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FORMULATION OF A METHOD FOR PREDICTING COUPLED CONVECTIVE AND RADIATIVE HEAT TRANSFER ABOUT A BLUNT BODY

Prepared under Contract No. NAS 8-20082 by
L. W. Spradley and C. D. Engel

LOCKHEED MISSILES AND SPACE COMPANY

For

NASA-GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama

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FOREWORD

This memorandum was prepared by personnel of the Thermal Environment Section of the Lockheed Missiles & Space Company's Huntsville Research & Engineering Center for the Aero-Astrodynamics Laboratory of the NASA Marshall Space Flight Center. The work was done under Contract NAS8-20082, (Subcontract NSL PO 5-09287) under Schedule Order No. 76, Task B, Amendment 1. NASA technical coordinator for this study was Mr. Homer Wilson of the Thermal Environment Branch of the Aero-Astrodynamics Laboratory.

SUMMARY

This Technical Memorandum presents the formulation of a method for obtaining solutions to the radiation-coupled blunt body flow problem. The primary objective of the study is to develop a method for obtaining heating rate distributions about a blunt body in a hypersonic flow field for the case where convective and radiative transport mechanisms are coupled. A numerical iteration procedure is developed to obtain solutions to the system of nonlinear partial differential equations governing a thin shock layer which is completely viscous and radiating. Initial results of a digital computer implementation of the method are given with convective heating rates being compared to heating rates found in open literature.

CONTENTS

NOMENCLATURE

a_i	velocity profile coefficients
B_ν	Planckian radiation intensity
B_i	velocity profile boundary conditions
E	radiant emission rate
$\mathcal{E}_1$	exponential integral
f	velocity function, u/u_δ
g	enthalpy function H/H_δ
H	total enthalpy
h	static enthalpy
I_1	integral of f^2
I_2	integral of f
I_ν	local radiation intensity
k	thermal conductivity
N	number density
P	static pressure
Pr	Prandtl number
$\dot{q}_c$	convective energy flux
$\dot{q}_r$	radiative energy flux
R	radius of curvature
Re	Reynolds number, $\rho_{\delta,0} U_\infty R/\mu_{\delta,0}$
r	defined in Figure 1a

T	temperature
U_{∞}	freestream velocity
u	velocity component parallel to body
V	velocity in rectangular coordinate system
v	velocity component normal to surface
x, y	body-oriented coordinates
δ	shock detachment distance
$\tilde{\delta}$	transformed shock detachment distance
ϵ	difference between body and shock angle
η	Dorodnitzyn variable
θ	body angle
κ	body curvature
$\tilde{\kappa}$	$1 + \kappa y$
α_{ν}	absorption coefficient
ℓ	direction cosine
μ	viscosity
ν	frequency
ξ	nondimensional x-coordinate
ρ	density
$\bar{\rho}$	density ratio across shock, $\rho_{\infty}/\rho_{\delta,0}$
Ω	solid angle
τ_{ν}	optical depth at frequency ν
ϕ	shock angle
ω	vorticity

Subscripts

a	sea level quantities
w	wall quantities
δ	quantities immediately behind the shock
∞	freestream condition
0	stagnation line
n	normal component
t	tangential component

Superscripts

$'$	dimensional quantities

Section 1
INTRODUCTION

A superorbital vehicle entering a planetary atmosphere encounters severe convective and radiative heating. To survive in a high temperature environment, an entry vehicle is usually designed with an ablative blunted nose. Hence the job of the designer is to determine the magnitude of the heating rates for a blunted nose entry vehicle and to determine what specific blunted nose shape to use.

The purpose of this study is to formulate a method to calculate the coupled convective and radiative heating rates for an arbitrarily shaped axisymmetric blunted nose vehicle in an arbitrary planetary atmosphere. The regime of atmospheric flight was restricted to the laminar continuum regime by starting with the basic steady Navier-Stokes equations and assuming laminar flow. The steady Navier-Stokes equations were reduced to a form consistent with the flow behavior around a blunted nose in a hypersonic stream by assuming that the shock layer was thin, and the viscous layer and the shock detachment distance were of the same order of magnitude. Further, this formulation is confined to the flight regime where thermodynamic equilibrium can be applied. This restriction can easily be released by adding a species continuity equation to the set of conservation equations.

This study provides a formulization of the approach Hoshizaki and co-workers[1, 2, 3*] have taken in the solution of the viscous radiative coupled blunt body problem. This method provides an advantage in calculating the radiative coupling between the viscous and inviscid regions. The advantage of this method is that it eliminates the necessity of matching the frequency dependent radiation flux at a viscous-inviscid boundary.

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*References - Page 40.

A numerical method, similar to the method used by Wilson and co-workers[4], has been developed to provide a solution to the viscous radiative blunt body problem. The momentum equation is solved using a modified Karman-Pohlhausen integral method while the energy equation is solved using a successive approximation technique. The present formulation includes mass injections effects while neglecting diffusion of chemical species. The solution yields the shock shape, complete details of the shock layer structure, and more significantly, convective and radiative heating rates to the body surface.

Section 2
FORMULATION OF EQUATIONS

The equations used in this analysis are a simplified set obtained from the total conservation equations for a multi-component continuum gas (Tsien[5] and Scala[6]). In this analysis the shock layer is assumed to be in thermodynamic equilibrium. This assumption is realistic for many blunt body hypersonic flow problems. For the cases where a finite rate or quasi-equilibrium analysis is necessary, an equation for conservation of species can easily be added to the basic equations. The transformation and method of solution of the species continuity equation is essentially the same as the energy equation.

The following assumptions are made in order to obtain a set of equations for use in obtaining a solution to the viscous radiation-coupled blunt body problem.

- the shock layer is assumed to be thin, $\delta'/R' << 1$
- the thickness of the viscous layer, and the shock detachment distance are taken to be of the same order of magnitude

The order of magnitude analysis of Hoshizaki and Wilson[1] is used to reduce the equations to a mathematically consistent system correct to the order of the density ratio across the shock. Terms which are of $O(\bar{\rho}^2)$ and higher have been consistently neglected. In performing this order of magnitude analysis, the variables are nondimensionalized according to the scheme shown in Equation (5). At any point in this analysis where a term is neglected by order of magnitude assumption, the variables are first nondimensionalized using Equation (5). The equations in this analysis are thus valid for a thin, completely viscous shock layer

The equations are written in the body oriented coordinate system shown in Figure 1a. The complete set of governing equations correct to $O(\bar{\rho})$ can be written as follows.

X-Momentum

$$\frac{r'}{r'_w}\,\rho'\,\tilde{\kappa}'\left[u'\,\frac{\partial u'}{\partial x'} + \tilde{\kappa}'\,v'\,\frac{\partial u'}{\partial y'} + \kappa'\,u'\,v'\right]$$

$$= -\frac{\partial P'}{\partial x'} + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w}\,\tilde{\kappa}'^2\,\mu'\,\frac{\partial u'}{\partial y'} - \kappa'\,\mu'\,u'\right] \tag{1}$$

Inviscid Y-Momentum

$$\frac{r'}{r'_w}\,\rho'\left[u'\,\frac{\partial v'}{\partial x'} + v'\,\frac{\partial v'}{\partial y'} - \frac{\kappa'}{\tilde{\kappa}'}\,u'^2\right] = -\frac{\partial P'}{\partial y'} \tag{2}$$

Continuity

$$\frac{\partial}{\partial x'}\,(r'\,\rho'\,u') + \frac{\partial}{\partial y'}\,(\tilde{\kappa}'\,r'\,\rho'\,v') = 0 \tag{3}$$

Energy

$$\frac{r'}{r'_w}\,\rho'\left[u'\,\frac{\partial H'}{\partial x'} + \tilde{\kappa}'\,v'\,\frac{\partial H'}{\partial y'}\right] - \frac{\partial}{\partial y'}\left[\tilde{\kappa}'\,\frac{r'}{r'_w}\,\frac{\mu'}{Pr}\,\frac{\partial h'}{\partial y'}\right]$$

$$= \frac{\partial}{\partial y'}\left[\left(\frac{r'}{r'_w} + 2\kappa'\,y'\right)\frac{\mu'}{\kappa'}\,u'\,\frac{\partial u'}{\partial y'}\right] - \frac{\partial}{\partial y'}\left[\kappa'\,\mu'\,u'^2\right] - \tilde{\kappa}'\,\frac{r'}{r'_w}\,E' \tag{4}$$

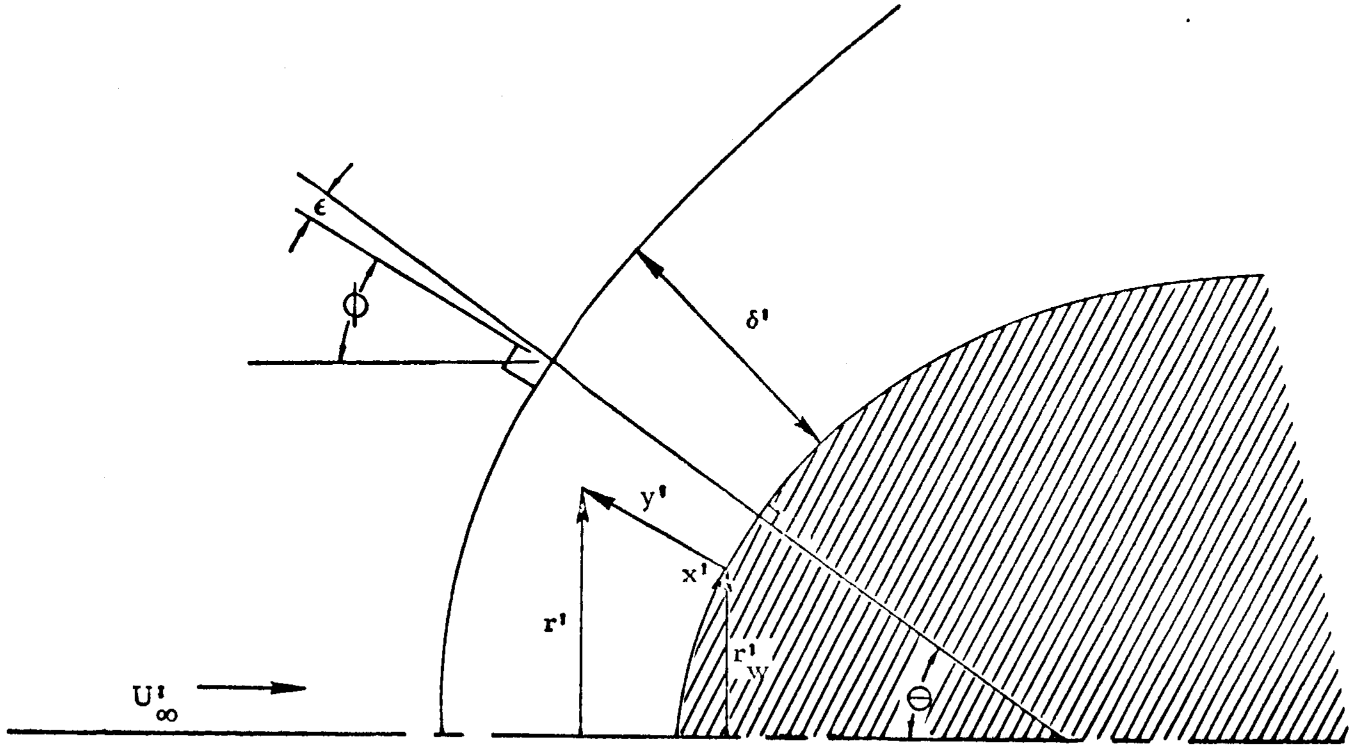

Figure 1a - Body-Oriented Coordinate System

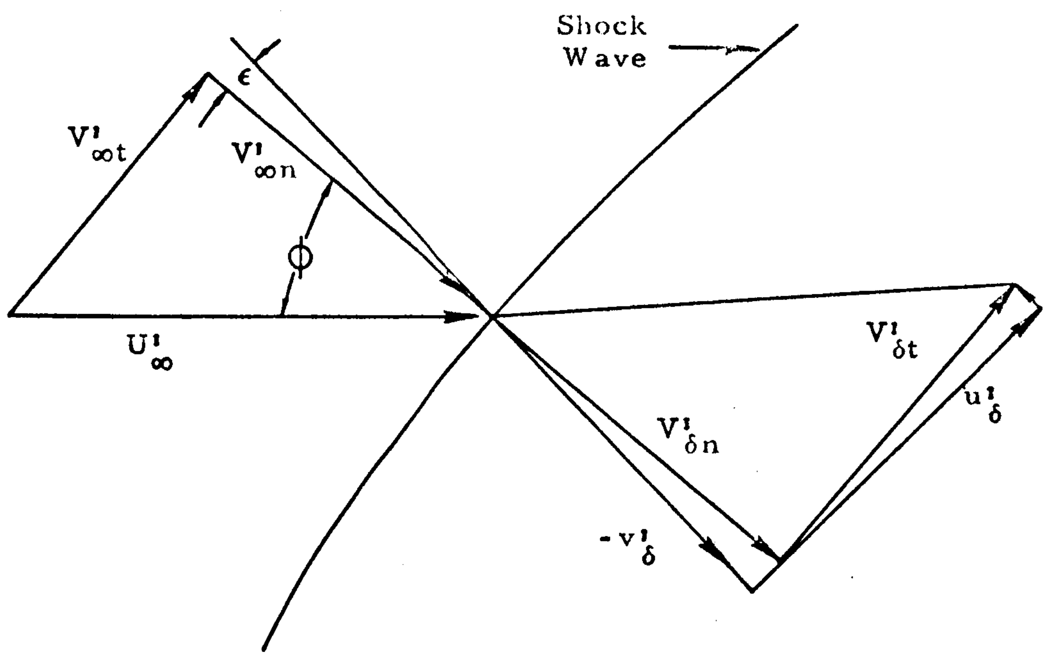

Figure 1b - Resolution of Velocity Components in a Body Oriented Coordinate System

The problem now is to develop a method for obtaining a solution to this system of coupled nonlinear parabolic partial differential equations. The solution will be a complete description of the shock layer flow field including shock shape, and convective and radiative heating to the body surface.

Solutions to the momentum equation will be obtained using an integral method similar to the Karman-Pohlhausen technique for boundary layer analysis. The momentum equation is integrated across the shock layer to obtain a differential equation in one variable. The shock layer velocity profile is then assumed to be representable by polynomials with sufficient boundary conditions being specified to uniquely define this polynomial. This type of method has been used by Maslen and Moeckel[7] and by Hoshizaki and Wilson[1] and this analysis closely follows their work.

The energy equation will be solved by a finite-difference, successive approximations technique. An initial guess is made for the dimensionless shock layer enthalpy profile and an iteration loop set up to successively approximate this profile until satisfactory convergence is obtained. Details of the iteration procedure are given in Section 3.

The following scheme is used in performing the nondimensionalization of Equations (1) through (4).

$$\xi = \frac{x'}{R'} \qquad y = \frac{y'}{R'} \qquad u = \frac{u'}{U'_\infty} \qquad v = \frac{v'}{U'_\infty}$$

$$\rho = \frac{\rho'}{\rho'_{\delta,0}} \qquad \mu = \frac{\mu'}{\mu'_{\delta,0}} \qquad \delta = \frac{\delta'}{R'} \qquad \tilde{\delta} = \frac{\tilde{\delta}'}{R'}$$

$$r = \frac{r'}{R'} \qquad \kappa = \kappa' R' \qquad P = \frac{P'}{\rho'_\infty {U'_\infty}^2} \qquad (\rho v)_w = \frac{(\rho' v')_w}{\rho'_\infty U'_\infty}$$

$$H = \frac{H'}{H'_\delta} \qquad h = \frac{h'}{H'_\delta} \qquad E = \frac{E' R'}{\rho' (U'_\infty)^3} \qquad H'_\delta = \frac{1}{2} {U'_\infty}^2 \tag{5}$$

The conservation equations as stated in Equations (1) through (4) are in the body oriented (x, y) coordinate system (Figure 1a). They will actually be solved in a dimensionless (ξ, η) coordinate system, where ξ is defined in Equation (5) and η is the Dorodnitsyn variable

$$\eta = \frac{\int_0^{y'} \frac{r'}{r'_w} \rho' \, dy'}{\int_0^{\delta'} \frac{r'}{r'_w} \rho' \, dy'} = \frac{1}{\tilde{\delta}'} \int_0^{y'} \frac{r'}{r'_w} \rho' \, dy' \quad . \tag{6}$$

The introduction of this coordinate transformation has a two-fold purpose:

- it reduces the effects of shock layer density variations
- it allows the use of a shock layer coordinate which ranges over the unit interval (0, 1).

The following two sub-sections present a mathematical analysis of the conservation equations for the purpose of obtaining a system of equations suitable for numerical solution on a digital computer. In Section 2.1, the x-momentum equation is integrated across the shock layer and transformed to dimensionless coordinates to obtain a form suitable for solution by a modified Karman-Pohlhausen integral technique. In Section 2.2, which begins on page 24, the energy equation is transformed and manipulated for solution by successive approximations. The radiation flux term in Equation (4) is formulated in terms of the local radiation intensity, Planckian intensity, spectral absorption coefficients and optical depth.

2.1 MOMENTUM EQUATION

To solve the momentum equation by a modified Karman-Pohlhausen method, an integral of this equation is required. Before integrating across the shock layer a more suitable form can be obtained by manipulation of Equation (1).

From the continuity Equation (3) we have

$$\frac{\partial}{\partial x'} \left[\frac{r}{r'_w} \rho' u' r'_w\right] = - \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \tilde{\kappa}' \rho' v'\right] r'_w .$$

Expanding the left-hand side we have

$$\frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'\right] + \frac{1}{r'_w} \frac{d r'_w}{dx} \rho' u' \frac{r'}{r'_w} = - \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \tilde{\kappa}' \rho' v'\right] . \qquad (A.1)$$

Now note that

$$\frac{r'}{r'_w} \rho' u' \frac{\partial u'}{\partial x'} = \frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'^2\right] - u'^2 \frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'\right] . \qquad (A.2)$$

Putting Equation (A.1) into Equation (A.2) yields

$$\frac{r'}{r'_w} \rho' u' \frac{\partial u'}{\partial x'} = \frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'^2\right] + \frac{r'}{r'_w} \rho' u'^2 \frac{1}{r'_w} \frac{d r'_w}{dx'}$$

$$+ u' \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \tilde{\kappa}' \rho' v'\right] . \qquad (A.3)$$

Consider the left-hand side of Equation (1). Substituting Equation (A.3) into Equation (1) we obtain

$$\widetilde{\kappa}' \frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'^2 \right] + \widetilde{\kappa}' \frac{r'}{r'_w} \rho' u'^2 \frac{1}{r'_w} \frac{d r'_w}{dx'}$$

$$+ \widetilde{\kappa}' u' \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \widetilde{\kappa}' \rho' v' \right] + \widetilde{\kappa}'^2 \rho' v' \frac{r'}{r'_w} \frac{\partial u'}{\partial y'}$$

$$+ \widetilde{\kappa}' \kappa' \rho' u' v' \frac{r'}{r_w} \quad . \tag{A.4}$$

Now note that

$$\widetilde{\kappa}' \rho' v' \frac{r'}{r'_w} \frac{\partial u'}{\partial y'} + \kappa' \rho' u' v' \frac{r'}{r'_w} = \rho' v' \frac{r'}{r'_w} \frac{\partial}{\partial y'} (\widetilde{\kappa}' u') \quad .$$

Using this result in Equation (A.4) we get

$$\widetilde{\kappa}' \frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'^2 \right] + \widetilde{\kappa}' \rho' u'^2 \frac{r'}{r'_w} \frac{1}{r_w} \frac{d r'_w}{dx'}$$

$$+ \widetilde{\kappa}' u' \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \widetilde{\kappa}' \rho' v' \right] + \widetilde{\kappa}' \rho' v' \frac{r'}{r'_w} \frac{\partial}{\partial y'} (\widetilde{\kappa}' u') \tag{A.5}$$

Using the following identity

$$\widetilde{\kappa}' u' \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \widetilde{\kappa}' \rho' v' \right] + \widetilde{\kappa}' \rho' v' \frac{r'}{r'_w} \frac{\partial}{\partial y'} (\widetilde{\kappa}' u') - \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \widetilde{\kappa}'^2 \rho' u' v \right]$$

Equation (A.5) becomes

$$\widetilde{\kappa}' \frac{\partial}{\partial x'} \left[\frac{r'}{r'_w} \rho' u'^2 \right] + \widetilde{\kappa}' \frac{r'}{r'_w} \frac{1}{r'_w} \frac{d r'_w}{dx'} \rho' u'^2 + \frac{\partial}{\partial y'} \left[\frac{r'}{r'_w} \widetilde{\kappa}'^2 \rho' u' v' \right] \tag{A.6}$$

By definition $\widetilde{\kappa}' = 1 + \kappa' \, y'$. Equation (A.6) can thus be written

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho' u'^2\right] + \kappa' \, y' \, \frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho' u'^2\right]$$

$$+ \frac{\widetilde{\kappa}'}{r'_w}\,\frac{r'}{r'_w}\,\frac{d\,r'_w}{dx'}\,\rho' u'^2 + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w}\,\widetilde{\kappa}'^2 \rho' u' v'\right] \qquad \text{(A.7)}$$

The following two identities are now cited:

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho' u'^2 \,\kappa' y'\right] = \kappa' \, y' \, \frac{\partial}{\partial x'}\left[\rho' u'^2 \,\frac{r'}{r'_w}\right] + \rho' u'^2 \,\frac{r'}{r'_w}\, y' \,\frac{d\kappa'}{dx'}$$

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho' u'^2 \,\kappa' y'\right] = y' \,\frac{r'}{r'_w}\,\frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right] + \kappa' y' \rho' u'^2 \,\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\right]$$

Combining these two identities yields the following:

$$\kappa' \, y' \,\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho' u'^2\right] = y' \,\frac{r'}{r'_w}\,\frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right] + \kappa' y' \rho' u'^2 \,\frac{\partial}{\partial x}\left[\frac{r'}{r'_w}\right]$$

$$- \rho' u'^2 \,\frac{r'}{r'_w}\, y' \,\frac{d\kappa'}{dx'} \quad .$$

Putting this result into Equation (A.7) we obtain the following form for the left-hand side of Equation (1).

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho' u'^2\right] + y' \,\frac{r'}{r'_w}\,\frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right] + y' \rho' u'^2 \left[\kappa' \,\frac{\partial}{\partial x}\left(\frac{r'}{r'_w}\right)\right.$$

$$\left. - \frac{r'}{r'_w}\,\frac{d\kappa'}{dx'}\right] + \frac{\widetilde{\kappa}'}{r'_w}\,\frac{r'}{r'_w}\,\rho' u'^2 \,\frac{d\,r'_w}{dx'} + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w}\,\widetilde{\kappa}'^2 \rho' u' v'\right] \qquad \text{(A.8)}$$

Substituting Equation (A.8) for the left-hand side of Equation (1) leads to the following form of the x-momentum equation:

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\rho' u'^2\right] + y' \frac{r'}{r'_w}\frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right] +$$

$$y' \rho' u'^2 \left[\kappa' \frac{\partial}{\partial x'}\left(\frac{r'}{r'_w}\right) - \frac{r'}{r'_w}\frac{d\kappa'}{dx'}\right] + \frac{\tilde{\kappa}'}{r'_w}\frac{r'}{r'_w}\rho' u'^2 \frac{d\, r'_w}{dx}$$

$$+ \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w}\tilde{\kappa}'^2 \rho' u' v'\right] = -\frac{\partial P'}{\partial x'} + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w}\tilde{\kappa}'^2 \mu' \frac{\partial u'}{\partial y'} - \kappa' \mu' u'\right] \quad \text{(A.9)}$$

Now consider the right-hand side of the x-momentum equation [Equation (A.9)]. The y-momentum equation can be used to manipulate the pressure gradient term into a form more convenient for integration.

The y-momentum equation [Equation (2)], correct to $O(\bar{\rho})$ with respect to terms in the x-momentum equation is

$$\frac{\partial P'}{\partial y'} = \kappa' \rho' u'^2 \quad .$$

Note the following identities, correct to $O(\bar{\rho})$

$$\frac{\partial}{\partial y'}\left[y' \frac{\partial P'}{\partial x'}\right] = \frac{\partial P'}{\partial x'} + y' \frac{\partial}{\partial y'}\left[\frac{\partial P'}{\partial x'}\right]$$

$$= \frac{\partial P'}{\partial x'} + y' \frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right]$$

$$= \frac{\partial P'}{\partial x'} + y' \frac{r'}{r'_w}\frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right] \quad .$$

Thus,

$$\frac{\partial P'}{\partial x'} = \frac{\partial}{\partial y'}\left[y' \frac{\partial P'}{\partial x'}\right] - y' \frac{r'}{r'_w} \frac{\partial}{\partial x'}\left[\kappa' \rho' u'^2\right] . \tag{A.10}$$

Putting Equation (A.10) into the right-hand side of Equation (A.9) we get

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w} \rho' u'^2\right] + y' \rho' u'^2 \left[\kappa' \frac{d}{dx'}\left(\frac{r'}{r'_w}\right) - \frac{r'}{r'_w} \frac{d\kappa'}{dx'}\right]$$

$$+ \frac{\tilde{\kappa}'}{r'_w} \frac{r'}{r'_w} \rho' u'^2 \frac{d r'_w}{dx'} + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w} \tilde{\kappa}'^2 \rho' u' v'\right]$$

$$= - \frac{\partial}{\partial y'}\left[y' \frac{\partial P'}{\partial x'}\right] + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w} \tilde{\kappa}'^2 \mu' \frac{\partial u'}{\partial y'} - \kappa' \mu' u\right] . \tag{A.11}$$

For consistency we now discard those terms of Equation (A.11) which are of $O(\bar{\rho}^2)$ and higher. Performing this order of magnitude analysis leads to the to the following form of the x-momentum equation.

$$\frac{\partial}{\partial x'}\left[\frac{r'}{r'_w} \rho' u'^2\right] - y' \rho' u'^2 \frac{d\kappa'}{dx'} + \frac{\tilde{\kappa}'}{r'_w} \frac{r'}{r'_w} \frac{d r'_w}{dx'} \rho' u'^2$$
$$+ \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w} \tilde{\kappa}'^2 \rho' u' v'\right] = \frac{\partial}{\partial y'}\left[y' \frac{\partial P'}{\partial x'}\right] + \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w} \tilde{\kappa}'^2 \mu' \frac{\partial u'}{\partial y'} - \kappa' \mu' u'\right] \tag{A.12}$$

Equation (A.12) is now in a suitable form for integration across the shock layer. Performing the integration along a ray line from the body surface ($y' = 0$) to the shock ($y' = \delta'$) we obtain

$$\int_0^{\delta'} \frac{\partial}{\partial x'}\left[\frac{r'}{r'_w}\,\rho'\,u'^2\right] dy' - \frac{d\kappa'}{dx'}\int_0^{\delta'} y'\,\rho'\,u'^2\,dy'$$

$$+ \frac{1}{r'_w}\,\frac{d\,r'_w}{dx'}\int_0^{\delta'} \widetilde{\kappa}'\,\frac{r'}{r'_w}\,\rho'\,u'^2\,dy' + \left[\frac{r'}{r'_w}\,\widetilde{\kappa}'^2\,\rho'\,u'\,v'\right]_{y'=0}^{y'=\delta'}$$

$$= \left[-y'\,\frac{\partial P'}{\partial x'}\right]_{y'=0}^{y'=\delta'} + \left[\frac{r'}{r'_w}\,\widetilde{\kappa}'^2\,\mu'\,\frac{\partial u'}{\partial y'} - \kappa'\,\mu'\,u'\right]_{y'=0}^{y'=\delta'} \quad . \qquad \text{(A.13)}$$

Leibnitz's rule is now employed to interchange the order of integration and differentiation in the first term of Equation (A.13). The following approximation is also introduced.[1]

$$\int_0^{\delta'} \widetilde{\kappa}'\,\frac{r'}{r'_w}\,\rho'\,u'^2\,dy' \approx \widetilde{\kappa}'_\delta \int_0^{\delta'} \frac{r'}{r'_w}\,\rho'\,u'^2\,dy'$$

Equation (A.13) then becomes

$$\frac{d}{dx'}\int_0^{\delta'} \frac{r'}{r'_w}\,\rho'\,u'^2\,dy' - \frac{r'_\delta}{r'_w}\,\rho'_\delta\,u'^{\,2}_\delta\,\frac{d\delta'}{dx'} - \frac{d\kappa'}{dx'}\int_0^{\delta'} y'\,\rho'\,u'^2\,dy'$$

$$+ \frac{\widetilde{\kappa}'_\delta}{r'_w}\,\frac{d\,r'_w}{dx'}\int_0^{\delta'} \frac{r'}{r'_w}\,\rho'\,u'^2\,dy' + \widetilde{\kappa}'^{\,2}_\delta\,\frac{r'_\delta}{r'_w}\,\rho'_\delta\,u'_\delta\,v'_\delta$$

$$= \delta'\left(\frac{\partial P'}{\partial x'}\right)_{\delta'} + \frac{r'_\delta}{r'_w}\,\widetilde{\kappa}'^{\,2}_\delta\,\mu'_\delta\left(\frac{\partial u'}{\partial y'}\right)_{\delta'} - \kappa'\,\mu'_\delta\,u'_\delta - \mu'_w\left(\frac{\partial u'}{\partial y'}\right)_w \quad . \qquad \text{(A.14)}$$

The dimensionless variables, Equation (5), are now introduced into Equation (A.14). In terms of these new variables, the x-momentum equation becomes

$$
(\rho'_{\delta,0}\, U'^{\,2}_{\infty})\ \frac{d}{d\xi}\int_0^{\delta} \frac{r}{r_w}\ \rho\, u^2\, dy\ -\ (\rho'_{\delta,0}\, U'^{\,2}_{\infty})\ \frac{r_\delta}{r_w}\ u_\delta^2\ \frac{d\delta}{d\xi}
$$

$$
-\ (\rho'_{\delta,0}\, U'^{\,2}_{\infty})\ \frac{d\kappa}{d\xi}\int_0^{\delta} y\,\rho\, u^2\, dy + (\rho'_{\delta,0}\, U'^{\,2}_{\infty})\ \frac{\tilde{\kappa}_\delta}{r_w}\ \frac{d\, r_w}{d\xi}\int_0^{\delta}\frac{r}{r_w}\ \rho u^2\, dy
$$

$$
+\ (\rho'_{\delta,0}\, U'^{\,2}_{\infty})\ \tilde{\kappa}_\delta^2\ \rho_\delta\, u_\delta\, v_\delta\ \frac{r_\delta}{r_w}\ =\ -\ (\rho'_{\infty}\, U'^{\,2}_{\infty})\ \delta \left(\frac{\partial P}{\partial \xi}\right)_\delta \qquad \text{(A.15)}
$$

$$
+\ \frac{U'_\infty\, \mu'_{\delta,0}}{R'}\ \tilde{\kappa}^2\ \frac{r_\delta}{r_w}\ \mu_\delta \left(\frac{\partial u}{\partial y}\right)_\delta\ -\ \frac{\kappa}{R'}\mu'_{\delta,0}\, U'_\infty\, \mu_\delta\, u_\delta\ -\ \frac{U'_\infty \mu'_{\delta,0}}{R'}\ \mu_w \left(\frac{\partial u}{\partial y}\right)_w .
$$

Now define

$$
f = \frac{u}{u_\delta} \qquad \text{(A.16)}
$$

divide by $\rho'_{\delta,0}\, U'^{\,2}_{\infty}$, and perform the coordinate transformation defined in Equation (6). Under this transformation Equation (A.15) becomes

$$
\frac{d}{d\xi}\left[\ u_\delta^2 \int_0^1 f^2\, d\eta\ \right] - \frac{r_\delta}{r_w}\ \rho_\delta\, u_\delta^2\ \frac{d\delta}{d\xi}\ -\ \frac{d\kappa}{d\xi}\ \tilde{\delta}^2\ u_\delta^2 \int_0^1 f^2\left[\int_0^{\eta}\frac{d\bar{\eta}}{\rho}\right] d\eta
$$

$$
+\ \frac{\tilde{\kappa}_\delta}{r_w}\ \frac{dr_w}{d\xi}\ u_\delta^2\ \tilde{\delta}\int_0^1 f^2\, d\eta\ +\ \tilde{\kappa}_\delta^2\ \frac{r_\delta}{r_w}\ u_\delta\, v_\delta\, \rho_\delta = \frac{-\delta\, \rho'_\infty}{\rho'_{\delta,0}} \left(\frac{\partial P}{\partial \xi}\right)_\delta\ +
$$

$$\frac{\mu'_{\delta,0}}{\rho'_{\delta,0}\, U'_\infty\, R} \left[\frac{\mu_\delta\, u_\delta}{\tilde{\delta}} \quad \tilde{\kappa}_\delta \left(\frac{r_\delta}{r_w}\right)^2 \quad \rho_\delta \left(\frac{\partial f}{\partial \eta}\right)_{\eta=1} - \kappa \mu_\delta\, u_\delta \right.$$

$$\left. - \frac{\mu_w\, u_\delta}{\tilde{\delta}} \quad \rho_w \left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} \right] \tag{A.17}$$

For convenience we define

$$I_1 = \tilde{\delta} \int_0^1 f^2\, d\eta \tag{A.18}$$

and set

$$\bar{\rho} = \frac{\rho'_\infty}{\rho'_{\delta,0}} \;, \qquad Re = \frac{\rho'_{\delta,0}\, U'_\infty\, R'}{\mu'_{\delta,0}} \quad . \tag{A.19}$$

Equation (A.17) can then be written

$$\frac{d}{d\xi}\left[u_\delta^2\, I_1\right] - \frac{r_\delta}{r_w}\, \rho_\delta\, u_\delta^2\, \frac{d\delta}{d\xi} - \tilde{\delta}^2\, u_\delta^2\, \frac{d\kappa}{d\xi} \int_0^1 f^2 \left[\int_0^\eta \frac{d\bar{\eta}}{\rho}\right] d\eta$$

$$+ \frac{\tilde{\kappa}_\delta}{r_w}\, \frac{d\, r_w}{d\xi}\, u_\delta^2\, I_1 + \tilde{\kappa}_\delta\, \frac{r_\delta}{r_w}\, u_\delta\, \rho_\delta\, v_\delta = -\delta\, \bar{\rho} \left(\frac{\partial P}{\partial \xi}\right)_\delta +$$

$$\frac{u_\delta}{Re} \left[(\rho\, \mu)_\delta\, \frac{\tilde{\kappa}_\delta}{\tilde{\delta}} \left(\frac{r_\delta}{r_w}\right)^2 \left(\frac{\partial f}{\partial \eta}\right)_{\eta=1} - \kappa\, \mu_\delta - \frac{(\rho\mu)_w}{\tilde{\delta}} \left(\frac{\partial f}{\partial \eta}\right)_{\eta=0} \right] \tag{A.20}$$

Rearranging Equation (A.20) we get the following final form of the momentum integro-differential equation.

$$u_\delta \frac{dI_1}{d\xi} + \left[2 \frac{du_\delta}{d\xi} + \tilde{\kappa}_\delta \frac{u_\delta}{r_w} \frac{dr_w}{d\xi}\right] I_1 = \frac{r_\delta}{r_w} \rho_\delta u_\delta \frac{d\delta}{d\xi}$$

$$+ \tilde{\delta}^2 u_\delta \frac{d\kappa}{d\xi} \int_0^1 f^2 \left[\int_0^\eta \frac{d\bar{\eta}}{\rho}\right] d\eta - \tilde{\kappa}_\delta \left(\frac{r_\delta}{r_w}\right) \rho_\delta v_\delta - \delta \bar{\rho} \frac{1}{u_\delta} \left(\frac{\partial P}{\partial \xi}\right)_\delta$$

$$+ \frac{1}{\tilde{\delta}\, Re} \left[\tilde{\kappa}_\delta (\rho\, \mu)_\delta \left(\frac{r_\delta}{r_w}\right)^2 \left(\frac{\partial f}{\partial \eta}\right)_{\eta=1} - \tilde{\delta}\, \kappa\, \mu_\delta - (\rho\, \mu)_w \left(\frac{\partial f}{\partial \eta}\right)_{\eta=0}\right] \quad (A.21)$$

<u>Velocity Profile and Boundary Conditions</u>

To approximate the velocity function by a fifth order polynomial

$$f(\xi, \eta) = \sum_{i=0}^{5} a_i(\xi)\, \eta^i \quad (A.22)$$

six boundary conditions are needed to uniquely determine the coefficients a_i. The boundary conditions chosen are

- $u = 0$ at $y = 0$
- $u = u_\delta$ at $y = \delta$
- $\frac{\partial^2 u}{\partial y^2} = 0$ at $y = \delta$
- Momentum equation evaluated at body surface
- Total mass balance
- $\omega = \omega_\delta$ at $y = \delta$.

The first three conditions transform directly to the (ξ, η) coordinate system.

(1) $f(0) = 0$

(2) $f(1) = 1$

(3) $f''(1) = 0$

Evaluating Equation (1) at the body surface we have

$$\left(\frac{\partial P'}{\partial x'}\right)_w + (\rho' v')_w = \frac{\partial}{\partial y'}\left[\frac{r'}{r'_w}\,\tilde{\kappa}'^2\,\mu'\,\frac{\partial u'}{\partial y'}\right]_w - \kappa'\,\frac{\partial}{\partial y'}\left[\mu'\,u'\right]_w \quad .$$

Transforming to (ξ, η) coordinates we have the following condition correct to $O(\bar{\rho})$.

$$(4)\quad f''(0) = \frac{1}{u_\delta}\left(\frac{\partial P}{\partial \xi}\right)_\delta \frac{\bar{\rho}\,Re\,\tilde{\delta}^2}{\rho_w\,(\rho\,\mu)_w} - f'(0)\,\frac{\tilde{\delta}}{\rho_w}\left[\kappa + \frac{\sin\theta}{r_w} - \frac{Re\,\bar{\rho}\,(\rho v)_w}{\mu_w}\right]$$

$$= B_1 - B_0\,f'(0)\,.$$

Forming an overall mass balance in the shock layer we have

$$\int_0^{\delta'} r'\,\rho'\,u'\,dy' = \frac{1}{2}\,\rho'_\infty\,U'_\infty\,r'^2_\delta + r'_w\int_0^{x'}(\rho'\,v')_w\,dx'\,.$$

Transform to (ξ, η) coordinates to get the following condition

$$(5)\quad \int_0^1 f\,d\eta = \frac{r_w}{u_\delta}\,\frac{\bar{\rho}}{\tilde{\delta}}\,\frac{1}{2}\left(\frac{r_\delta}{r_w}\right)^2 + \frac{\bar{\rho}}{\tilde{\delta}}\,\frac{1}{u_\delta}\int_0^{\xi}(\rho\,v)_w\,d\xi = B_2\,.$$

Evaluating the vorticity behind a curved shock for the axisymmetric case[8] we get

$$\left(\frac{\partial u'}{\partial y'} + \frac{\kappa'}{\tilde{\kappa}'} u'\right)_\delta = U'_\infty \frac{(1-\bar{\rho})^2}{\bar{\rho}} \frac{\kappa'}{\tilde{\kappa}_\delta} \sin\phi .$$

In the (ξ, η) coordinate system, this expression becomes

$$(6) \quad f'(1) = \left(\frac{-\kappa}{\tilde{\kappa}_\delta}\right)\left(\frac{r_w}{r_\delta}\right) \frac{\tilde{\delta}}{\rho_\delta} \left[1 + \frac{(1-\bar{\rho})^2}{\bar{\rho}} \frac{\sin\phi}{u_\delta}\right] = B_3 .$$

Applying the boundary conditions 1 - 6 to the polynomial defined in Equation (A.22) we obtain a system of linear algebraic equations for the a_i.

$$\begin{aligned}
a_0 &= 0 \\
a_1 + a_2 + a_3 + a_4 + a_5 &= 1 \\
2a_2 + 6a_3 + 12a_4 + 20a_5 &= 0 \\
B_0 a_1 + 2a_2 &= B_1 \\
1/2\, a_1 + 1/3\, a_2 + 1/4\, a_3 + 1/5\, a_4 + 1/6\, a_5 &= B_2 \\
a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5 &= B_3 .
\end{aligned} \qquad \text{(A.23)}$$

The system (A.23) has the following solution:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \frac{1}{12-B_0} \begin{pmatrix} -60 & 0 & -1 & 120 & 12 \\ 0 & 30 & 60 & -60B_0 & -6B_0 \\ 480 & -100 & -12 & 180B_0 - 720 & 22B_0 - 120 \\ -660 & 105 & 10 & 960 - 180B_0 & 180 - 25B_0 \\ 252 & -36 & -3 & 60B_0 - 360 & 9B_0 - 72 \end{pmatrix} \begin{pmatrix} 1 \\ B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix} \qquad \text{(A.24)}$$

Thus the coefficients a_i, and hence the velocity profile, is expressed in terms of quantities at the body surface and immediately behind the shock, and the shock stand-off distance.

Shock Boundary Conditions

The shock boundary conditions chosen here are the Rankine Hugoniot equations and when written in rectangular coordinates are:

Continuity $$\rho'_\infty V'_{\infty n} = \rho'_\delta V'_{\delta n} \tag{A.25}$$

Momentum

(normal) $$\rho'_\infty V'^{2}_{\infty n} + P'_\infty = \rho'_\delta V'^{2}_{\delta n} + P'_\delta \tag{A.26}$$

(tangential) $$V'_{\infty t} = V'_{\delta t} \tag{A.27}$$

Energy $$\frac{1}{2} V'^{2}_{\infty n} + h'_\infty = \frac{1}{2} V'_{\delta n} + h'_\delta \quad . \tag{A.28}$$

Using Figure 1b, page 5, the above equations can be written in body oriented coordinates. From geometry we have

$$v'_\delta = V'_{\delta t} \sin\epsilon - V'_{\delta n} \cos\epsilon \tag{A.29}$$

$$u'_\delta = V'_{\delta t} \cos\epsilon + V'_{\delta n} \sin\epsilon \tag{A.30}$$

where

$$V'_{\infty n} = U'_\infty \cos\phi$$

$$V'_{\delta n} = \bar{\rho}\, U'_\infty \cos\phi$$

$$V'_{\delta t} = V'_{\infty t} = U'_\infty \sin\phi$$

Substituting for $V'_{\infty n}$, $V'_{\delta n}$, and $V'_{\infty t}$, we obtain

$$v'_\delta = U'_\infty \sin\phi \sin\epsilon - \bar{\rho}\, U'_\infty \cos\phi \sin\epsilon \tag{A.31}$$

$$u'_\delta = U'_\infty \sin\phi \cos\epsilon + \bar{\rho}\, U'_\infty \cos\phi \cos\epsilon \ . \tag{A.32}$$

The pressure behind the shock can be obtained by using the normal momentum equation and substituting for $V'_{\infty n}$ and $V'_{\delta n}$.

$$\rho'_\infty (U'_\infty \cos\phi)^2 + P'_\infty = \rho'_\delta (\bar{\rho}\, U'_\infty \cos\phi)^2 + P'_\delta \ . \tag{A.33}$$

Non-dimensionalizing Equations (A.31), (A.32) and (A.33) and dropping a term of order $(\bar{\rho})^2$ in Equation (A.33), we have the shock-boundary conditions.

$$v_\delta = \sin\phi \sin\epsilon - \bar{\rho} \cos\phi \cos\epsilon \tag{A.34}$$

$$u_\delta = \sin\phi \cos\epsilon + \bar{\rho} \cos\phi \cos\epsilon \tag{A.35}$$

$$P_\delta = (1 - \bar{\rho}) \cos^2\phi \ . \tag{A.36}$$

Pressure Profile

A shock layer pressure profile is necessary in order to determine the thermodynamic and transport properties of the shock layer gas. For this purpose the inviscid y-momentum equation correct to $O(\bar{\rho})$ with respect to the normal pressure gradient is used. The pressure profile is obtained by integrating Equation (2) under the assumption of a linear variation in the normal velocity component. The Rankine-Hugoniot equation (A.36) is used for evaluating the pressure immediately behind the shock.

Introduce the dimensionless variables, definition (5), into Equation (2) to obtain

$$\frac{r}{r_w} \rho \left[u \frac{\partial v}{\partial \xi} + v \frac{\partial v}{\partial y} - \frac{\kappa}{\tilde{\kappa}} u^2 \right] = - \bar{\rho} \frac{\partial P}{\partial y} . \qquad \text{(A.37)}$$

Integration of Equation (A.37) along a ray line in the shock layer yields

$$- \bar{\rho} \int_\delta^y \frac{\partial P}{\partial y} dy = \int_\delta^y \frac{r}{r_w} \rho u \frac{\partial v}{\partial \xi} dy + \int_\delta^y \frac{r}{r_w} \rho v \frac{\partial v}{\partial y} dy$$

$$- \frac{\kappa}{\tilde{\kappa}} \int_\delta^y \frac{r}{r_w} \rho u^2 dy . \qquad \text{(A.38)}$$

The following approximation is now introduced[1]

$$v = v_\delta \eta . \qquad \text{(A.39)}$$

Using Equation (A.39) we can evaluate Equation (A.38) in (ξ, η) coordinates to obtain

$$P - P_\delta = \frac{\tilde{\delta}}{\bar{\rho}} u_\delta \frac{\partial v_\delta}{\partial \xi} \int_\eta^1 \eta f d\eta + \frac{v_\delta^2}{\bar{\rho}} \int_\eta^1 \eta \rho d\eta$$

$$- \frac{\kappa \tilde{\delta} u_\delta^2}{\bar{\rho}} \int_\eta^1 f^2 d\eta . \qquad \text{(A.40)}$$

A more convenient expression for computing the pressure profile can be obtained by employing the following relation[9]

$$\frac{h'}{H'_\delta} \approx \frac{\rho'_{\delta,0}}{\rho'} \quad . \tag{A.41}$$

and in (ξ, η) coordinates we have

$$\frac{h'}{H'_\delta} = g - u_\delta^2 f^2 \quad .$$

Thus

$$\frac{\rho'_{\delta,0}}{\rho'} \approx g - u_\delta^2 f^2 \, . \tag{A.42}$$

Substituting Equation (A.42) into Equation (A.40) we get the following expression for the shock layer pressure profile

$$P = P_\delta + \frac{\tilde{\delta}}{\bar{\rho}} u_\delta \frac{\partial v_\delta}{\partial \xi} \int_\eta^1 \eta \, f \, d\eta + \frac{v_\delta^2}{\bar{\rho}} \int_\eta^1 \frac{\eta}{g - u_\delta^2 f^2} \, d\eta$$

$$- \frac{\kappa \tilde{\delta} u_\delta^2}{\bar{\rho}} \int_\eta^1 f^2 \, d\eta \, . \tag{A.43}$$

Shock Detachment Distance

The method adapted here is to determine the shock detachment distance simultaneously with the solution of the momentum equation. A suitable expression for δ in terms of the shock layer properties can be obtained from the Dorodnitzyn transformation, Equation (6)

$$\eta = \frac{1}{\tilde{\delta}'} \int_0^{y'} \frac{r'}{r'_w} \rho' \, dy' .$$

Solving for y we get

$$y = \tilde{\delta} \int_0^{\eta} \frac{r_w}{r} \frac{\rho'_{\delta,0}}{\rho'} \, d\eta .$$

When $\eta = 1$ we have $y = \delta$, thus

$$\delta = \tilde{\delta} \int_0^{1} \frac{r_w}{r} \frac{\rho'_{\delta,0}}{\rho'} \, d\eta . \qquad (A.44)$$

A more convenient expression can be obtained by employing the geometric identity

$$r = r_w + y \sin\theta$$

and manipulating to obtain the following relation for the shock detachment distance.

$$\delta = \tilde{\delta} \int_0^{1} \frac{\rho'_{\delta,0}}{\rho'} \, d\eta - \frac{\sin\theta}{2 r_w} \left[\tilde{\delta} \int_0^{1} \frac{\rho'_{\delta,0}}{\rho'} \, d\eta \right]^2 . \qquad (A.45)$$

2.2 ENERGY EQUATION

The energy equation, Equation (4), will now be transformed to the (ξ, η) coordinate system and manipulated into a form suitable for solution by a successive approximations technique.

Before transforming, the following identity correct to $O(\bar{\rho})$ is introduced.

$$\frac{1}{\tilde{\kappa}'}\left[\frac{r'}{r'_w} + 2\kappa' y'\right] = \tilde{\kappa}' \frac{r'}{r_w} \tag{B.1}$$

Using this identity, Equation (4) can be written

$$\frac{r'}{r'_w}\rho'\left[u'\frac{\partial H'}{\partial x'} + \tilde{\kappa}' v' \frac{\partial H'}{\partial y'}\right] - \frac{\partial}{\partial y'}\left[\tilde{\kappa}' \frac{r'}{r'_w} \frac{\mu'}{Pr} \frac{\partial h'}{\partial y'}\right]$$
$$= \frac{\partial}{\partial y'}\left[\tilde{\kappa}' \mu' \frac{r'}{r'_w} u' \frac{\partial u'}{\partial y'}\right] - \frac{\partial}{\partial y'}\left[\kappa' \mu' u'^2\right] - \tilde{\kappa}' \frac{r'}{r'_w} E' \tag{B.2}$$

For convenience, we will transform the left-hand side and right-hand side of Equation (B.2) separately and combine the results.

Consider the left-hand side of Equation (B.2). In terms of the dimensionless variables, Equation (5), and upon division by the common term

$$\rho'_{\delta,o} U'_\infty H'_\delta u_\delta \frac{1}{R'} \tag{B.3}$$

we have

$$\frac{r}{r_w}\rho\left[\frac{u}{u_\delta}\frac{\partial H}{\partial \xi} + \tilde{\kappa}\frac{v}{u_\delta}\frac{\partial H}{\partial y}\right] - \frac{1}{Re}\frac{\partial}{\partial y}\left[\frac{\tilde{\kappa}}{u_\delta}\frac{r}{r_w}\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right]. \tag{B.4}$$

Transforming to (ξ, η) coordinates and employing the chain rule for differentiation, Equation (B.4) becomes

$$\frac{r}{r_w}\rho\frac{u}{u_\delta}\left[\frac{\partial H}{\partial \xi}+\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial \xi}\right]+\tilde{\kappa}\frac{r}{r_w}\frac{v}{u_\delta}\frac{\partial H}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$-\frac{1}{Re}\frac{\partial}{\partial \eta}\left[\frac{\tilde{\kappa}}{u_\delta}\frac{r}{r_w}\frac{\mu}{Pr}\frac{\partial h}{\partial \eta}\frac{\partial \eta}{\partial y}\right]\frac{\partial \eta}{\partial y}. \tag{B.5}$$

For convenience, multiply (B.5) by

$$\frac{r_w\,\tilde{\delta}}{r\rho H_\delta} \tag{B.6}$$

to obtain

$$\tilde{\delta}\frac{u}{u_\delta}\frac{\partial}{\partial \xi}\left(\frac{H}{H_\delta}\right)+\tilde{\delta}\frac{u}{u_\delta}\frac{\partial}{\partial \eta}\left(\frac{H}{H_\delta}\right)\frac{\partial \eta}{\partial \xi}+\tilde{\kappa}\,\tilde{\delta}\frac{v}{u_\delta}\frac{\partial}{\partial \eta}\left(\frac{H}{H_\delta}\right)\frac{\partial \eta}{\partial y}$$

$$-\frac{1}{Re}\frac{\partial}{\partial \eta}\left[\frac{\tilde{\kappa}}{u_\delta}\frac{r}{r_w}\frac{\mu}{Pr}\frac{\partial}{\partial \eta}\left(\frac{h}{H_\delta}\right)\frac{\partial \eta}{\partial y}\right].$$

Define

$$g = \frac{H}{H_\delta} \tag{B.7}$$

and using definition (A.16), we get

$$\tilde{\delta}\, f \frac{\partial g}{\partial \xi} + \left[\tilde{\delta}\, f \frac{\partial \eta}{\partial \xi} + \tilde{\kappa} \frac{v}{u_\delta} \frac{r}{r_w} \rho\right] \frac{\partial g}{\partial \eta}$$

$$- \frac{1}{Re} \frac{\partial}{\partial \eta} \left[\frac{\tilde{\kappa}}{u_\delta} \frac{r}{r_w} \frac{\mu}{Pr} \left(\frac{\partial g}{\partial \eta} - u_\delta^2 \frac{U'^2_\infty}{H'_\delta} f \frac{\partial f}{\partial \eta}\right) \frac{\partial \eta}{\partial y}\right] . \tag{B.8}$$

A more convenient form can be obtained by elimination of the normal component of velocity. A suitable form can be obtained by integrating the continuity equation. From Equation (3) we have

$$\tilde{\kappa}' \rho' \frac{r'}{r'_w} v' = - \frac{1}{r'_w} \int_0^{y'} \frac{\partial}{\partial x'} \left[r' \rho' u'\right] dy' + (\rho' v')_w \quad .$$

Non-dimensionalization yields

$$\tilde{\kappa} \rho \frac{r}{r_w} \frac{v}{u_\delta} = - \frac{1}{u_\delta r_w} \int_0^{y} \frac{\partial}{\partial \xi} \left[\frac{r}{r_w} \rho \frac{u}{u_\delta} (u_\delta r_w)\right] dy + \frac{\bar{\rho} (\rho v)_w}{u_\delta}$$

$$= - \frac{1}{u_\delta r_w} \frac{\partial}{\partial \xi} \left[u_\delta r_w\right] \int_0^{y} \frac{r}{r_w} \rho \frac{u}{u_\delta} dy - \int_0^{y} \frac{\partial}{\partial \xi} \left[\frac{r}{r_w} \rho \frac{u}{u_\delta}\right] dy$$

$$+ \frac{\bar{\rho}(\rho v)_w}{u_\delta} \quad .$$

Transforming to (ξ, η) coordinates and using the identity (A.16) we get

$$\tilde{\kappa} \rho \frac{r}{r_w} \frac{v}{u_\delta} = - \frac{1}{u_\delta r_w} \frac{\partial}{\partial \xi} \left[u_\delta r_w\right] \tilde{\delta} \int_0^{\eta} f \, d\eta - \tilde{\delta} \int_0^{\eta} \frac{\partial f}{\partial \xi} d\eta + \frac{\bar{\rho} (\rho v)_w}{u_\delta} \quad .$$

Define

$$I_2 = \tilde{\delta} \int_0^{\eta} f \, d\eta \qquad \text{(B.9)}$$

and obtain the following expression for the normal velocity component.[2, 3]

$$\tilde{\kappa} \rho \frac{r}{r_w} \frac{v}{u_\delta} = - \frac{1}{r_w u_\delta} \frac{\partial}{\partial \xi} (r_w u_\delta) I_2 - \frac{\partial I_2}{\partial \xi} - \tilde{\delta} f \frac{\partial \eta}{\partial \xi} + \frac{\bar{\rho}(\rho v)_w}{u_\delta} \qquad \text{(B.10)}$$

Substitute Equation (B.10) into Equation (B.8) to obtain

$$\tilde{\delta} f \frac{\partial g}{\partial \xi} - \left[\frac{\partial I_2}{\partial \xi} + \frac{1}{u_\delta r_w} \frac{\partial}{\partial \xi} (u_\delta r_w) I_2 - \frac{\bar{\rho} (\rho v)_w}{u_\delta} \right] \frac{\partial g}{\partial \eta}$$

$$- \frac{1}{Re} \frac{\partial}{\partial \eta} \left[\frac{\tilde{\kappa}}{u_\delta} \frac{r}{r_w} \frac{\mu}{Pr} \left(\frac{\partial g}{\partial \eta} - u_\delta^2 \frac{U_\infty'^2}{H_\delta'} f \frac{\partial f}{\partial \eta} \right) \frac{\partial \eta}{\partial y} \right] .$$

Multiplication by $(-Re\, u_\delta\, \tilde{\delta})$ and manipulation yields the following form for the left-hand side of the energy equation.

$$\frac{\partial}{\partial \eta} \left[\frac{\rho\mu}{Pr} \left(\frac{r}{r_w} \right)^2 \tilde{\kappa} \frac{\partial g}{\partial \eta} \right] + \tilde{\delta}\, Re \left[u_\delta \frac{\partial I_2}{\partial \xi} + \left(\frac{du_\delta}{d\xi} + \frac{u_\delta}{r_w} \frac{dr_w}{d\xi} \right) I_2 \right.$$

$$\left. - \bar{\rho} (\rho v)_w \right] \frac{\partial g}{\partial \eta} - Re\, \tilde{\delta}^2 u_\delta f \frac{\partial g}{\partial \xi} - \frac{\partial}{\partial \eta} \left[\frac{\rho\mu}{Pr} \left(\frac{r}{r_w} \right)^2 \tilde{\kappa}\, u_\delta^2 \frac{U_\infty'^2}{H_\delta'} f \frac{\partial f}{\partial \eta} \right] \qquad \text{(B.11)}$$

Now consider the right-hand side of the energy equation [Equation (B.2)].

Introducing the dimensionless variables, Equation (5), and dividing out the factor (B.3), we get

$$\frac{U'_\infty \mu'_{\delta,o}}{H'_\delta \rho'_{\delta,o} R' u_\delta}\left[\frac{\partial}{\partial y}\left(\widetilde{\kappa}\,\mu\,\frac{r}{r_w}\,u\,\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial y}\,(\kappa\,\mu\,u^2)\right]\;-\frac{U'^{\,2}_\infty \rho'_\infty \widetilde{\kappa}}{H'_\delta \rho'_{\delta,o} u_\delta}\,\frac{r}{r_w}\;E. \quad (B.12)$$

Transform Equation (B.12) to the (ξ,η) coordinate system and obtain

$$\frac{U'_\infty \mu'_{\delta,o}}{H'_\delta \rho'_{\delta,o} R' u_\delta}\left[\frac{\partial}{\partial \eta}\left(\kappa\,\mu\,\frac{r}{r_w}\,u\,\frac{\partial u}{\partial \eta}\,\frac{\partial \eta}{\partial y}\right)\frac{\partial \eta}{\partial y}-\frac{\partial}{\partial \eta}\,(\kappa\,\mu\,u^2)\,\frac{\partial \eta}{\partial y}\right]-\frac{U'^{\,2}_\infty \rho'_\infty \widetilde{\kappa}}{H'_\delta \rho'_{\delta,o} u_\delta}\,\frac{r}{r_w}\;E. \quad (B.13)$$

Following the procedure for the left-hand side, we must now multiply by expression (B.6) to get

$$\frac{U'_\infty \mu'_{\delta,o}}{H'_\delta \rho'_{\delta,o} R' u_\delta}\left[\frac{\partial}{\partial \eta}\left(\widetilde{\kappa}\,\mu\,\frac{r}{r_w}\,u\,\frac{\partial u}{\partial \eta}\,\frac{\partial \eta}{\partial y}\right)-\frac{\partial}{\partial \eta}\,(\kappa\,\mu\,u^2)\right]-\frac{U'_\infty \rho'_\infty \widetilde{\kappa}\,\widetilde{\delta}}{H'_\delta\,\rho\,\rho'_{\delta,o} u_\delta}\;E\,.$$

and then manipulate as in Equation (B.11) to obtain the following form for the right-hand side of the energy equation.

$$-\frac{\partial}{\partial \eta}\left[\widetilde{\kappa}\,\rho\,\mu\left(\frac{r}{r_w}\right)^2 u_\delta^2\,\frac{U'^{\,2}_\infty}{H'_\delta}\;f\,\frac{\partial f}{\partial \eta}\right]+\frac{U'^{\,2}_\infty}{H'_\delta}\,u_\delta^2\,\kappa\,\frac{\partial}{\partial \eta}\,(\mu\,f^2)$$

$$+\,\widetilde{\kappa}\,\widetilde{\delta}^2\,\mathrm{Re}\,\overline{\rho}\left(\frac{\rho_\delta}{\rho}\right)\frac{U'^{\,2}_\infty}{H'_\delta}\;E \qquad (B.14)$$

Combine Equations (B.11) and (B.14) and we have the following form for the transformed energy equation.

$$\frac{\partial}{\partial \eta}\left[\frac{\rho\mu}{Pr}\left(\frac{r}{r_w}\right)^2 \tilde{\kappa}\,\frac{\partial g}{\partial \eta}\right] + \tilde{\delta}\,Re\left[u_\delta \frac{\partial I_2}{\partial \xi} + \left(\frac{du_\delta}{d\xi} + \frac{u_\delta}{r_w}\,\frac{dr_w}{d\xi}\right) I_2 - \bar{\rho}\,(\rho v)_w\right]\frac{\partial g}{\partial \eta} =$$

$$Re\,\tilde{\delta}^2\,u_\delta\,f\,\frac{\partial g}{\partial \xi} + \frac{\partial}{\partial \eta}\left[(\rho\mu)\tilde{\kappa}\left(\frac{r}{r_w}\right)^2\left(\frac{1}{Pr} - 1\right)\frac{{U'_\infty}^2}{H'_\delta}\,u_\delta^2\,f\,\frac{\partial f}{\partial \eta}\right] \tag{B.15}$$

$$+\frac{{U'_\infty}^2}{H'_\delta}\,u_\delta^2\,\tilde{\delta}\,\kappa\,\frac{\partial}{\partial \eta}(\mu\,f^2) - \tilde{\kappa}\,\tilde{\delta}^2\,Re\,\bar{\rho}\left(\frac{\rho_\delta}{\rho}\right)\frac{{U'_\infty}^2}{H'_\delta}\;E\;.$$

This equation has the form

$$\frac{\partial}{\partial \eta}\left[\frac{1}{\Gamma_1}\;\frac{\partial g}{\partial \eta}\right] + \left[\frac{1}{\Gamma_1}\;\frac{\partial g}{\partial \eta}\right]\Gamma_2 = \Gamma_3 + \frac{\partial \Gamma_4}{\partial \eta} + \frac{\partial \Gamma_5}{\partial \eta} \tag{B.16}$$

where

$$\frac{1}{\Gamma_1} = \left(\frac{\rho\,\mu}{Pr}\right)\tilde{\kappa}\left(\frac{r}{r_w}\right)^2$$

$$\Gamma_2 = \Gamma_1\,\tilde{\delta}\,Re\left[u_\delta\,\frac{\partial I_2}{\partial \xi} + \left(\frac{du_\delta}{d\xi} + \frac{u_\delta}{r_w}\,\frac{dr_w}{d\xi}\right) I_2 - \bar{\rho}\,(\rho\,v)_w\right]$$

$$\Gamma_3 = Re\,\tilde{\delta}^2\,u_\delta\,f\,\frac{\partial g}{\partial \xi} + \tilde{\delta}^2\,Re\,\tilde{\kappa}\,\bar{\rho}\left(\frac{\rho_\delta}{\rho}\right)\frac{{U'_\infty}^2}{H'_\delta}\;E$$

$$\Gamma_4 = (\rho\,\mu)\,\tilde{\kappa}\left(\frac{r}{r_w}\right)^2\left(\frac{1}{Pr} - 1\right)\frac{{U'_\infty}^2}{H'_\delta}\,u_\delta^2\,f\,\frac{\partial f}{\partial \eta}$$

$$\Gamma_5 = \frac{{U'_\infty}^2}{H'_\delta}\,u_\delta^2\,\tilde{\delta}\,\kappa\,\mu\,f^2 \tag{B.17}$$

If the Γ_i were known as functions of the shock layer coordinate η, then Equation (B.16) could be formally solved. The formal solution in terms of the Γ_i is

$$g = g_w + \int_0^{\eta} \Gamma_1 \exp\left(-\int_0^{\eta} \Gamma_2 \, d\eta\right)\left[\int_0^{\bar{\eta}} (\Gamma_3 - \Gamma_2\Gamma_4 - \Gamma_2\Gamma_5) \exp\left(\int_0^{\bar{\eta}} \Gamma_2 \, d\bar{\eta}\right) d\bar{\eta} \right. \tag{B.18}$$

$$\left. + \Gamma_4 \exp\left(\int_0^{\bar{\eta}} \Gamma_2 \, d\bar{\eta}\right) - \Gamma_4(0) + \Gamma_5 \exp\left(\int_0^{\bar{\eta}} \Gamma_2 \, d\bar{\eta}\right) + C_1\right] d\eta$$

where C_1 is determined from the boundary condition $g(1) = 1$.

Convective Heating Rate

An expression for the convective heating rate to the body surface will now be obtained in the (ξ, η) coordinate system.

The convective heat transfer rate to a wall is expressed by Fourier's equation

$$\dot{q}'_c = - k'_w \left(\frac{\partial T'}{\partial y'}\right)_w \tag{B.19}$$

where k' is the total thermal conductivity.

Writing Equation (B.19) in terms of total enthalpy we get

$$\dot{q}'_c = - \left(\frac{\mu'}{Pr}\right)_w \left(\frac{\partial H'}{\partial y'}\right)_w \tag{B.20}$$

where

$$Pr = \frac{C'_p \, \mu'}{k'} \ . \tag{B.21}$$

In terms of the dimensionless variables we have

$$q'_c = -\left(\frac{\mu}{Pr}\right)_w \frac{\mu'_{\delta,0} H'_\delta}{R'} \left(\frac{\partial H}{\partial y}\right)_w .$$

Transforming to (ξ, η) coordinates and using definition (B.7) we get

$$q'_c = -\left(\frac{\rho\,\mu}{Pr}\right)_w \frac{U'^{\,2}_\infty \mu'_{\delta,0}}{2 R' \tilde{\delta}} \left(\frac{\partial g}{\partial \eta}\right)_w . \tag{B.22}$$

With the aid of definitions (A.19) we obtain the following form for the convective heating rate to the body surface

$$q'_c = -\left(\frac{\rho\,\mu}{Pr}\right)_w \frac{\rho'_\infty U'^{\,3}_\infty}{2 \tilde{\delta} \operatorname{Re} \bar{\rho}} \left(\frac{\partial g}{\partial \eta}\right)_w . \tag{B.23}$$

<u>Radiative Energy Flux</u>

The radiative term E that appears in the energy equation, Equation (4), represents the volumetric rate of emission or absorption of energy by the shock layer gas due to radiation. Two assumptions are made here in order to evaluate this term in a practical manner.

- the shock layer geometry is approximated by a semi-infinite plane slab
- the shock layer is assumed to be locally one-dimensional in that radiation transport influence is allowed in only one direction

It has been shown that this one-dimensional plane slab model can be useful in obtaining quantitatively valid results.[2] Without the one-dimensional approximation the integro-differential equations defining E are more difficult to evaluate and the conservation equations used in this analysis lose their parabolic nature due to upstream feedback effects. This analysis follows Vincenti and Kruger[10] with the exception that the grey-gas approximation is not made here.

The radiative flux divergence can be written

$$-\dot{q}'_r = -E' = \int_0^\infty \alpha_\nu \left[\int_\Omega I_\nu \, d\Omega - 4\pi B_\nu \right] d\nu \tag{B.24}$$

where the inner integration is over a surface solid angle Ω.

To carry out the solid angle integration we consider the intensity I_ν at a fixed point y and in a direction defined by γ and ℓ, the angle and direction cosine of the direction of propagation. The element of solid angle is then

$$d\Omega = - d\ell \, d\gamma$$

Envoking the one-dimensional plane slab model, the integration can be readily carried out to yield

$$- E' = \int_0^\infty - 2\pi \, \alpha_\nu \left[\int_1^{-1} I_\nu \, d\ell - 4\pi B_\nu \right] d\nu \tag{B.25}$$

The local intensity I_ν is given by

$$I_\nu = I_\nu(\tau_{\nu,\,s}) \exp(-\tau_\nu) + \int_0^{\tau_{\nu,\,s}} B_\nu \exp(-\tau_\nu) \, d\tau_\nu \tag{B.26}$$

where $I_\nu(\tau_{\nu,\,s})$ is a boundary value on the local intensity and τ_ν is the optical depth defined by

$$\tau_\nu = \int_0^y \alpha_\nu \, dy$$

In terms of the direction cosine ℓ and the optical depth τ_ν, Equation (B.26) can be written

$$I_\nu = I_\nu(\tau_{\nu, s}) \exp\left(- \frac{\tau_\nu - \tau_{\nu, s}}{\ell}\right) - \int_{\tau_\nu}^{\tau_{\nu, s}} B_\nu \exp\left(- \frac{\tau_\nu - \hat{\tau}_\nu}{\ell}\right) \frac{d\hat{\tau}_\nu}{\ell} \qquad \text{(B.27)}$$

We have now to substitute Equation (B.27) into Equation (B.25) and formally integrate with respect to ℓ. To obtain the boundary values on I_ν, we assume that the body surface acts as a perfect absorber and that the gas outside the shock does not emit, absorb, or reflect radiant energy. The integral in Equation (B.25) can now be written

$$\int_1^{-1} I_\nu \, d\ell = -\int_1^0 \left[\int_{\tau_\nu}^0 B_\nu \exp -\left(\frac{\tau_\nu - \hat{\tau}_\nu}{\ell}\right) \frac{d\hat{\tau}_\nu}{\ell}\right] d\ell - \int_0^{-1} \left[\int_{\tau_\nu}^{\tau_{\nu, s}} B_\nu \exp -\left(\frac{\tau_\nu - \hat{\tau}_\nu}{\ell}\right) \frac{d\hat{\tau}_\nu}{\ell}\right] d\ell$$

Interchanging the order of integration[10] we can write

$$\int_{-1}^{1} I_\nu \, d\ell = -\int_0^{\tau_\nu} B_\nu \int_0^1 \ell^{-1} \exp\left(- \frac{\tau_\nu - \hat{\tau}_\nu}{\ell}\right) d\ell \, d\hat{\tau}_\nu - \int_{\tau_\nu}^{\tau_{\nu, s}} B_\nu \int_0^1 \ell^{-1} \exp\left(- \frac{\hat{\tau}_\nu - \tau_\nu}{\ell}\right) d\ell \, d\hat{\tau}_\nu \qquad \text{(B.28)}$$

The exponential integral function $\mathcal{E}_1$ can be defined as

$$\mathcal{E}_1(t) = \int_0^1 \ell^{-1} \exp(-t/\ell) \, d\ell \tag{B.29}$$

With this definition, Equation (B.28) becomes

$$\int_{-1}^{1} I_\nu \, d\ell = -\int_0^{\tau_\nu} B_\nu \mathcal{E}_1(\tau_\nu - \hat{\tau}_\nu) \, d\hat{\tau}_\nu - \int_{\tau_\nu}^{\tau_{\nu,s}} B_\nu \mathcal{E}_1(\hat{\tau}_\nu - \tau_\nu) \, d\hat{\tau}_\nu \tag{B.30}$$

Substituting the result (B.30) into Equation (B.25) we obtain the following expression for the radiative flux divergence.

$$- E' = \int_0^\infty 2\pi \alpha_\nu \left[\int_0^{\tau_\nu} B_\nu \; \mathcal{E}_1 \; (\tau_\nu - \hat{\tau}_\nu) \, d\hat{\tau}_\nu \right.$$

$$\left. + \int_{\tau_\nu}^{\tau_{\nu,s}} B_\nu \; \mathcal{E}_1 \; (\hat{\tau}_\nu - \tau_\nu) \, d\hat{\tau}_\nu - 2 \; B_\nu \right] d\nu \tag{B.31}$$

This equation completes the formulation of the radiation-coupled hypersonic flow problem. The conservation equations (1 through 4) can be solved to yield convective and radiative heat transfer providing the thermodynamic, transport and radiative properties of the gas are known.

Section 3
NUMERICAL SOLUTION

The viscous radiation-coupled flow field equations are solved numerically by means of a complex iteration scheme on a digital computer. The momentum equation is solved by an integral method while the energy equation is solved by successive approximations. The basic idea of the iteration is to assume sufficient parameters to allow a solution to be computed, compare the computed parameters to the assumed values, and iterate until convergence is obtained.

The total iteration procedure is started by first obtaining a stagnation point solution. Symmetry conditions plus the assumption of a symmetric concentric shock at the stagnation point provide necessary boundary conditions to obtain a stagnation point solution. The procedure for obtaining a solution around the body involves another iteration loop. An initial estimate is made for the shock shape as a function of the streamwise coordinate ξ. With the shock shape determined and the stagnation point solution known, the conservation equations can be solved to obtain a new shock shape. The solution around the body can then be obtained by iterating on the shock shape until convergence is achieved.

Stagnation Point Solution

The momentum and energy equations, Equations (A.21) and (B.15), take a much simplified form at the stagnation point due to symmetry conditions. At the body point $\xi = 0$ we have

$$u_\delta = 0 \quad , \frac{\partial I_1}{\partial \xi} = 0 \quad , \quad \frac{\partial \delta}{\partial \xi} = 0 \quad , \quad \frac{\partial I_2}{\partial \xi} = 0 \quad , \quad \frac{\partial g}{\partial \xi} = 0 \; .$$

The stagnation point solution is obtained by means of a double iteration loop.

A brief outline of this procedure follows.

1. A total enthalpy profile is assumed known.

2. An initial estimate is made for the shock detachment distance parameters, δ, $\tilde{\delta}$.

3. The velocity profile boundary conditions B_i can now be determined and hence the profile coefficients a_i [Equation (A.24)]. The shock layer velocity function f is then given by Equation (A.22).

4. The pressure variation through the shock layer is now given by Equation (A.43) utilizing the Rankine-Hugoniot relations [Equations (A.35) and (A.36)].

5. The shock layer gas properties are now determined as a function of enthalpy and pressure.

6. The momentum integral I_1 can be computed from Equation (A.21). Note that this is reduced to an algebraic equation for I_1 at the stagnation point due to symmetry conditions.

7. The transformed shock standoff distance $\tilde{\delta}$ is now determined from Equation (A.18). This computed value of $\tilde{\delta}$ is compared with the assumed value and the iteration loop returns to "3." until convergence is achieved.

8. The physical standoff distance δ is now given by Equation (A.45).

9. Since all the parameters Γ [Equation (B.17)] are known, the energy equation (B.18) can be evaluated to yield a new total enthalpy profile. This computed value is compared to "1." and iterated until the enthalpy profiles agree to a specified numerical tolerance.

10. The coupled convective and radiative heating rates are now output from Equations (B.23) and (B.31).

Total Axisymmetric Solution

The solution procedure for $x > 0$ is similar to the stagnation point iteration with the following exceptions.

- A complete shock shape is assumed. An initial iterative estimate is made for the shock curvature as a function of the streamwise coordinate ξ. Hence Equation (A.45) is not evaluated during the downstream solution. This relation is used to check the assumed shock shape after each iteration cycle.

- The streamwise derivatives are now non-zero and are computed using finite-difference approximations.

The outer loop on the numerical solution procedure is the shock shape iteration. The approach of specifying boundary values at the body surface and behind the shock allows a numerically stable solution to the system of parabolic partial differential equations.[2] It should be noted that the accuracy of the solution for $\xi > 0$ is based on the initial value solution at the stagnation point. This solution assumes a symmetric concentric shock at the stagnation point.

The total solution to the conservation equations [Equations (1) through (4)] thus yields a complete description of the shock layer structure with radiation-coupling. The convective and radiative heating rates are obtained from Equations (B.23) and (B.31).

Section 4
APPLICATIONS

This section provides a brief description of some feasible applications of the method formulated in the preceding sections. It is hoped that from this description the reader will better appreciate the wide range of problems that may be solved using this blunt body technique.

The computational procedure presented in the Numerical Solution section has been programmed using Fortran IV language. Initial results of stagnation line solutions for air are given in Appendix A. These stagnation line solutions were obtained considering only convective heat transfer and assuming thermodynamic equilibrium in real air with no diffusion or mass injection.

The formulated method can be used to solve blunt body flow field problems in which any of the following capabilities are required.

- a.* Arbitrary axisymmetric blunt body
- b.* Stagnation line solutions
- c.* Total axisymmetric solutions (complete flow field description)
- d.* Coupled convective and radiative energy flux
- e. Arbitrary atmospheres
- f.* Non-uniform initial flow field (plume application)
- g.* Mass injection of atmospheric species
- h. Mass injection of ablation products
- i. Binary or multi-component diffusion
- j.* Complete thermodynamic equilibrium
- k. Finite rate, quasi-thermodynamic equilibrium, or frozen flow chemistry (including electron densities and ionization effects)

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*Capabilities included in the present computer program.

To exercise some of the above capabilities, absorption cross sections, transport properties and thermodynamic properties or reaction rates are needed to a reasonable accuracy. If such data are available, the method presented in this report can be used to solve a wide variety of blunt body flow problems.

The present plan is to incorporate the NASA/Lewis Thermochemical code[11] into the existing computer program. A digital code for calculation of thermodynamic data[12], also developed by the Lewis Research Center, is being considered for use in providing necessary data for chemical species at high temperature. This capability will allow the use of thermodynamic properties for arbitrary gas mixtures and will provide a method of calculating species number densities needed in the radiation flux calculations. A frozen flow capability will also be available. Computer solutions will then provide a means of developing heating flux correlation equations for various planetary atmospheres of current interest. Future plans include development of the additional capabilities stated above.

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Appendix A

INITIAL RESULTS USING THE FORMULATED METHOD

INITIAL RESULTS USING THE FORMULATED METHOD

Laminar stagnation point convective heating rates have been calculated assuming no radiative losses using the method formulated in the main text. Calculations were made for velocities from 10,000 to 50,000 ft/sec at an altitude of 180 kft for comparison with existing methods. The present analysis assumed equilibrium chemistry and used shock density ratios from Reference A-1, viscosities behind the shock from Reference A-2, and equilibrium thermodynamic and transport properties in the shock layer from Reference A-3. Figure A-1 shows good agreement between values of the heating rate parameter, $\dot{q}\sqrt{R\ P_a/P_\delta}$, calculated using the present method and values of the parameter both calculated (Reference A-4), and measured (References A-5 and A-6). The present theoretical calculations and the theoretical calculations of Fenster (Reference A-4) using equilibrium chemistry are shown in Figure A-1 to agree well up to 40,000 ft/sec. Above 40,000 ft/sec the present analysis provides better agreement with experimental data.

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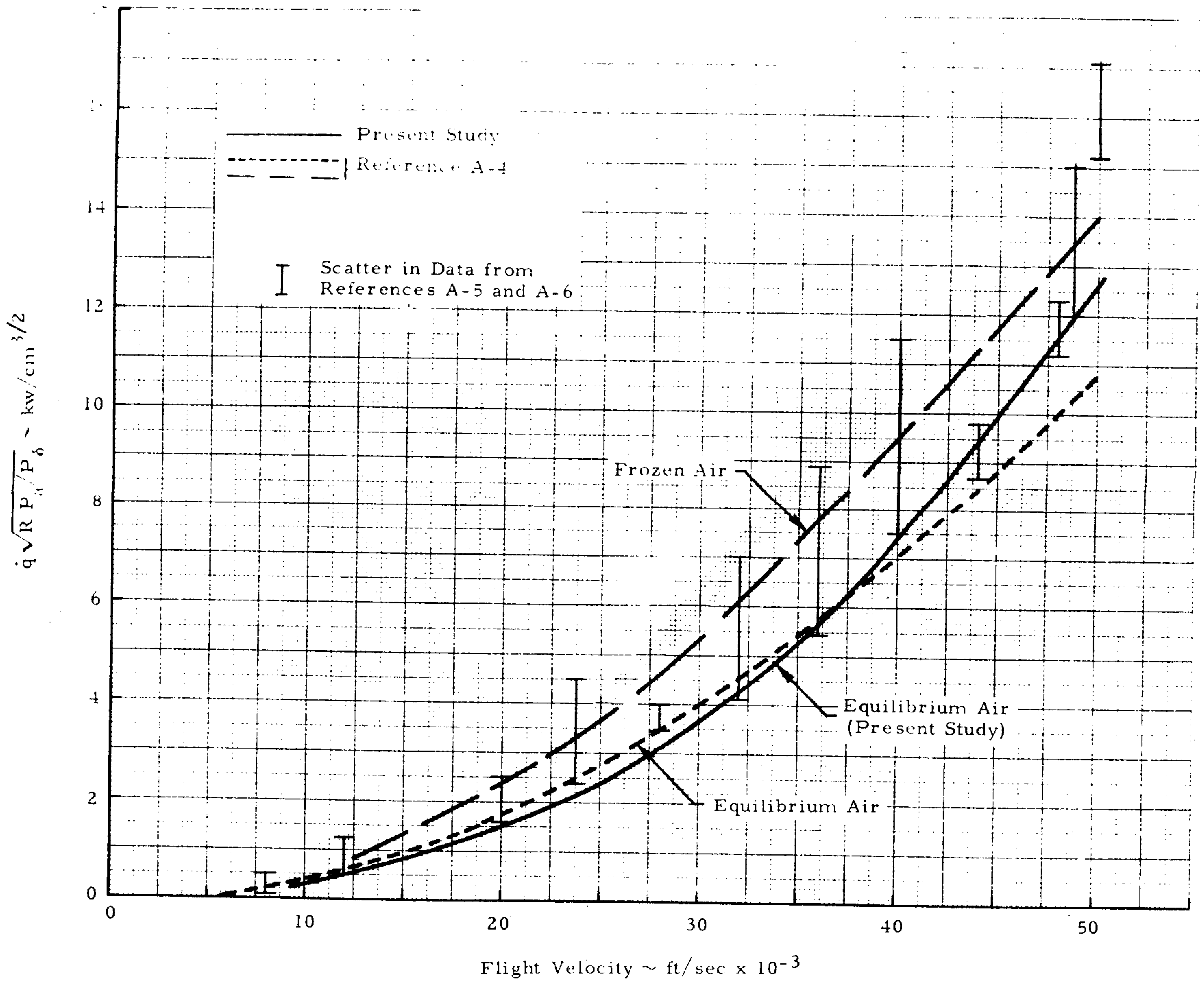

Figure A-1 - Heating Rate Parameter vs Flight Velocity in Air