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APPROXIMATE SOLUTION OF SYSTEM OF SINGULAR INTEGRAL EQUATIONS

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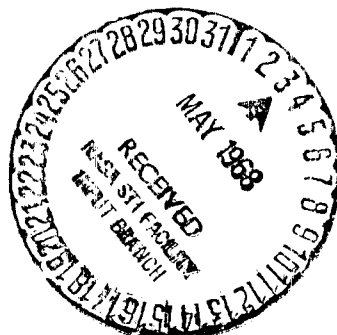
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by
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F. Erdogan^{**}

ABSTRACT

Using the properties of the related orthogonal polynomials, approximate solution of a system of simultaneous singular integral equations is obtained, in which the essential features of the singularity of the unknown functions are preserved. In the system of integral equations of first kind, the fundamental solution is the weight function of the Chebishev polynomials of first or second kind. In the system of singular integral equations of second kind with constant coefficients, the elements of the fundamental matrix are the weights of Jacobi polynomials. The approximate solution is expressed as the fundamental function, representing the singular behavior of the unknown functions, multiplied by series of proper orthogonal polynomials with unknown coefficients. The techniques of deriving the system of algebraic equations to determine these coefficients is described.

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1. INTRODUCTION

The system of singular integral equations of the form

$$\sum_1^M a_{ij}(t) \phi_j(t) + \int_{-1}^1 \sum_1^M b_{ij}(\tau) \phi_j(\tau) \frac{d\tau}{\tau-t} \\ + \int_{-1}^1 \sum_1^M k_{ij}(t,\tau) \phi_j(\tau) d\tau = f_i(t)$$

$$(i = 1, \dots, M), \quad (-1 < t < 1) \quad (1)$$

may be found in the formulation of many boundary value problems containing geometric singularities. In (1), the functions a_{ij} , b_{ij} and f_i are given on $(-1,1)$ and satisfy a Hölder condition, the kernels k_{ij} are also known and satisfy a Hölder condition in each of the variables t and τ , and the unknown functions ϕ_j are likewise required to satisfy a Hölder condition. In the known physical problems of practical interest, a_{ij} and b_{ij} are constants and M , the number of the unknown functions, is 2 or 3. Among the physical examples, we may mention the elastostatic problems in shells, composite materials and layered media containing cuts, diffusion problems in nonhomogeneous media containing partially insulated line barriers, wave diffraction problems in homogeneous media containing rigid line barriers or cuts, and contact problems in the presence of friction.

The general theory of the system of integral equations (1) is given in [1], where a standard technique of reducing it to a system of Fredholm integral equations is discussed. However, this technique is based on the assumption that the fundamental matrix corresponding to the solution of the dominant system is somehow obtained. Aside from the difficulty encountered in obtaining the fundamental matrix, for which no general method is given, in practice, the direct method of reducing the system of singular integral equations to a system of Fredholm equations is rather cumbersome. For these reasons, the development of an approximate method preserving the correct nature of singularities of the unknown functions seems to be very desirable.

In physical problems, the ends ∓ 1 are points of geometric singularity. Usually the investigation of the behavior of the unknown functions in the neighborhood of these singular points is one of the main objectives in solving the problem. Generally, physical arguments provide sufficient information about such behavior to fix the index for each unknown function. Invariably, these arguments simply amount to stating that if the unknown function is a potential (e.g., temperature, displacement, velocity potential), it has to be bounded at the singular points, and if it is a flux-type quantity (e.g., stress, heat flux, velocity), its value at singular points would be infinite but integrable. Thus, in the case of a single unknown function, solving the dominant equation and fixing the index by means of

physical arguments, one may easily obtain the fundamental function of the integral equation.

In a singular integral equation with constant coefficients, the fundamental function turns out to be the weight function of some well-known orthogonal polynomials. For example, in the integral equation of first kind, the fundamental function is the weight of Chebishev polynomials of first and second kind for the values of the index $\kappa = 1$ and $\kappa = -1$, respectively, and in the singular integral equation of second kind, the fundamental function is the weight of Jacobi polynomials. Thus, using the properties of the related orthogonal polynomials, an approximate solution of the integral equation may be obtained in which the essential features of the singularity of the unknown function is preserved. In [2], by an indirect use of the properties of Chebishev polynomials, the approximate solution of singular integral equation of first kind has been obtained. The problem has also been considered in [3] in rather general terms. More notable and somewhat more detailed applications using the properties of Jacobi polynomials appeared in recent papers [4] and [5]. In [5], a special case of (1) is considered in integrated form. Another technique to obtain an approximate solution to a singular integral equation of first kind is described in [6], in which instead of the proper orthogonal polynomials a power series is used and the unknown coefficients are obtained by the method of least squares.

In what follows, we will first consider the system of singular integral equations of first kind. In some physical problems, the kernels k_{ij} contain logarithmic singularities. For better accuracy in the subsequent quadratures involving k_{ij} , we will assume that the terms containing these singularities are also separated and remaining Fredholm kernels are bounded. We will then consider the system of integral equations of second kind with constant coefficients, obtain the fundamental matrix and describe two different approaches to find an approximate solution.

2. SINGULAR INTEGRAL EQUATIONS OF FIRST KIND, $\kappa = -1$

Consider the system of singular integral equations (1) in which the coefficients a_{ij} are zero, b_{ij} are given constants, the kernels k_{ij} contain no weak singularities but may have logarithmic singularities having also constant coefficients which are assumed to be separated. We thus have

$$\int_{-1}^1 \sum_{j=1}^M b_{ij} \phi_j(\tau) \frac{d\tau}{\tau-t} + \int_{-1}^1 \sum_{j=1}^M c_{ij} \phi_j(\tau) \log |\tau-t| d\tau + \int_{-1}^1 \sum_{j=1}^M k_{ij}(t,\tau) \phi_j(\tau) d\tau = f_i(t), \quad (i = 1, \dots, M), \quad (2)$$

$$(-1 < t < 1)$$

where the matrix b_{ij} is nonsingular, k_{ij} are known and bounded in the closed interval $-1 \leq (t, \tau) \leq 1$, and f_i are given functions satisfying the Hölder condition

$$|f(t) - f(\tau)| \leq A|t-\tau|^\mu, \quad (0 < \mu \leq 1), \quad (-1 \leq (t, \tau) \leq 1)$$

Considering the dominant part of each equation in (2), the fundamental functions for $\psi_k(t) = \sum_{kj} b_{kj} \phi_j(t)$ may easily be determined to be

$$R_k(t) = (1-t)^{1/2 + \lambda_k'} (1+t)^{-1/2 + \lambda_k''}, \quad (\lambda_k', \lambda_k'' = 0, \mp 1,) \quad (3)$$

The arbitrary integers λ_k' , λ_k'' are determined through physical arguments concerning the singular behavior of $\psi_k(t)$ at ∓ 1 . If ψ_k is bounded or infinite but integrable at ∓ 1 , we have

$$-1 < \frac{1}{2} + \lambda_k' < 1, \quad -1 < -\frac{1}{2} + \lambda_k'' < 1$$

The sum $-(\lambda_k' + \lambda_k'') = \kappa_k$ is known as the index of the corresponding equation and will have a value of $-1, 0$ or 1 . Thus, if the functions ψ_k are bounded and have the same behavior at ∓ 1 , $\kappa = -1$ and the fundamental functions are

$$R_k(t) = (1-t^2)^{1/2} \quad (4)$$

In this case, in addition to the system of equations (2), the solution must satisfy the following consistency conditions:

$$\int_{-1}^1 [f_k(t) - \int_{-1}^1 \sum_{j=1}^M (k_{ij}(t, \tau) + c_{ij} \log |t-\tau|) \phi_j(\tau) d\tau]$$

$$\frac{dt}{R_k(t)} = 0, \quad (k = 1, \dots, M) \quad (5)$$

$$(6)$$

Noting that the fundamental function $R_k(t)$ is the weight of the Chebishev polynomials of second kind, to obtain an approximate solution for (2), we will proceed as follows. Letting

$$\phi_i(t) = (1-t^2)^{1/2} \sum_0^N A_{in} U_n(t), \quad (i = 1, \dots, M) \quad (6)$$

we observe that the singular nature of the functions ϕ_k is preserved, where $U_n(t)$ is the Chebishev polynomial of second kind and A_{in} ($i = 1, \dots, M; n = 0, \dots, N$) are constant coefficients. Now, using the property [7]

$$\int_{-1}^1 \frac{U_n(\tau)(1-\tau^2)^{1/2}}{\tau-t} d\tau = -\pi T_{n+1}(t) \quad (7)$$

and defining

$$\int_{-1}^1 U_n(\tau)(1-\tau^2)^{1/2} \log |t-\tau| d\tau = V_n(t) \quad (8)$$

$$\int_{-1}^1 k_{ij}(t,\tau) U_n(\tau) (1-\tau^2)^{1/2} d\tau = g_{ijn}(t)$$

from (2), we obtain

$$\sum_{j=1}^M \sum_{n=0}^N [-\pi b_{ij} A_{jn} T_{n+1}(t) + c_{ij} A_{jn} V_n(t) + g_{ijn}(t) A_{jn}] = f_i(t), \quad (i = 1, \dots, M), \quad (-1 < t < 1) \quad (9)$$

where $T_n(t)$ is the Chebishev polynomial of first kind and $V_n(t)$

is given by

$$\begin{aligned}
 & - \frac{\pi}{2} (t^2 + \frac{1}{2} + \log 2), \quad (n = 0) \\
 V_n(t) = & \frac{\pi}{2} \left(\frac{T_n(t)}{n} - \frac{T_{n+2}(t)}{n+2} \right), \quad (n = 1, 3, 5, \dots) \quad (10) \\
 & \frac{\pi}{2} \left(\frac{T_n(t)}{n} - \frac{T_{n+2}(t)}{n+2} \right) + (-1)^{\frac{n+2}{n}} \frac{2\pi(n+1)}{(n+1)^2 - 1}, \\
 & \quad (n = 2, 4, \dots)
 \end{aligned}$$

To obtain the unknown constants A_{jn} , in (9), we multiply both sides by $T_k(t)(1-t^2)^{-1/2}$ and integrate from -1 to 1. Using the orthogonality relation

$$\int_{-1}^1 T_n(t) T_k(t) (1-t^2)^{-1/2} dt = \begin{cases} 0, & (n \neq k) \\ \pi, & (n = k = 0) \\ \frac{\pi}{2}, & (n = k > 0) \end{cases} \quad (11)$$

and defining

$$\begin{aligned}
 \alpha_{nk} &= \int_{-1}^1 V_n(t) T_k(t) (1-t^2)^{-1/2} dt \\
 \beta_{ijnk} &= \int_{-1}^1 g_{ijn}(t) T_k(t) (1-t^2)^{-1/2} dt \\
 f_{ik} &= \int_{-1}^1 f_i(t) T_k(t) (1-t^2)^{-1/2} dt
 \end{aligned} \quad (12)$$

from (9) we obtain

$$-\frac{\pi^2}{2} \sum_{j=1}^M b_{ij} A_{jk-1} + \sum_{j=1}^M \sum_{n=0}^N (c_{ij} A_{jn} \alpha_{nk} + \beta_{ijnk} A_{jn}) = f_{ik}$$

$$(i = 1, \dots, M), (k = 0, 1, \dots, N), (A_{j,-1} = 0) \quad (13)$$

Equation (13) provides a system of algebraic equations to determine the coefficients A_{jn} ($j = 1, \dots, M, n = 0, 1, \dots, N$).

In (13), the matrix (α_{nk}) is bi-diagonal and is given by

$$\alpha_{nk} = 0 \quad (k \neq n, k \neq n + 2)$$

$$= \frac{\pi}{2} \left(\frac{3}{2} + \log 2 \right), \quad (k = n = 0)$$

$$= \frac{\pi^2}{4n}, \quad (k = n > 0)$$

$$= -\frac{\pi^2}{4(n+2)}, \quad (k = n + 2, n \geq 0) \quad (14)$$

Noting that $A_{jk-1} = 0$ for $k = 0$, $T_0(t) = 1$, and considering the definitions (6), (8) and (12), it is seen that the first set of M equations obtained from (13) by letting $k = 0$ is equivalent to the consistency conditions given by (5). Hence, using the foregoing technique, the conditions of consistency are satisfied without any reference to or requirement of certain oddness - evenness properties of the functions ϕ_i , f_i and k_{ij} .

In the procedure outlined above, the main bulk of the numerical work lies in the evaluation of the integrals in (8) and (12), which may be considerable if one considers the fact

that the kernels k_{ij} are, in most cases, either very complicated functions or given in terms of improper integrals. However, note that all the integrals in (8) and (12) are of Gauss-Chebichev type and may easily be evaluated by using the proper quadrature formulas given by

$$\int_{-1}^1 h(t)(1-t^2)^{-1/2} dt \approx \sum_{i=1}^p w_i h(t_i) \quad (15)$$

$$w_i = \frac{\pi}{p}, \quad t_i = \cos \theta_i, \quad \theta_i = \frac{(2i-1)\pi}{2p}$$

with the related polynomial $T_n(t) = T_n(\cos \theta) = \cos n\theta$,

$$\int_{-1}^1 h(t)(1-t^2)^{1/2} dt \approx \sum_{i=1}^p w_i h(t_i) \quad (16)$$

$$w_i = \frac{\pi}{p+1} \sin^2 \theta_i, \quad t_i = \cos \theta_i, \quad \theta_i = \frac{(i+1)\pi}{p+1}$$

with the related polynomial $U_n(t) = U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$. In practice, this basic simplicity of evaluating the coefficients α_{nk} and β_{ijnk} would make it possible to take into account large number of terms in the series (6) resulting in a high degree of accuracy for the approximate solution.

3. SINGULAR INTEGRAL EQUATIONS OF THE FIRST KIND, $\kappa = 1$

If the unknown functions $\phi_j(t)$ are infinite but integrable at ∓ 1 , the index of the equations and the fundamental func-

tions will be

$$\kappa = 1, R_k(t) = (1-t^2)^{-1/2} \quad (17)$$

Observing that $R_k(t)$ is the weight of the Chebishev polynomials of the first kind, $T_n(t)$, the approximate solution of (2) may be expressed as

$$\phi_j(t) \approx (1-t^2)^{-1/2} \sum_0^N B_{jn} T_n(t), \quad (j = 1, \dots, M) \quad (18)$$

Using the relation, [7]

$$\int_{-1}^1 T_n(\tau)(1-\tau^2)^{-1/2} \frac{d\tau}{\tau-t} = \begin{cases} 0, & (n = 0) \\ \pi U_{n-1}(t), & (n > 0) \end{cases} \quad (19)$$

and defining

$$\begin{aligned} \int_{-1}^1 T_n(\tau)(1-\tau^2)^{-1/2} \log |t-\tau| d\tau &= W_n(t) \\ \int_{-1}^1 k_{ij}(t,\tau) T_n(\tau)(1-\tau^2)^{-1/2} d\tau &= h_{ijn}(t) \\ \gamma_{nk} &= \int_{-1}^1 W_n(t) U_k(t)(1-t^2)^{1/2} dt \\ \rho_{ijnk} &= \int_{-1}^1 h_{ijn}(t) U_k(t)(1-t^2)^{1/2} dt \\ F_{ik} &= \int_{-1}^1 f_i(t) U_k(t)(1-t^2)^{1/2} dt \end{aligned} \quad (20)$$

from (2), multiplying both sides by $U_k(t)(1-t^2)^{1/2}$ and integrating, we obtain

$$\frac{\pi^2}{2} \sum_{j=1}^M b_{ij} B_{jk+1} + \sum_{j=1}^M \sum_{n=0}^N (c_{ij} B_{jn} \gamma_{nk} + \rho_{ijnk} B_{jn}) = F_{ik}$$

(i = 1, ---, M), (k = 0, ---, N) (21)

where the orthogonality relation

$$\int_{-1}^1 U_n(t) U_k(t)(1-t^2)^{1/2} dt = \begin{cases} 0 & (n \neq k) \\ \frac{\pi}{2} & (n = k) \end{cases}$$

has been used. In (20), the functions W_n and the bi-diagonal matrix γ_{nk} may be evaluated, giving

$$W_n(t) = \begin{cases} -\pi \log 2, & (n = 0) \\ \frac{\pi}{n} T_n(t), & (n = 1, 3, 5, \dots) \\ \frac{\pi}{n} T_n(t) - \frac{2\pi}{n} (-1)^{n/2}, & (n = 2, 4, \dots) \end{cases}$$

(22)

$$\begin{aligned} \gamma_{nk} &= -\frac{\pi^2}{2} \log 2, \quad (n = k = 0) \\ &= 0, \quad (n \neq k, n \neq k + 2) \\ &= \frac{\pi^2}{4n}, \quad (n = k > 0) \\ &= -\frac{\pi^2}{4n}, \quad (n = k + 2, k \geq 0) \end{aligned}$$

In the case of $\kappa = 1$, generally there are some additional (physical) conditions which are not met by the solution given by (18) and (21). In the function-theoretical approach, the solution of the system of integral equation contains a set of arbitrary (complex) constants which are determined by using the additional conditions. In most physical problems, these conditions consist of specifying first and second moments of the functions $\phi_i(t)$ over the interval $(-1,1)$, i.e.,

$$\int_{-1}^1 \phi_i(t) dt = P_i, \int_{-1}^1 t\phi_i(t) dt = M_i, (i = 1, \dots, M) \quad (23)$$

where P_i, M_i are given constants. Substituting from (18) into (23) and using the orthogonality relation (11), it is seen that

$$B_{i0} = \frac{P_i}{\pi}, B_{i1} = \frac{2M_i}{\pi}, (i = 1, \dots, M) \quad (24)$$

The remaining $M \times (N-1)$ constants B_{ij} may then be obtained from (21) by letting $k = 2, 3, \dots, N$. The conditions given in the form other than (23) may be handled in a similar way. If the problem possesses any kind of symmetry, one may retain only even or odd terms in (6) and (18) which may result in considerable reduction in numerical work.

4. GENERAL REMARKS

Referring to (3), if the index of the equations is zero, that is, if the function is bounded at one end and unbounded at the other, one may try to extend the definition of the functions and the kernels outside the interval of $(-1,1)$ and reduce the problem to one having an index $\kappa = 1$. Or, observing that the fundamental function for $\kappa = 0$, $(1-t)^{1/2} (1+t)^{-1/2}$, is the weight of Jacobi polynomials $P_n^{(1/2, -1/2)}(t)$, using the properties of these polynomials, a procedure similar to that given above may be followed to obtain an approximate solution.

In practice, the quantities representing the strength of the singularities at ∓ 1 have great physical significance and, referring to (6) and (18), may easily be expressed as

$$\lim_{t \rightarrow \bar{+}1} \phi_i(t)(1-t^2)^{-1/2} \approx \sum_0^N A_{in} U_n(1) = \sum_{n=0}^N (n+1) A_{in}, (\kappa=-1)$$

(25)

$$\lim_{t \rightarrow \bar{+}1} \phi_i(t)(1-t^2)^{1/2} \approx \sum_0^N B_{in} T_n(1) = \sum_{n=0}^N B_{in}, (\kappa=1)$$

If the kernels $k_{ij}(t, \tau)$ in (1) are simply Hölder-continuous having no further restrictions on them, i.e., if they contain weak as well as logarithmic singularities, the technique described above is still applicable. In this case, however,

special attention must be paid to quadratures involving k_{ij} .

If the coefficient matrices (b_{ij}) and (c_{ij}) in (2) are not constants but known functions, the matrix (b_{ij}) being nonsingular in the interval $(-1,1)$, by defining $\psi_i(t) = \sum b_{ij}(t) \phi_j(t)$, the system of equations (2) may be reduced to one of constant coefficients. In this case, however, the coefficients α_{nk} in (13) and γ_{nk} in (21) may not be obtainable in closed form.

Finally, no major difficulty would arise in the application of this method to the system of equations with mixed indices, i.e., the case in which $\kappa = -1$ for some and $\kappa = +1$ for the rest of the unknown functions ϕ_j .

5. SYSTEM OF SINGULAR INTEGRAL EQUATIONS OF SECOND KIND

Consider the following system of integral equations of second kind

$$A\phi(t) + \frac{1}{\pi i} \int_a^b B\phi(\tau) \frac{d\tau}{\tau-t} + \int_a^b K(t,\tau) \phi(\tau) d\tau = f(t) \quad (26)$$

$$(a < t < b)$$

where the coefficient matrices $A = (a_{ij})$, $B = (b_{ij})$ are constant, the matrices $A \mp B$ are nonsingular, and the known functions $f = (f_i)$ and $K = (k_{ij})$ are Hölder-continuous in the interval (a,b) . The solution satisfying a Hölder condition and having certain singular behavior at a and b is sought. To

solve the problem, we first find the fundamental matrix by solving the homogeneous dominant system

$$A\phi(t) + \frac{1}{\pi i} \int_a^b B\phi(\tau) \frac{d\tau}{\tau-t} = 0, \quad (a < t < b) \quad (27)$$

Defining the matrix $\phi(z) = (\phi_j)$ by

$$\phi_j(z) = \frac{1}{2\pi i} \int_a^b \frac{\phi_j(t) dt}{t-z}, \quad (j = 1, \dots, M) \quad (28)$$

equation (27) may be written as

$$(A+B) \phi^+(t) = (A-B) \phi^-(t), \quad (a < t < b) \quad (29)$$

where ϕ^+ and ϕ^- correspond to the boundary values of the sectionally holomorphic functions $\phi_j(z)$. We now look for the solution of the homogeneous Riemann-Hilbert problem (29) in the form

$$\phi(z) = r w(z), \quad w^+(t) = \lambda w^-(t) \quad (30)$$

where w is a scalar function and the column matrix r and the coefficient λ are constants. Substituting (30) into (29), we obtain the following eigenvalue problem

$$(A-B)r = \lambda(A+B)r \quad (31)$$

Let the eigenvalues and eigenvectors obtained from (31) be $\lambda_1, \dots, \lambda_M$ and $(r_{j1}), \dots, (r_{jM})$. Corresponding to each λ_k , we

obtain a fundamental function $w_k(z)$ as follows

$$w_k^+(t) = \lambda_k w_k^-(t) \quad (k = 1, \dots, M) \tag{32}$$

$$w_k(z) = (z-a)^{\alpha_k} (z-b)^{\beta_k}$$

where

$$\alpha_k = \alpha_k' + i\beta_k' + \gamma_k', \quad \beta_k = \alpha_k'' + i\beta_k'' + \gamma_k''$$

$$\alpha_k' + i\beta_k' = -\frac{\log \lambda_k}{2\pi i}, \quad \alpha_k'' + i\beta_k'' = \frac{\log \lambda_k}{2\pi i}$$

$$-1 < \alpha_k' + \gamma_k' < 1, \quad -1 < \alpha_k'' + \gamma_k'' < 1$$

and the integers γ_k', γ_k'' are chosen in such a way that the behavior of the fundamental functions $w_k(z)$ at a and b is compatible with the expected singular behavior of the unknown functions $\phi_j(t)$, (i.e., either bounded or infinite but integrable). The constant

$$\kappa_k = -(\alpha_k + \beta_k) = -(\gamma_k' + \gamma_k'')$$

is known as the index of $w_k(z)$. In the physically important problems, the eigenvalues λ_k are real and negative, giving

$$\alpha_k = \mp \frac{1}{2} + \frac{\log |\lambda_k|}{2\pi} i, \quad \beta_k = \mp \frac{1}{2} - \frac{\log |\lambda_k|}{2\pi} i,$$

$$\kappa_k = -1, 0, 1$$

The square matrix

$$W = (w_{kj}(z)) = (r_{kj} w_j(z)) \quad (33)$$

is often referred to as the fundamental matrix of the boundary value problem (29). Once W is found, the general solution of (27) and (29) may be obtained as

$$\phi_k(z) = \sum_{j=1}^M r_{kj} w_j(z) P_j(z) \quad (34)$$

$$\phi_k(t) = \phi^+(t) - \phi^-(t), \quad (k = 1, \dots, M)$$

where $P_j(z)$, ($j = 1, \dots, M$), are arbitrary polynomials compatible with the behavior of ϕ_k at infinity.

It is easy to see that

$$(A+B) W^+(t) = (A-B) W^-(t) \quad (35)$$

If we now consider the nonhomogeneous problem

$$A\phi(t) + \frac{1}{\pi i} \int_a^b B\phi(\tau) \frac{d\tau}{\tau-t} = g(t) \quad (36)$$

$$(A+B) \phi^+(t) = (A-B) \phi^-(t) + g(t), \quad (a < t < b)$$

by using (35), we obtain

$$(W^{-1}\phi)^+ = (W^{-1}\phi)^- + (MW^+)^{-1} g, \quad M = A + B \quad (37)$$

From (37), the general solution of (36) may be obtained as

$$\phi(z) = \frac{W(z)}{2\pi i} \int_a^b (MW^+)^{-1}(\tau) g(\tau) \frac{d\tau}{\tau-z} + W(z) P(z) \quad (38)$$

where the elements of the matrix $P(z) = (P_n(z))$ are arbitrary polynomials compatible with the behavior of $\phi(z)$ at infinity. In the special case of real and negative eigenvalues λ_k , we have,

(a) $\kappa = 1$, $\phi_k(t)$ is infinite but integrable at $t = a$, $t = b$; for a solution vanishing at infinity $P_n = C_n = \text{constant}$; the constants C_1, \dots, C_M may be obtained from the physical conditions as explained in Section 3.

(b) $\kappa = 0$, $\phi_k(t)$ is bounded at one end and infinite and integrable at the other; (38) gives the solution with $P(z) = 0$.

(c) $\kappa = -1$, $\phi_k(t)$ is bounded at both ends; (38) gives the solution with $P(z) = 0$, provided the following consistency conditions are satisfied:

$$\int_a^b (MW^+)^{-1} g(t) dt = 0 \quad (39)$$

We now return to the integral equation (26) and first describe a direct method to obtain an approximate solution. Let

$$\int_a^b \sum_{j=1}^M k_{ij}(t, \tau) \phi_j(\tau) d\tau = \sum_{i=1}^N c_{in} p_n(t) \quad (40)$$

($i = 1, \dots, M$), ($a < t < b$)

where the known functions $p_n(t)$ form a complete system in (a,b) and c_{in} are undetermined constants*. Comparing (26) and (36), it is seen that, aside from the unknown constants c_{in} , the matrix $g(t)$ is known:

$$g_k(t) = f_k(t) - \sum_{n=1}^N c_{kn} p_n(t) \quad (41)$$

Thus, the solution of (26) can be obtained in closed form in terms of the undetermined constants c_{in} from (38) and

$$\phi_k(t) = \phi_j^+(t) - \phi_j^-(t) \quad (42)$$

The system of equations to obtain c_{in} is obtained by substituting $\phi_k(t, c_{in})$ into (40) and using a weighted residual method.

* Noting that $\phi(\tau)$ contains the functions $w_k(\tau)$ as multiplying factors, and in practice $p_n(t)$ is usually taken as a simple polynomial, there may be a question about the regularity of the left-hand side of (40). However, according to a theorem of Riesz [8], "every L^2 kernel $k(t, \tau)$ can be approximated (in the mean) as closely as we wish by means of a kernel of finite rank", i.e., $k(t, \tau) = \sum_{i=1}^N r_i(t) s_i(\tau)$, r_i, s_i being L^2 functions, $-1 < t, \tau < 1$. If we now assume that $\phi(\tau)$ may be expressed as $\phi(\tau) = \sum A_n w(\tau) p_n^{(\alpha, \beta)}(\tau)$ and the functions $r_i(t)$ are regular, the left-hand side of (40) becomes a series in $r_i(t)$ the coefficients of which contain the constants A_n and the expansion coefficients, B_{in} of $s_i(\tau)$ in a series of Jacobi polynomials $p_n^{(\alpha, \beta)}(\tau)$, $s_i(\tau) = \sum B_{in} p_n^{(\alpha, \beta)}(\tau)$. In physical problems of interest, since k_{ij} is either bounded or has at most a logarithmic singularity in the interval (a,b) , in principle, the assumption (40) seems to be justified.

$$\int_a^b \left[\int_a^b \sum_{j=1}^M k_{ij}(t, \tau) \phi_j(\tau, c_{in}) d\tau - \sum_{j=1}^N c_{in} p_n(t) \right] v_{im}(t) dt = 0$$

$$(i = 1, \dots, M), (m = 1, \dots, N) \quad (43)$$

where the weight functions $v_{im}(t)$ are such that either (43) is equivalent to a least square method, or $v_{im}(t)$, ($m = 1, 2, \dots$) form a complete system in (a, b) meaning that as $N \rightarrow \infty$, the approximate solution $\phi_j(t, c_{i1}, \dots, c_{iN})$ would be expected to approach the exact solution. However, in practice, one may also choose the weights as delta-functions, $v_{im}(t) = \delta(t - t_m)$, ($a < t_m < b$, $m = 1, \dots, N$), simplifying the numerical work considerably at the expense of somewhat slower convergence. In the case of $\kappa = -1$, to take into account the additional M (consistency) equations (39), the number of equations in (43) should be reduced by M by taking ($m = 1, \dots, N-1$).

In the singular integral equations of second kind with constant coefficients, the fundamental functions $w_k(t)$ are weights of Jacobi polynomials $P_n^{(\alpha_k, \beta_k)}(t)$ if the interval (a, b) is normalized to be $(-1, 1)$. Hence, using the properties of the Jacobi polynomials [4, 7, 9] and following a procedure similar to that of Sections 2 and 3 of this paper, a second approximate method to solve the system of equations (26) may also be developed. To do this, we first premultiply (26) by A^{-1} (assuming that A is nonsingular) and diagonalize the matrix $A^{-1}B = D = (d_{ij})$. Let λ_i , r_i , R be, respectively, the eigenvalues, the

eigenvectors and the modal matrix of D, i.e.,

$$|D - \lambda I| = (-1)^M \prod_1^M (\lambda - \lambda_j),$$

$$D r_j = \lambda_j r_j, \quad (j = 1, \dots, M), \quad R = (r_{ij})$$

Substituting

$$\phi(t) = R\psi(t), \quad \Lambda = (\lambda_j \delta_{ij})$$

and using the property $DR = R\Lambda$, it is seen that (26) may be written as

$$\begin{aligned} \psi(t) + \frac{1}{\pi i} \int_a^b \Lambda \psi(\tau) \frac{d\tau}{\tau-t} + \int_a^b R^{-1} A^{-1} K(t,\tau) R \psi(\tau) d\tau \\ = R^{-1} A^{-1} f(t), \quad (a < t < b) \end{aligned} \quad (44)$$

The dominant system in (44) is uncoupled and may easily be solved to give the fundamental functions, which, assuming that the interval (a,b) is normalized to be $(-1,1)$, may be written as

$$w_k(t) = (1+t)^{\alpha_k} (1-t)^{\beta_k}, \quad (k = 1, \dots, M), \quad (-1 < t < 1) \quad (45)$$

where

$$\alpha_k = -\frac{1}{2\pi i} \log \left(\frac{1+\lambda_k}{1-\lambda_k} \right) + \gamma_k'$$

$$\beta_k = \frac{1}{2\pi i} \log \left(\frac{1+\lambda_k}{1-\lambda_k} \right) + \gamma_k''$$

and the integers γ_k^I and γ_k^{II} are chosen in such a way that the singular behavior of $w_k(t)$ at $\bar{\tau} = 1$ is compatible with that of $\psi_k(t)$. Thus, the corresponding index is $\kappa_k = -\alpha_k - \beta_k \geq -1$.

The approximate solution of (44) may now be written as

$$\psi_k(t) = \sum_{n=0}^N c_{kn} w_k(t) P_n^{(\alpha_k, \beta_k)}(t), \quad (k = 1, \dots, M),$$

$$(-1 < t < 1) \tag{46}$$

where the constants c_{kn} are unknown. Defining the following matrices

$$R^{-1} A^{-1} K(t, \tau) R = H(t, \tau) = (h_{km}(t, \tau)), \quad (k, m = 1, \dots, M)$$

$$\tag{47}$$

$$R^{-1} A^{-1} f(t) = g(t) = (g_k(t)), \quad (k = 1, \dots, M)$$

and substituting (46) into (44), we obtain

$$\sum_{n=0}^N c_{kn} (w_k(t) P_n^{(\alpha_k, \beta_k)}(t) + \frac{\lambda_k}{\pi i} \int_{-1}^1 w_k(\tau) P_n^{(\alpha_k, \beta_k)}(\tau) \frac{d\tau}{\tau-t})$$

$$+ \int_{-1}^1 \sum_{m=1}^M h_{km}(t, \tau) \sum_{j=0}^N c_{mj} w_m(\tau) P_j^{(\alpha_m, \beta_m)}(\tau) d\tau = g_k(t)$$

$$(-1 < t < 1), \quad (k = 1, \dots, M) \tag{48}$$

Now, assuming that the kernels $h_{km}(t, \tau)$, $(k, m = 1, \dots, M)$ are square integrable in $-1 < t, \tau < 1$, they may be approximated by*

*See the footnote on page 20.

$$h_{km}(t, \tau) = \sum_{p=0}^P \omega_{kmp}(\tau) P_p^{(-\alpha_k, -\beta_k)}(t) \quad (49)$$

where, using the orthogonality condition [9]

$$\int_{-1}^1 P_n^{(\alpha, \beta)}(t) P_j^{(\alpha, \beta)}(t) w(t) dt = 0, \quad (n \neq j) \quad (50)$$

$$= \theta_n^{(\alpha, \beta)} = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{n! (2n+\alpha+\beta+1) \Gamma(n+\alpha+\beta+1)} \quad (n = j)$$

the functions ω_{kmp} are found to be

$$\omega_{kmp}(\tau) = \frac{p! (2p+\kappa_k+1) \Gamma(p+1+\kappa_k)}{2^{1+\kappa_k} \Gamma(p+1-\alpha_k) \Gamma(p+1-\beta_k)} \cdot \int_{-1}^1 h_{km}(t, \tau) P_p^{(-\alpha_k, -\beta_k)}(t) (1+t)^{-\alpha_k} (1-t)^{-\beta_k} dt$$

Using the relation [4],

$$P_n^{(\alpha_k, \beta_k)}(t) w_k(t) + \frac{\lambda_k}{\pi i} \int_{-1}^1 P_n^{(\alpha_k, \beta_k)}(\tau) w_k(\tau) \frac{d\tau}{\tau-t} = (-1)^{\kappa_k + \gamma_k'} 2^{-\kappa_k} P_{n-\kappa_k}^{(-\alpha_k, -\beta_k)}(t) \quad (51)$$

from (48) and (49), we find

$$\sum_{n=0}^N c_{kn} (-1)^{\kappa_k + \gamma_k} 2^{-\kappa_k} P_{n-\kappa_k}^{(-\alpha_k, -\beta_k)}(t) + \sum_{m=1}^M \sum_{p=0}^P \sum_{j=0}^N d_{kmpj} c_{mj} P_p^{(-\alpha_k, -\beta_k)}(t) = g_k(t) \quad (-1 < t < 1), \quad (k = 1, \dots, M) \quad (52)$$

where

$$d_{kmpj} = \int_{-1}^1 \omega_{kmp}(\tau) w_m(\tau) P_j^{(\alpha_m, \beta_m)}(\tau) d\tau$$

Finally, noting that in (52) the Jacobi polynomials appearing in each row have the same weight

$$w_{-k}(t) = (1+t)^{-\alpha_k} (1-t)^{-\beta_k}$$

using (50), the system of equations to determine the unknown coefficients c_{kn} may be obtained as

$$(-1)^{\kappa_k + \gamma_k} 2^{-\kappa_k} c_{k, p + \kappa_k} P_p^{(-\alpha_k, -\beta_k)} + \sum_{m=1}^M \sum_{j=0}^N d_{kmpj} P_p^{(-\alpha_k, -\beta_k)} c_{mj} = G_{kp} \quad (k = 1, \dots, M), \quad (p = 0, 1, \dots, N) \quad (53)$$

$$G_{kp} = \int_{-1}^1 g_k(t) P_p^{(-\alpha_k, -\beta_k)}(t) w_{-k}(t) dt$$

We again observe that, since $P_0^{(\alpha, \beta)}(t) = 1$, in the case of $\kappa_k = -1$, the equations corresponding to $p = 0$ in (53) are equivalent to the following consistency conditions

$$\int_{-1}^1 [g_k(t) - \int_{-1}^1 \sum_{m=1}^M h_{km}(t, \tau) \psi_m(\tau) d\tau] \frac{dt}{w_k(t)} = 0 \quad (k = 1, \dots, M)$$

Of the two methods described above, the first which leads to (43) avoids numerical work with Jacobi polynomials and seems to be simpler than the second method leading to (53).

Finally, we note that, from the physical view point, singular behaviors of the solutions of integral equations of first and second kind are quite different. In the case of integral equations of first kind, the fundamental function is the weight of Chebishev polynomials and goes to zero or infinity smoothly as the singular point is approached. The fundamental functions of the integral equations of second kind, which are the weights of Jacobi polynomials, also go to zero or infinity as the variable approaches the singular point. However, in the latter case, since the exponents α_k and β_k are complex (see: (32)), in the neighborhood of the singular point, the solution exhibits wild oscillations [10], affecting also other physical quantities. Hence, particularly in searching for an approximate solution, this point requires careful consideration. For example, the solution of the system of integral equations given in [11] is, in this respect, incorrect. In fact, the dominant part of

the system considered in [11] is identical to that of the integral equations found in [12], and the method of [12] or of this paper may be used to obtain an approximate solution of the system in [11] with the correct singular behavior.

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