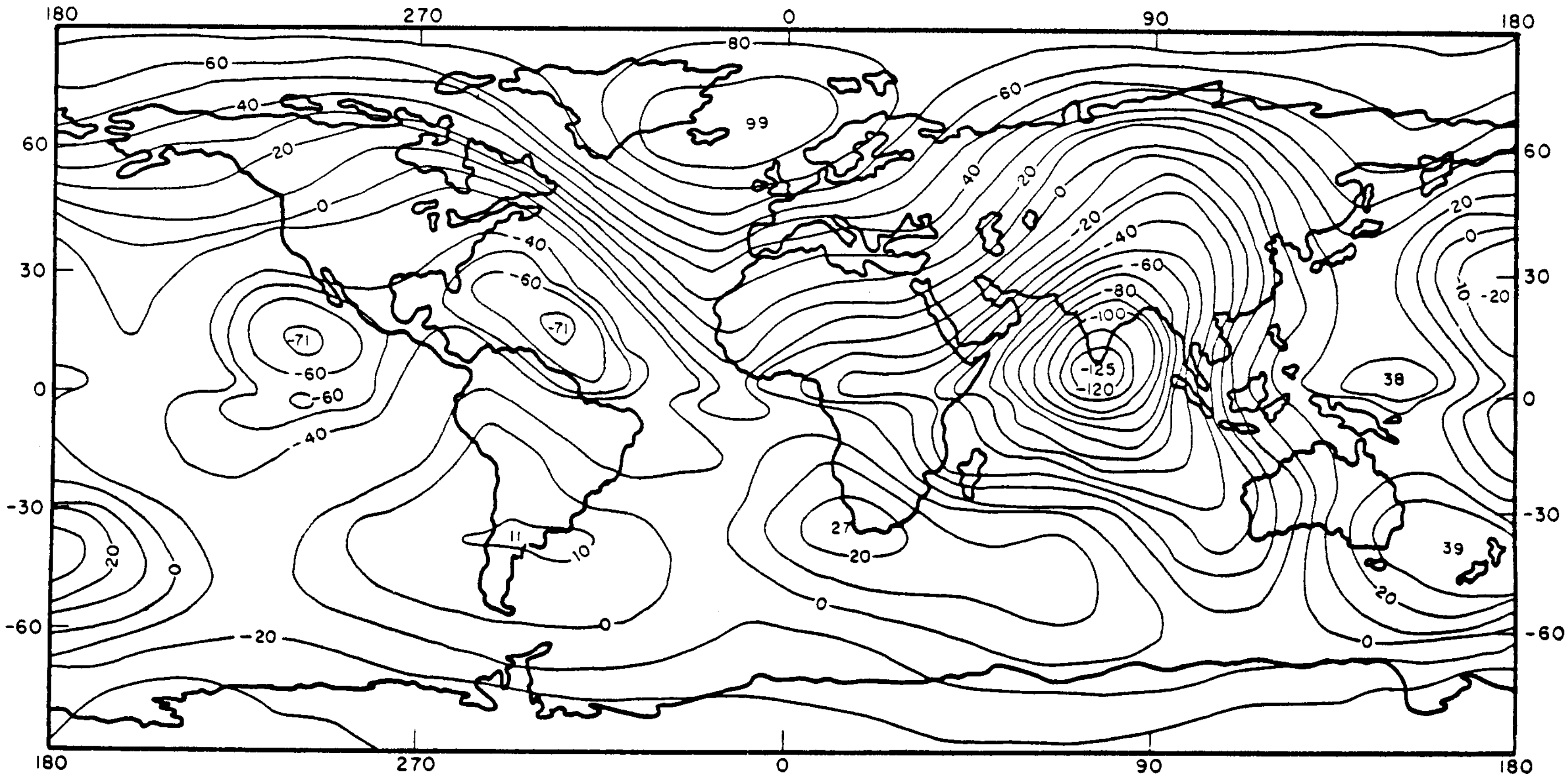


Fig. 7. Geoidal undulations in meters referred to the international reference ellipsoid, computed using mean coefficients up to $P_{88}$ as defined in the text. Contour interval, 10 meters. (June 1966.)

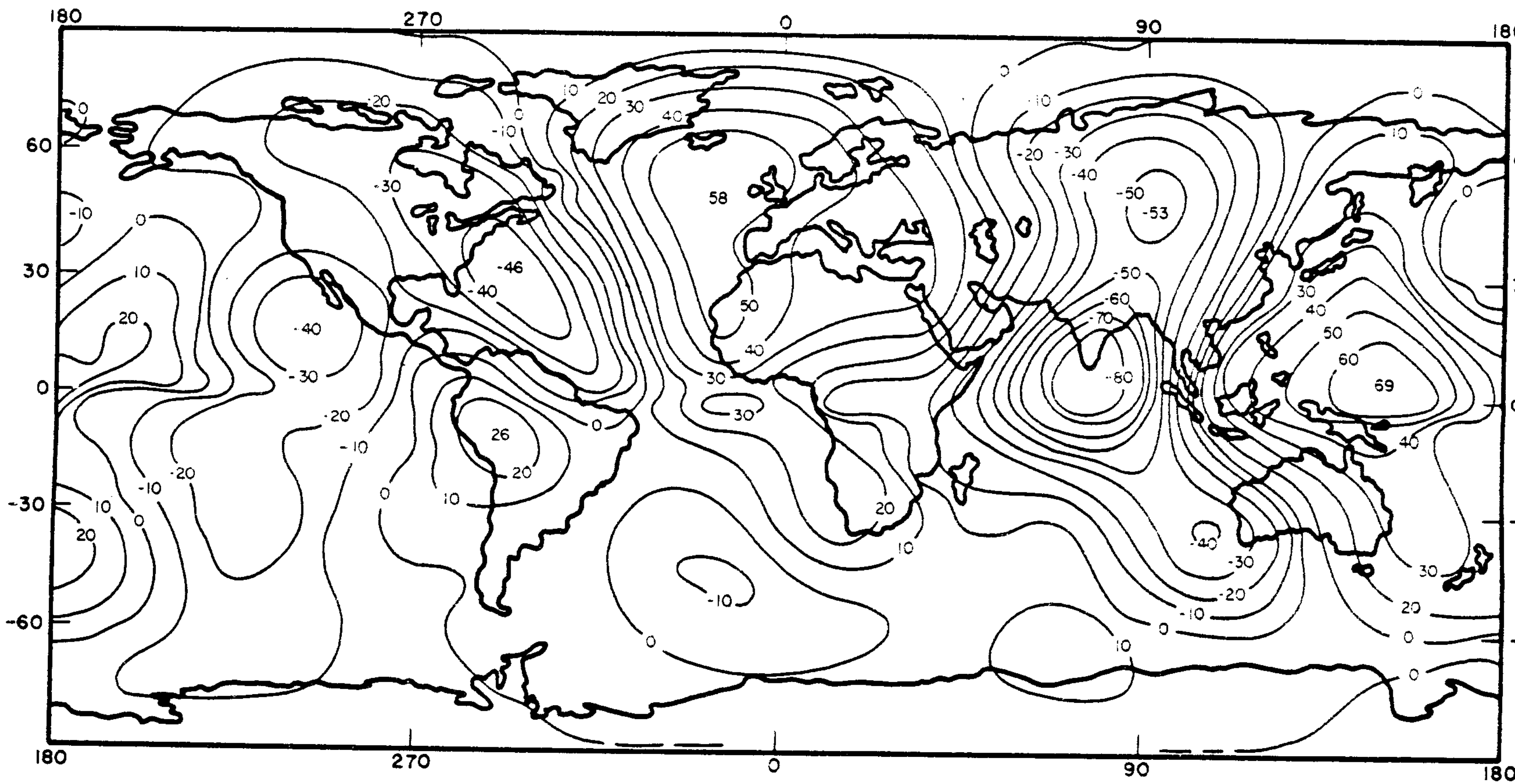

Fig. 8. Geoidal undulations in meters referred to a best-fitting satellite spheroid, as defined in the text; computed using mean tesseral coefficients to $P_{88}$, as defined in the text. Contour interval, 10 meters. (June 1966.)

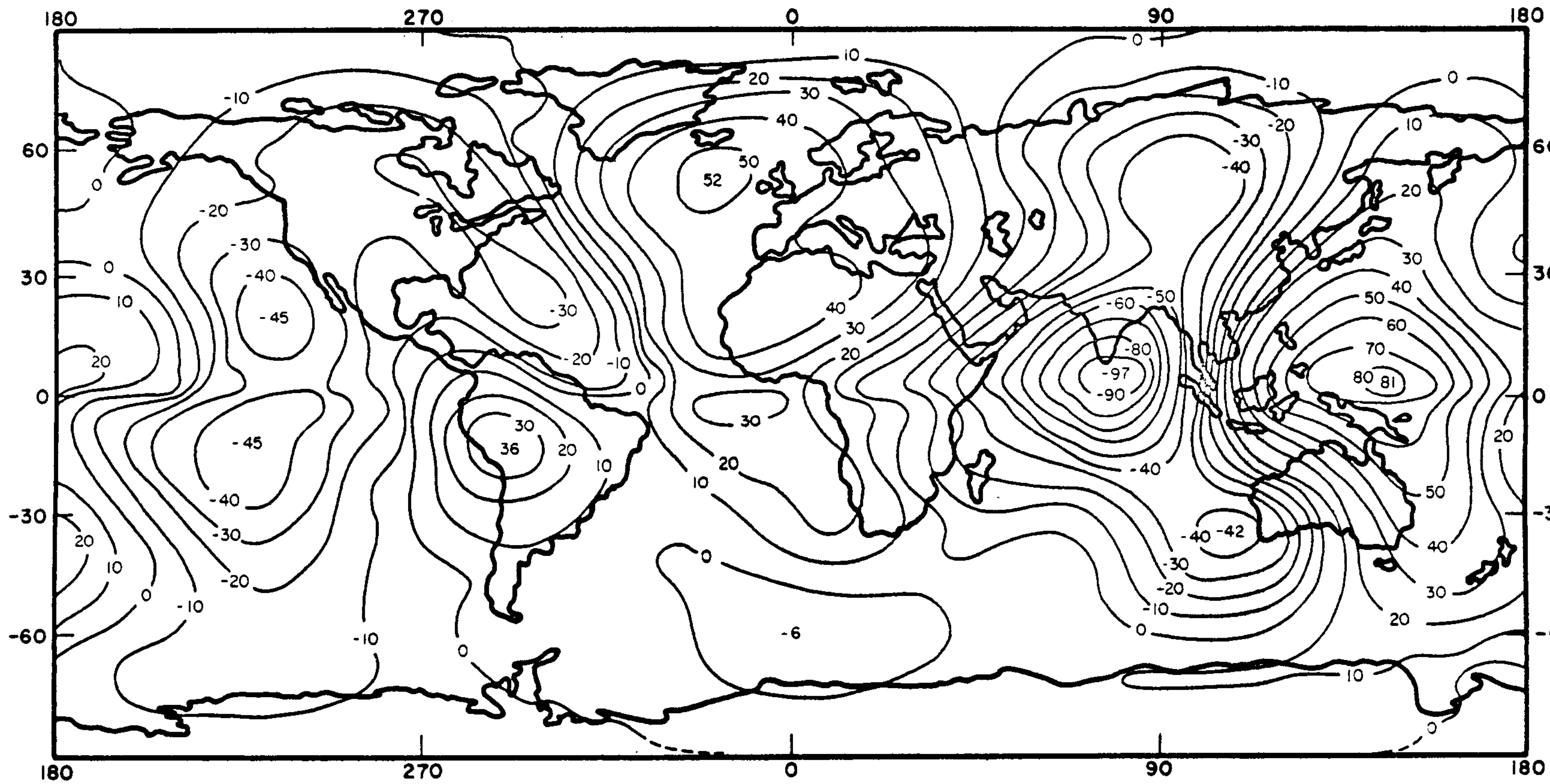


Fig. 9. Geoidal undulations in meters referred to a best-fitting satellite spheroid as defined in the text, computed using Gaposhkin's (1966) tesseral harmonic coefficients to $P_{88}$. Contour interval, 10 meters. (June 1966.)

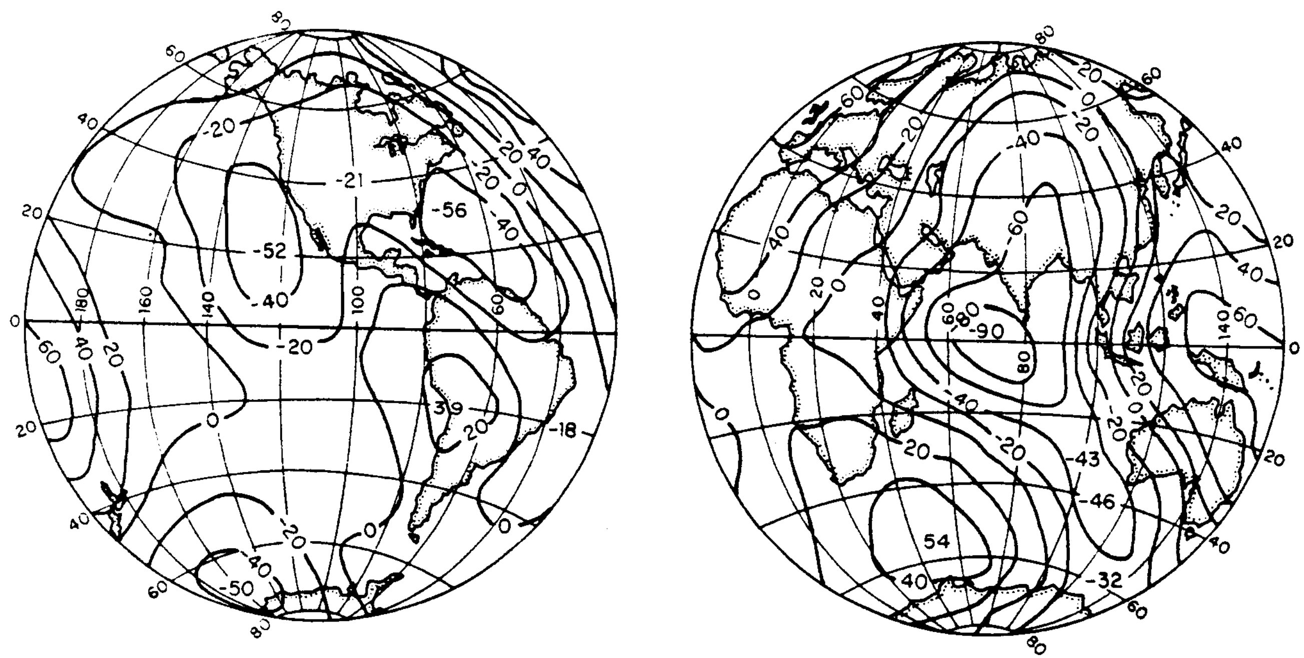


Fig. 10. Geoid heights in meters referred to an ellipsoid of flattening 1/298.25, calculated by Stokes' formula from 300-nautical-mile mean anomalies based on a combination of satellite data and terrestrial gravimetry. (Taken from Kaula, 1966.)

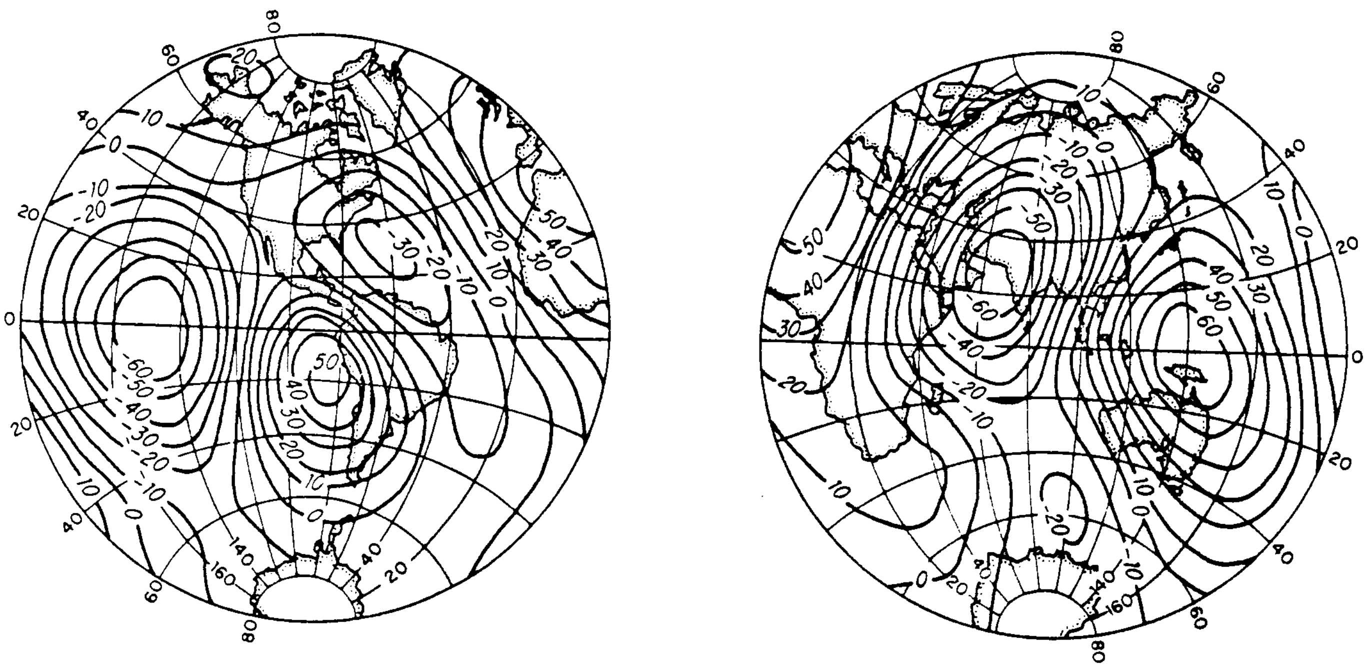

Fig. 11. Geoidal undulations in meters referred to an ellipsoid of flattening 1/298.24. Computed by Uotila (1962) using free-air anomalies.

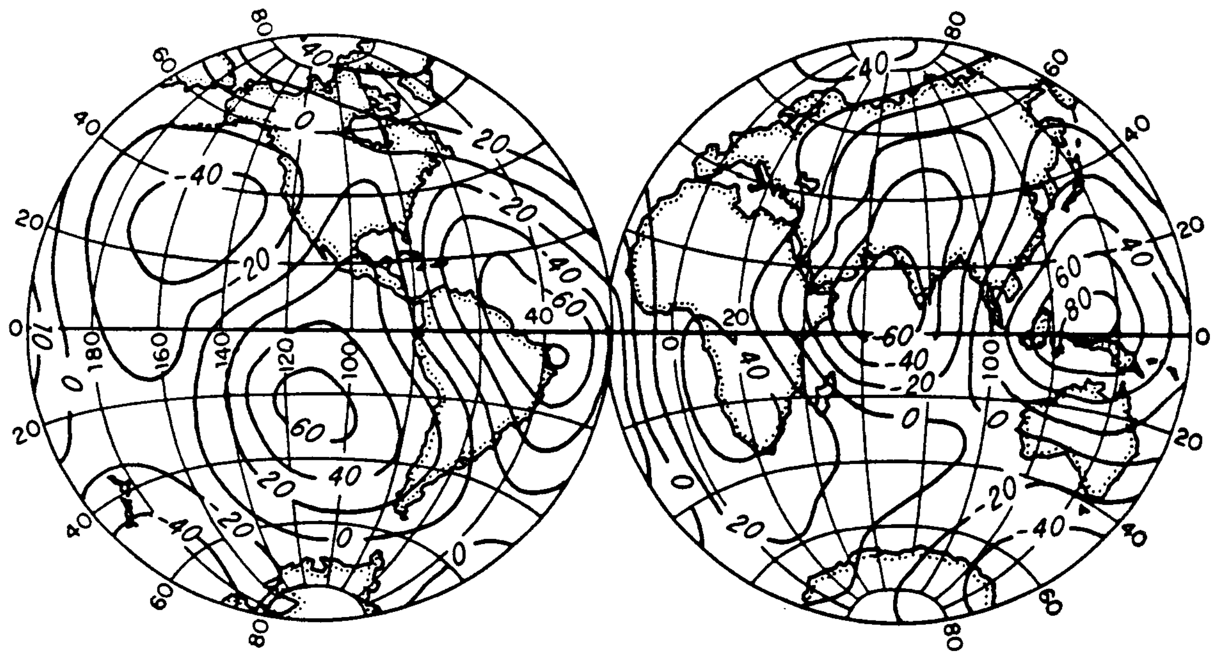

Fig. 12. Geoidal undulations in meters referred to the Russian reference ellipsoid. Computed by Zhongolovich (1952) using surface gravity information.

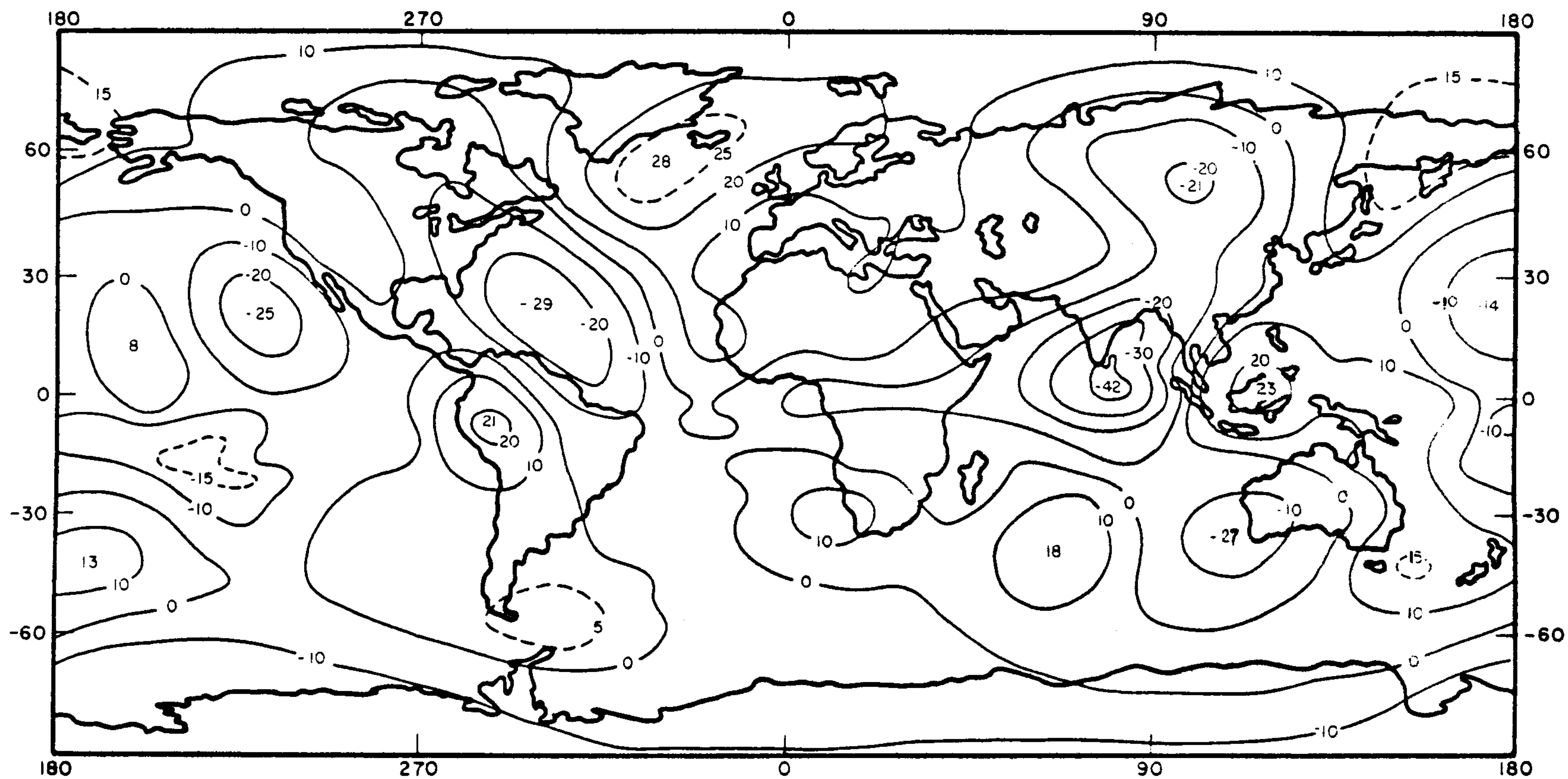


Fig. 13. Free-air gravity anomalies in milligals referred to the international gravity formula, computed using Kozai's (1964) zonals and Gaposhkin's (1966) tesserals to $P_{88}$. Contour interval between solid lines, 10 milligals; between solid and broken lines, 5 milligals. (June 1966.)

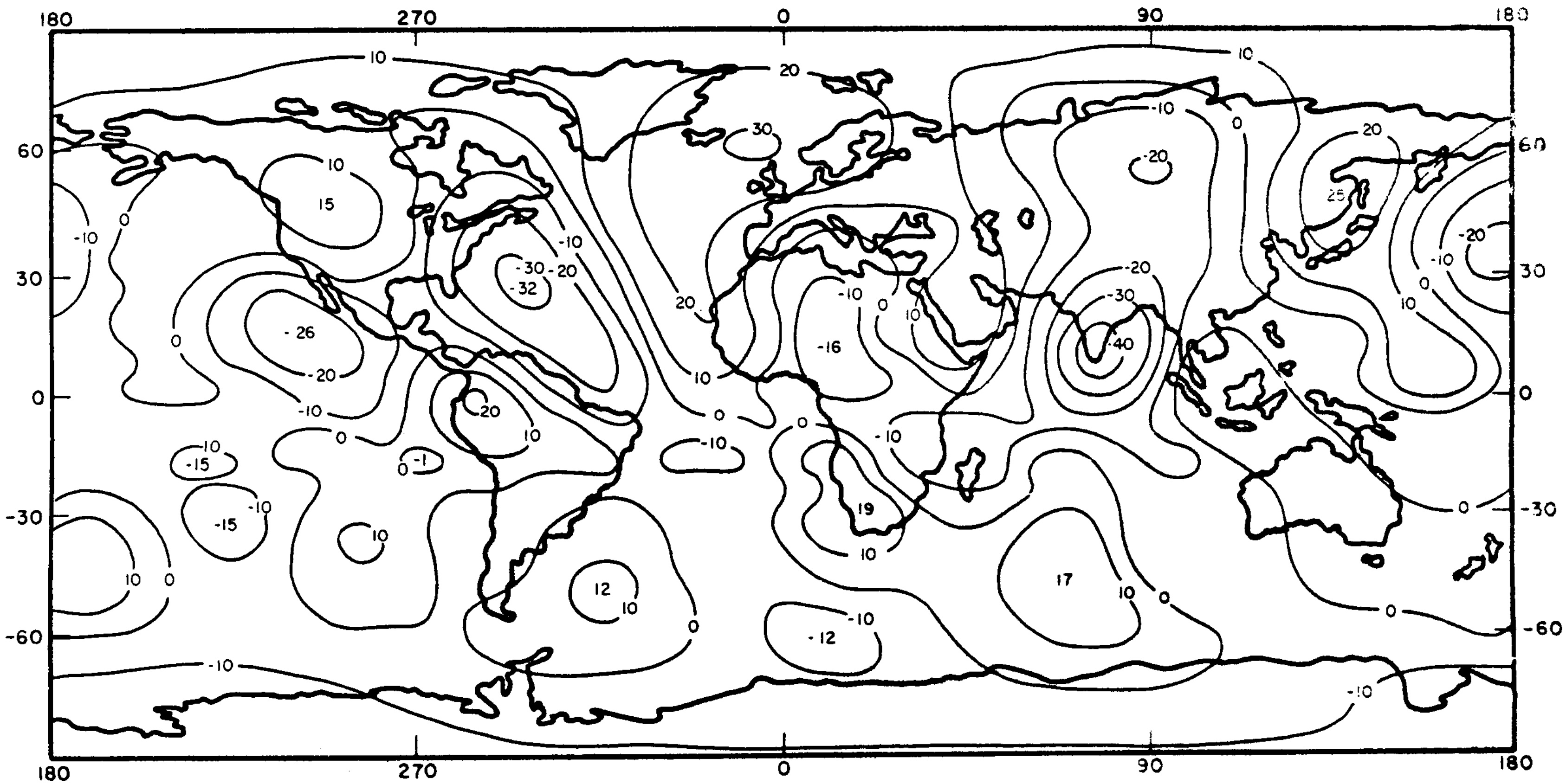


Fig. 14. Gravity anomalies in milligals referred to the international gravity formula, computed using Smith's (1963, 1965) zonals and Guier and Newton's (1965) tesserals to $P_{88}$. Contour interval, 10 milligals. (June 1966.)

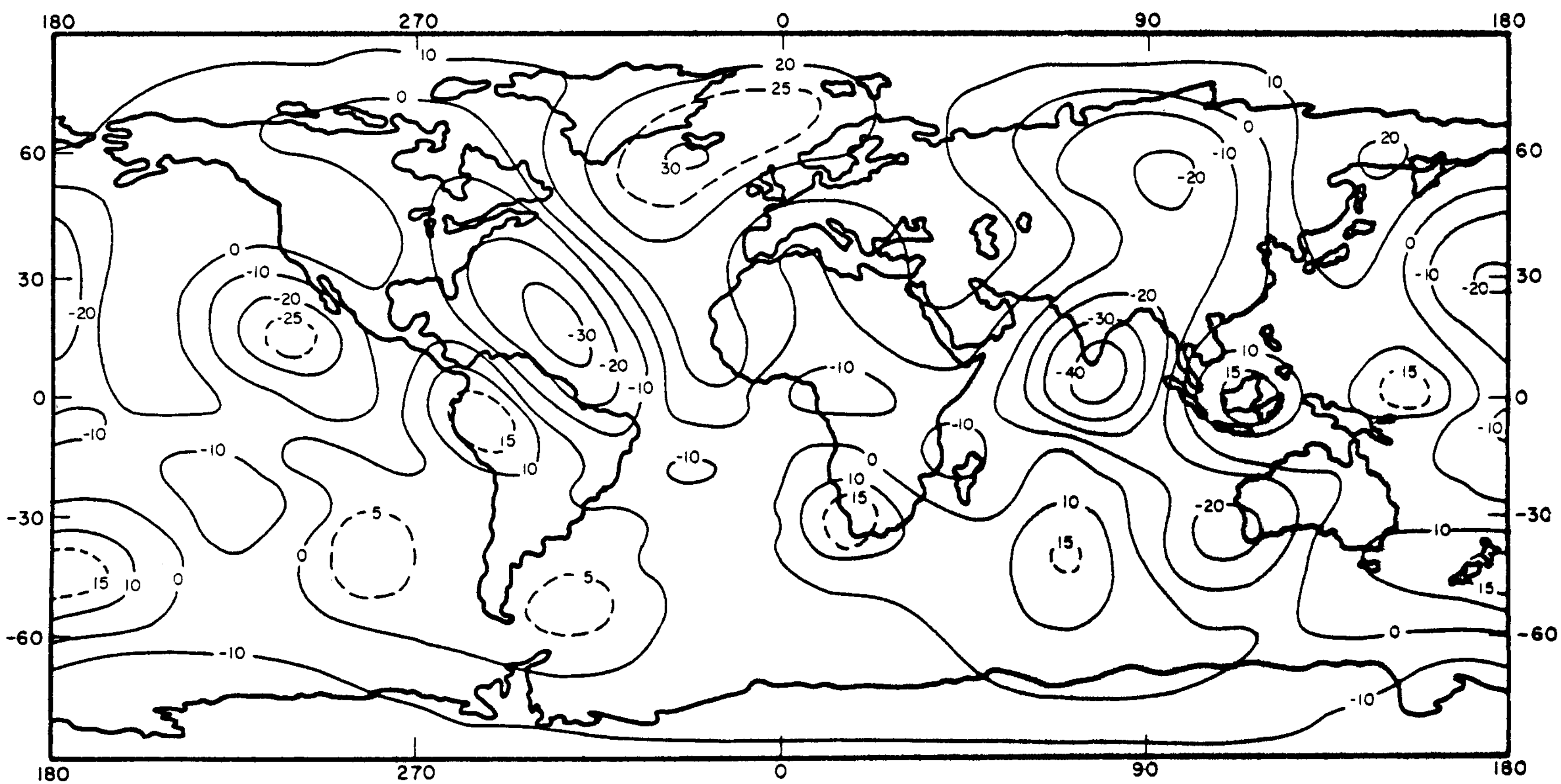


Fig. 15. Gravity anomalies in milligals referred to the international gravity formula, computed using mean zonals and tesserals up to $P_{88}$, as defined in the text. Contour interval between solid lines, 10 milligals; between solid and broken lines, 5 milligals. (June 1966.)

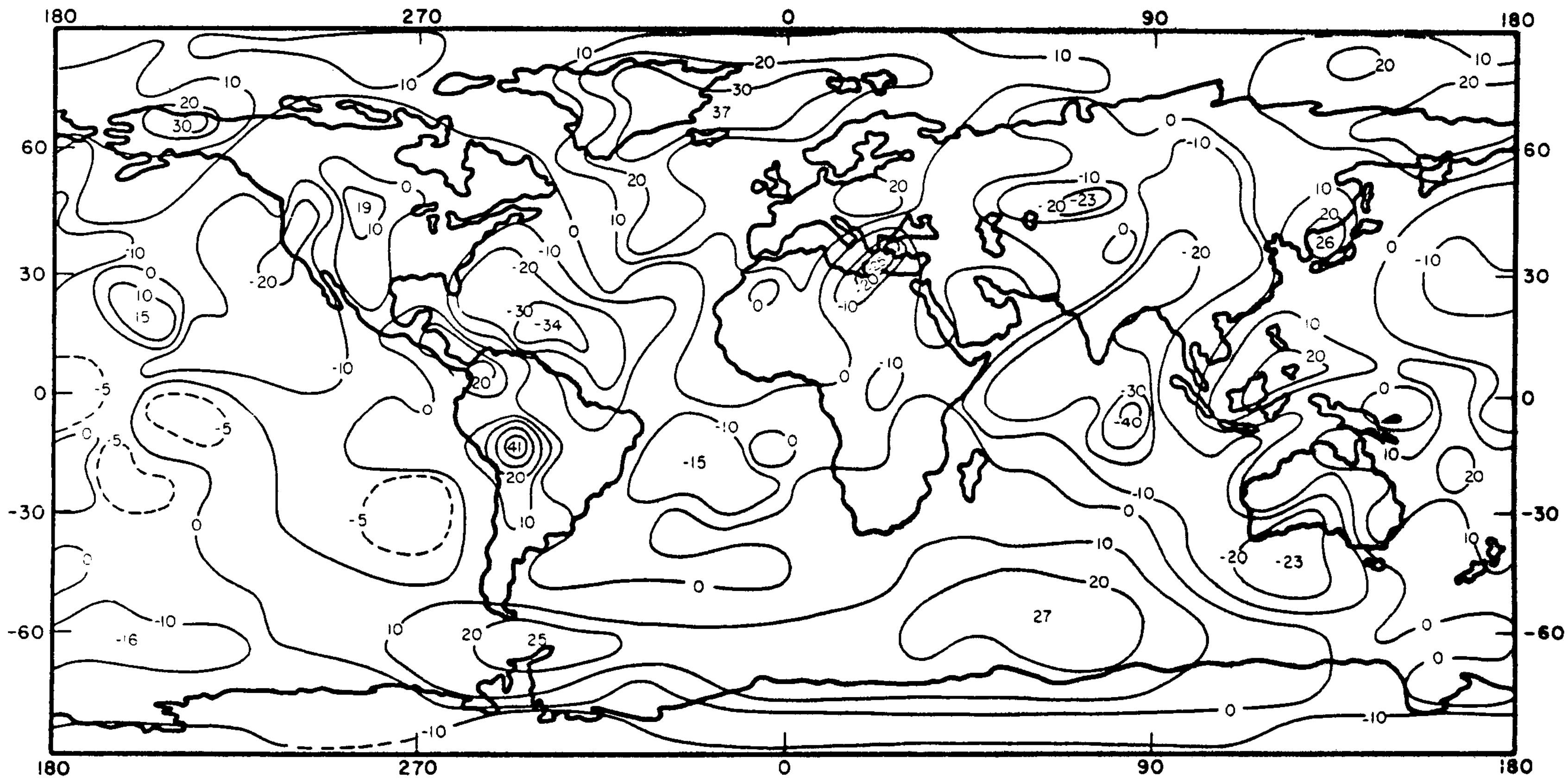


Fig. 16. Gravity anomalies in milligals (taken from Kaula, 1966); results combine available surface gravity measurement with the satellite information. Contour interval between solid lines, 10 milligals; between solid and broken lines, 5 milligals. (June 1966.)

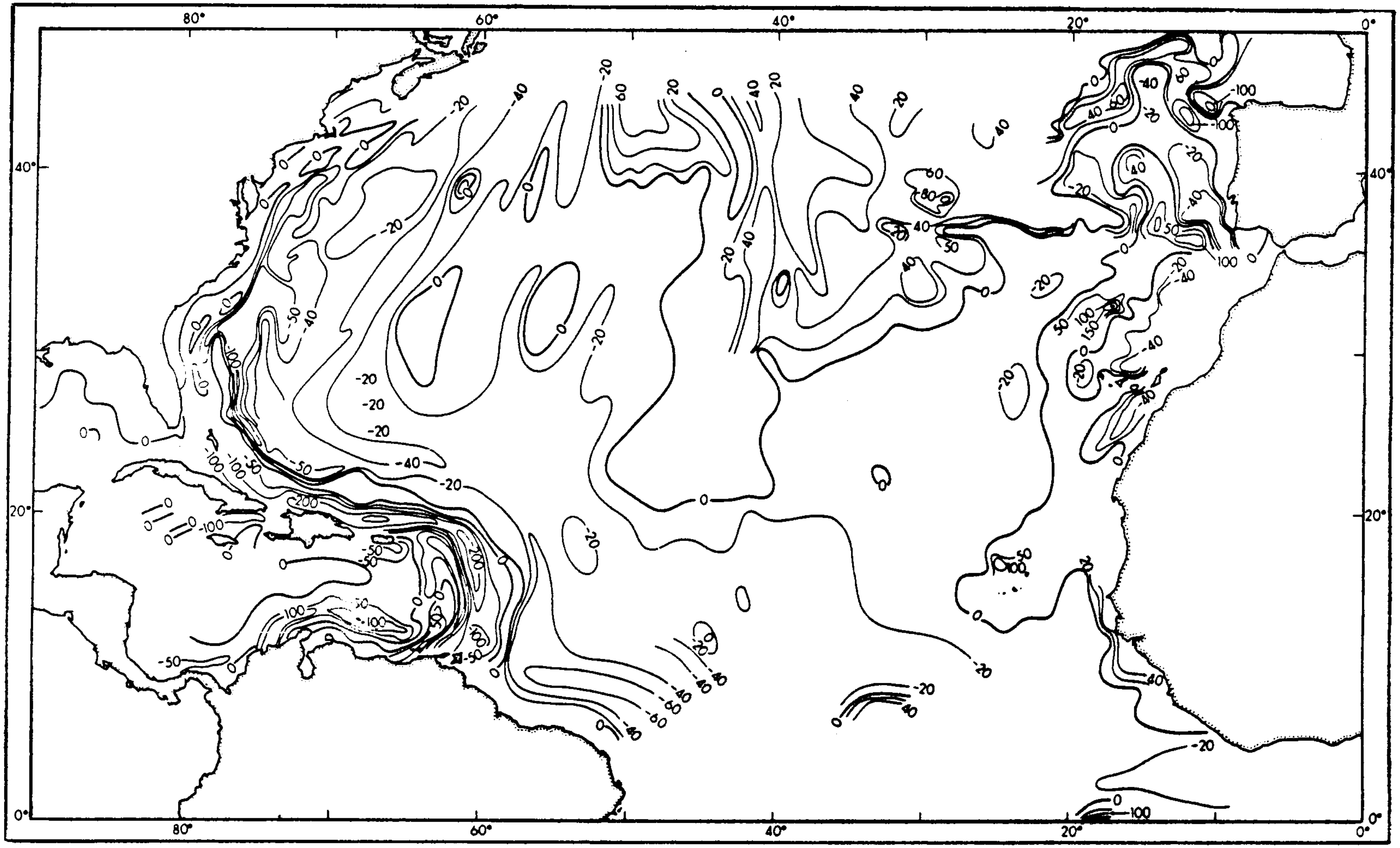


Fig. 17. Free-air gravity anomaly map for the North Atlantic Ocean.

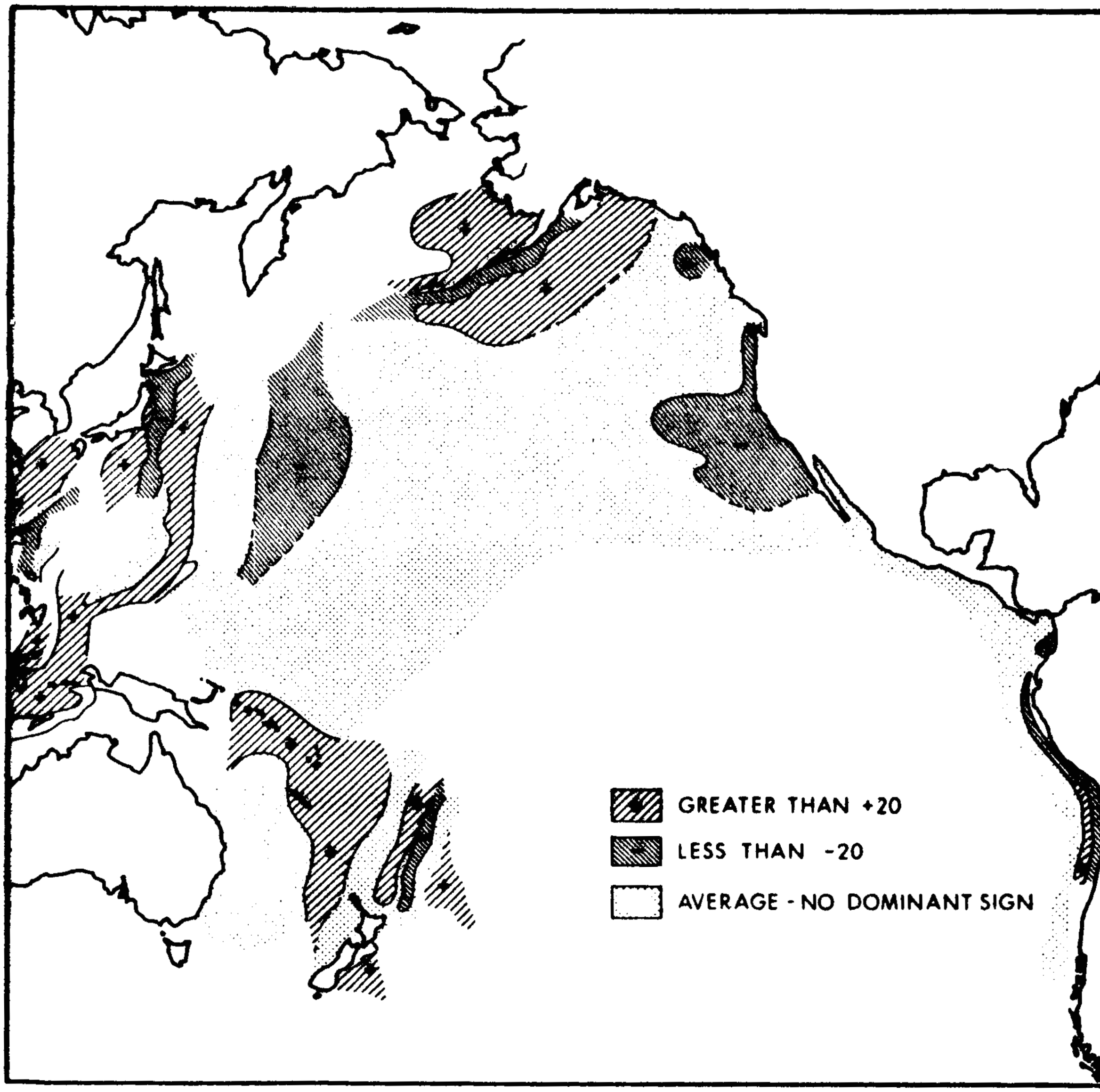


Fig. 18. Regional variations in free-air gravity anomaly values in the Pacific Ocean.