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by Michael D. Greenberg and Alvin L. Kaskel

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THERM ADVANCED RESEARCH, INC.
Ithaca, N. Y.
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SUMMARY

The inviscid flow field induced by a rotor in ground effect is calculated based upon an actuator disk model of the rotor, for the case of a constant circulation distribution over the blade radius. The governing nonlinear integral equations are solved by a systematic iterative scheme which is similar to the Newton-Raphson method for the solution of nonlinear algebraic equations. Numerical results are presented for both the ground-effect case and the out-of-ground-effect limit.

INTRODUCTION

Several important problems arise in connection with a rotor hovering in ground effect, such as downwash impingement, and the effect of rotor-ground interference on the blade loading.

In the present paper, we are concerned with the problem of downwash impingement. Of the various aspects of this problem, we will confine our attention to the calculation of the inviscid flow field. This is of special importance since it is required as input for the subsequent calculation of the ground boundary layer and particle entrainment.

A numerical investigation of the inviscid flow field induced by a finite-bladed rotor has been carried out at the Cornell Aeronautical Laboratory, over the past several years, by W. G. Brady, P. Crimi, F. A. DuWaldt, and A. Sowyrda. Initially, they represented the rotor wake by discrete (finite core) vortex rings released periodically from the edge of the rotor disk (References 1,2). More recently, they have used a wake model based upon distorted continuous helices emanating from the blade tips (Reference 3).

In contrast, we will consider the axisymmetric flow field associated with an actuator disk representation of the rotor. The governing nonlinear integral equations will be solved by a systematic iterative procedure which is based upon the Newton-Raphson method for the solution of nonlinear algebraic equations (Reference 4). The mathematical treatment is somewhat general and could, we believe, be applied to other nonlinear free-boundary problems.

The nonlinear actuator disk, in the absence of ground effect, has already been treated in an important paper by T. Y. Wu (Reference 5), although numerical results are not yet available. The work of H. R. Chaplin (Reference 6) should also be noted, even though it deals with the <u>shrouded</u> disk, since that problem is fundamentally similar to the one treated here. Both Wu and Chaplin employ iterative schemes which differ appreciably from the one developed in the present paper.

The authors would like to thank their colleagues, Messrs. J. C. Erickson, Jr. and G. R. Hough, for many helpful discussions during the course of this work.

PRINCIPAL NOMENCLATURE

a_j	coefficients in expansion of slipstream vorticity
b_j	coefficients in expansion of slipstream radius
C	loading coefficient
C_T	thrust coefficient, thrust/$\tilde{\rho}(\Omega R)^2(\pi R^2)$
f_j	matching functions for slipstream shape
F	function in dynamic equation, $C - C^2/4T^2(x)$
F_∞	F with $T(x)$ replaced by T_∞
g_j	matching functions for slipstream vorticity
G	Green's function for L, over the infinite domain $-\infty < x < \infty$, $r < \infty$
h_j	modified f_j's for ground-effect case
L	linear differential operator, $\nabla^2 - r^{-2}$
M	number of shape collocation points
n	iteration index
N	number of gamma collocation points
p	static pressure
q	fluid velocity, $(u^2 + v^2 + w^2)^{\frac{1}{2}}$
$Q_{\pm\frac{1}{2}}$	Legendre functions of second kind and degree $\pm\frac{1}{2}$
t, T	slipstream radius, with arguments ξ and x respectively
u, v, w	x, r, θ fluid velocity components
U	free-stream speed

x, r, θ	cylindrical coordinates
X	x-location of ground plane
α, β	damping factors
γ, γ_s	slipstream circulation per unit x-length and arc-length, respectively
γ_∞	asymptotic value of γ
Γ	blade circulation distribution
ζ	meridional velocity, $(u^2 + v^2)^{\frac{1}{2}}$
λ	advance ratio, $U/\Omega R$
ξ, ρ	dummy x, r variables, respectively
$\tilde{\rho}$	fluid mass density
Ψ	stream function
Ψ_∞	value of Ψ on the slipstream
$\tilde{\omega}_{1,2}$	Legendre function arguments
Ω	blade rotational velocity, radians per unit time
$(\ )'$	prime denotes perturbational quantity
$(\ )_x$	subscripted variable denotes partial differentiation with respect to that variable
$(\ )^{(n)}$	n^{th} iterate
$\sqrt{\ }$	d(arc-length)/dx , $[1 + (dT/dx)^2]^{\frac{1}{2}}$

NOTE: Prior to equation (21), all quantities are in dimensional form. Starting with equation (21), they are nondimensionalized as follows: lengths with respect to R ; velocities, γ and γ_s with respect to ΩR ; Γ with respect to ΩR^2 ; and Ψ with respect to ΩR^3 . However, for notational simplicity we omit any explicit reminder of nondimensionalization, such as primes or asterisks.

THEORETICAL DEVELOPMENT

Out-of-Ground-Effect Limit; Actuator Disk Theory

Governing Nonlinear Differential Equation. Let us consider a propeller of blade radius R and negligible hub radius, operating relative to a uniform free stream U . We consider the blade number to be infinite, the so-called "actuator disk" model, and view the steady axisymmetric flow from a Newtonian x,r,θ coordinate system; see Fig. 1 (where we have sketched only one of the infinitely

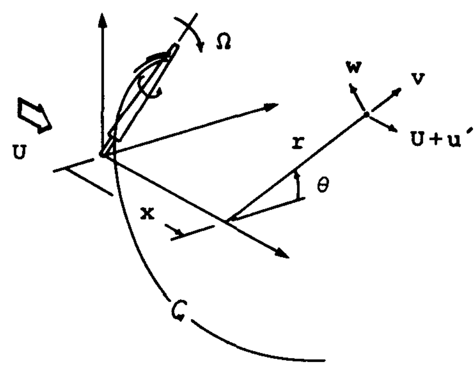

Figure 1. Coordinates and Geometry

many blades). The static actuator disk, i.e. for $U = 0$, is equivalent to our hovering rotor out of ground effect. We will, however, retain an arbitrary U in our analysis since it presents no additional difficulty and, at the same time, extends the applicability of our solution from the static condition, up to the light loading limit (where $U \gg$ the perturbational velocities).

The flow field is defined by the x,r,θ velocity components u,v,w respectively, or - equivalently - by w and a stream function Ψ , such that

$$u = U + u' = \Psi_r/r \tag{1}$$

$$v = v' = -\Psi_x/r \tag{2}$$

where the primed terms are perturbational quantities, and subscripts denote partial differentiation.

It has been shown by Wu (Reference 4), that Ψ must satisfy the following nonlinear partial differential equation,

$$\Psi_{rr} - r^{-1}\Psi_r + \Psi_{xx} = -(\Omega r^2 + wr)\ d(wr)/d\Psi \qquad (3)$$

Briefly, this may be derived by computing the circulation about an elemental meridional area dxdr, in two different ways: According to Stokes' theorem it may be computed as the θ component of vorticity, $v_x - u_r$, times the area dxdr, or, alternatively, as the line integral of "$\underline{q}\cdot d\underline{r}$" around the circumference of the element. Equating these two results produces (3).

<u>Conversion to an Integral Equation</u>. It will be convenient to convert (3) to an integral equation. With $L \equiv \nabla^2 - r^{-2}$ and $\Psi = Ur^2/2 + \Psi'$, we can express (3) in the form

$$L(\Psi'/r) = -(\Omega r + w)\ d(wr)/d\Psi \qquad (4)$$

Noting that L is linear (the nonlinearity being confined to the <u>right</u> hand side) we apply the method of Green's functions: Specifically, we seek the Green's function G as the solution of the associated equation

$$L(G/r) = -\delta(x-\xi)\delta(r-\rho) \qquad (5)$$

with the δ's denoting Dirac delta functions. Multiplying (5) through by $rJ_1(\bar{r}r)\exp(-i\bar{x}x)$ and integrating on r from $0\to\infty$ and on x from $-\infty\to\infty$, we obtain

$$\rho J_1(\bar{r}\rho)e^{-i\bar{x}\xi}/(\bar{x}^2 + \bar{r}^2) \qquad (6)$$

as the Hankel-Fourier transform of G/r, where J_1 denotes the Bessel function of the first kind and order one. Carrying out the Fourier inversion using the calculus of residues, and the Hankel inversion with the help of formula (2) on page 389 of Reference 7, we obtain

$$G(\xi,\rho;x,r) = r^{\frac{1}{2}}\rho^{\frac{1}{2}}Q_{\frac{1}{2}}(\tilde{\omega})/2\pi \qquad (7)$$

where $Q_{\frac{1}{2}}$ is the Legendre function of second kind and degree $\frac{1}{2}$, with argument

$$\tilde{\omega} = 1 + [(\xi-x)^2 + (\rho-r)^2]/2\rho r \qquad (8)$$

This is equivalent to the forms given by Wu and Chaplin. Physically, we may identify G as the stream function induced at a field point x,r,θ by a ring

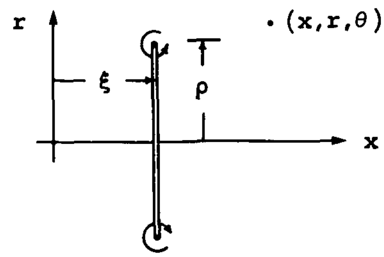

Figure 2. Interpretation of the Green's Function

vortex of unit strength, as shown in Fig. 2 .

With the Green's function in hand, we may re-express (3) in the form

$$\Psi(x,r) = Ur^2/2 + \iint_D \frac{r^{\frac{1}{2}}\rho^{\frac{1}{2}}}{2\pi} Q_{\frac{1}{2}}(\tilde{\omega})(\Omega\rho + w)\, \frac{d(w\rho)}{d\Psi}\, d\rho\, d\xi \tag{9}$$

This is a nonlinear integral equation in the two unknowns Ψ and D . The region D is clearly the slipstream, since w - and hence $d(w\rho)/d\Psi$ - is zero outside the slipstream, by application of Kelvin's theorem.

<u>Reduction for Uniform Circulation Distribution</u>. For the case of uniform blade circulation distribution we have

$$\begin{aligned} w\rho &= \text{constant} = -\Gamma/2\pi \quad , \quad \text{inside } D \\ &= 0 \quad , \quad \text{outside } D \end{aligned} \tag{10}$$

where Γ is the strength of the "hub" vortex, coinciding with the positive x axis. Converting the ρ,ξ integration variables to Ψ,ξ according to $d\rho d\xi = d\Psi d\xi/(\partial\Psi/\partial\rho)$, the Ψ integration can be carried out explicitly since the $d(w\rho)/d\Psi$ term in the integrand is zero except at the hub and tip; $\rho = 0, R$. Of these two contributions, the hub portion is zero since $\rho^{\frac{1}{2}}Q_{\frac{1}{2}} = 0$ at $\rho = 0$. The resulting integral equation, then, is

$$\Psi(x,r) = Ur^2/2 + \frac{r^{\frac{1}{2}}}{4\pi^2} \int_0^\infty \left\{ \Omega\Gamma t^{3/2} - \frac{\Gamma^2}{4\pi}\, t^{-\frac{1}{2}} \right\} Q_{\frac{1}{2}}(\tilde{\omega}_1)\, \frac{d\xi}{\partial\Psi/\partial\rho}\Big|_{\rho=t} \tag{11}$$

where $t(\xi)$ will denote the slipstream radius, and $\tilde{\omega}_1$ is identical to $\tilde{\omega}$, with ρ replaced by $t(\xi)$.

Vortex Sheet Interpretation. Although we can work directly with (11), we prefer to re-express the integral term in terms of an equivalent vortex representation of the slipstream; specifically, a distribution of ring vortices*, of circulation $\gamma(\xi)$ per unit ξ-length, over the slipstream surface $\rho = t(\xi)$.

According to our physical interpretation of the Green's function G, we can therefore express $\Psi(x,r)$ in the form

$$\Psi(x,r) = Ur^2/2 + \int_0^\infty G(\xi,t;x,r)\,\gamma(\xi)\,d\xi \tag{12}$$

We can establish the equivalence between (11) and (12) as follows:

Applying the Bernoulli equation to streamline A (see Fig. 3) between the points "∞" and (ξ,t^+), we have

$$p_A + \tfrac{1}{2}\tilde{\rho}q_A^2 = p_\infty + \tfrac{1}{2}\tilde{\rho}U^2 \tag{13}$$

where $\tilde{\rho}$ is the fluid mass density and $q^2 \equiv u^2 + v^2 + w^2$.

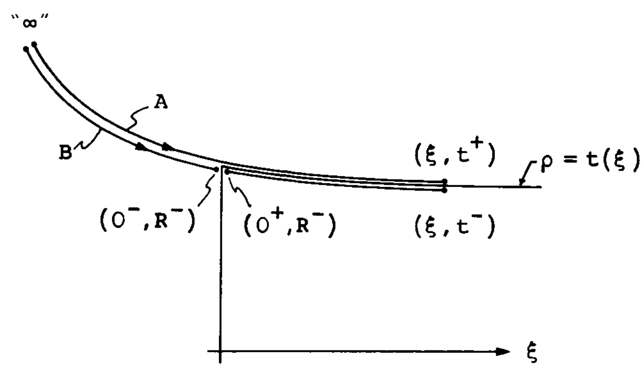

Figure 3. Application of Bernoulli Equation

* There will also be a distribution of vortices, over the slipstream surface, which are oriented axially. These contribute to w but not to Ψ, and will not directly concern us.

If we also apply it to streamline B , from "∞" to $(0^-,R^-)$ and then from $(0^+,R^-)$ to (ξ,t^-) , we find that

$$p_B + \tfrac{1}{2}\tilde{\rho}q_B^2 = p_\infty + \tfrac{1}{2}\tilde{\rho}U^2 + \tfrac{1}{2}\tilde{\rho}w^2(0^+,R^-) + \Delta p \tag{14}$$

where Δp is the pressure jump across the propeller plane at $\rho = R^-$. Now, the slipstream vorticity drifts freely so that we must have $p_B - p_A = 0$. Subtracting (13) from (14), then,

$$0 = p_B - p_A = \Delta p - \tfrac{1}{2}\tilde{\rho}(q_B^2 - q_A^2) + \tfrac{1}{2}\tilde{\rho}w^2(0^+,R^-) \tag{15}$$

The first and last terms on the right side are simply

$$\begin{aligned}\Delta p &= d(\text{thrust})/2\pi\rho d\rho \\ &= \tilde{\rho}(\Omega\rho - \Gamma/4\pi\rho)\Gamma d\rho/2\pi\rho d\rho\end{aligned} \tag{16}$$

at $\rho = R^-$, according to the Kutta-Joukowski formula, and

$$\tfrac{1}{2}\tilde{\rho}w^2(0^+,R^-) = \tfrac{1}{2}\tilde{\rho}(\Gamma/2\pi R)^2 \tag{17}$$

according to (10). To evaluate the middle term in (15), we note that

$$\begin{aligned}q_B^2 - q_A^2 &= (u^2 + v^2 + w^2)_B - (u^2 + v^2 + w^2)_A \\ &= \zeta_B^2 - \zeta_A^2 + \Gamma^2/4\pi^2t^2\end{aligned} \tag{18}$$

where we have defined the "meridional" velocity, $\zeta \equiv (u^2 + v^2)^{\frac{1}{2}}$, and have used the fact that $w_B = \Gamma/2\pi t$ and $w_A = 0$ from (10). Finally,

$$\begin{aligned}\zeta_B^2 - \zeta_A^2 &= (\zeta_B - \zeta_A)(\zeta_B + \zeta_A) \\ &= (\gamma_s)(2\zeta) = 2\gamma u\end{aligned} \tag{19}$$

where γ_s denotes the slipstream circulation per unit <u>arc</u>-length along the slipstream. Combining (15)-(19), we may express the force-free condition on the slipstream in the simple form

$$\gamma u = \frac{\Omega\Gamma}{2\pi} - \frac{\Gamma^2}{8\pi^2 t^2} \tag{20}$$

If we solve (20) for γ, noting that $u = \Psi_\rho/\rho$ at $\rho = t$, we find that the integral term in (12) is, in fact, identical to the one in (11), thus establishing the validity of our vortex sheet representation.

<u>The Final Integral Equations</u>. First, let us non-dimensionalize as follows: lengths with respect to R; velocities, γ and γ_s with respect to ΩR; Γ with respect to ΩR^2; and Ψ with respect to ΩR^3. <u>For notational simplicity we will omit any explicit reminder of nondimensionalization, such as asterisks or primes, in the remainder of the report.</u> Equations (12) and (20), for example, may therefore be rewritten as

$$\Psi(x,r) = \lambda r^2/2 + \int_0^\infty G(\xi;t;x,r)\,\gamma(\xi)\,d\xi \tag{21}$$

$$\gamma u = \frac{C}{2} - \frac{C^2}{8t^2} \tag{22}$$

respectively, where we have defined an advance ratio $\lambda \equiv U/\Omega R$, and a "loading coefficient" $C \equiv \Gamma/\pi$.

Whereas the integral equation (21) contains both the kinematics <u>and</u> dynamics, we prefer to express these conditions separately. The <u>kinematic</u> condition on the slipstream is that it be a streamline (more precisely, an axisymmetric stream surface). Setting $r = T(x)$* in (21), we have

$$\Psi_\infty = \lambda T^2/2 + \int_0^\infty G(\xi,t;x,T)\,\gamma\,d\xi \tag{23}$$

where we have set $\Psi(0,1) = \Psi[\infty,T(\infty)] \equiv \Psi(\infty,T_\infty) \equiv \Psi_\infty$.

The <u>dynamic</u> condition is given by (22). Changing the independent variable from ξ to x, and noting that $u = \Psi_r/r$ at $r = T$, with Ψ_r obtained from (21), we obtain

---

* It will be convenient to express the slipstream radius as t or T depending on whether the argument is the integration variable ξ or the field point x, respectively.

$$\left\{\frac{c}{2} - \frac{c^2}{8T^2}\right\}\frac{T}{\gamma} - \lambda T = \int_0^\infty G_T(\xi,t;x,T)\,\gamma\,d\xi \tag{24}$$

Equations (23) and (24), then, constitute two coupled nonlinear integral equations in the two unknowns γ and T, and are to be satisfied over the extent of the slipstream, $0 < x < \infty$.

The kernels are as follows:

$$\begin{aligned} G &= T^{\frac{1}{2}} t^{\frac{1}{2}} Q_{\frac{1}{2}}(\tilde{\omega}_2)/2\pi \\ &= O(\ell n\,|\xi - x|) \quad \text{as} \quad \xi \to x \\ &= O(\xi^{-3}) \quad \text{as} \quad \xi \to \infty \end{aligned} \tag{25}$$

where $\tilde{\omega}_2$ is identical to $\tilde{\omega}$, with ρ and r replaced by t and T respectively. Using the relation,

$$dQ_{\frac{1}{2}}(z)/dz = [zQ_{\frac{1}{2}}(z) - Q_{-\frac{1}{2}}(z)]/2(z^2-1) \tag{26}$$

we may express

$$\begin{aligned} G_T &= [AQ_{\frac{1}{2}}(\tilde{\omega}_2) + BQ_{-\frac{1}{2}}(\tilde{\omega}_2)]/(\tilde{\omega}_2^2-1) \\ &= O(\xi - x)^{-1} \quad \text{as} \quad \xi \to x \\ &= O(\xi^{-3}) \quad \text{as} \quad \xi \to \infty \end{aligned} \tag{27}$$

where

$$\begin{aligned} A &= [T^2 - t^2 + (\xi - x)^2]/8\pi T^{\frac{1}{2}} t^{3/2} \\ B &= [t^2 - T^2 + (\xi - x)^2]/8\pi T^{3/2} t^{\frac{1}{2}} \end{aligned} \tag{28}$$

Finally, we point out that the integral in (24) is to be interpreted in the Cauchy principal value sense.

Asymptotic Behavior of the Unknowns. Before proceeding with the detailed solution of (23) and (24) let us examine the equations at $x = 0$ and ∞.

As $x \to \infty$, it is known that $T(x) \sim$ constant $\equiv T_\infty$, say, and $\gamma(x) \sim$ constant $\equiv \gamma_\infty$. With these quantities constant at $x = \infty$, the integrals in (23) and (24) can be evaluated analytically. Instead of pursuing the details of the integration, let us use the known fact (e.g. Reference 8) that an infinite solenoid of constant radius and constant vortex strength, T_∞ and γ_∞ in our case, induces a velocity field given by

$$
\begin{aligned}
(u',v',w') &= (0\,,0\,,0) && , \quad r > T_\infty \\
&= (\gamma_\infty/2\,,0\,,0) && , \quad r = T_\infty \\
&= (\gamma_\infty\,,0\,,0) && , \quad r < T_\infty
\end{aligned}
\qquad (29)
$$

Now, since $2\pi\Psi(\xi,\rho)$ is the mass flow through the disk $r<\rho$ at $x=\xi$, we see - by virtue of (29) - that (23) must reduce to

$$\Psi_\infty = (\pi T_\infty^2)(\lambda+\gamma_\infty)/2\pi = T_\infty^2(\lambda+\gamma_\infty)/2 \qquad (30)$$

at $x=\infty$. Noting that (24) is merely a re-statement of (22), we see also that (24) must reduce to

$$\gamma_\infty\left\{\lambda+\frac{\gamma_\infty}{2}\right\} = \frac{C}{2} - \frac{C^2}{8T_\infty^2} \equiv \frac{F_\infty}{2} \qquad (31)$$

at $x=\infty$, where we have introduced the quantity $F(x) \equiv C - C^2/4T^2(x)$ for convenience. Equation (31) can be solved for γ_∞ in the form

$$\gamma_\infty = (\lambda^2+F_\infty)^{\frac{1}{2}} - \lambda \qquad (32)$$

It is interesting to note that (30) and (32) constitute two equations in the three unknowns Ψ_∞ , T_∞ and γ_∞ so that the final slipstream contraction cannot be computed (in terms of the operating conditions λ and C) simply by investigation of the asymptotic behavior, but must await the complete solution of the governing equations (23) and (24).

Now let us see what can be said about the behavior of the unknowns at the lip of the slipstream; i.e. as $x\to 0$ through positive values. Consider, first, the static case, where $\lambda=0$. Anticipating a flow field as sketched in Fig. 4, it is clear that the flow around the lip implies a square-root singularity in the circulation, so that $\gamma_s(x) = O(x^{-\frac{1}{2}})$ as $x\to 0$. This singularity should be present even when $\lambda>0$, and will vanish only in the light loading limit where the slipstream-induced jet velocity is negligible compared to the free stream.

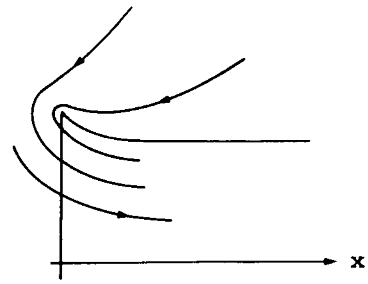

Figure 4. Flow Field for the Static Condition

Solution Based Upon the Newton-Raphson Method. With an exact solution of the highly nonlinear, coupled, integral equations (23) and (24) apparently out of the question, we will develop an iterative solution as follows. Starting with

$$T^{(0)}(x) = \text{constant} = 1 \tag{33}$$

$$\gamma^{(0)}(x) = \text{constant} = \gamma_\infty^{(0)} \tag{34}$$

we determine an improved slipstream shape, $T^{(1)}$, from (23); with $T = T^{(1)}$ we then determine an improved vortex distribution, $\gamma^{(1)}$, from (24); with $\gamma = \gamma^{(1)}$, $T^{(2)}$ is then computed from (23), and so on, until suitable convergence is attained. Our notation is to be interpreted in the obvious way. For example, $\gamma_\infty^{(0)}$ is given by (32) with F_∞ replaced by $F_\infty^{(0)}$ which, in turn, is defined according to the same formula as $F(x)$ but with $T(x)$ replaced by $T_\infty^{(0)}$.

In order to carry out the solution of (23) for $T^{(n+1)}$ at each step we linearize all the terms in (23) about the previous iterate, $T^{(n)}$. Similarly, to solve (24) for $\gamma^{(n+1)}$ we expand the nonlinear term about $\gamma^{(n)}$. Specifically,

$$\begin{aligned}\Psi_\infty^{(n+1)} &= T_\infty^{(n+1)2}(\lambda + \gamma_\infty^{(n)})/2 \\ &\approx [T_\infty^{(n)\,2} + 2T_\infty^{(n)}(T_\infty^{(n+1)} - T_\infty^{(n)})]\,(\lambda + \gamma_\infty^{(n)})/2\end{aligned} \tag{35}$$

$$T^{(n+1)2} \approx T^{(n)\,2} + 2T^{(n)}(T^{(n+1)} - T^{(n)}) \tag{36}$$

$$G^{(n+1)} \approx G^{(n)} + G_T^{(n)}(T^{(n+1)} - T^{(n)}) + G_t^{(n)}(t^{(n+1)} - t^{(n)}) \tag{37}$$

in (23), where $G^{(n)}$ denotes $G(\xi, t^{(n)}; x, T^{(n)})$, and

$$1/\gamma^{(n+1)} \approx 1/\gamma^{(n)} - (1/\gamma^{(n)})^2(\gamma^{(n+1)} - \gamma^{(n)}) \qquad (38)$$

in (24). This "stepwise linearization" is, basically, analogous to the Newton-Raphson method for the solution of algebraic equations (Reference 4). We emphasize that (35)-(38) tend to equalities as the (presumably convergent) iteration proceeds, and therefore in no way compromise the full nonlinearity of (23) and (24).

Whereas the two "correction" terms in (38), for example, are supplied automatically by the mathematics, it is instructive to interpret them physically. Multiplying (23) through by 2π , for convenience, and taking $n = 0$ for definiteness, the integral term is expanded, according to (38), in the form

$$2\pi \int_0^\infty G^{(1)} \gamma^{(0)} d\xi \approx 2\pi \int_0^\infty G^{(0)} \gamma^{(0)} d\xi + 2\pi \int_0^\infty G_T^{(0)} (T^{(1)} - T^{(0)})\gamma^{(0)} d\xi$$
$$+ 2\pi \int_0^\infty G_t^{(0)} (t^{(1)} - t^{(0)})\gamma^{(0)} d\xi \equiv ① + ② + ③ \qquad (39)$$

Now, ① is easily identified as the mass flow rate induced through the disk AB (see Fig. 5) by $\gamma^{(0)}$ on $t^{(0)}$, whereas we really want the flow induced through AC by $\gamma^{(0)}$ on $t^{(1)}$ if (23) is to be an equality at that particular

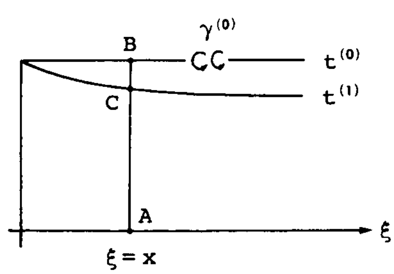

Figure 5. Interpretation of Correction Terms

value of x . The next term, ② , does in fact partially correct this by deducting (approximately) the flow through the annulus BC . To see this, let us re-express

$$② = \underbrace{-\,2\pi T^{(0)}(T^{(0)} - T^{(1)})}_{(i)} \cdot \underbrace{\frac{1}{T^{(0)}}\int_0^\infty G_T^{(0)}\gamma^{(0)}\,d\xi}_{(ii)} \tag{40}$$

where (i) approximates the area of the annulus BC , and (ii) is the x-velocity induced at B by $\gamma^{(0)}$ on $t^{(0)}$.

The last term, ③ , supplies an additional correction which is not, however, as easily interpreted in physical terms.

To provide a measure of control over the convergence of the iteration we introduce "damping factors" α and β so that the right hand side of (37) is replaced by

$$G^{(n)} + \alpha G_T^{(n)}(T^{(n+1)} - T^{(n)}) + \beta G_t^{(n)}(t^{(n+1)} - t^{(n)}) \tag{41}$$

Based upon numerical results, we have found that if $\alpha+\beta$ is too small, $T^{(1)}$ will be overcontracted and the iteration will diverge. An optimum is obtained, with regard to rapid convergence, when $\alpha+\beta$ is increased to approximately 1.8, independent of the disk loading. Curiously, the details of the iteration are quite insensitive as to how the "1.8" is divided between α and β . Consequently, we will take $\beta = 0$, from here on, for simplicity.

Actually, it is not surprising that with $\beta = 0$ the optimum $\alpha \approx 1.8$ since (ii) in (40) is the x-velocity computed right <u>on</u> the slipstream at B (Fig. 5) whereas the desired velocity just <u>inside</u> the slipstream is approximately twice as large*.

To proceed with the solution we expand

$$T^{(n)}(x) = 1 + \sum_{j=1}^{M} [f_j(x) - f_j(0)]b_j^{(n)} \tag{42}$$

$$\gamma^{(n)}(x) = \sqrt{1 + (dT^{(n)}/dx)^2}\;\gamma_s^{(n)}(x)$$

$$= \text{"}\sqrt{}^{(n)}\text{"}\;\gamma_\infty^{(n)}\left\{1 + \sum_{j=1}^{N} g_j(x)\,a_j^{(n)}\right\} \tag{43}$$

---

* Recall from (29) that at $x=\infty$, u' inside the slipstream is exactly twice as large as it is <u>on</u> the slipstream. This is, in fact, a good approximation for <u>finite</u> x as well.

where the f_j's and g_j's are suitably chosen "matching functions" which tend to zero at infinity. The form of these expressions guarantees satisfaction of the required end conditions, $T^{(n)}(0) = 1$ and $\gamma^{(n)}(\infty) = \gamma_\infty^{(n)}$. In addition, at least one of the g_j's include an $x^{-\frac{1}{2}}$ factor, to ensure the required square-root singularity at the lip.

Using the above expressions, our "kinematic" equation (23) can be re-written in the form

$$\int_0^\infty \left\{ -G^{(n)} + \alpha(T^{(n)} - 1)G_T^{(n)} \right\} \gamma^{(n)} d\xi + T_\infty^{(n)}(\lambda + \gamma_\infty^{(n)})(1 - T_\infty^{(n)}/2)$$

$$- \lambda T^{(n)}(1 - T^{(n)}/2) = \sum_{j=1}^{M} \left\{ \alpha[f_j(x) - f_j(0)] \int_0^\infty G_T^{(n)} \gamma^{(n)} d\xi \right.$$

$$\left. + \lambda T^{(n)}[f_j(x) - f_j(0)] + T_\infty^{(n)}(\lambda + \gamma_\infty^{(n)}) f_j(0) \right\} b_j^{(n+1)} \tag{44}$$

and our "dynamic" equation (24) can be expressed as

$$\left\{ \frac{F^{(n+1)}(2\gamma_s^{(n)} - \gamma_\infty^{(n+1)})}{2\sqrt{}^{(n+1)}\gamma_s^{(n)2}} - \lambda \right\} \frac{T^{(n+1)}}{\gamma_\infty^{(n+1)}} - \int_0^\infty G_T^{(n+1)} \sqrt{}^{(n+1)} d\xi$$

$$= \sum_{j=1}^{N} \left\{ \frac{F^{(n+1)} T^{(n+1)} g_j}{2\sqrt{}^{(n+1)}\gamma_s^{(n)2}} + \int_0^\infty G_T^{(n+1)} \sqrt{}^{(n+1)} g_j d\xi \right\} a_j^{(n+1)} \tag{45}$$

where it is understood that the "$\sqrt{}$" terms are evaluated at ξ or x depending on whether they are under an integral sign or not, respectively; similarly for g_j, γ and γ_s.

Our solution proceeds as follows: Starting with $n = 0$, we require the satisfaction of (44) at M "collocation" points $x_1, \ldots, x_M$. This produces M simultaneous linear algebraic equations which are then solved for the unknown coefficients $b_1^{(1)}, \ldots, b_M^{(1)}$. Next, we require the satisfaction of (45) at N collocation points (which need not coincide with the M points used to solve (44)) and hence compute $a_1^{(1)}, \ldots, a_N^{(1)}$. The process is then repeated for $n = 1, 2, \ldots$ until suitable convergence is attained.

We point out that instead of solving (23) and (24) successively for T and γ, we could have solved them <u>simultaneously</u>, at each step. Although this might lead to convergence in fewer iterations, the overall computing time would almost certainly be greater, however, since it takes approximately twice as long to generate an $(M+N)$th order set of linear algebraic equations as it does to generate M and Nth order sets separately.

Interpretation of the Loading Coefficient. Before discussing our numerical results, let us clarify the physical significance of our "loading coefficient", $C = \Gamma/\pi$. Defining the thrust coefficient C_T as the thrust divided by $\tilde{\rho}(\Omega R)^2(\pi R^2)$, we may use the Kutta-Joukowski formula to express

$$C_T = \frac{1}{\pi}\int_0^1 \left\{ r - \frac{\Gamma(r)}{4\pi r} \right\} \Gamma(r)\, dr \tag{46}$$

Now, in our analysis we have considered the blade circulation distribution $\Gamma(r)$ = constant = Γ over $0 < r < 1$. For this case, the swirl term $(\Gamma/4\pi r)$ in the integrand causes the integral to diverge. In reality, however, $\Gamma(r)$ will drop to zero at a finite radius, say ϵ , where $0 < \epsilon < 1$. Replacing the lower integration limit by ϵ , the integration in (46) may be carried out, to give

$$C_T = C\,[\,1 - \epsilon^2 + (C/2)\ell n\,\epsilon\,]/2 \tag{47}$$

For typical values of C and ϵ , $\epsilon^2 - (C/2)\ell n\,\epsilon$ is quite small compared to unity, so that the loading coefficient C is approximately twice the thrust coefficient C_T .

Numerical Results. As an illustration, let us consider the static case $\lambda = 0$, with a loading coefficient $C = 0.02$.

We define our collocation scheme by choosing $M = 7$, with the corresponding "shape collocation points",

$$x_j = 0.03,\ 0.1,\ 0.25,\ 0.5,\ 0.9,\ 1.5,\ 2.5$$

for $j = 1, \ldots, 7$ respectively; and $N = 9$, with the corresponding "gamma collocation points",

$$x_j = 0.02,\ 0.05,\ 0.1,\ 0.18,\ 0.3,\ 0.5,\ 0.85,\ 1.4,\ 2.5$$

for $j = 1, \ldots, 9$. We emphasize that there is little point in choosing collocation points further downstream than $x = 2.5$, say, since (as we will see in the subsequent Figures) the flow at that station is essentially identical to that in the ultimate jet. In fact, it can be expected to lead to an ill-conditioned set of equations since our expression (43) for γ automatically satisfies the dynamic equation at infinity. As a final word of caution we note that $x=0$ must not be included as a gamma collocation point since the dynamic equation is not satisfied at $x=0$.

As our "matching functions" we choose

$$f_j(x) = e^{-jx} \tag{48}$$

$$\begin{aligned} g_j(x) &= x^{-\frac{1}{2}}e^{-3x} \qquad , \quad j=1 \\ &= x^{-0.84+0.51j}e^{-3x} \quad , \quad j \geq 2 \end{aligned} \tag{49}$$

as shown in Fig. 6 . These were arrived at by trial and error, and appear to be equally suitable for all values $\lambda \geq 0$ and $C > 0$.

Starting with $T^{(0)} = 1$ and $\gamma_s^{(0)} = 0.1411$ (from (31) and (32)), and setting the damping factor $\alpha = 1.8$, the iteration is found to be rapidly convergent, as shown in Figs. 7-12 . As our convergence criterion, we required the iteration to continue until $T^{(n+1)}$ and $\gamma_s^{(n+1)}$ agreed with their previous values, $T^{(n)}$ and $\gamma_s^{(n)}$, to within 0.6% at each of the x values listed in Figs. 8-12 . Although it took five iterations to achieve this condition, it is seen that even the second iterate provides a fairly good uniform approximation to the solution.

However, it remains to show that the converged results do, in fact, represent the solution - since we only required satisfaction of the equations at several discrete collocation points. To settle this point, we have included a numerical check in the program (Appendix), which actually compares the left and right hand sides of the kinematic and dynamic equations (23) and (22). The results of this check indicate (Fig. 13) uniformly good agreement.

The flow field has also been computed, and is shown in Fig. 14 . It is important to note that for the static condition the streamline pattern is virtually independent of C , at least over the range of values which are of practical interest. To see this, consider the governing equations (23) and (24). For $\lambda = 0$, the C dependence cancels out of (23) since both Ψ_∞ and γ are proportional to γ_∞ which, in turn, contains the C dependence. Turning to the dynamic equation (24), we see that if we discard the swirl term $C^2/8T^2$, γ will (for $\lambda = 0$) simply be proportional to $C^{\frac{1}{2}}$. With the swirl term omitted, then, it follows that the streamline pattern will be completely independent of C although, of course, the velocities will be proportional to $C^{\frac{1}{2}}$. With the swirl term included, this result is no longer true in an exact sense. However, for practical values of C , $C^2/8T^2 \ll C/2$ in (24), so that our statement nevertheless remains true in an approximate sense. To verify this numerically, we re-computed our numerical example with C increased by eight times, i.e. with $C = 0.16$, and found the streamlines to be virtually unchanged!

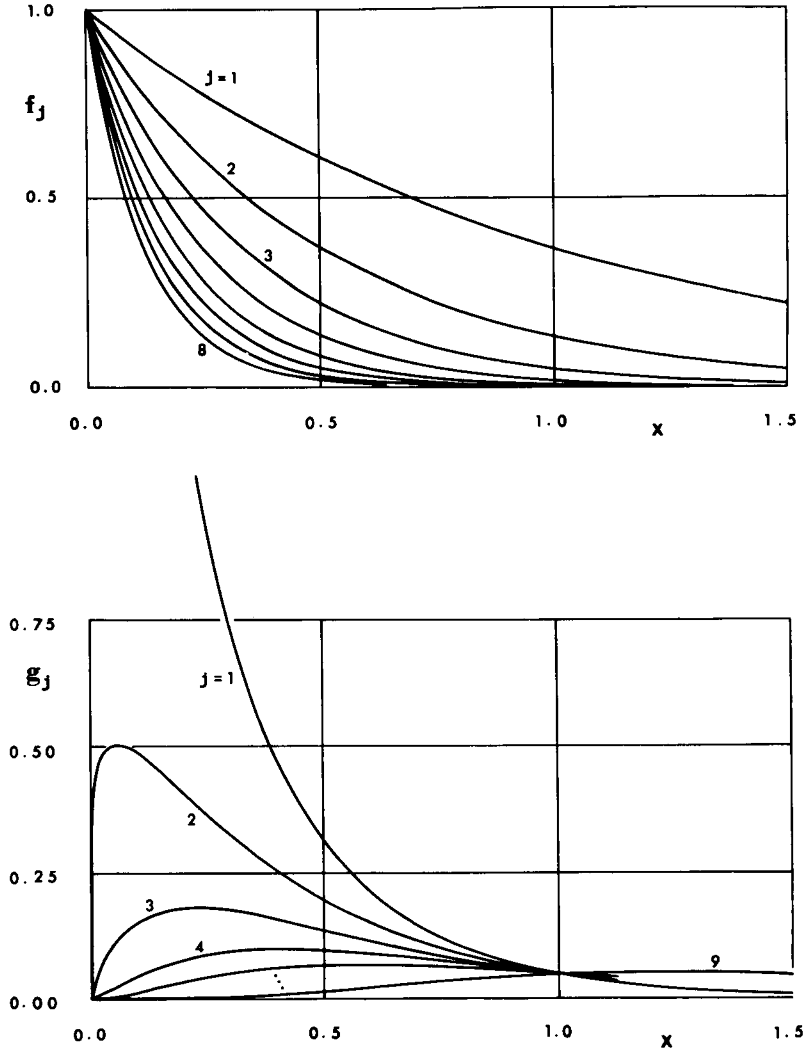

Figure 6. The Matching Functions

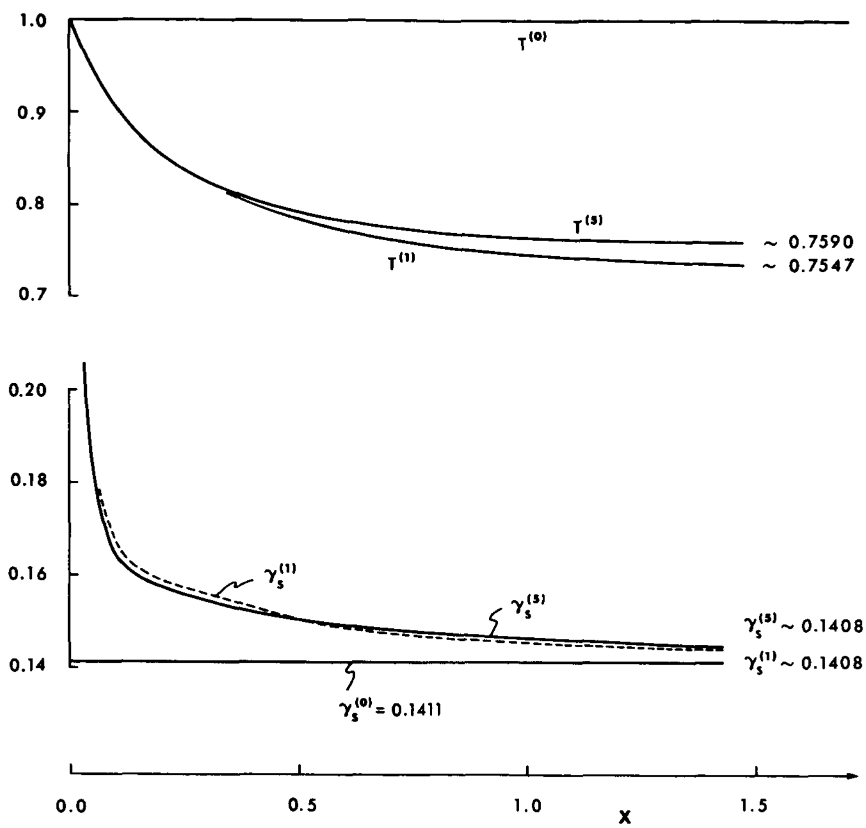

Figure 7. Successive Iterates for Numerical Example

ITERATION NO. 1

SHAPE COEFFICIENTS		GAMMA COEFFICIENTS		X	SLIPSTREAM RADIUS, T	SLIPSTREAM CIRCULATION GAMMA SUB S
B SUB 1 =	-.6159	A SUB 1 =	.0933	.000	1.0000	INFINITY
B SUB 2 =	5.1913	A SUB 2 =	.7019	.010	.9873	.2792
B SUB 3 =	-19.5532	A SUB 3 =	-7.6336	.030	.9645	.2102
B SUB 4 =	39.7679	A SUB 4 =	26.9076	.050	.9446	.1875
B SUB 5 =	-43.3664	A SUB 5 =	-45.3438	.070	.9272	.1762
B SUB 6 =	23.7830	A SUB 6 =	54.8806	.100	.9051	.1675
B SUB 7 =	-4.9614	A SUB 7 =	-59.9943	.150	.8763	.1615
		A SUB 8 =	44.1277	.200	.8544	.1590
		A SUB 9 =	-13.0430	.250	.8371	.1574
				.300	.8230	.1560
				.400	.8010	.1531
				.500	.7846	.1505
				.600	.7722	.1486
				.700	.7631	.1473
				.800	.7564	.1465
				.900	.7515	.1460
				1.000	.7478	.1457
				1.250	.7411	.1448
				1.500	.7359	.1436
				1.750	.7319	.1423
				2.000	.7295	.1412
				2.250	.7290	.1405
				2.500	.7300	.1401
				2.750	.7321	.1401
				3.000	.7347	.1401
				4.000	.7451	.1406
				5.000	.7508	.1408
				6.000	.7532	.1408
				7.000	.7542	.1408
				8.000	.7545	.1408
				INFINITY	.7547	.1408

UNIFORM ACCURACY OF .60000 PERCENT IS NOT ATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION DISTRIBUTIONS.
AT MOST 6 MORE ITERATION(S) WILL BE ATTEMPTED.

Figure 8. Computer Print-Out for Numerical Example; Iteration No. 1

ITERATION NO. 2

SHAPE COEFFICIENTS		GAMMA COEFFICIENTS		X	SLIPSTREAM RADIUS, T	SLIPSTREAM CIRCULATION GAMMA SUB S
B SUB 1 =	.1830	A SUB 1 =	.1394	.000	1.0000	INFINITY
B SUB 2 =	-2.0314	A SUB 2 =	-.9968	.010	.9857	.2849
B SUB 3 =	9.8344	A SUB 3 =	2.7803	.030	.9606	.2012
B SUB 4 =	-24.6488	A SUB 4 =	-8.3600	.050	.9394	.1790
B SUB 5 =	34.7424	A SUB 5 =	46.0351	.070	.9213	.1693
B SUB 6 =	-25.5606	A SUB 6 =	-127.7391	.100	.8988	.1628
B SUB 7 =	7.7311	A SUB 7 =	171.6096	.150	.8700	.1587
		A SUB 8 =	-110.4082	.200	.8486	.1567
		A SUB 9 =	27.7593	.250	.8320	.1552
				.300	.8186	.1538
				.400	.7986	.1515
				.500	.7848	.1499
				.600	.7753	.1488
				.700	.7686	.1480
				.800	.7640	.1473
				.900	.7607	.1466
				1.000	.7583	.1460
				1.250	.7550	.1448
				1.500	.7540	.1441
				1.750	.7544	.1437
				2.000	.7550	.1435
				2.250	.7555	.1433
				2.500	.7557	.1430
				2.750	.7555	.1426
				3.000	.7551	.1422
				4.000	.7527	.1411
				5.000	.7511	.1408
				6.000	.7504	.1408
				7.000	.7501	.1408
				8.000	.7500	.1408
				INFINITY	.7499	.1408

UNIFORM ACCURACY OF .60000 PERCENT IS NOT ATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION DISTRIBUTIONS.
AT MOST 5 MORE ITERATION(S) WILL BE ATTEMPTED.

Figure 9. Computer Print-Out for Numerical Example; Iteration No. 2

ITERATION NO. 3

SHAPE COEFFICIENTS		GAMMA COEFFICIENTS		X	SLIPSTREAM RADIUS, T	SLIPSTREAM CIRCULATION GAMMA SUB S
B SUB 1 =	.0287	A SUB 1 =	.1420	.000	1.0000	INFINITY
B SUB 2 =	-.4209	A SUB 2 =	-1.0209	.010	.9862	.2885
B SUB 3 =	2.8568	A SUB 3 =	3.2076	.030	.9618	.2034
B SUB 4 =	-8.9825	A SUB 4 =	-10.2344	.050	.9410	.1805
B SUB 5 =	15.5383	A SUB 5 =	46.0189	.070	.9233	.1704
B SUB 6 =	-13.3981	A SUB 6 =	-112.1238	.100	.9011	.1635
B SUB 7 =	4.5818	A SUB 7 =	137.3062	.150	.8728	.1591
		A SUB 8 =	-81.8722	.200	.8517	.1572
		A SUB 9 =	19.3796	.250	.8353	.1558
				.300	.8222	.1545
				.400	.8026	.1521
				.500	.7890	.1503
				.600	.7795	.1491
				.700	.7729	.1482
				.800	.7684	.1475
				.900	.7652	.1469
				1.000	.7629	.1464
				1.250	.7595	.1453
				1.500	.7578	.1444
				1.750	.7570	.1437
				2.000	.7567	.1432
				2.250	.7566	.1428
				2.500	.7566	.1424
				2.750	.7566	.1421
				3.000	.7565	.1417
				4.000	.7562	.1410
				5.000	.7560	.1408
				6.000	.7559	.1408
				7.000	.7559	.1408
				8.000	.7558	.1408
				INFINITY	.7558	.1408

UNIFORM ACCURACY OF .60000 PERCENT IS NOT ATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION DISTRIBUTIONS.
AT MOST 4 MORE ITERATION(S) WILL BE ATTEMPTED.

Figure 10. Computer Print-Out for Numerical Example; Iteration No. 3

ITERATION NO. 4

SHAPE COEFFICIENTS		GAMMA COEFFICIENTS		X	SLIPSTREAM RADIUS, T	SLIPSTREAM CIRCULATION GAMMA SUB S
B SUB 1 =	.0535	A SUB 1 =	.1427	.000	1.0000	INFINITY
B SUB 2 =	-.7122	A SUB 2 =	-.9527	.010	.9864	.2910
B SUB 3 =	4.1058	A SUB 3 =	2.4768	.030	.9625	.2049
B SUB 4 =	-11.5929	A SUB 4 =	-6.3161	.050	.9421	.1817
B SUB 5 =	18.3790	A SUB 5 =	34.8208	.070	.9245	.1714
B SUB 6 =	-14.8818	A SUB 6 =	-95.2418	.100	.9026	.1642
B SUB 7 =	4.8908	A SUB 7 =	124.5816	.150	.8745	.1596
		A SUB 8 =	-77.8530	.200	.8535	.1575
		A SUB 9 =	19.1372	.250	.8371	.1559
				.300	.8240	.1546
				.400	.8044	.1522
				.500	.7909	.1505
				.600	.7814	.1492
				.700	.7748	.1483
				.800	.7702	.1476
				.900	.7669	.1469
				1.000	.7646	.1464
				1.250	.7612	.1452
				1.500	.7596	.1443
				1.750	.7591	.1438
				2.000	.7591	.1433
				2.250	.7592	.1429
				2.500	.7592	.1425
				2.750	.7592	.1422
				3.000	.7592	.1418
				4.000	.7586	.1410
				5.000	.7582	.1408
				6.000	.7580	.1408
				7.000	.7579	.1408
				8.000	.7579	.1408
				INFINITY	.7578	.1408

UNIFORM ACCURACY OF .60000 PERCENT IS NOT ATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION DISTRIBUTIONS.
AT MOST 3 MORE ITERATION(S) WILL BE ATTEMPTED.

Figure 11. Computer Print-Out for Numerical Example; Iteration No. 4

ITERATION NO. 5

SHAPE COEFFICIENTS	
B SUB 1 =	.0465
B SUB 2 =	-.6370
B SUB 3 =	3.7753
B SUB 4 =	-10.8371
B SUB 5 =	17.4260
B SUB 6 =	-14.2465
B SUB 7 =	4.7138

GAMMA COEFFICIENTS	
A SUB 1 =	.1434
A SUB 2 =	-.9322
A SUB 3 =	2.2818
A SUB 4 =	-5.2265
A SUB 5 =	31.3735
A SUB 6 =	-88.9891
A SUB 7 =	118.0818
A SUB 8 =	-74.2536
A SUB 9 =	18.3213

X	SLIPSTREAM RADIUS, T	SLIPSTREAM CIRCULATION GAMMA SUB S
.000	1.0000	INFINITY
.010	.9866	.2925
.030	.9629	.2059
.050	.9427	.1824
.070	.9253	.1720
.100	.9036	.1648
.150	.8756	.1600
.200	.8547	.1577
.250	.8384	.1561
.300	.8253	.1547
.400	.8057	.1523
.500	.7921	.1505
.600	.7826	.1492
.700	.7760	.1483
.800	.7714	.1476
.900	.7681	.1470
1.000	.7658	.1464
1.250	.7623	.1452
1.500	.7608	.1444
1.750	.7602	.1438
2.000	.7601	.1433
2.250	.7601	.1429
2.500	.7602	.1425
2.750	.7602	.1421
3.000	.7602	.1418
4.000	.7597	.1410
5.000	.7593	.1408
6.000	.7591	.1408
7.000	.7591	.1408
8.000	.7590	.1408
INFINITY	.7590	.1408

UNIFORM ACCURACY OF .60000 PERCENT IS ATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION DISTRIBUTIONS.
ITERATION IS TERMINATED. DYNAMIC AND KINEMATIC CONDITIONS ON STREAMLINE WILL BE CHECKED AND FLOW PATTERN COMPUTED.

Figure 12. Computer Print-Out for Numerical Example; Iteration No. 5

NUMERICAL CHECK OF DYNAMIC (FORCE-FREE) AND KINEMATIC (STREAMLINE) CONDITIONS ON SLIPSTREAM

			DYNAMIC, U*GAMMA SHOULD EQUAL F/2		KINEMATIC, TOTAL (CAP) PSI ON SLIPSTREAM SHOULD EQUAL ITS VALUE AT X=INFINITY	
X	TOTAL X-VEL. ON SLIPSTREAM, U(X)	SLIPSTREAM VORTICITY PER UNIT X-LENGTH, GAMMA(X)	LHS	RHS	LHS	RHS
.01	.02116	.47632	.01008	.00995	.04048	.04056
.02	.02771	.35891	.00995	.00995	.04061	.04056
.03	.03275	.30479	.00998	.00995	.04063	.04056
.04	.03664	.27202	.00997	.00994	.04062	.04056
.05	.03983	.24963	.00994	.00994	.04060	.04056
.10	.05051	.19676	.00994	.00994	.04060	.04056
.15	.05602	.17734	.00993	.00993	.04055	.04056
.20	.05910	.16799	.00993	.00993	.04054	.04056
.30	.06244	.15897	.00993	.00993	.04059	.04056
.40	.06430	.15429	.00992	.00992	.04059	.04056
.50	.06550	.15146	.00992	.00992	.04057	.04056
.75	.06697	.14809	.00992	.00992	.04056	.04056
1.00	.06769	.14645	.00991	.00991	.04058	.04056
1.25	.06826	.14524	.00991	.00991	.04058	.04056
1.50	.06868	.14437	.00992	.00991	.04057	.04056
2.00	.06913	.14330	.00991	.00991	.04058	.04056
3.00	.06948	.14181	.00985	.00991	.04051	.04056
5.00	.06982	.14084	.00983	.00991	.04039	.04056
10.00	.07017	.14081	.00988	.00991	.04045	.04056
INF.	.07040	.14081	.00991	.00991	.04056	.04056

Figure 13. Computer Print-Out for Numerical Example; Numerical Check

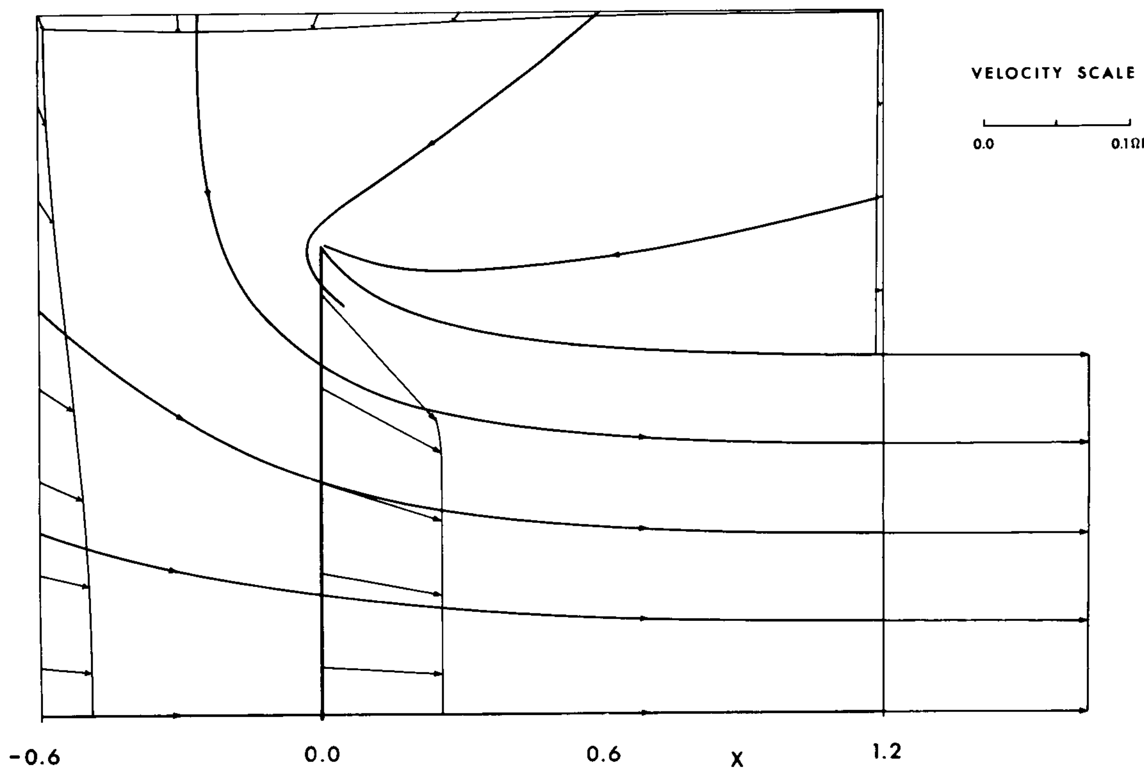

Figure 14. Resulting Flow Field for Numerical Example

Two points are of special interest with regard to Fig. 14 . First, we point out that at $x=1.2$ the (meridional) velocity inside the slipstream is almost constant, and is only about one percent smaller than its ultimate value at $x=\infty$.

Second, we see that the x-component of velocity is almost exactly constant over the actuator disk - out to about $r=0.9$ where it starts to drop off. As r increases further, the trend must reverse since the axial velocity must $\to\infty$ as $r\to 1^-$ by virtue of the square-root singularity in γ at the lip. Now, it is known (e.g. Reference 8) that in the light loading limit (i.e. the linearized actuator disk theory), the axial component of the induced velocity is exactly constant over the disk radius. The fact that this result is born out over most of the disk radius in our example, which is at the other (nonlinear) extreme, leads us to wonder whether the axial component of the induced velocity is in fact exactly constant over the disk radius (for our case $\Gamma(r)$ = constant), for any condition between (and including) the lightly loaded and static limits.

If this were true, it would imply that our initial slipstream contraction must be purely radial! This follows immediately from the fact that the meridional velocity is infinite just inside the lip, due to the square-root singularity in γ . Its inclination must therefore be radial if the axial velocity is to remain constant at $x=0$ as $r\to 1^-$.

Flow visualization studies (References 9,10) (for a finite blade number, of course) do indicate a strong radial flow in the tip region. Instead of purely radial flow, in fact, Reference 10 reports a slight upstream inclination of the flow at the tip, so that a reverse flow exists over approximately the outer 5% of the blade radius. We must note, however, that the existing $\Gamma(r)$ in Reference 10 is undoubtedly quite unlike our prescribed distribution, $\Gamma(r)\equiv$ constant , especially near the tip.

On the other hand, if the axial component of the induced velocity is not exactly constant over the disk radius then the strongly nonuniform axial inflow in the tip region is in fact correct, and may have a bearing on the well-known double bump in spanwise loading which has been observed near the tip of a number of helicopter rotors.

In any case, it seems clear that our results (Figs. 7-14) are quite accurate except, possibly, in the immediate tip region. The details in this region remain to be clarified.

Inclusion of the Ground Effect

The Integral Equations. Three changes are required in the integral equations (23) and (24) in order to accommodate the effect of a ground plane at $x=X$; see Fig. 15 . First of all they contain an arbitrary advance ratio λ ,

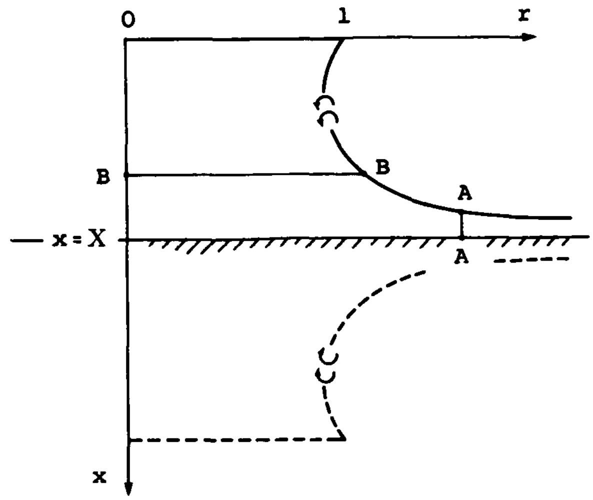

Figure 15. Ground Effect Model

whereas in the ground effect case we limit ourselves to static hover, so that $\lambda = 0$. In addition, we change the upper integration limits to X , and modify the Green's function so that it satisfies the additional boundary condition $u = 0$ at the ground plane. Specifically, we now have

$$G = T^{\frac{1}{2}}t^{\frac{1}{2}}[Q_{\frac{1}{2}}(\tilde{\omega}_2) - Q_{\frac{1}{2}}(\tilde{\tilde{\omega}}_2)]/2\pi \tag{50}$$

where $\tilde{\tilde{\omega}}_2$ is identical to $\tilde{\omega}_2$, with $(x-\xi)^2$ replaced by $(x-2X+\xi)^2$. Interpreted in terms of a vortex model, this amounts to adding an image system as indicated by the dashed lines in Fig. 15 .

Analogous to equations (27) and (28), we now have

$$G_T = [AQ_{\frac{1}{2}}(\tilde{\omega}_2) + BQ_{-\frac{1}{2}}(\tilde{\omega}_2)]/(\tilde{\omega}_2^2-1) - [\tilde{\tilde{A}}Q_{\frac{1}{2}}(\tilde{\tilde{\omega}}_2) + \tilde{\tilde{B}}Q_{-\frac{1}{2}}(\tilde{\tilde{\omega}}_2)]/(\tilde{\tilde{\omega}}_2^2-1) \tag{51}$$

where $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ are identical to A and B , with $(x-\xi)^2$ replaced by $(x-2X+\xi)^2$.

We observe that as $X \to \infty$, $\tilde{\tilde{\omega}}_2$ and $Q_{\pm\frac{1}{2}}(\tilde{\tilde{\omega}}_2)$ all tend to zero so that we do recover our previous out-of-ground-effect equations.

Asymptotic Behavior Near the Ground Plane. As $x \to X^-$, we see from (19) and (22) that $\gamma_s \zeta \sim c/2$ since $t \to \infty$ in the denominator of the last term in (22). In addition, we must have $\zeta \sim \gamma_s/2$, the drift velocity induced on the slipstream by the image vortex sheet. Combining these results, it follows that $\gamma_s \sim c^{\frac{1}{2}}$.

To obtain the asymptotic behavior of T, we apply the continuity equation at an arbitrary station AA, as shown in Fig. 15. The "control area" is $(2\pi T)(X-x)$ and the velocity through it is $\sim \gamma_s$; $\gamma_s/2$ due to the slipstream vorticity, and $\gamma_s/2$ due to the image vorticity. Continuity therefore requires that $\gamma_s(2\pi T)(X-x) =$ constant and, recalling that $\gamma_s \sim c^{\frac{1}{2}}$, it follows that $T(x) \sim O(X-x)^{-1}$.

In order to incorporate this behavior explicitly, we expand

$$T^{(n)}(x) = 1 + \sum_{j=1}^{M} [h_j(x) - h_j(0)]\, b_j^{(n)} \tag{52}$$

$$\gamma^{(n)}(x) = \sqrt{\Gamma^{(n)}}\, \gamma_s^{(n)}(x) = \sqrt{\Gamma^{(n)}}\, c^{\frac{1}{2}} \left\{ 1 + \sum_{j=1}^{N} g_j(x)\, a_j^{(n)} \right\} \tag{53}$$

where $h_j(x) = f_j(x)/(X-x)$ and the g_j's tend to zero as $x \to X$; c.f. equations (42) and (43) for the out-of-ground-effect limit.

The Final Equations. Linear algebraic equations analogous to (44) and (45) can be obtained almost exactly as before. The only difference is in the expression of "Ψ_∞". Since $2\pi\Psi_\infty$ is the mass flow rate through any disk BB (see Fig. 15) it is also, by continuity, the flow rate through the asymptotic station AA; namely, $\gamma_s(2\pi T)(X-x)$. Instead of (35), therefore, we have

$$\Psi_\infty^{(n+1)} = c^{\frac{1}{2}} \sum_{j=1}^{M} f_j(X)\, b_j^{(n+1)} \tag{54}$$

Using $\beta = 0$ again, we find that our kinematic equation can be reduced to

$$\int_0^X \left\{ -G^{(n)} + \alpha(T^{(n)} - 1) G_T^{(n)} \right\} \gamma^{(n)}\, d\xi$$
$$= \sum_{j=1}^{M} \left\{ \alpha[h_j(x) - h_j(0)] \int_0^X G_T^{(n)} \gamma^{(n)}\, d\xi - c^{\frac{1}{2}} f_j(X) \right\} b_j^{(n+1)} \tag{55}$$

for $n = 0, 1, \ldots$, and for the dynamic equation we obtain

$$\left[\frac{\gamma_s^{(n)}}{c^{1/2}} - \frac{1}{2}\right]\frac{T^{(n+1)}F^{(n+1)}}{\sqrt{\Gamma}^{(n+1)}\gamma_s^{(n)2}} - \int_0^X G_T^{(n+1)}\sqrt{\Gamma}^{(n+1)}\,d\xi$$

$$= \sum_{j=1}^{N}\left\{\frac{T^{(n+1)}F^{(n+1)}g_j}{2\sqrt{\Gamma}^{(n+1)}\gamma_s^{(n)2}} + \int_0^X G_T^{(n+1)}\sqrt{\Gamma}^{(n+1)}g_j\,d\xi\right\}a_j^{(n+1)} \tag{56}$$

Our procedure is the same as it was with equations (44) and (45); starting with $n=0$ we compute the $b_j^{(1)}$'s by satisfying (55) at M collocation points, the $a_j^{(1)}$'s by satisfying (56) at N collocation points, and so on, in turn, until convergence is attained.

Numerical Results. Let us consider, for example, the case where $X=1$ and $c=0.02$.

We fix our collocation scheme by choosing $M=6$, with the corresponding shape collocation points

$$x_j = 0.03,\ 0.1,\ 0.25,\ 0.45,\ 0.65,\ 0.9$$

for $j=1, \ldots, 6$ respectively; and $N=1$, for simplicity, with the corresponding gamma collocation point $x_1 = 0.1$.

As our matching functions we choose

$$f_j(x) = x^j \quad , \quad j = 1, \ldots, 6 \tag{57}$$

$$g_j(x) = x^{-1/2}(X-x)^2 \quad , \quad j = 1 \tag{58}$$

The form of g_1 ensures the satisfaction of both end conditions; $\gamma_s = O(x^{-1/2})$ as $x \to 0$ and $\gamma_s \sim c^{1/2}$ as $x \to X$.

Starting with $a_1^{(0)} = 0$, $b_2^{(0)} = 0.1$ and $b_j^{(0)} = 0$ for $j \neq 2$, the iteration is found to be more slowly convergent than in the out-of-ground-effect case. The streamline pattern and slipstream shape corresponding to the eighth iterate, which appears to have settled down to within about one percent, are shown in Fig. 16; the shape collocation points are indicated (on the slipstream) by dots. The corresponding slipstream vorticity is defined by the value $a_1^{(8)} = 0.022$, so that

$$\gamma_s^{(8)} = c^{1/2}[1 + 0.022x^{-1/2}(1-x)^2] \tag{59}$$

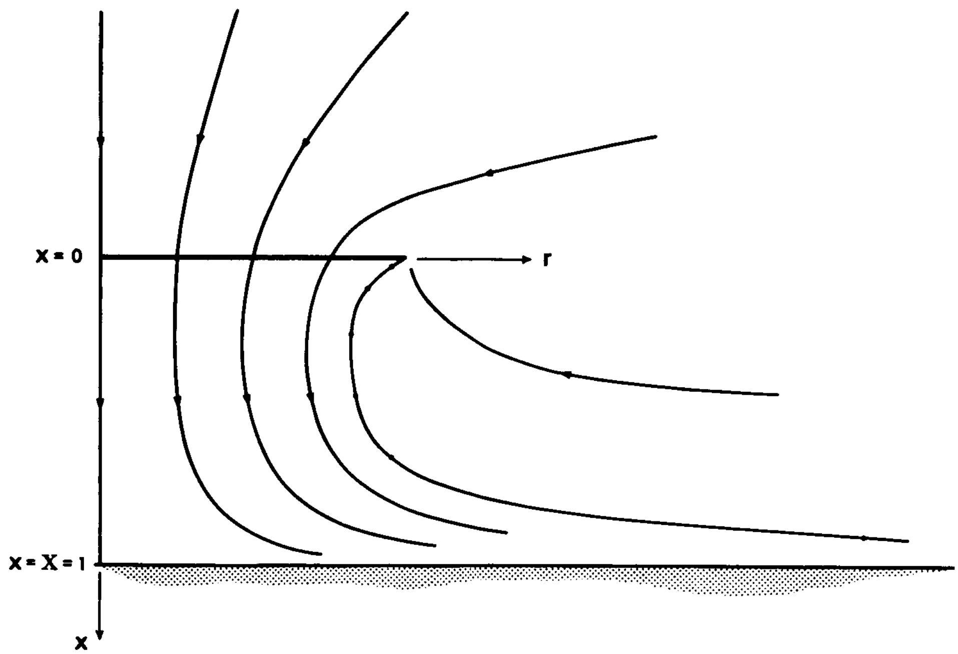

Figure 16. Resulting Flow Field for Numerical Example

Qualitatively, the results in Fig. 16 appear to be quite reasonable compared with the smoke visualization studies of Fradenburgh (Reference 11) - except for the absence of a dead-air dome beneath the hub, predicted by Heyson (Reference 12) and observed by Fradenburgh. This is to be expected, however, since our blade circulation is assumed to be constant all the way down to $r = 0^+$, so that there are no trailing vortices of reverse strength emitted over the inboard portion of the blades.

Quantitatively, we hesitate to claim a level of accuracy comparable to that obtained in the out-of-ground-effect case since only a single gamma collocation point was used. We did, in fact, run cases with $N = 6$ or more, but unsatisfactory "wiggles" began to appear in both γ_s and T. We attribute this to our inability to prescribe a sufficiently "natural" family of g_j matching functions.

We point out that our previous statement "For the static condition the streamline pattern is virtually independent of C", pertaining to the out-of-ground-effect case, is equally valid for the ground-effect case.

CONCLUSIONS

The inviscid flow field induced by a rotor in ground effect is found, based upon an actuator disk model with a constant circulation distribution. The governing nonlinear integral equations are solved by a systematic iterative scheme which is similar to the Newton-Raphson method for the solution of nonlinear algebraic equations.

First, the out-of-ground-effect limit is considered in detail. The iteration is found to be rapidly convergent and the results are shown to be quite accurate, except possibly in the immediate neighborhood of the blade tips. Specifically, there is some question as to whether or not the axial inflow should be constant over the blade radius or, equivalently, whether or not the initial slipstream contraction should be purely radial. This point is of some importance because of the square-root singularity in the slipstream vorticity at the "lip" of the slipstream. That is, the axial inflow (which is crucial from the point of view of blade design) through the tip portion of the blade will be bounded if the initial contraction is purely radial, and unbounded if it is not! In any case, it seems clear that our results are quite accurate except, possibly, in the immediate neighborhood of the blade tip.

Results for the ground effect case look entirely reasonable compared with the flow visualization studies of Fradenburgh (Reference 11), although we cannot claim a level of accuracy as high as in the out-of-ground-effect case.

In either case, in ground effect or not, it is shown that for the static condition the streamline pattern is virtually independent of the thrust coefficient

More precisely, it is exactly independent of the thrust coefficient - if the effects of swirl are neglected.

REFERENCES

1. Sowyrda, A., and DuWaldt, F. A., Representation of Wakes of Propellers in Ground Effect by Ring Vortex Systems, Cornell Aeronautical Laboratory Report No. BB-1665-S-1, April 1963 (DDC No. AD 602 988).

2. Brady, W. G., and Crimi, P., Representation of Propeller Wakes by Systems of Finite Core Vortices, Cornell Aeronautical Laboratory Report No. BB-1665-S-2, February 1965 (DDC No. AD 612 007).

3. DuWaldt, F. A., Representation of Propeller Wakes by Systems of Finite Core Vortices, Cornell Aeronautical Laboratory Report No. BB-1665-S-3, November 1966.

4. Scarborough, J. B., Numerical Mathematical Analysis, Sixth Edition, Johns Hopkins Press, Baltimore, 1966.

5. Wu, T. Y., Flow Through a Heavily Loaded Actuator Disk, Schiffstechnik, Vol. 9, No. 47, pp. 134-138, 1962.

6. Chaplin, H. R., A Method for Numerical Calculation of Slipstream Contraction of a Shrouded Impulse Disc in the Static Case With Application to Other Axisymmetric Potential Flow Problems, DTMB Aero Report 1077, 1964.

7. Watson, G. N., A Treatise on the Theory of Bessel Functions, Second Edition, Cambridge University Press, 1958.

8. Hough, G. R., and Ordway, D. E., The Generalized Actuator Disk, Developments in Theoretical and Applied Mechanics, Vol. II. Pergamon Press, Oxford/London/Edinburgh/New York/Paris/Frankfurt, 1965.

9. Taylor, M. K., A Balsa-Dust Technique for Air-Flow Visualization and its Application to Flow Through Model Helicopter Rotors in Static Thrust, NACA TN 2220, 1950.

10. Adams, G. N., Propeller Research at Canadair Limited, Proceedings of CAL/USAAVLABS Symposium on Aerodynamic Problems Associated with V/STOL Aircraft (Buffalo, New York, June 1966), Vol. 1.

11. Fradenburgh, E. A., Flow Field Measurements for a Hovering Rotor Near the Ground, Paper presented at the American Helicopter Society Fifth Annual Western Forum, Los Angeles, September 1958.

12. Heyson, Harry H., Induced Velocity Near a Rotor and its Application to Helicopter Problems, Proceedings of the 14th Annual National Forum, American Helicopter Society, April 1958.

APPENDIX

Listing of the Computer Codes

On the following pages are Fortran listings of the two computer codes, ROTORIGE (for the in-ground-effect case) and ROTOROGE (for the out-of-ground-effect case). Actually, the ROTORIGE code is presented for the special case where $N = 1$, to be consistent with the numerical example in the text, and does not apply for $N \geqslant 2$.

Of the input variables (see the Common Statements in ROTORIGE) only "ACC" requires further description. If, for example, ACC = 0.01 the iteration will proceed until both $T^{(n+1)}$ and $\gamma^{(n+1)}$ agree with their previous values, $T^{(n)}$ and $\gamma^{(n)}$, at each of the print-out x's (e.g. Fig. 8) to within one percent or better - or until n = "NITER" , whichever occurs first.

Regarding the speed of the calculation, we point out that for the numerical examples presented in the text the machine time per iteration (on the CDC 1604) was 30 seconds for ROTORIGE, and one minute and 30 seconds for ROTOROGE.

```
      PROGRAM ROTORIGE
C
C     ROTOR IN GROUND EFFECT (NAS1-6349)
C     FORMULATION USES ONE (1) COLLOCATION POINT FOR THE
C     DYNAMIC (GAMMA) EQUATION.
C
C     DEFINITION OF INPUT VARIABLES
C         IM = MONTH OF YEAR IN INTEGER FORM.
C         ID = DAY OF MONTH IN INTEGER FORM.
C         IY = LAST TWO DIGITS OF YEAR IN INTEGER FORM.
C         MM = NO. OF COLLOCATION POINTS FOR SHAPE EQUATION.
C         NN = NO. OF COLLOCATION POINTS FOR GAMMA EQUATION.
C     NITER = MAXIMUM NO. OF ITERATIONS TO BE ATTEMPTED.
C         CC = LOADING COEFFICIENT.
C       CAPX = NON-DIMENSIONAL DISTANCE FROM GROUND PLANE TO PROPELLER PLANE.
C        ACC = CONVERGENCE CRITERION IN PERCENT/100
C         XS = NON-DIMENSIONAL AXIAL COORDINATES OF SHAPE COLLOCATION POINTS.
C         XG = NON-DIMENSIONAL AXIAL COORDINATES OF GAMMA COLLOCATION POINTS.
C          B = INITIAL SHAPE COEFFICIENTS.
C          A = INITIAL GAMMA COEFFICIENTS.
C
      EXTERNAL FJ
      TYPE REAL LHS
      COMMON/ADDED/FZ(50)
      COMMON/COEFS/A(50),B(50)
      COMMON/INPUT/CC,ALPHA,CAPX
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      COMMON/MATRIX/RHS(50,50),LHS(50,1)
      COMMON/NUMBERXS/NXS,NXG
      COMMON/PRINT/XS(50),XG(50),XP(50),SR(50),SC(50),ITER,IM,ID,IY
      COMMON/SAVE/NP,INDEX,SRS(50),SCS(50)
 1000 FORMAT(4I5)
 1010 FORMAT(10F8.5)
 1020 FORMAT(1H0,28HINITIAL CONDITIONS - AT MOST,I3,14H ITERATION(S) ,
     *       18HWILL BE ATTEMPTED.)
 1030 FORMAT(1H0,19HUNIFORM ACCURACY OF,F8.5,16H PERCENT IS NOT ,
     *       47HATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION ,
     *       14HDISTRIBUTIONS.,/,8H AT MOST,I3,19H MORE ITERATION(S) ,
     *       18HWILL BE ATTEMPTED.)
 1040 FORMAT(1H0,19HUNIFORM ACCURACY OF,F8.5,12H PERCENT IS ,
     *       47HATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION ,
     *       14HDISTRIBUTIONS.,/,25H ITERATION IS TERMINATED.)
 1050 FORMAT(1H0,19HUNIFORM ACCURACY OF,F8.5,16H PERCENT IS NOT ,
     *       47HATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION ,
     *       14HDISTRIBUTIONS.,/,25H ITERATION IS TERMINATED.)
      READ 1000,IM,ID,IY
   10 READ 1000,MM,NN,NITER             $ IF (MM) 150,150,20
   20 READ 1010,CC,CAPX,ACC
      READ 1010,(XS(I),I=1,MM)          $ READ 1010,(XG(I),I=1,NN)
      NXS=MM                            $ NXG=NN
      READ 1010,(B(I),I=1,NXS)          $ READ 1010,(A(I),I=1,NXG)
      ALPHA=1.8
      DO 30 I=1,NXS
      FZ(I)=FJ(0.0,I)
   30 CONTINUE
      ITER=0                            $ ACCP=100.0*ACC
      INDEX=1                           $ CALL OUTPUT
      INDEX=2                           $ PRINT 1020,NITER
      DO 120 ITER=1,NITER
      N=NXS
```

```
      DO 40 I=1,N
      ZZ=XS(I)                                  $ CALL TOPBOT(ZZ)
      CALL LIMITCHK                             $ CALL BEQNS(I)
   40 CONTINUE
      CALL MATINV(RHS,N,LHS,1,DET)
      DO 50 J=1,N
      B(J)=LHS(J,1)
   50 CONTINUE
      N=NXG
      DO 60 I=1,N
      ZZ=XG(I)                                  $ CALL TOPBOT(ZZ)
      CALL LIMITCHK                             $ CALL AEQNS(I)
   60 CONTINUE
      CALL ACOEFS
      CALL OUTPUT
      DO 90 I=2,NP
      CACC=ACC*SC(I)                            $ TACC=ACC*SR(I)
      IF (ABSF(SC(I)-SCS(I))-CACC) 80,80,100
   80 IF (ABSF(SR(I)-SRS(I))-TACC) 90,90,100
   90 CONTINUE
      GO TO 130
  100 LEFT=NITER-ITER                           $ IF (LEFT) 140,140,110
  110 PRINT 1030,ACCP,LEFT
  120 CONTINUE
  130 PRINT 1040,ACCP                           $ GO TO 10
  140 PRINT 1050,ACCP                           $ GO TO 10
  150 END



      SUBROUTINE ACOEFS
      COMMON/ASPEC/AJ(2),AK(2),AL(2),FZZ,GZZ,AZZ,BZZ
      COMMON/COEFS/A(50),B(50)
      TOP=-AK(1)+SQRTF(AK(1)**2-4.0*AJ(1)*AL(1))
      BOT=2.0*AL(1)                             $ A(1)=TOP/BOT
      END



      SUBROUTINE AEQNS(I)
      EXTERNAL AJINT,AKINT,ALINT
      EXTERNAL GJ
      COMMON/ASPEC/AJ(2),AK(2),AL(2),FZZ,GZZ,AZZ,BZZ
      COMMON/COEFS/A(50),B(50)
      COMMON/INPUT/CC,ALPHA,CAPX
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      CALL SHAPE(ZZ,TT)                         $ CALL FACTOR(ZZ,FZZ)
      GZZ=GJ(ZZ)                                $ FF=CC-(CC/(2.0*TT))**2
      AJ(I)=-FF*TT/(2.0*CC*FZZ)                 $ AK(I)=AL(I)=0.0
      CALL ONEINTGL(AJINT,AJ(I))                $ CALL ONEINTGL(AKINT,AK(I))
      CALL ONEINTGL(ALINT,AL(I))
      END



      FUNCTION AJINT(Z)
      COMMON/ASPEC/AJ(2),AK(2),AL(2),FZZ,GZZ,AZZ,BZZ
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      CALL SHAPE(Z,T)                           $ CALL FACTOR(Z,FZ)
      CALL GEES(1,TT,T,ZZ,Z,G,GTT)              $ AJINT=GTT*FZ
      END
```

```
      FUNCTION AKINT(Z)
      EXTERNAL GJ
      COMMON/ASPEC/AJ(2),AK(2),AL(2),FZZ,GZZ,AZZ,BZZ
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      CALL SHAPE(Z,T)                 $ CALL FACTOR(Z,FZ)
      CALL GEES(1,TT,T,ZZ,Z,G,GTT)    $ AKINT=GTT*FZ*(GZZ+GJ(Z))
      END


      FUNCTION ALINT(Z)
      EXTERNAL GJ
      COMMON/ASPEC/AJ(2),AK(2),AL(2),FZZ,GZZ,AZZ,BZZ
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      CALL SHAPE(Z,T)                 $ CALL FACTOR(Z,FZ)
      CALL GEES(1,TT,T,ZZ,Z,G,GTT)    $ ALINT=GTT*FZ*GZZ*GJ(Z)
      END


      SUBROUTINE BEQNS(I)
      EXTERNAL BINT,FJ
      TYPE REAL LHS
      COMMON/ADDED/FZ(50)
      COMMON/INPUT/CC,ALPHA,CAPX
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      COMMON/MATRIX/RHS(50,50),LHS(50,1)
      NN=N+1                          $ CALL SHAPE(ZZ,TT)
      GTTG=GGM=0.0
      NGEES=1                         $ CALL ONEINTGL(BINT,GTTG)
      NGEES=3                         $ CALL ONEINTGL(BINT,GGM)
      DO 30 J=1,NN
      K=J                             $ IF (J-NN) 10,20,20
   10 TERM=FJ(ZZ,K)/(CAPX-ZZ)-FZ(K)/CAPX
      RHS(I,J)=-SQRTF(CC)*FJ(CAPX,J)+ALPHA*TERM*GTTG
      GO TO 30
   20 RHS(I,J)=GGM+ALPHA*(TT-1.0)*GTTG
   30 CONTINUE
      LHS(I,1)=RHS(I,NN)
      END


      FUNCTION BINT(Z)
      EXTERNAL FJ
      COMMON/ADDED/FZ(50)
      COMMON/INPUT/CC,ALPHA,CAPX
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      CALL SHAPE(Z,T)                 $ CALL VORTEX(Z,GAM)
      CALL FACTOR(Z,FAC)              $ GAM=FAC*GAM
      CALL GEES(NGEES,TT,T,ZZ,Z,G,GTT)
      GO TO (10,30,20),NGEES
   10 BINT=GTT*GAM                    $ GO TO 30
   20 BINT=-G*GAM
   30 END


      SUBROUTINE FACTOR(X,FAC)
      EXTERNAL FJ,FPJ
      COMMON/COEFS/A(50),B(50)
      COMMON/INPUT/CC,ALPHA,CAPX
      COMMON/NUMBERXS/NXS,NXG
      TP=0.0                          $ BOT=CAPX-X
      DO 10 I=1,NXS
      TP=TP+B(I)*(FPJ(X,I)/BOT+FJ(X,I)/BOT**2)
   10 CONTINUE
      FAC=SQRTF(1.0+TP**2)
      END
```

```
      FUNCTION FJ(X,J)
      FJ=X**J
      END


      FUNCTION FPJ(X,J)
      IF (J-1) 10,10,20
   10 FPJ=1.0                          $ GO TO 30
   20 FPJ=J*X**(J-1)
   30 END


      SUBROUTINE GEES(NUM,TT,T,ZZ,Z,G,GTT)
      COMMON/INPUT/CC,ALPHA,CAPX
      PI=3.1415927
      DX=(ZZ-Z)**2                     $ DT=(TT-T)**2
      DZ=(ZZ-2.0*CAPX+Z)**2            $ TERM=SQRTF(TT*T)
      ARGA=1.0+(DT+DX)/(2.0*TERM**2)   $ CALL QPMHALF(ARGA,QPA,QMA)
      ARGB=1.0+(DT+DZ)/(2.0*TERM**2)   $ CALL QPMHALF(ARGB,QPB,QMB)
      GO TO (10,20,10),NUM
   10 DT=TT**2-T**2                    $ BOT=8.0*PI*TERM
      BOTA=ARGA**2-1.0                 $ BOTB=ARGB**2-1.0
      AA=(DX+DT)*QPA/(BOT*T)           $ BB=(DX-DT)*QMA/(BOT*TT)
      EE=(DZ+DT)*QPB/(BOT*T)           $ DD=(DZ-DT)*QMB/(BOT*TT)
      GTT=(AA+BB)/BOTA-(EE+DD)/BOTB    $ GO TO (30,20,20),NUM
   20 G=TERM*(QPA-QPB)/(2.0*PI)
   30 END


      FUNCTION GGG(Z)
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      CALL SHAPE(Z,T)                  $ CALL VORTEX(Z,G)
      CALL FACTOR(Z,FAC)               $ G=FAC*G
      CALL GEES(2,TT,T,ZZ,Z,GG,DUM)
      GGG=GG*G
      END


      FUNCTION GJ(X)
      COMMON/INPUT/CC,ALPHA,CAPX
      GJ=(CAPX-X)**2/SQRTF(X)
      END


      SUBROUTINE LIMITCHK
      COMMON/LIMITS/TOP(5),BOT(5),NGD(5)
      DO 40 I=1,5
      IF (BOT(I)-TOP(I)) 40,40,10
   10 IF (I-5) 20,30,30
   20 BOT(I+1)=BOT(I)
   30 BOT(I)=TOP(I)
   40 CONTINUE
      END
```

```
      SUBROUTINE MATINV(A,N,B,M,DETERM)
      DIMENSION IPIVOT(50),A(50,50),B(50,1),INDEX(50,2),PIVOT(50)
      DETERM=1.0
      DO 10 J=1,N
      IPIVOT(J)=0
   10 CONTINUE
      DO 200 I=1,N
      AMAX=0.0
      DO 60 J=1,N
      IF (IPIVOT(J)-1) 20,60,20
   20 DO 50 K=1,N
      IF (IPIVOT(K)-1) 30,50,240
   30 IF (ABSF(AMAX)-ABSF(A(J,K))) 40,50,50
   40 IROW=J                          $ ICOLUM=K
      AMAX=A(J,K)
   50 CONTINUE
   60 CONTINUE
      IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1 $ IF (IROW-ICOLUM) 70,110,70
   70 DETERM=-DETERM
      DO 80 L=1,N
      SWAP=A(IROW,L)                  $ A(IROW,L)=A(ICOLUM,L)
      A(ICOLUM,L)=SWAP
   80 CONTINUE
      IF (M) 110,110,90
   90 DO 100 L=1,M
      SWAP=B(IROW,L)                  $ B(IROW,L)=B(ICOLUM,L)
      B(ICOLUM,L)=SWAP
  100 CONTINUE
  110 INDEX(I,1)=IROW
      INDEX(I,2)=ICOLUM               $ PIVOT(I)=A(ICOLUM,ICOLUM)
      DETERM=DETERM*PIVOT(I)          $ A(ICOLUM,ICOLUM)=1.0
      DO 120 L=1,N
      A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  120 CONTINUE
      IF (M) 150,150,130
  130 DO 140 L=1,M
      B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
  140 CONTINUE
  150 DO 200 L1=1,N
      IF(L1-ICOLUM) 160,200,160
  160 T=A(L1,ICOLUM)                  $ A(L1,ICOLUM)=0.0
      DO 170 L=1,N
      A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  170 CONTINUE
      IF (M) 200,200,180
  180 DO 190 L=1,M
      B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  190 CONTINUE
  200 CONTINUE
      DO 230 I=1,N
      L=N+1-I
      IF (INDEX(L,1)-INDEX(L,2)) 210,230,210
  210 JROW=INDEX(L,1)                 $ JCOLUM=INDEX(L,2)
      DO 220 K=1,N
      SWAP=A(K,JROW)                  $ A(K,JROW)=A(K,JCOLUM)
      A(K,JCOLUM)=SWAP
  220 CONTINUE
  230 CONTINUE
  240 RETURN
      END
```

```
      SUBROUTINE NGAUSS(B,A,FX,NTIME,INTEGRAL)
      TYPE REAL INTEGRAL
      DIMENSION R(5),U(5)
      DATA (R=0.1477621124,0.1346333597,0.1095431813,0.07472567458,
     *        0.03333567215),
     *     (U=0.0744371695,0.2166976971,0.3397047841,0.4325316833,
     *        0.4869532643)
      INTEGRAL=0.0
      DO 20 J=1,NTIME
      XL=A+(J-1)*(B-A)/NTIME              $ XU=B-(NTIME-J)*(B-A)/NTIME
      D=XU-XL                             $ S=(XU+XL)/2.0
      TEMP=0.0
      DO 10 K=1,5
      TEMP=TEMP+R(K)*(FX(S+D*U(K))+FX(S-D*U(K)))
   10 CONTINUE
      TEMP=TEMP*D                         $ INTEGRAL=INTEGRAL+TEMP
   20 CONTINUE
      END


      SUBROUTINE ONEINTGL(FA,A)
      COMMON/LIMITS/TOP(5),BOT(5),NGD(5)
      DO 20 I=1,5
      IF (ABSF(TOP(I)-BOT(I))-0.0000001) 20,20,10
   10 CALL NGAUSS(TOP(I),BOT(I),FA,NGD(I),AA)
      A=A+AA
   20 CONTINUE
      END


      SUBROUTINE OUTPUT
      COMMON/ADDED/FZ(50)
      COMMON/COEFS/A(50),B(50)
      COMMON/INPUT/CC,ALPHA,CAPX
      COMMON/INTGND/NGEES,TT,ZZ,N,K
      COMMON/NUMBERXS/NXS,NXG
      COMMON/PRINT/XS(50),XG(50),XP(50),SR(50),SC(50),ITER,IM,ID,IY
      COMMON/SAVE/NP,INDEX,SRS(50),SCS(50)
 1000 FORMAT(1H1,25X,32HR O T O R   I N   G R O U N D   ,
     *       35HE F F E C T   ( N A S 1 - 6 3 4 9 ),5X,1H(,I2,1H/,I2,
     *       1H/,I2,1H))
 1010 FORMAT(1H0,35X,29HDAMPING COEFFICIENTS, ALPHA =,F5.2,9H , BETA =,
     *       4H 0.0,/,36X,
     *       24HLOADING COEFFICIENT, C =,F7.4,/,36X,
     *       16HHUB RADIUS = 0.0)
 1020 FORMAT(1H0,21X,29HSHAPE COLLOCATION POINTS, M =,I3,12X,
     *       29HGAMMA COLLOCATION POINTS, N =,I3,/)
 1030 FORMAT(1H ,28X,5HX SUB,I3,2H =,F8.4,26X,5HX SUB,I3,2H =,F8.4)
 1040 FORMAT(1H ,72X,5HX SUB,I3,2H =,F8.4)
 1050 FORMAT(1H ,28X,5HX SUB,I3,2H =,F8.4)
 1060 FORMAT(1H0,13HITERATION NO.,I3,/)
 1070 FORMAT(1H ,79X,35HSLIPSTREAM   SLIPSTREAM CIRCULATION,/,7X,
     *       18HSHAPE COEFFICIENTS,14X,18HGAMMA COEFFICIENTS,11X,1HX,
     *       11X,30HRADIUS, T          GAMMA SUB S,/)
 1071 FORMAT(1H ,4X,10HB SUB  1 =,F12.4,10X,10HA SUB  1 =,F12.4,8X,
     *       4H.000,11X,6H1.0000,F18.4)
 1072 FORMAT(1H ,4X,10HB SUB  1 =,F12.4,10X,10HA SUB  1 =,F12.4,8X,
     *       4H.000,11X,6H1.0000,10X,8HINFINITY)
 1080 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,10X,5HA SUB,I3,2H =,F12.4,
     *       F12.3,F17.4,F18.4)
 1090 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,10X,5HA SUB,I3,2H =,F12.4,
     *       6HCAP X=,F6.3,9X,8HINFINITY,F18.4)
 1100 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,10X,5HA SUB,I3,2H =,F12.4)
 1110 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,32X,F12.3,F17.4,F18.4)
```

```
1120 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,32X,
    *        6HCAP X=,F6.3,9X,8HINFINITY,F18.4)
1130 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4)
1140 FORMAT(1H ,36X,5HA SUB,I3,2H =,F12.4,F12.3,F17.4,F18.4)
1150 FORMAT(1H ,36X,5HA SUB,I3,2H =,F12.4,
    *        6HCAP X=,F6.3,9X,8HINFINITY,F18.4)
1160 FORMAT(1H ,36X,5HA SUB,I3,2H =,F12.4)
1170 FORMAT(1H ,58X,F12.3,F17.4,F18.4)
1180 FORMAT(1H ,58X,6HCAP X=,F6.3,9X,8HINFINITY,F18.4)
     NP=26
     DO 60 I=2,NP
     GO TO (5,4),INDEX
   4 SRS(I)=SR(I)                        $ SCS(I)=SC(I)
     GO TO 50
   5 IF (I-5) 20,20,10
  10 IF (I-22) 30,30,40
  20 J=2*I-3                             $ XP(I)=0.01*J
     GO TO 50
  30 J=I-4                               $ XP(I)=0.05*J
     GO TO 50
  40 J=I-22                              $ XP(I)=0.9+0.02*J
  50 CALL SHAPE(XP(I),SR(I))             $ CALL VORTEX(XP(I),SC(I))
  60 CONTINUE
  65 PRINT 1000,IM,ID,IY                 $ PRINT 1010,ALPHA,CC
     PRINT 1020,NXS,NXG                  $ IF (NXS-NXG) 70,70,80
  70 M=NXS                               $ GO TO 90
  80 M=NXG
  90 PRINT 1030,(I,XS(I),I,XG(I),I=1,M)
     M=M+1                               $ IF (NXS-NXG) 100,120,110
 100 PRINT 1040,(I,XG(I),I=M,NXG)        $ GO TO 120
 110 PRINT 1050,(I,XS(I),I=M,NXS)
 120 PRINT 1060,ITER                     $ PRINT 1070
     IF (NXS-NXG) 130,140,140
 130 IF (NXG-NP) 150,150,160
 140 IF (NXS-NP) 150,150,170
 150 M=NP+1                              $ GO TO 180
 160 M=NXG                               $ GO TO 180
 170 M=NXS
 180 KOUNT=1                             $ IF (ITER) 181,181,182
 181 TEMP=SQRTF(CC)                      $ PRINT 1071,B(1),A(1),TEMP
     GO TO 183
 182 PRINT 1072,B(1),A(1)
 183 DO 390 I=2,M
     IF (I-NXS) 190,190,220
 190 IF (I-NXG) 200,200,210
 200 IF (I-NP) 250,250,260
 210 IF (I-NP) 290,290,300
 220 IF (I-NXG) 230,230,240
 230 IF (I-NP) 330,330,340
 240 IF (I-NP) 370,370,380
 250 PRINT 1080,I,B(I),I,A(I),XP(I),SR(I),SC(I)
     GO TO 390
 260 GO TO (270,280),KOUNT
 270 PRINT 1090,I,B(I),I,A(I),CAPX,TEMP
     KOUNT=2                             $ GO TO 390
 280 PRINT 1100,I,B(I),I,A(I)            $ GO TO 390
 290 PRINT 1110,I,B(I),XP(I),SR(I),SC(I)
     GO TO 390
 300 GO TO (310,320),KOUNT
 310 PRINT 1120,I,B(I),CAPX,TEMP
     KOUNT=2                             $ GO TO 390
 320 PRINT 1130,I,B(I)                   $ GO TO 390
 330 PRINT 1140,I,A(I),XP(I),SR(I),SC(I)
     GO TO 390
 340 GO TO (350,360),KOUNT
```

```
350 PRINT 1150,I,A(I),CAPX,TEMP
    KOUNT=2                                 $ GO TO 390
360 PRINT 1160,I,A(I)                       $ GO TO 390
370 PRINT 1170,XP(I),SR(I),SC(I)            $ GO TO 390
380 PRINT 1180,CAPX,TEMP
390 CONTINUE
    END


    SUBROUTINE QPNHALF(Z,QPH,QMH)
    TYPE REAL KPRIMESQ,K
    KPRIMESQ=1.0-(2.0/(Z+1.0))              $ A1=KPRIMESQ
    A2=A1*A1                                $ A3=A2*A1
    A4=A2*A2                                $ ALO=LOGF(1.0/KPRIMESQ)
    K=SQRTF(2.0/(Z+1.0))                    $ B=SQRTF(2.0*(Z+1.0))
    ELE=1.0000000000+.4432514l463*A1+.06260601220*A2
   *                 +.04757383546*A3+.01736506451*A4+
   *                 (.24998368310*A1+.09200180037*A2+
   *                  .04069697526*A3+.00526449639*A4)*ALO
    ELK=1.38629436112+.09666344259*A1+.03590092383*A2
   *                 +.03742563713*A3+.01451196212*A4+
   *     (.5000000000+.12498593597*A1+.06880248576*A2
   *                 +.03328355346*A3+.00441787012*A4)*ALO
    QPH=Z*K*ELK-B*ELE                       $ QMH=K*ELK
    END


    SUBROUTINE SHAPE(X,S)
    EXTERNAL FJ
    COMMON/ADDED/FZ(50)
    COMMON/COEFS/A(50),B(50)
    COMMON/INPUT/CC,ALPHA,CAPX
    COMMON/NUMBERXS/NXS,NXG
    S=1.0                                   $ BOT=CAPX-X
    DO 10 J=1,NXS
    S=S+(FJ(X,J)/BOT-FZ(J)/CAPX)*B(J)
 10 CONTINUE
    END


    SUBROUTINE TOPBOT(ZZ)
    COMMON/INPUT/CC,ALPHA,CAPX
    COMMON/LIMITS/TOP(5),BOT(5),NGD(5)
    TEMP=CAPX-0.12
    DO 10 I=1,5
    TOP(I)=BOT(I)=0.0                       $ NGD(I)=1
 10 CONTINUE
    IF (ABSF(ZZ)-0.0000001) 70,70,20
 20 IF (ZZ) 70,70,30
 30 IF (ABSF(ZZ-0.12)-0.0000001) 120,120,40
 40 IF (ZZ-0.12) 80,120,50
 50 IF (ABSF(ZZ-TEMP)-0.0000001) 120,120,60
 60 IF (ZZ-TEMP) 120,120,130
 70 TOP(1)=BOT(2)=0.1                       $ TOP(2)=BOT(3)=CAPX-0.1
    TOP(3)=CAPX                             $ NGD(2)=2
    GO TO 170
 80 IF (ZZ-0.02) 90,90,100
 90 EPS=ZZ                                  $ GO TO 110
100 EPS=0.02
110 TOP(1)=BOT(2)=ZZ-EPS                    $ TOP(2)=BOT(3)=ZZ+EPS
    TOP(3)=BOT(4)=CAPX-0.1                  $ TOP(4)=CAPX
    NGD(2)=3                                $ GO TO 170
```

```
120 TOP(1)=BOT(2)=0.10                    $ TOP(2)=BOT(3)=ZZ-0.02
    TOP(3)=BOT(4)=ZZ+0.02                 $ TOP(4)=BOT(5)=CAPX-0.1
    TOP(5)=CAPX                           $ NGD(3)=3
    GO TO 170
130 IF (TEMP-0.02) 140,140,150
140 EPS=TEMP                              $ GO TO 160
150 EPS=0.02
160 TOP(1)=BOT(2)=0.1                     $ TOP(2)=BOT(3)=ZZ-EPS
    TOP(3)=BOT(4)=ZZ+EPS                  $ TOP(4)=CAPX
    NGD(3)=3
170 END


    SUBROUTINE VORTEX(X,V)
    EXTERNAL GJ
    COMMON/COEFS/A(50),B(50)
    COMMON/INPUT/CC,ALPHA,CAPX
    V=SQRTF(CC)*(1.0+A(1)*GJ(X))
    END
       END         ROTORIGE
       FINIS
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      PROGRAM ROTOROGE
C
C     ROTOR OUT OF GROUND EFFECT (NAS1-6349)
C
      EXTERNAL FJ
      TYPE REAL LHS,LAMBDA
      COMMON/ADDED/FZ(50),TI
      COMMON/COEFS/A(50),B(50)
      COMMON/INPUT/LAMBDA,CC,ALPHA,BETA
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      COMMON/MATRIX/RHS(50,50),LHS(50,1)
      COMMON/NUMBERXS/NXS,NXG
      COMMON/PRINT/XS(50),XG(50),XP(50),SR(50),SC(50),ITER,IM,ID,IY
      COMMON/SAVE/NP,INDEX,SRS(50),SCS(50)
 1000 FORMAT(4I5)
 1010 FORMAT(10F8.5)
 1020 FORMAT(1H0,28HINITIAL CONDITIONS - AT MOST,I3,14H ITERATION(S) ,
     *       18HWILL BE ATTEMPTED.)
 1030 FORMAT(1H0,19HUNIFORM ACCURACY OF,F8.5,16H PERCENT IS NOT ,
     *       47HATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION ,
     *       14HDISTRIBUTIONS.,/,8H AT MOST,I3,19H MORE ITERATION(S) ,
     *       18HWILL BE ATTEMPTED.)
 1040 FORMAT(1H0,47HRESULTS OF THIS ITERATION INDICATE AN IMPROPER ,
     *       42HSTREAMLINE SHAPE - ITERATION IS TERMINATED)
 1050 FORMAT(1H0,47HRESULTS OF THIS ITERATION INDICATE AN IMPROPER ,
     *       50HCIRCULATION DISTRIBUTION - ITERATION IS TERMINATED)
 1060 FORMAT(1H0,19HUNIFORM ACCURACY OF,F8.5,12H PERCENT IS ,
     *       47HATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION ,
     *       14HDISTRIBUTIONS.)
 1070 FORMAT(1H0,19HUNIFORM ACCURACY OF,F8.5,16H PERCENT IS NOT ,
     *       47HATTAINED FOR SLIPSTREAM RADIUS AND CIRCULATION ,
     *       14HDISTRIBUTIONS.)
 1080 FORMAT(1H ,47HITERATION IS TERMINATED. DYNAMIC AND KINEMATIC ,
     *       41HCONDITIONS ON STREAMLINE WILL BE CHECKED.)
      READ 1000,IM,ID,IY
   10 READ 1000,MM,NN,NITER              $ IF (MM) 190,190,20
   20 READ 1010,CC,LAMBDA,ACC
      READ 1010,(XS(I),I=1,MM)           $ READ 1010,(XG(I),I=1,NN)
      NXS=MM                             $ NXG=NN
      ALPHA=1.8                          $ BETA=0.0
      DO 30 I=1,NXS
      B(I)=0.0                           $ FZ(I)=FJ(0.0,I)
   30 CONTINUE
      DO 40 I=1,NXG
      A(I)=0.0
   40 CONTINUE
      CALL SHAPE(2,0,0,TI)               $ FI=CC-(CC/(2.0*TI))**2
      GI=SQRTF(LAMBDA**2+FI)-LAMBDA
      ITER=0                             $ ACCP=100.0*ACC
      INDEX=1                            $ CALL OUTPUT
      M=NP+1                             $ SC(M)=SCS(M)=GI
      INDEX=2                            $ PRINT 1020,NITER
      DO 130 ITER=1,NITER
      TIS=TI                             $ N=NXS
      DO 50 I=1,N
      ZZ=XS(I)                           $ CALL TOPBOT(ZZ)
      CALL LIMITCHK                      $ CALL BEQNS(I)
   50 CONTINUE
      CALL MATINV(RHS,N,LHS,1,DET)
      DO 60 J=1,N
```

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      B(J)=LHS(J,1)
   60 CONTINUE
      CALL SHAPE(2.0,0.TI)              $ FI=CC-(CC/(2.0*TI))**2
      GI=SQRTF(LAMBDA**2+FI)-LAMBDA     $ N=NXG
      DO 70 I=1,N
      ZZ=XG(I)                          $ CALL TOPBOT(ZZ)
      CALL LIMITCHK                     $ CALL AEQNS(I)
   70 CONTINUE
      CALL MATINV(RHS,N,LHS,1,DET)
      DO 80 J=1,N
      A(J)=LHS(J,1)
   80 CONTINUE
      CALL OUTPUT
      CALL SHAPE(1,0.05,TTEST)          $ IF (TTEST-1.0) 90,140,140
   90 CALL VORTEX(0.01,GI,GTEST)        $ IF (GTEST-GI) 150,150,100
  100 SR(M)=TI                          $ SRS(M)=TIS
      DO 102 I=2,M
      CACC=ACC*SC(I)                    $ TACC=ACC*SR(I)
      IF (ABSF(SC(I)-SCS(I))-CACC) 101,101,110
  101 IF (ABSF(SR(I)-SRS(I))-TACC) 102,102,110
  102 CONTINUE
      GO TO 160
  110 LEFT=NITER-ITER                   $ IF (LEFT) 170,170,120
  120 PRINT 1030,ACCP,LEFT
  130 CONTINUE
  140 PRINT 1040                        $ GO TO 10
  150 PRINT 1050                        $ GO TO 10
  160 PRINT 1060,ACCP                   $ GO TO 180
  170 PRINT 1070,ACCP                   $ GO TO 180
  180 PRINT 1080                        $ CALL DYKICHCK
      GO TO 10
  190 END



      SUBROUTINE AEQNS(I)
      EXTERNAL GJ
      TYPE REAL LHS,LAMBDA
      COMMON/INPUT/LAMBDA,CC,ALPHA,BETA
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      COMMON/LIMITS/TOP(7),BOT(7),NGD(7)
      COMMON/MATRIX/RHS(50,50),LHS(50,1)
      DIMENSION XI(400),GTTR(400)
      DIMENSION R(5),U(5)
      DATA (R=0.1477621124,0.1346333597,0.1095431813,0.07472567458,
     *        0.03333567215),
     *     (U=0.0744371695,0.2166976971,0.3397047841,0.4325316833,
     *        0.4869532643)
      CALL SHAPE(1,ZZ,TT)               $ CALL FACTOR(ZZ,FAC)
      CALL VORTEX(ZZ,GI,GAM)            $ CON=2.0*GAM**2*FAC
      FF=CC-(CC/(2.0*TT))**2
      NN = N+1
      KK = 0
      DO 70 J=1,7
      IF(ABSF(TOP(J)-BOT(J))-0.0000001)70,70,10
   10 NTIME = NGD(J)
      B = TOP(J)   $   A = BOT(J)
      DO 60 II=1,NTIME
      XL = A+(II-1)*(B-A)/NTIME   $   XU = B-(NTIME-II)*(B-A)/NTIME
      D = XU-XL   $    S = (XU+XL)/2.0
      DO 50 L=1,10
```

```
      KK = KK+1
      IF(L-5)20,20,30
   20 JJ = L      $     XI(KK) = S+D*U(JJ)
      GO TO 40
   30 JJ = 11-L     $      XI(KK) = S-D*U(JJ)
   40 CALL SHAPE(1,XI(KK),T)
      CALL FACTOR(XI(KK),ROOT)
      CALL GEES(1,TT,T,ZZ,XI(KK),G,GT,GTT)
      GTTR(KK) = D*R(JJ)*GTT*ROOT
   50 CONTINUE
   60 CONTINUE
   70 CONTINUE
      DO 140 J=1,NN
      K = J      $    IF(J-NN)80,90,90
   80 RHS(I,J)=FF*TT*GJ(ZZ,J)/CON       $ GO TO 100
   90 RHS(I,J)=TT*(LAMBDA+FF*GI/CON-FF/(FAC*GAM))/GI
  100 DO 130 L=1,KK
      IF(J-NN)110,120,120
  110 RHS(I,J) = RHS(I,J)+GTTR(L)*GJ(XI(L),J)
      GO TO 130
  120 RHS(I,J) = RHS(I,J)+GTTR(L)
  130 CONTINUE
  140 CONTINUE
      LHS(I,1)=-RHS(I,NN)
      END



      SUBROUTINE BFONS(I)
      EXTERNAL BINT,FJ
      TYPE REAL LHS,LAMBDA
      COMMON/ADDED/FZ(50),TI
      COMMON/INPUT/LAMBDA,CC,ALPHA,BETA
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      COMMON/MATRIX/RHS(50,50),LHS(50,1)
      NN=N+1                              $ CALL SHAPE(1,ZZ,TT)
      GTTG=GGM=0.0
      NGEES=1                             $ CALL ONEINTGL(BINT,GTTG)
      NGEES=3                             $ CALL ONEINTGL(BINT,GGM)
      ADD=GGM+ALPHA*(TT-1.0)*GTTG
      DO 30 J=1,NN
      K=J                                 $ IF (J-NN) 10,20,20
   10 CON=FJ(ZZ,J)-FZ(J)
      RHS(I,J)=LAMBDA*TT*CON+(LAMBDA+GI)*TI*FZ(J)+ALPHA*CON*GTTG
      GO TO 30
   20 RHS(I,J)=(LAMBDA+GI)*TI*(1.0-0.5*TI)-LAMBDA*TT*(1.0-0.5*TT)+ADD
   30 CONTINUE
      LHS(I,1)=RHS(I,NN)
      END



      FUNCTION BINT(Z)
      EXTERNAL FJ
      COMMON/ADDED/FZ(50),TI
      COMMON/INPUT/LAMBDA,CC,ALPHA,BETA
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      CALL SHAPE(1,Z,T)                   $ CALL VORTEX(Z,GI,GAM)
      CALL FACTOR(Z,FAC)                  $ GAM=FAC*GAM
      CALL GEES(NGEES,TT,T,ZZ,Z,G,GT,GTT)
      GO TO (10,30,20),NGEES
   10 BINT=GTT*GAM                        $ GO TO 30
   20 BINT=-G*GAM
   30 END
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      SUBROUTINE DYKICHCK
      EXTERNAL GTG,GGG
      TYPE REAL LAMBDA
      COMMON/INPUT/LAMBDA,CC,ALPHA,BETA
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      COMMON/NUMBERXS/NXS,NXG
      COMMON/PRINT/XS(50),XG(50),XP(50),SR(50),SC(50),ITER,IM,ID,IY
      DIMENSION X(19)
      DATA (X=0.01,0.02,0.03,0.04,0.05,0.10,0.15,0.20,0.30,0.40,0.50,
     *        0.75,1.00,1.25,1.50,2.00,3.00,5.00,10.00)
 1000 FORMAT(1H1,22X,40HR O T O R   O U T   O F   G R O U N D   ,
     *       35HE F F E C T   ( N A S 1 - 6 3 4 9 ),5X,1H(,I2,1H/,I2,
     *       1H/,I2,1H))
 1010 FORMAT(1H0,35X,29HDAMPING COEFFICIENTS, ALPHA =,F5.2,9H , BETA =,
     *       F5.2,/,36X,
     *       24HLOADING COEFFICIENT, C =,F7.4,/,36X,
     *       23HADVANCE RATIO, LAMBDA =,F7.4,/,36X,
     *       36HNO. OF SHAPE COLLOCATION POINTS, M =,I3,/,36X,
     *       36HNO. OF GAMMA COLLOCATION POINTS, N =,I3,/,36X,
     *       16HHUB RADIUS = 0.0)
 1020 FORMAT(1H0,//,45H NUMERICAL CHECK OF DYNAMIC (FORCE-FREE) AND ,
     *       47HKINEMATIC (STREAMLINE) CONDITIONS ON SLIPSTREAM,//)
 1030 FORMAT(1H ,50X,8HDYNAMIC,,25X,10HKINEMATIC,,//,54X,
     *       20HU*GAMMA SHOULD EQUAL,11X,19HTOTAL (CAP) PSI ON ,
     •       10HSLIPSTREAM,/,9X,36HTOTAL X-VEL.    SLIPSTREAM VORTICITY,
     *       18X,3HF/2,15X,36HSHOULD EQUAL ITS VALUE AT X=INFINITY,
     *       /,8X,36HON SLIPSTREAM,    PER UNIT X-LENGTH,,9X,8H--------,
     *       14H--------------,5X,31H-------------------------------,
     *       7H-------,/,4H   X,9X,4HU(X),14X,8HGAMMA(X),17X,3HLHS,11X,
     *       3HRHS,17X,3HLHS,13X,3HRHS,/)
 1040 FORMAT(1H ,F5.2,F12.5,F20.5,F22.5,F14.5,F20.5,F16.5)
 1050 FORMAT(1H ,5H INF.,F12.5,F20.5,F22.5,F14.5,F20.5,F16.5)
      PRINT 1000,IM,ID,IY
      PRINT 1010,ALPHA,BETA,CC,LAMBDA,NXS,NXG
      PRINT 1020                        $ PRINT 1030
      CALL SHAPE(2.0,0,TI)              $ CB=0.5*TI**2*(LAMBDA+GI)
      DO 60 I=1,20
      IF (I-20) 20,10,10
   10 U=LAMBDA+0.5*GI                   $ GAM=GI
      CA=0.5*(CC-(CC/(2.0*TI))**2)
      PSI=CB                            $ GO TO 30
   20 ZZ=X(I)                           $ CALL SHAPE(1,ZZ,TT)
      CA=0.5*(CC-(CC/(2.0*TT))**2)
      CALL VORTEX(ZZ,GI,GAM)            $ CALL FACTOR(ZZ,FAC)
      GAM=FAC*GAM                       $ CALL TOPBOT(ZZ)
      CALL LIMITCHK                     $ U=LAMBDA
      PSI=0.5*LAMBDA*TT**2              $ CALL TWOINTGL(GTG,GGG,U,PSI)
   30 UG=U*GAM                          $ IF (I-20) 40,50,50
   40 PRINT 1040,ZZ,U,GAM,UG,CA,PSI,CB
      GO TO 60
   50 PRINT 1050,U,GAM,UG,CA,PSI,CB
   60 CONTINUE
      END


      SUBROUTINE FACTOR(X,FAC)
      EXTERNAL FPJ
      COMMON/COEFS/A(50),B(50)
      COMMON/NUMBERXS/NXS,NXG
      TP=0.0
      DO 10 I=1,NXS
      TP=TP+B(I)*FPJ(X,I)
   10 CONTINUE
      FAC=SQRTF(1.0+TP**2)
      END
```

```
      FUNCTION FJ(X,J)
      FJ=EXPF(-J*X)
      END


      FUNCTION FPJ(X,J)
      FPJ=-J*EXPF(-J*X)
      END


      SUBROUTINE GEES(NUM,TT,T,ZZ,Z,G,GT,GTT)
      PI=3.1415927                         $ TERM=SQRTF(TT*T)
      DEL=(ZZ-Z)**2                        $ ATOP=DEL+TT**2-T**2
      BTOP=DEL+T**2-TT**2                  $ ABOT=8.0*PI*TERM*T
      BBOT=8.0*PI*TERM*TT
      A=ATOP/ABOT                          $ B=BTOP/BBOT
      ARG=1.0+((TT-T)**2+(ZZ-Z)**2)/(2.0*TT*T)
      CALL QPMHALF(ARG,QPH,QMH)            $ GO TO (30,20,10),NUM
   10 G=TERM*QPH/(2.0*PI)
   20 GT=(B*QPH+A*QMH)/(ARG**2-1.0)
   30 GTT=(A*QPH+B*QMH)/(ARG**2-1.0)
      END


      FUNCTION GGG(Z)
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      CALL SHAPE(1,Z,T)                    $ CALL VORTEX(Z,GI,G)
      CALL FACTOR(Z,FAC)                   $ G=FAC*G
      CALL GEES(3,TT,T,ZZ,Z,GG,DUM,DUMM)
      GGG=GG*G
      END


      FUNCTION GJ(X,J)
      IF (J-1) 10,10,20
   10 GJ=EXPF(-3.0*X)/SQRTF(X)           $ GO TO 30
   20 GJ=X**(-0.84+0.51*J)*EXPF(-3.0*X)
   30 END


      FUNCTION GTG(Z)
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      CALL SHAPE(1,Z,T)                    $ CALL VORTEX(Z,GI,G)
      CALL FACTOR(Z,FAC)                   $ G=FAC*G
      CALL GEES(1,TT,T,ZZ,Z,DUM,DUMM,GTT)
      GTG=GTT*G/TT
      END


      SUBROUTINE LIMITCHK
      COMMON/LIMITS/TOP(7),BOT(7),NGD(7)
      DO 40 I=1,7
      IF (BOT(I)-TOP(I)) 40,40,10
   10 IF (I-7) 20,30,30
   20 BOT(I+1)=BOT(I)
   30 BOT(I)=TOP(I)
   40 CONTINUE
      END
```

```
      SUBROUTINE MATINV(A,N,B,M,DETERM)
      DIMENSION IPIVOT(50),A(50,50),B(50,1),INDEX(50,2),PIVOT(50)
      DETERM=1.0
      DO 10 J=1,N
      IPIVOT(J)=0
   10 CONTINUE
      DO 200 I=1,N
      AMAX=0.0
      DO 60 J=1,N
      IF (IPIVOT(J)-1) 20,60,20
   20 DO 50 K=1,N
      IF (IPIVOT(K)-1) 30,50,240
   30 IF (ABSF(AMAX)-ABSF(A(J,K))) 40,50,50
   40 IROW=J                          $ ICOLUM=K
      AMAX=A(J,K)
   50 CONTINUE
   60 CONTINUE
      IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1 $ IF (IROW-ICOLUM) 70,110,70
   70 DETERM=-DETERM
      DO 80 L=1,N
      SWAP=A(IROW,L)                  $ A(IROW,L)=A(ICOLUM,L)
      A(ICOLUM,L)=SWAP
   80 CONTINUE
      IF (M) 110,110,90
   90 DO 100 L=1,M
      SWAP=B(IROW,L)                  $ B(IROW,L)=B(ICOLUM,L)
      B(ICOLUM,L)=SWAP
  100 CONTINUE
  110 INDEX(I,1)=IROW
      INDEX(I,2)=ICOLUM               $ PIVOT(I)=A(ICOLUM,ICOLUM)
      DETERM=DETERM*PIVOT(I)          $ A(ICOLUM,ICOLUM)=1.0
      DO 120 L=1,N
      A(ICOLUM,L)=A(ICOLUM,L)/PIVOT(I)
  120 CONTINUE
      IF (M) 150,150,130
  130 DO 140 L=1,M
      B(ICOLUM,L)=B(ICOLUM,L)/PIVOT(I)
  140 CONTINUE
  150 DO 200 L1=1,N
      IF(L1-ICOLUM) 160,200,160
  160 T=A(L1,ICOLUM)                  $ A(L1,ICOLUM)=0.0
      DO 170 L=1,N
      A(L1,L)=A(L1,L)-A(ICOLUM,L)*T
  170 CONTINUE
      IF (M) 200,200,180
  180 DO 190 L=1,M
      B(L1,L)=B(L1,L)-B(ICOLUM,L)*T
  190 CONTINUE
  200 CONTINUE
      DO 230 I=1,N
      L=N+1-I
      IF (INDEX(L,1)-INDEX(L,2)) 210,230,210
  210 JROW=INDEX(L,1)                 $ JCOLUM=INDEX(L,2)
      DO 220 K=1,N
      SWAP=A(K,JROW)                  $ A(K,JROW)=A(K,JCOLUM)
      A(K,JCOLUM)=SWAP
  220 CONTINUE
  230 CONTINUE
  240 RETURN
      END
```

```
      SUBROUTINE NGAUSS(B,A,FX,NTIME,INTEGRAL)
      TYPE REAL INTEGRAL
      DIMENSION R(5),U(5)
      DATA (R=0.1477621124,0.1346333597,0.1095431813,0.07472567458,
     *        0.03333567215),
     *     (U=0.0744371695,0.2166976971,0.3397047841,0.4325316833,
     *        0.4869532643)
      INTEGRAL=0.0
      DO 20 J=1,NTIME
      XL=A+(J-1)*(B-A)/NTIME              $ XU=B-(NTIME-J)*(B-A)/NTIME
      D=XU-XL                             $ S=(XU+XL)/2.0
      TEMP=0.0
      DO 10 K=1,5
      TEMP=TEMP+R(K)*(FX(S+D*U(K))+FX(S-D*U(K)))
   10 CONTINUE
      TEMP=TEMP*D                         $ INTEGRAL=INTEGRAL+TEMP
   20 CONTINUE
      END


      SUBROUTINE ONEINTGL(FA,A)
      COMMON/LIMITS/TOP(7),BOT(7),NGD(7)
      DO 20 I=1,7
      IF (ABSF(TOP(I)-BOT(I))-0.0000001) 20,20,10
   10 CALL NGAUSS(TOP(I),BOT(I),FA,NGD(I),AA)
      A=A+AA
   20 CONTINUE
      END


      SUBROUTINE OUTPUT
      TYPE REAL LAMBDA
      COMMON/ADDED/FZ(50),TI
      COMMON/COEFS/A(50),B(50)
      COMMON/INPUT/LAMBDA,CC,ALPHA,BETA
      COMMON/INTGND/NGEES,TT,ZZ,N,K,GI
      COMMON/NUMBERXS/NXS,NXG
      COMMON/PRINT/XS(50),XG(50),XP(50),SR(50),SC(50),ITER,IM,ID,IY
      COMMON/SAVE/NP,INDEX,SRS(50),SCS(50)
 1000 FORMAT(1H1,22X,40HR O T O R   O U T   O F   G R O U N D   ,
     *        35HE F F E C T   ( N A S 1 - 6 3 4 9 ),5X,1H(,I2,1H/,I2,
     *        1H/,I2,1H))
 1010 FORMAT(1H0,35X,29HDAMPING COEFFICIENTS, ALPHA =,F5.2,9H , BETA =,
     *        F5.2,/,36X,
     *        24HLOADING COEFFICIENT, C =,F7.4,/,36X,
     *        23HADVANCE RATIO, LAMBDA =,F7.4,/,36X,
     *        16HHUB RADIUS = 0.0)
 1020 FORMAT(1H0,21X,29HSHAPE COLLOCATION POINTS, M =,I3,12X,
     *        29HGAMMA COLLOCATION POINTS, N =,I3,/)
 1030 FORMAT(1H ,28X,5HX SUB,I3,2H =,F8.4,26X,5HX SUB,I3,2H =,F8.4)
 1040 FORMAT(1H ,72X,5HX SUB,I3,2H =,F8.4)
 1050 FORMAT(1H ,28X,5HX SUB,I3,2H =,F8.4)
 1060 FORMAT(1H0,13HITERATION NO.,I3,/)
 1070 FORMAT(1H ,79X,35HSLIPSTREAM   SLIPSTREAM CIRCULATION,/,7X,
     *        18HSHAPE COEFFICIENTS,14X,18HGAMMA COEFFICIENTS,11X,1HX,
     *        11X,30HRADIUS, T              GAMMA SUB S,/)
 1071 FORMAT(1H ,4X,10HB SUB  1 =,F12.4,10X,10HA SUB  1 =,F12.4,8X,
     *        4H.000,11X,6H1.0000,F18.4)
 1072 FORMAT(1H ,4X,10HB SUB  1 =,F12.4,10X,10HA SUB  1 =,F12.4,8X,
     *        4H.000,11X,6H1.0000,10X,8HINFINITY)
 1080 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,10X,5HA SUB,I3,2H =,F12.4,
     *        F12.3,F17.4,F18.4)
 1090 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,10X,5HA SUB,I3,2H =,F12.4,5X,
     *        8HINFINITY,F16.4,F18.4)
```

```
1100 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,10X,5HA SUB,I3,2H =,F12.4)
1110 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,32X,F12.3,F17.4,F18.4)
1120 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4,37X,8HINFINITY,F16.4,F18.4)
1130 FORMAT(1H ,4X,5HB SUB,I3,2H =,F12.4)
1140 FORMAT(1H ,36X,5HA SUB,I3,2H =,F12.4,F12.3,F17.4,F18.4)
1150 FORMAT(1H ,36X,5HA SUB,I3,2H =,F12.4,5X,8HINFINITY,F16.4,F18.4)
1160 FORMAT(1H ,36X,5HA SUB,I3,2H =,F12.4)
1170 FORMAT(1H ,58X,F12.3,F17.4,F18.4)
1180 FORMAT(1H ,63X,8HINFINITY,F16.4,F18.4)
     NP=30
     DO 60 I=2,NP
     GO TO (5,4),INDEX
   4 SRS(I)=SR(I)                     $ SCS(I)=SC(I)
     GO TO 55
   5 IF (I-5) 30,30,10
  10 IF (I-10) 40,40,20
  20 IF (I-17) 45,45,25
  25 IF (I-25) 50,50,51
  30 J=2*I-3                          $ XP(I)=0.01*J
     GO TO 55
  40 J=I-4                            $ XP(I)=0.05*J
     GO TO 55
  45 J=I-7                            $ XP(I)=0.10*J
     GO TO 55
  50 J=I-13                           $ XP(I)=0.25*J
     GO TO 55
  51 J=I-22                           $ XP(I)=1.00*J
  55 CALL SHAPE(I,XP(I),SR(I))        $ CALL VORTEX(XP(I),GI,SC(I))
  60 CONTINUE
  65 PRINT 1000,IM,ID,IY              $ PRINT 1010,ALPHA,BETA,CC,LAMBDA
     PRINT 1020,NXS,NXG               $ IF (NXS-NXG) 70,70,80
  70 M=NXS                            $ GO TO 90
  80 M=NXG
  90 PRINT 1030,(I,XS(I),I,XG(I),I=1,M)
     M=M+1                            $ IF (NXS-NXG) 100,120,110
 100 PRINT 1040,(I,XG(I),I=M,NXG)     $ GO TO 120
 110 PRINT 1050,(I,XS(I),I=M,NXS)
 120 PRINT 1060,ITER                  $ PRINT 1070
     IF (NXS-NXG) 130,140,140
 130 IF (NXG-NP) 150,150,160
 140 IF (NXS-NP) 150,150,170
 150 M=NP+1                           $ GO TO 180
 160 M=NXG                            $ GO TO 180
 170 M=NXS
 180 KOUNT=1                          $ IF (ITER) 181,181,182
 181 PRINT 1071,B(1),A(1),GI          $ GO TO 183
 182 PRINT 1072,B(1),A(1)
 183 DO 390 I=2,M
     IF (I-NXS) 190,190,220
 190 IF (I-NXG) 200,200,210
 200 IF (I-NP) 250,250,260
 210 IF (I-NP) 290,290,300
 220 IF (I-NXG) 230,230,240
 230 IF (I-NP) 330,330,340
 240 IF (I-NP) 370,370,380
 250 PRINT 1080,I,B(I),I,A(I),XP(I),SR(I),SC(I)
     GO TO 390
 260 GO TO (270,280),KOUNT
 270 PRINT 1090,I,B(I),I,A(I),TI,GI
     KOUNT=2                          $ GO TO 390
 280 PRINT 1100,I,B(I),I,A(I)         $ GO TO 390
 290 PRINT 1110,I,B(I),XP(I),SR(I),SC(I)
     GO TO 390
 300 GO TO (310,320),KOUNT
 310 PRINT 1120,I,B(I),TI,GI
     KOUNT=2                          $ GO TO 390
```

```
320 PRINT 1130,I,B(I)                    $ GO TO 390
330 PRINT 1140,I,A(I),XP(I),SR(I),SC(I)
    GO TO 390
340 GO TO (350,360),KOUNT
350 PRINT 1150,I,A(I),TI,GI
    KOUNT=2                              $ GO TO 390
360 PRINT 1160,I,A(I)                    $ GO TO 390
370 PRINT 1170,XP(I),SR(I),SC(I)         $ GO TO 390
380 PRINT 1180,TI,GI
390 CONTINUE
    END

    SUBROUTINE QPMHALF(Z,QPH,QMH)
    TYPE REAL KPRIMESQ,K
    KPRIMESQ=1.0-(2.0/(Z+1.0))           $ A1=KPRIMESQ
    A2=A1*A1                             $ A3=A2*A1
    A4=A2*A2                             $ ALO=LOGF(1.0/KPRIMESQ)
    K=SQRTF(2.0/(Z+1.0))                 $ B=SQRTF(2.0*(Z+1.0))
    ELE=1.00000000000+.44325141463*A1+.06260601220*A2
   *                  +.04757383546*A3+.01736506451*A4+
   *                  (.24998368310*A1+.09200180037*A2+
   *                   .04069697526*A3+.00526449639*A4)*ALO
    ELK=1.386294361 12+.09666344259*A1+.03590092383*A2
   *                  +.03742563713*A3+.01451196212*A4+
   *     (.50000000000+.12498593597*A1+.06880248576*A2
   *                  +.03328355346*A3+.00441787012*A4)*ALO
    QPH=Z*K*ELK-B*ELE                    $ QMH=K*ELK
    END

    SUBROUTINE SHAPE(KODE,X,S)
    EXTERNAL FJ
    COMMON/ADDED/FZ(50),TI
    COMMON/COEFS/A(50),B(50)
    COMMON/NUMBERXS/NXS,NXG
    GO TO (10,30),KODE
 10 S=TI
    DO 20 J=1,NXS
    S=S+B(J)*FJ(X,J)
 20 CONTINUE
    GO TO 50
 30 S=1.0
    DO 40 J=1,NXS
    S=S-B(J)*FZ(J)
 40 CONTINUE
 50 END

    SUBROUTINE TOPBOT(ZZ)
    COMMON/LIMITS/TOP(7),BOT(7),NGD(7)
    DO 10 I=1,7
    TOP(I)=BOT(I)=0.0                    $ NGD(I)=1
 10 CONTINUE
    NGD(4)=3
    IF (ABSF(ZZ)-0.0000001) 80,80,20
 20 IF (ZZ) 70,80,30
 30 IF (ABSF(ZZ-0.2)-0.0000001) 100,100,40
 40 IF (ZZ-0.2) 100,100,50
 50 IF (ABSF(ZZ-3.0)-0.0000001) 140,140,60
 60 IF (ZZ-3.0) 140,140,150
 70 NGD(4)=2                             $ EPS=0.1
    GO TO 90
 80 EPS=0.05
```

```
 90 TOP(4)=BOT(5)=EPS                        $ TOP(5)=BOT(6)=1.0
    TOP(6)=BOT(7)=10.0                       $ TOP(7)=100.0
    GO TO 180
100 IF (ZZ-0.05) 110,110,120
110 EPS=ZZ                                   $ GO TO 130
120 EPS=0.05
130 TOP(3)=BOT(4)=ZZ-EPS                     $ TOP(4)=BOT(5)=ZZ+EPS
    GO TO 170
140 TOP(2)=BOT(3)=0.15                       $ TOP(3)=BOT(4)=ZZ-0.05
    GO TO 160
150 TOP(1)=BOT(2)=0.15                       $ TOP(2)=BOT(3)=ZZ-1.0
    TOP(3)=BOT(4)=ZZ-0.05
160 TOP(4)=BOT(5)=ZZ+0.05
170 TOP(5)=BOT(6)=ZZ+1.0                     $ TOP(6)=BOT(7)=ZZ+10.0
    TOP(7)=ZZ+100.0
180 END


    SUBROUTINE TWOINTGL(FA,FB,A,B)
    COMMON/LIMITS/TOP(7),BOT(7),NGD(7)
    DO 20 I=1,7
    IF (ABSF(TOP(I)-BOT(I))-0.0000001) 20,20,10
 10 CALL NGAUSS(TOP(I),BOT(I),FA,NGD(I),AA)
    CALL NGAUSS(TOP(I),BOT(I),FB,NGD(I),BB)
    A=A+AA                                   $ B=B+BB
 20 CONTINUE
    END


    SUBROUTINE VORTEX(X,GI,V)
    EXTERNAL GJ
    COMMON/COEFS/A(50),B(50)
    COMMON/NUMBERXS/NXS,NXG
    V=1.0
    DO 10 J=1,NXG
    V=V+A(J)*GJ(X,J)
 10 CONTINUE
    V=GI*V
    END
       END          ROTOROGE
       FINIS
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