MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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CONDITIONS FOR ASYMPTOTIC STABILITY OF THE DISCRETE, MINIMUM VARIANCE, LINEAR, ESTIMATOR

by

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ABSTRACT

Stability of the discrete, homogeneous, linear, minimum variance estimation formulas is investigated. Sufficient conditions for uniform asymptotic stability in the large are derived. The conditions, if satisfied, also imply stochastic controllability and observability of the plant.

CONDITIONS FOR ASYMPTOTIC STABILITY OF THE DISCRETE, MINIMUM VARIANCE, LINEAR, ESTIMATOR

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I. Introduction

Stability of the minimum variance linear estimation formulas is essential in cases for which the estimation takes place over extended periods of time. For continuous systems, Kalman [2] has provided conditions which insure stability of the homogeneous filter equations. It appears however, that a similar treatment of the discrete estimation formulas is not available in the control literature. In a recent paper, Sorenson [4] derives upper and lower bounds on the discrete error covariance matrix, but does not consider stability. It is the purpose of this paper to derive the conditions which insure that the discrete minimum variance estimations formulas are uniformly, asymptotically stable in the large.

Before proceeding to the derivation, it is necessary to define certain notation. The matrix norm $||A||$ is A^TA. Matrix I is the unit matrix. Given two symmetric matrices B and C, of equal dimension, the inequality $B \geq C$ implies that the difference B-C is non-negative definite. Similarly $B > C$ implies that B-C is positive definite. Finally, it is tacitly assumed that all matrices are bounded from above in norm.

II Filter Equations and Conditions for Stability

Consider a discrete time stochastic process whose state vector $x(k)$ satisfies the recursion relation

$$x(k+1) = \Phi(k+1,k)x(k) + G(k)v(k); \quad ||\Phi(k+1,k)|| \geq \delta_1 I > 0 \tag{2-1}$$

where $v(k)$ is a white random vector sequence with

$$E[v(k)] = 0 \qquad E[v(k)v^T(j)] = \Delta_{kj}Q(k) \; ; \; Q(k) \geq \delta_2 I > 0 \tag{2-2}$$

and Δ_{kj} is the Kronecker delta. Linear measurement vectors $z(k)$ are available as outputs and the $z(k)$ are defined as

$$z(k) = H(k)x(k)+w(k) \tag{2-3}$$

where $w(k)$ is a white random vector sequence with

$$E[w(k)] = 0 \qquad E[w(k)w^T(j)] = \Delta_{kj}R(k) \; ; \; R(k) \geq \delta_3 I > 0 \tag{2-4}$$

For the system (2-1) through (2-4) the discrete minimum variance linear estimate of $x(k)$ is determined by equations

$$\hat{x}(k) = P(k)[P'(k)^{-1}\Phi(k,k-1)\hat{x}(k-1)+H^T(k)R(k)^{-1}z(k)] \; ; \; \hat{x}(0)=E[x(0)] \tag{2-5}$$

$$P(k) = [P'(k)^{-1}+H^T(k)R(k)^{-1}H(k)]^{-1} \; ; \; P(0)=E[x(0)x^T(0)] \tag{2-6}$$

$$P'(k+1) = \Phi(k+1,k)P(k)\Phi^T(k+1,k)+G(k)Q(k)G^T(k) \tag{2-7}$$

where $\hat{x}(k)$ is the minimum variance estimate, $P(k)$ is the estimation error covariance matrix after processing the measurement $z(k)$ and $P'(k+1)$ is the extrapolated error covariance matrix.

Now the system

$$y(k) = P(k)P'(k)^{-1}\Phi(k,k-1)y(k-1) \tag{2-8}$$

represents the homogeneous part of (2-5). If there exist real scalar functions $V(y(k),k)$, $\gamma_1(||y(k)||)$, $\gamma_2(||y(k)||)$ and $\gamma_3(||y(k)||)$ such that for some finite $N \geq 0$ *

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* Defining these conditions over N steps is equivalent to Kalman's definitions, over a finite interval, in the continuous case [2]

$$0 < \gamma_1(||y(k)||) \leq V(y(k),k) \leq \gamma_2(||y(k)||) \qquad y(k) \neq 0 \tag{2-9}$$

$$V(y(k),k)-V(y(k-N),k-N) \leq \gamma_3(||y(k)||) < 0 \quad k \geq N, y(k) \neq 0 \tag{2-10}$$

and in addition

$$\gamma_1(0) = \gamma_2(0) = 0 \qquad \lim_{\rho \to \infty} \gamma_1(\rho) = \infty \tag{2-11}$$

then the system (2-8) is uniformly asymptotically stable in the large [3].

In the sequel it will be shown that if there are real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that the conditions

$$\alpha_1 I \geq \sum_{i=k-N}^{k-1} \Phi(k,i+1)G(i)Q(i)G^T(i)\Phi^T(k,i+1) \geq \alpha_2 I \quad 0<\alpha_2,\ \alpha_1<\infty \tag{2-12}$$

$$\beta_1 I \leq \sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k) \leq \beta_2 I \quad 0<\beta_1,\ \beta_2<\infty \tag{2-13}$$

hold for all $k \geq N$; then the function $V_p(y(k),k)$ defined as

$$V_p(y(k),k) = y^T(k)P(k)^{-1}y(k) \tag{2-14}$$

satisfies conditions (2-9),(2-10) and (2-11). Thus if the system (2-1) through (2-4) satisfies conditions (2-12) and (2-13), the homogeneous system (2-8) is uniformly asymptotically stable in the large. Note that (2-12) and (2-13) imply that the system is stochastically controllable and observable [1].

III. A Lower Bound

Since $\hat{x}(k)$ given by (2-5,6,7) is the minimum variance linear estimate, the estimate $\tilde{x}(k)$ defined as *

$$\tilde{x}(k) = \tilde{P}(k)\left[\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}z(i)\right] \qquad k \geq N \tag{3-1}$$

$$\tilde{P}(k) = \left[\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k)\right]^{-1} \qquad k \geq N \tag{3-2}$$

* It can be shown that $\tilde{x}(k)$ is the maximum likelihood estimate for the $x(k)$ system with no process noise ($Q(k)=0$), and ignoring all data before k-N.

has an error covariance matrix satisfying the inequality

$$\mathrm{cov}\left\{\tilde{x}(k)-x(k)\right\} \equiv E\left\{[\tilde{x}(k)-x(k)][\tilde{x}(k)-x(k)]^T\right\} \geq P(k) \tag{3-3}$$

Now, with (2-1) and (2-3), the measurement z(i) may be written as

$$z(i)=H(i)\Phi(i,k)x(k)+w(i)-H(i)\Phi(i,k)\sum_{j=i}^{k-1} \Phi(k,j+1)G(j)v(j) \tag{3-4}$$

where, as usual, the state transition matrix has the properties

$$\Phi(k,i)=\Phi(k,k-1)\Phi(k-1,k-2)\ldots.\Phi(i+1,i)$$

$$\Phi(i,k)=\Phi(k,i)^{-1} \; ; \quad \Phi(k,k)=I \tag{3-5}$$

Combining (3-4),(3-3) and (3-1) yields

$$\mathrm{cov}\left\{x(k)-\tilde{x}(k)\right\} = \mathrm{cov}\left\{\tilde{P}(k)\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}w(i)\right\} \tag{3-6}$$

$$+\mathrm{cov}\left\{\tilde{P}(k)\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k)\sum_{j=i}^{k-1}\Phi(k,j+1)G(j)v(j)\right\}$$

Adding non-negative definite terms on the right of (3-6), by altering the lower limit of j, produces the inequality

$$\mathrm{cov}\left\{x(k)-\tilde{x}(k)\right\} \leq \mathrm{cov}\left\{\tilde{P}(k)\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}w(i)\right\} \tag{3-7}$$

$$+\mathrm{cov}\left\{\tilde{P}(k)\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k)\sum_{j=k-N}^{k-1} \Phi(k,j+1)G(j)v(j)\right\}$$

$$= \left[\sum_{i=k-n}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k)\right]^{-1}$$

$$+\sum_{i=k-N}^{k-1} \Phi(k,i+1)G(i)Q(i)G^T(i)\Phi^T(k,i+1)$$

Applying (2-12), (2-13) and (3-3) obtains an upper bound on P(k)

$$P(k) \leq \left[\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k)\right]^{-1} \tag{3-8}$$

$$+\sum_{i=k-N}^{k-1} \Phi(k,i+1)G(i)Q(i)G^T(i)\Phi^T(k,i+1) \leq \left(\frac{1}{\beta_1}+\alpha_1\right)I$$

and thus a lower bound on $V_p(y(k),k)$ is

$$V_p(y(k),k)=y^T(k)P(k)^{-1}y(k) \geq \left(\frac{\beta_1}{1+\alpha_1\beta_1}\right)||y(k)||^2 \equiv \gamma_1(||y(k)||) \tag{3-9}$$

IV. <u>An Upper Bound</u>.

Define the inverse of $P(k)$ as $W(k)$ and from (2-6)

$$W(k)\equiv P(k)^{-1}=P'(k)^{-1}+H^T(k)R(k)^{-1}H(k) \tag{4-1}$$

Similarly, define $W_1(k)$ and $W_1'(k)$ as

$$W_1(k) = W(k)-H^T(k)R(k)^{-1}H(k) = P'(k)^{-1} \tag{4-2}$$

$$W_1'(k+1) = \Phi^T(k+1,k)^{-1}W(k)\Phi(k+1,k)^{-1} \tag{4-3}$$

and with the help of (2-6) and (2-7), (4-2) and (4-3) become

$$W_1(k) =[W_1'(k)^{-1}+G(k-1)Q(k-1)G^T(k-1)]^{-1} \tag{4-4}$$

$$W_1'(k+1) = \Phi^T(k+1,k)^{-1}W_1(k)\Phi(k+1,k)^{-1}+\Phi^T(k+1,k)^{-1}H^T(k)R(k)^{-1}\cdot H(k)\Phi(k+1,k)^{-1} \tag{4-5}$$

By noting the similarity between (4-4), (4-5) and (2-6), (2-7) it is seen that $W_1(k)$ may be interpreted as the estimation error covariance matrix for a system described by the equations

$$x(k+1) = \Phi^T(k+1,k)^{-1}x(k)+\Phi^T(k+1,k)^{-1}H^T(k)s(k) \tag{4-6}$$

$$z(k) = G^T(k-1)x(k)+m(k-1) \tag{4-7}$$

$$\begin{aligned} E[s(k)] &= 0 \qquad E[s(k)s^T(j)] = \Delta_{kj}R(k)^{-1} \\ E[m(k)] &= 0 \qquad E[m(k)m^T(j)] = \Delta_{kj}Q(k)^{-1} \end{aligned} \tag{4-8}$$

Applying the results of Section III to this system, it is clear that

$$W_1(k) \leq [\sum_{i=k-N}^{k} \Phi(i,k)^{-1}G(i-1)Q(i-1)G^T(i-1)\Phi^T(i,k)^{-1}]^{-1} \tag{4-9}$$

$$+\sum_{i=k-N}^{k-1} \Phi^T(k,i+1)^{-1}\Phi^T(i+1,i)^{-1}H^T(i)R(i)^{-1}H(i)\Phi(i+1,i)^{-1}\Phi(k,i+1)^{-1}$$

or

$$W_1(k) \leq [\sum_{i=k-N-1}^{k-1} \Phi(k,i+1)G(i)Q(i)G^T(i)\Phi^T(k,i+1)]^{-1} \tag{4-10}$$

$$+\sum_{i=k-N}^{k-1} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k)$$

and according to (4-2)

$$W(k) \leq [\sum_{i=k-N}^{k-1} \Phi(k,i+1)G(i)Q(i)G^T(i)\Phi^T(k,i+1)]^{-1} \tag{4-11}$$

$$+\sum_{i=k-N}^{k} \Phi^T(i,k)H^T(i)R(i)^{-1}H(i)\Phi(i,k) \leq \left(\frac{1}{\alpha_2}+\beta_2\right) I$$

Therefore an upper bound on $V_p(y(k),k)$ is

$$V_p(y(k),k)=y^T(k)W(k)y(k)\leq\left(\frac{1}{\alpha_2}+\beta_2\right)||y(k)||^2\equiv\gamma_2(||y(k)||) \tag{4-12}$$

To this point, it has been shown that $V_p(y(k),k)$ satisfies conditions (2-9) and (2-11). Condition (2-10) is yet to be satisfied.

V. A Relevant Control Problem

Consider the system

$$y(k) = \Phi(k,k-1)y(k-1)+u(k) \tag{5-1}$$

where $u(k)$ is a control input. If the cost function J is defined as

$$J = \sum_{i=k-N}^{k} [y^T(i)H^T(i)R(i)^{-1}H(i)y(i)+u^T(i)P'(i)^{-1}u(i)] \tag{5-2}$$

then it will be useful to determine the optimal control sequence $u^*(k)$ which minimizes (5-2). Define large vectors Y and U as

$$Y = \begin{bmatrix} y(k) \\ y(k-1) \\ \vdots \\ y(k-N) \end{bmatrix} ; \qquad U = \begin{bmatrix} u(k) \\ u(k-1) \\ \vdots \\ u(k-N) \end{bmatrix} \tag{5-3}$$

and matrices M,B,L thus (5-4)

$$M = \begin{bmatrix} R(k) & & 0 \\ & R(k-1) & \\ 0 & & \ddots R(k-N) \end{bmatrix}; B= \begin{bmatrix} P'(k) & & 0 \\ & P'(k-1) & \\ 0 & & \ddots P'(k-N) \end{bmatrix}; L= \begin{bmatrix} H(k) & & 0 \\ & H(k-1) & \\ 0 & & \ddots H(k-N) \end{bmatrix}$$

so the cost J may be written as

$$J = Y^T L^T M^{-1} L Y + U^T B^{-1} U \tag{5-5}$$

Further, if matrices C and D are

$$C = \begin{bmatrix} \Phi(k,k-N-1) \\ \Phi(k-1,k-N-1) \\ \cdot \\ \cdot \\ \Phi(k-N,k-N-1) \end{bmatrix} \quad D = \begin{bmatrix} I & \Phi(k,k-1) & \Phi(k,k-2) & \cdots & \Phi(k,k-N) \\ & I & \Phi(k-1,k-2) & \cdot & \Phi(k-1,k-N) \\ & & I & \ddots & \vdots \\ & 0 & & \ddots & \\ & & & & I \end{bmatrix} \tag{5-6}$$

and the initial condition is defined as

$$y_0 = y(k-N-1) \tag{5-7}$$

then

$$Y = Cy_0 + DU \tag{5-8}$$

and

$$J = [U^T D^T + y_0^T C^T] L^T M^{-1} L [Cy_0 + DU] + U^T B^{-1} U \tag{5-9}$$

The first and second derivatives of J with respect to U(gradient vector and hessian matrix)are

$$\frac{\partial J}{\partial U} = 2[U^T D^T + y_0^T C^T] L^T M^{-1} L D + 2U^T B^{-1} \tag{5-10}$$

$$\frac{\partial^2 J}{\partial U^2} = 2B^{-1} > 0 \tag{5-11}$$

Controls u(k) are unconstrained so (5-10) and (5-11) provide necessary and sufficient conditions for the optimal control sequence U*. Setting (5-10) equal to zero obtains

$$U^* = -[D^T L^T M^{-1} L D + B^{-1}]^{-1} D^T L^T M^{-1} L C y_0 \tag{5-12}$$

and substituting into (5-9) yields the minimum cost

$$J^* = y_0^T C^T L^T [M^{-1} - M^{-1} L [L^T M^{-1} L + D^{T-1} B^{-1} D^{-1}]^{-1} L^T M^{-1}] L C y_0$$

$$= y_0^T C^T L^T [M + LDBDL^T]^{-1} L C y_0 \tag{5-13}$$

Now, from the definitions of L, C, M and condition (2-13)

$$\beta_1 I \le C^T L^T M^{-1} L C \le \beta_2 I \tag{5-14}$$

Because R(k) in Eq. (2-4) is bounded below, from (5-14) there exists a real number β_3 such that

$$||LCy_0|| \ge \beta_3 ||y_0|| \qquad 0 < \beta_3 < \infty \tag{5-15}$$

Also, because the norms of $R(k)$, $\Phi(k,k-1)$, $P'(k)$, and $H(k)$ are bounded above, it follows that there are real numbers β_4 and β_5 such that

$$\beta_4 I \leq M+LDBDL^T \leq \beta_5 I \qquad 0 < \beta_4,\ \beta_5 < \infty \tag{5-16}$$

and therefore

$$\beta_4^{-1} I \geq [M+LDBDL^T]^{-1} \geq \beta_5^{-1} I \tag{5-17}$$

Equations (5-13), (5-15) and (5-17) combine to establish that the minimum cost is positive if the initial state is non-zero

$$J^* = y_0^T C^T L^T [M+LDBDL^T]^{-1} LCy_0 \geq \beta_3^2 \beta_5^{-1} ||y(k-N-1)||^2 \tag{5-18}$$

VI. Stability

In order to use the results of Section III to prove stability, the homogeneous equation (2-8) is written as two equations

$$y(k) = y'(k)+u(k) \tag{6-1}$$

$$y'(k) = \Phi(k,k-1)y(k-1) \tag{6-2}$$

where $u(k)$ is considered as a control input and

$$u(k) = [P(k)P'(k)-I]y'(k) \tag{6-3}$$

Applying (6-1) and (6-2), the function $V_p(y(k),k)$ becomes

$$V_p(y(k),k)=y^T(k)P(k)^{-1}y(k)=y^T(k)[P'(k)^{-1}+H^T(k)R(k)^{-1}H(k)\ ]y(k)$$

$$=y'^T(k)P'(k)^{-1}y'(k)-y^T(k)H^T(k)R(k)^{-1}H(k)y(k)$$
$$+2y^T(k)[P(k)^{-1}-P'(k)^{-1}]y(k)+y^T(k)P'(k)^{-1}y(k)-y'^T(k)P'(k)^{-1}y'(k)$$

$$=y'^T(k)P'(k)^{-1}y'(k)-y^T(k)H^T(k)R(k)^{-1}H(k)y(k)$$
$$-[y(k)-y'(k)]^T P'(k)^{-1}[y(k)-y'(k)]$$

$$=y'^T(k)[\Phi(k,k-1)P(k-1)\Phi^T(k,k-1)+G(k-1)Q(k-1)G^T(k-1)]^{-1}y'(k)$$
$$-y^T(k)H^T(k)R(k)^{-1}H(k)y(k)-u^T(k)P'(k)^{-1}u(k)$$

$$=y^T(k-1)[P(k-1)+\Phi(k-1,k)G(k-1)Q(k-1)G^T(k-1)\Phi^T(k-1,k)]^{-1}y(k-1)$$
$$-y^T(k)H^T(k)R(k)^{-1}H(k)y(k)-u^T(k)P'(k)^{-1}u(k)$$

$$\leq y^T(k-1)P(k-1)^{-1}y(k-1)-y^T(k)H^T(k)R(k)^{-1}H(k)y(k)-u^T(k)P'(k)^{-1}u(k) \quad (6-4)$$

hence

$$y^T(k)P(k)^{-1}y(k)-y^T(k-1)P(k-1)^{-1}y(k-1) \leq -y^T(k)H^T(k)R(k)^{-1}H(k)y(k)-u^T(k)P'(k)^{-1}u(k) \quad (6-5)$$

whence

$$y^T(k)P(k)^{-1}y(k)-y^T(k-N)P(k-N)^{-1}y(k-N) \leq -\sum_{i=k-N}^{k} [y^T(i)H^T(i)R(i)^{-1}H(i)y(i)+u^T(i)P'(i)^{-1}u(i)] \quad (6-6)$$

and from definitions (2-14) and (5-2)

$$V_p(y(k),k)- V_p(y(k-N),k-N) \leq -J \quad (6-7)$$

Since J* is the minimum cost, and satisfies (5-18)

$$V_p(y(k),k)-V_p(y(k-N),k-N)\leq -J\leq -J^*\leq -\beta_3^2\beta_5^{-1}||y(k-N-1)||^2 \quad (6-8)$$

Now define a matrix transformation $\theta(k,k-N-1)$ as

$$\theta(k,k-N-1)=[P(k)P'(k)^{-1}\Phi(k,k-1)][P(k-1)P'(k-1)^{-1}\Phi(k-1,k-2)]\ldots$$

$$\ldots[P(k-N)P'(k-N)\Phi(k-N,k-N-1)] \quad (6-9)$$

From (2-8) and the fact that all matrices in the product on the right of (6-8) are non-singular and bounded below in norm

$$||y(k-N-1)||=||\theta(k,k-N-1)^{-1}y(k)||\geq\beta_6||y(k)||$$

Finally (6-8) yields

$$V_p(y(k),k)-V_p(y(k-N),k-N)\leq -\beta_3^2\beta_5^{-1}\beta_6||y(k)||^2\equiv\gamma_3(||y(k)||) \quad (6-10)$$

and since $\gamma_3(||y(k)||)$ is negative for all non-zero vectors y(k), the system (2-8) is uniformly asymptotically stable in the large.

Conclusions

It has been shown that if a linear stochastic system is stochastically observable and controllable, then the corresponding discrete homogeneous minimum variance estimation equations are uniformly asymptotically stable in the large.

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