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PROPAGATION OF LOVE WAVES IN LAYERS WITH IRREGULAR BOUNDARIES

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INTRODUCTION

In a first approach to the study of Love Waves in the earth, the author has considered an earth model consisting of an elastic layer having an irregular boundary, overlying a rigid half-space [1].
The present work treats the same problem using the more realistic earth model in which the half-space is elastic.

DISCUSSION OF PROBLEM

We consider the field which results when a horizontally polarized shear wave, propagating in the plane portion of an elastic layer, is incident on the irregular portion

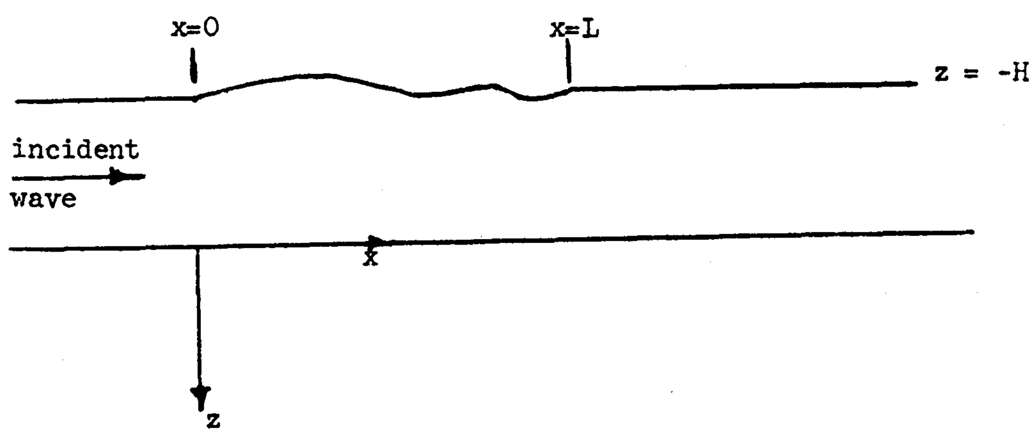

figure 2

The interface between the layer and half-space is given by $z = 0$ and the upper boundary may be described by $z = z_B$ where

$$z_B = -H + bh(x), \qquad h(x) = 0 \text{ for } x \leq 0, \quad x \geq L$$

$$h(x) = f(x) \text{ for } 0 \leq x \leq L$$

and b is the maximum amplitude of the boundary irregularities. For physical reasons we require that the scattered field have only outgoing waves at $x = \pm\infty$ and at $z = \infty$.

Assuming a harmonic time variation $e^{i\omega t}$ the equations of motion become

$$\frac{\partial^2 V_i}{\partial x^2} + \frac{\partial^2 V_i}{\partial z^2} + k_i^2 V_i = 0, \quad i = 1,2 \tag{1}$$

where the subscripts 1 and 2 refer to the layer and half-space respectively, and $k_i = \omega/c_i$, the c_i's being the shear wave velocities, and the V_i's the displacement components in the y direction.

The boundary condition on the traction free upper boundary may be written

$$\frac{\partial V_1}{\partial z} - bh' \frac{\partial V_1}{\partial x} = 0 \quad \text{on } z = z_B, \quad \text{where } h' \equiv \frac{dh}{dx} \tag{2}$$

The displacement and stress continuity on the lower boundary yields

$$\left.\begin{aligned} V_1 &= V_2 \\ \text{and } \mu_1 \frac{\partial V_1}{\partial z} &= \mu_2 \frac{\partial V_2}{\partial z} \end{aligned}\right\} \text{ on } z = 0 \tag{3}$$

where μ_1 and μ_2 are material constants.

METHOD OF SOLUTION

The incident wave which exists under the flat boundary may be written in the form

$$
\begin{aligned}
V_{1,in} &= A \cos\beta_1(z + H)\, e^{-i\alpha x} \\
V_{2,in} &= A\, e^{-\beta_2 z} \cos\beta_1 H\, e^{-i\alpha x}
\end{aligned}
\tag{4}
$$

with $\beta_1 = \left(k_1^2 - \alpha^2\right)^{\frac{1}{2}}$, $\beta_2 = \left(\alpha^2 - k_2^2\right)^{\frac{1}{2}}$ and α is a root of

$$
\tan\beta_1 H = \frac{\mu_2\beta_2}{\mu_1\beta_1} \tag{5}
$$

Since we are only concerned with propagating disturbances, we will consider only roots of (5) for which α is real, such roots exist only if $k_1 > k_2$ [2].

In order to arrive at the scattered field described qualitatively in the problem discussion above we assume a solution, which satisfies the wave equations (1), in the form of a contour integral in the complex ν plane given by

$$
\begin{aligned}
V_{1,\,scat.} &= \int_c \left[B(\nu)e^{i\sigma_1 z} + C(\nu)e^{-i\sigma_1 z}\right] e^{-i\nu x} d\nu, \text{ with } \sigma_1 = \left(k_1^2 - \nu^2\right)^{\frac{1}{2}} \\
V_{2,\,scat.} &= \int_c D(\nu)e^{-\sigma_2 z}\, e^{-i\nu x} d\nu, \text{ with } \sigma_2 = \left(\nu^2 - k_2^2\right)^{\frac{1}{2}}
\end{aligned}
\tag{6}
$$

where the contour c is shown in figure 2.

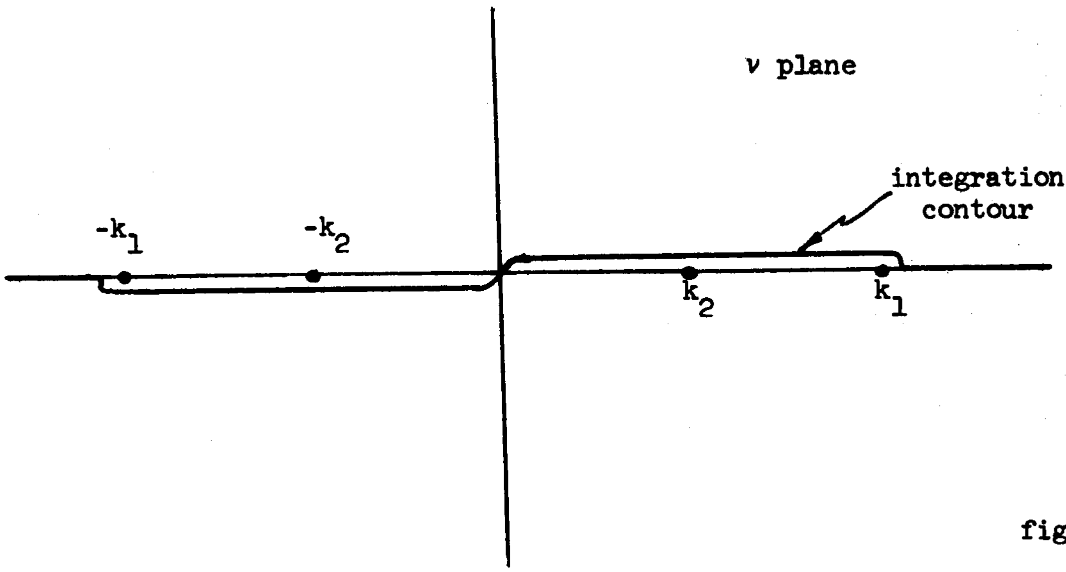

figure 2

Substituting (6) into the boundary conditions (3) it is found that the functions $B(\nu)$, $C(\nu)$ and $D(\nu)$ are related by

$$C = \frac{(\gamma-1)}{(\gamma+1)} B, \quad D = \frac{2\gamma}{\gamma+1} B \tag{7}$$

where $\gamma = \dfrac{i\mu_1 \xi_1}{\mu_2 \xi_2}$

Using (7) in (6) an expression for the total displacement fields in the layer and half-space may be written

$$V_1 = V_{1,in} + V_{1,scat} = A\cos\beta_1(z+H)e^{-i\alpha x} + \int_c \frac{2B(\nu)}{1+\gamma}\left[\gamma\cos\xi_1 z - i\sin\xi_1 z\right]e^{-i\nu x}d\nu$$

$$(8)$$

$$V_2 = V_{2,in} + V_{2,scat} = Ae^{-\beta_2 z}\cos\beta_2 H\, e^{-i\alpha x} + \int_c \frac{2\gamma B(\nu)}{1+\gamma}\, e^{-\xi_2 z}e^{-i\nu x}d\nu$$

These expressions for the displacement field satisfy the wave equation and the boundary conditions (3), it remains to determine $B(\nu)$ such that the boundary condition (2) is satisfied. Accordingly, inserting the first of (8) into (2), we arrive at the following form of the boundary condition (2),

$$
\begin{aligned}
& Ae^{-i\alpha x}\left(-\beta_1 \sin\beta_1 bh + i\alpha bh' \cos\beta_1 bh\right) \\
& - \int_c \frac{2B(\nu)}{1+\gamma}\left\{\xi_1\left[\gamma \sin\xi_1(-H+bh) + i\cos\xi_1(-H+bh)\right] + i\nu bh'\left[\gamma\cos\xi_1(-H+bh)\right.\right. \\
& \left.\left. - i\sin\xi_1(-H+bh)\right]\right\} e^{-i\nu x} d\nu = 0
\end{aligned} \tag{9}
$$

The solution to this integral equation is quite formidable, however if we restrict ourselves to boundaries having small irregularities, that is $b \ll 1$, we may apply a perturbation procedure to evaluate $B(\nu)$.

To carry out this perturbation we assume a series solution for $B(\nu)$ in the form

$$
B(\nu) = \sum_{n=1}^{\infty} B_n(\nu) b^n \tag{10}
$$

Inserting (10) into (9) and expanding the resulting equation in powers of b we obtain

$$
\begin{aligned}
& Ae^{-i\alpha x}\left\{-\beta_1\left[\beta_1 bh + \ldots\right] + i\alpha bh'\left[1 - (\beta_1 bh)^2/2 + \ldots\right]\right\} \\
& - \int_c \left[2/(1+\gamma)\right]\left(B_1(\nu)b + \ldots\right)\left\{\left[\gamma(\xi_1 bh + \ldots) + i(1-[\xi_1 bh]^2/2 + \ldots)\right]\xi_1 \cos\xi_1 H\right. \\
& - \left[\gamma\left(1-[\xi_1 bh]^2/2 + \ldots\right) - i(\xi_1 bh + \ldots)\right]\xi_1 \sin\xi_1 H - i\nu bh'\left[i(\xi_1 bh + \ldots)\right. \\
& \left. - \gamma\left(1-[\xi_1 bh]^2/2 + \ldots\right)\right]\cos\xi_1 H - \left[i\left(1-[\xi_1 bh]^2/2 + \ldots\right)\right. \\
& \left.\left. + \gamma\left(\xi_1 bh + \ldots\right)\right]\sin\xi_1 H\right\} e^{-i\nu x} d\nu = 0
\end{aligned} \tag{11}
$$

To first order in b we obtain

$$Ae^{-i\alpha x}\left(-\beta_1^2\, h + i\alpha h'\right) - \int_c \left[2/(1+\gamma)\right]B_1(\nu)\left(i\cos\xi_1 H-\gamma\sin\xi_1 H\right)\xi_1 e^{-i\nu x}d\nu = 0 \quad (12)$$

which may be inverted to yield

$$B_1(\nu) = \left[A(1+\gamma)/4\pi\xi_1(i\cos\xi_1 H-\gamma\sin\xi_1 H)\right]\int_{-\infty}^{\infty}(i\alpha h'-\beta_1^2 h)e^{i(\nu-\alpha)y}\,dy \quad (13)$$

Inserting (13) into (8) we obtain expressions for the displacement field to first order in b, these may be written

$$V_1 = A\cos\beta_1(z+H)e^{-i\alpha x} + b\int_{-\infty}^{\infty}(A/2\pi)(i\alpha h'-\beta_1^2 h)e^{-i\alpha y}\int_c \frac{\gamma\cos\xi_1 z-i\sin\xi_1 z}{\xi_1(i\cos\xi_1 H-\gamma\sin\xi_1 H)}e^{i\nu(y-x)}d\nu dy$$

(14)

$$V_2 = Ae^{-\beta_2 z}\cos\beta_2 He^{-i\alpha x} + b\int_{-\infty}^{\infty}(A/2\pi)(i\alpha h'-\beta_1^2 h)e^{-i\alpha y}\int_c \frac{\gamma e^{-\xi_2 z}}{\xi_1(i\cos\xi_1 H-\gamma\sin\xi_1 H)}e^{i\nu(y-x)}d\nu dy$$

Since the integrands for the contour integrals in the ν plane, appearing in (14), are not single valued, the contour c must be chosen to lie on the sheet which will yield the form of solution described in the problem discussion above. Accordingly, the ν plane is cut as shown in figure 3 with the contour lying on the sheet which maps into the right half ξ_2 plane under the mapping $\xi_2 = (\nu^2-k_2^2)^{\frac{1}{2}}$. To evaluate these integrals the cases $y>x$ and $y<x$ are considered separately.

Consider first the integrals

$$\int_c \frac{\gamma\cos\xi_1 z - i\sin\xi_1 z}{\xi_1(i\cos\xi_1 H - \gamma\sin\xi_1 H)} e^{i\nu(y-x)}\,d\nu$$

and (15)

$$\int_c \frac{\gamma e^{-\xi_2 z}}{\xi_1(i\cos\xi_1 H - \gamma\sin\xi_1 H)} e^{i\nu(y-x)}\,d\nu, \quad \text{with } y>x$$

To evaluate these integrals, the contour c is closed by arcs at infinity in the left and right upper half plane connected by a contour around the branch line, as shown in figure 3.

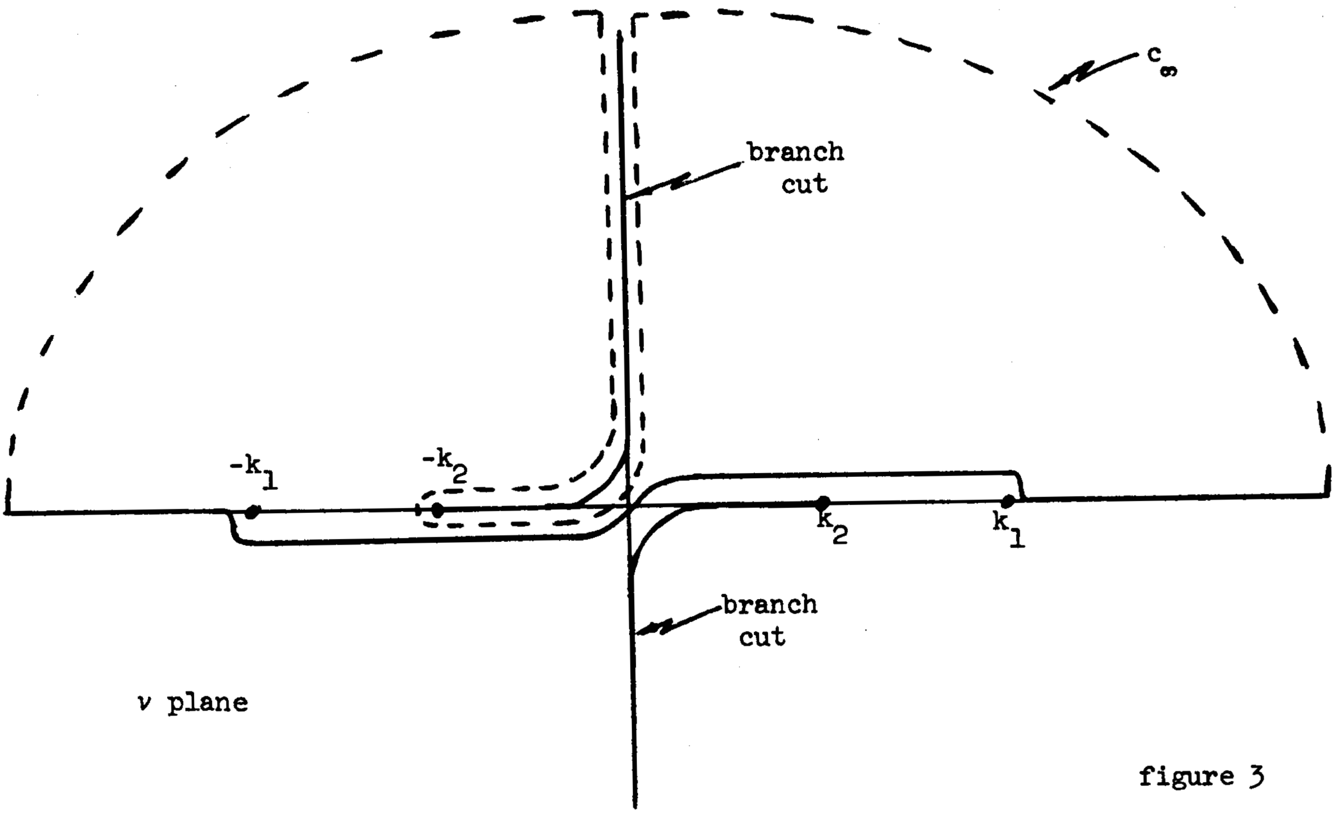

figure 3

The singularities of the integrands in (15) within this closed contour are poles which exist at the zeros of

$$i\cos\xi_1 H - \gamma\sin\xi_1 H = 0$$

For the sheet chosen all of these zeros, denoted by ν_m, lie on the real ν axis and satisfy $-k_1 \leq \nu_m \leq -k_2$ (1). Furthermore, if this relation

(1) See Appendix

is written in the form

$$\tan\xi_1 H = \frac{\mu_2\xi_2}{\mu_1\xi_1} \tag{16}$$

a sketch of the functions on the right and left hand side reveals that there exist N such zeros, where N is the integer part of the number

$$\left[\left(k_1^2 - k_2^2\right)^{\frac{1}{2}} H/\pi\right] + 1$$

If we let

$$G_1(\nu) = \frac{\gamma\cos\xi_1 z - i\sin\xi_1 z}{\xi_1(i\cos\xi_1 H - \gamma\sin\xi_1 H)} e^{i\nu(y-x)}$$

and

$$G_2(\nu) = \frac{\gamma e^{-\xi_2 z}}{\xi_1(i\cos\xi_1 H - \gamma\sin\xi_1 H)} e^{i\nu(y-x)}$$

(15) may be written

$$\int_c G_1(\nu)d\nu = 2\pi i \sum \text{Res}G_1 - \int_{\substack{\text{Branch}\\\text{line}}} G_1(\nu)d\nu - \int_{c_\infty} G_1(\nu)d\nu$$

and (17)

$$\int_c G_2(\nu)d\nu = 2\pi i \sum \text{Res}G_2 - \int_{\substack{\text{Branch}\\\text{line}}} G_2(\nu)d\nu - \int_{c_\infty} G_2(\nu)d\nu$$

The residues of G_1 and G_2 at the poles ν_m are given by

$$\text{Res}G_1 = \cos\xi_{1m}(H + z)e^{i\nu_m(y-x)}/\nu_m H$$

and $m = 1, 2 \ldots N$ (18)

$$\text{Res}G_2 = e^{-\xi_{2m}z}\cos\xi_{1m}H\, e^{i\nu_m(y-x)}/\nu_m H$$

Where ξ_{1m} and ξ_{2m} are ξ_1 and ξ_2 evaluated at ν_m.

The asymptotic approximation of the integral around the branch line in the first of (17) contributes to order $1/x^{\frac{3}{2}}$. In the second of (17) it contributes to order $1/x^{\frac{3}{2}}$ independent of z, and to order $e^{-k_2 z}/z^{\frac{1}{2}}$ independent of x. Therefore, if we restrict our attention to solutions far from the irregular portion of the boundary the contribution of the branch line integrals in (17) are small compared to the contribution of the residue term.

Furthermore, since the integrals over the arcs at infinity in (17) vanish, we obtain

$$\int_c G_1(\nu)d\nu = 2\pi i\sum_{m=1}^{N}\cos\xi_{1m}(H+z)e^{i\nu_m(y-x)}/\nu_m H$$

and (19)

$$\int_c G_2(\nu)d\nu = 2\pi i\sum_{m=1}^{N} e^{-\xi_{2m}z}\cos\xi_{1m}He^{i\nu_m(y-x)}/\nu_m H$$

with the zeros $\nu_m<0$.

Similarly, for $y<x$ we may close the contour in the lower half plane and proceeding as above we obtain

$$\int_c G_1(\nu)d\nu = -2\pi i \sum_{m=1}^{N} \cos\xi_{1m}(H+z)e^{i\nu_m(y-x)}/\nu_m H$$

$$\int_c G_2(\nu)d\nu = -2\pi i \sum_{m=1}^{N} e^{-\xi_{2m}z}\cos\xi_{1m}He^{i\nu_m(y-x)}/\nu_m H \tag{20}$$

with the zeros $\nu_m > 0$.

Inserting (19) and (20) into (14), one obtains the displacement fields

$$V_1 = A\cos\beta_1(z+H)e^{-i\alpha x} - ibA\sum_{m=1}^{N} \frac{\cos\xi_{1m}(H+z)}{\nu_m H}\left[e^{-i\nu_m x}\int_{-\infty}^{x}(i\alpha h' - \beta_1^2 h)e^{i(\nu_m-\alpha)y}dy\right.$$

$$\left. + e^{i\nu_m x}\int_{x}^{\infty}(i\alpha h' - \beta_1^2 h)e^{-i(\nu_m+\alpha)y}dy\right], \quad \nu_m > 0$$

and (21)

$$V_2 = Ae^{-\beta_2 z}\cos\beta_2 He^{-i\alpha x} - ibA\sum_{m=1}^{N} \frac{e^{-\xi_{2m}z}\cos\xi_{1m}H}{\nu_m H}\left[e^{-i\nu_m x}\int_{-\infty}^{x}(i\alpha h' - \beta_1^2 h)e^{i(\nu_m-\alpha)y}dy\right.$$

$$\left. + e^{i\nu_m x}\int_{x}^{\infty}(i\alpha h' - \beta_1^2 h)e^{-i(\nu_m+\alpha)y}dy\right], \quad \nu_m > 0$$

Since the upper boundary of the layer is given by

$$h(x) = 0 \quad \text{for} \quad x \le 0, \quad x \ge L$$

$$h(x) = f(x) \quad \text{for} \quad 0 \le x \le L$$

then

$$\int_0^L f'(x)e^{-ipx}dx = ip\int_0^L f(x)e^{-ipx}dx \qquad (22)$$

With the aid of (22), the solution (21) may be written

$$V_1 = A\cos\beta_1(z+H)e^{-i\alpha x} - ibA\sum_{m=1}^{N}\frac{\cos\xi_{1m}(H+z)}{\nu_m H}e^{i\nu_m x}(\alpha\nu_m - k_1^2)\int_0^L f(y)e^{-i(\alpha+\nu_m)y}dy$$

$$V_2 = A\,e^{-\beta_2 z}\cos\beta_2 H\,e^{-i\alpha x} - ibA\sum_{m=1}^{N}\frac{e^{-\xi_{2m}z}\cos\xi_{1m}H}{\nu_m H}e^{i\nu_m x}(\alpha\nu_m - k_1^2)\int_0^L f(y)e^{-i(\alpha+\nu_m)y}dy$$

for $x \ll 0$, and

$$V_1 = A\cos\beta_1(z+H)e^{-i\alpha x} - ibA\sum_{m=1}^{N}\frac{\cos\xi_{1m}(H+z)}{\nu_m H}e^{-i\nu_m x}(\alpha\nu_m - k_1^2)\int_0^L f(y)e^{-i(\alpha-\nu_m)y}dy$$

$$V_2 = A\,e^{-\beta_2 z}\cos\beta_2 He^{-i\alpha x} - ibA\sum_{m=1}^{N}\frac{e^{-\xi_{2m}z}\cos\xi_{1m}H}{\nu_m H}e^{-i\nu_m x}(\alpha\nu_m - k_1^2)\int_0^L f(y)e^{-i(\alpha-\nu_m)y}dy$$

for $x \gg L$.

APPENDIX

To show that on the sheet in the ν plane which maps into the right half ξ_2 plane, equation (16)

$$\tan \xi_1 H = \frac{\mu_2 \xi_2}{\mu_1 \xi_1}\ , \text{ where } \xi_1 = \left(k_1^2 - \nu^2\right)^{\frac{1}{2}} \text{ and } \xi_2 = \left(\nu^2 - k_2^2\right)^{\frac{1}{2}} \qquad (16)$$

has only real roots ν_m, m=1, 2, 3 ..., and these roots lie either in the interval $k_2 \leq \nu_m \leq k_1$ or $-k_1 \leq \nu_m \leq -k_2$.

We may demonstrate this by showing that the roots of (16) in the right half ξ_2 plane exist only for real ξ_2 which satisfies $0 \leq \xi_2 \leq \left(k_1^2 - k_2^2\right)^{\frac{1}{2}}$.

To do this let $\xi_n H = \alpha_n + i\beta_n$, n = 1, 2. Equation (16) then becomes

$$\frac{\tan \alpha_1 + i \tanh\beta_1}{1 - i \tan \alpha_1 \tanh\beta_1} = \frac{\mu_2(\alpha_2 + i\beta_2)}{\mu_1(\alpha_1 + i\beta_1)} \qquad (A\text{-}1)$$

and the real and imaginary parts of (A-1) yield

$$\tan \alpha_1 = \left[(\mu_2/\mu_1)\alpha_2 + \beta_1 \tanh\beta_1\right] / \left[\alpha_1 - (\mu_2/\mu_1)\beta_2 \tanh\beta_1\right] \qquad (A\text{-}2)$$

$$\tan \alpha_1 = \left[(\mu_2/\mu_1)\beta_2 - \alpha_1 \tanh\beta_1\right] / \left[\beta_1 + (\mu_2/\mu_1)\alpha_2 \tanh\beta_1\right] \qquad (A\text{-}3)$$

Equating (A-2) and (A-3) we obtain

$$\left[(\mu_2/\mu_1)\alpha_2 + \beta_1 \tanh\beta_1\right] / \left[\alpha_1 - (\mu_2/\mu_1)\beta_2 \tanh\beta_1\right] =$$

$$\left[(\mu_2/\mu_1)\beta_2 - \alpha_1 \tanh\beta_1\right] / \left[\beta_1 + (\mu_2/\mu_1)\alpha_2 \tanh\beta_1\right] \qquad (A\text{-}4)$$

Furthermore, since $\nu^2 = k_1^2 - \xi_1^2$ and $\nu^2 = \xi_2^2 - k_2^2$, we may write

$$k_1^2 - \xi_1^2 = \xi_2^2 - k_2^2$$

the real part of which yields $\alpha_1\beta_1 = -\alpha_2\beta_2$. (A-5)

Eliminating α_1 between (A-4) and (A-5) the result may be written

$$\beta_1^2\left[(\mu_2/\mu_1)\alpha_2+\beta_1\tanh\beta_1\right] / \left[\alpha_2+(\mu_2/\mu_1)\beta_1\tanh\beta_1\right]$$

$$= -\beta_2^2\left[(\mu_2/\mu_1)\beta_1+\alpha_2\tanh\beta_1\right] / \left[\beta_1+(\mu_2/\mu_1)\alpha_2\tanh\beta_1\right] \quad \text{(A-6)}$$

From the expression (A-6) it can be seen that if ξ_2 is not real, that is, if $\beta_2 \neq 0$, there exists no $\alpha_2 \geq 0$ which satisfies (A-6) since the left and right sides are always positive and negative respectively. Thus roots of (16), on the sheet of interest, exist only for ξ_2 real.

It remains to show that there are no roots for real $\xi_2 > (k_1^2 - k_2^2)^{\frac{1}{2}}$, that is, for $\beta_2 = 0$ and $\alpha_2 > (k_1^2 - k_2^2)^{\frac{1}{2}} H$. This can be seen by observing that $\xi_2 = \alpha_2 > (k_1^2 - k_2^2)^{\frac{1}{2}}$ implies $|\nu| > k_1$ or ξ_1 is pure imaginary, that is, $\alpha_1 = 0$, and $\xi_1 H = i\beta_1$. Equation (16) becomes

$$\tan i\beta_1 = \frac{\mu_2\alpha_2}{i\mu_1\beta_1}$$

or

$$-\mu_1\beta_1\tanh\beta_1 = \mu_2\alpha_2 \quad \text{(A-7)}$$

Since the left and right sides of (A-7) are negative and positive respectively, no roots exist.

Thus the only roots of (16) on the sheet of interest must satisfy $k_2 \leq \nu_m \leq k_1$ or $-k_1 \leq \nu_m \leq -k_2$. These are in fact the roots of the characteristic equation in the classical Love Wave problem.

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